Asymptotic study of regular planar graphs

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Chapter 1

Introduction

In this thesis we study enumerative properties of regular planar graphs and that of their embeddings on the sphere, usually called maps. Both topics are active areas in discrete mathematics with many important applications in physics, algorithmic, probability and algebraic geometry, among other disciplines. The language we will use is the one introduced by P. Flajolet and R. Sedgewick in their reference book Analytic Combinatorics [25].

There are many ways to enumerate discrete combinatorial structures. One can do it directly by finding a closed formula, by exhibiting a bijection with another class of structures, or if it holds, by exploiting its recursive nature, see [31, 62]. Those methods become very natural when seen through the prism of the symbolic method: to each class is associated a purely algebraic object, a generating function whose n-th coefficient counts the number of objects of size n. Combinatorial relations between classes are then directly translated into algebraic operations between their associated generating functions. Enumeration can then be understood as the computation of those coefficients. In the absence of a closed formula, one can try to design a recursive decomposition scheme to enumerate a combinatorial class via a bootstrapping-type method. In particular, when a generating function is defined implicitly as the solution of a Langrangian equation, a polynomial equation or a linear differential equation with polynomial coefficients, there exists effective methods for computing its coefficients.

A uniformly at random discrete structure of a certain size is the typical object taken uniformly at random among all the other objects with the same size in the class. In probabilistic combinatorics, one is interested in predicting the properties of such a typical object. In fact, as the sizes of the objects grow close to infinity, those predictions become deterministic in some sense. The asymptotic study of a random discrete structure is then the study of a typical object in the class, as its size goes to infinity. A classical method to compute the probability that a property holds asymptotically is to study the associated probability generating function. The symbolic method reveals then its usefulness when one considers generating functions as analytic objects, namely functions of the complex variable, so that one can use powerful methods coming from complex analysis. In particular, when such a function is analytic in a neighbourhood of the origin, one can estimate its coefficients via the analysis of its singularities. This thesis is organised as follows.

First, in Chapter 2 we will introduce the objects under study as well as some of the main notions of graph theory, enumeration the language of the so-called Symbolic Method. We will also introduce the main tools originating from complex analysis, mostly Singularity Analysis, that we will use to obtain asymptotic and probabilistic results. Finally, we will discuss the central graph-theoretic tools developped throughout this thesis, namely the recursive decompositions of a class of graph into smaller subclasses in which the graphs have higher connectivity.

In Chapter 3, we will then apply the connectivity-decomnpositon method in pair with singularity analysis to the enumeration of the family of labelled cubic planar graphs. The enumeration of labelled planar graphs has recently been the subject of much research; see [48] and [49] for two surveys on the area. While the enumeration of subclasses of planar graphs, such as trees, series-parallel graphs, etc., and more generally so-called subcritical classes of graphs is well understood and was solved several years ago [21, 30, 56], the problem of counting planar graphs was recently solved by Giménez and Noy [28, 29]. And that of the enumeration of cubic planar graphs was partially solved by Bodirsky, Kang, Löffler and McDiarmid [7].

On the other hand, the enumeration of cubic graphs that non-necessarily planar has a long history. Bruce Read [58] found a formula for the number of labelled cubic graphs using recurrence relations. Later Wormald [72] found a second order differential equation satisfied by their associated exponential generating function. Wormald's method cannot be adapted to the enumeration of cubic planar graphs, since one of the operations needed in his analysis does not preserve planarity.

In this chapter, we will instead apply the method introduced by Bodirsky, Kang, Löffler and McDiarmid in [7], and based on the decomposition of graphs in term of their connected components. We will then revisit and extend their work. In particular, we will exhibit full asymptotic estimates for the number of arbitrary, connected, 2-connected and 3-connected cubic planar graphs. It is noteworthy to point that the estimate for arbitrary graphs requires a special tool, that did not fully exist yet at the time of [7], the so-called *Dissymetry Theorem for tree-decomposable classes*. Then, using the same approach, we also give full asymptotic estimates on the number of arbitrary and connected cubic planar *multigraphs*.

Concerning the spherical embeddings of cubic planar graphs, Gao and Wormald enumerated three classes of simple cubic planar maps in [27]: arbitrary, 2-connected and triangle-free 3-connected. Their proofs are based on the duality between cubic maps and arbitrary triangulations (containing possibly loops and multiple edges), and on the decomposition of 3-connected triangulations from 4-connected triangulations. It is mentioned in [27] that it would be very interesting to find an alternative approach, specially since some of the results rely on heavy computations with `Maple`. We will hence reprove the first two results from [27], using a different scheme based on the connectivity-decompositon approach developed for the enumeration of cubic planar graphs.

Consequently, in Chapter 4 we will apply our asymptotic results to the analysis of random cubic planar graphs according to the uniform distribution. More precisely, let $\mathcal{G}$ be the class of labelled cubic planar graphs. For every $n \in \mathbb{N}$ we then denote by $\mathcal{G}_n$ the subclass of $\mathcal{G}$ consisting of the graphs with n vertices. If we now pick a graph uniformly at random in $\mathcal{G}_n$, then each graph has the same probability $1/g_n$ of being picked, where $g_n = |\mathcal{G}_n|$ is the number of labelled cubic planar graphs with n vertices. This random model was first analysed in [7]. In this chapter, we will extend their work and shed more light on the structure of random cubic planar graphs. We will in particular prove that the sequences of random variables $(A_n)_{n\geq 0}$, $(B_n)_{n\geq 0}$ and $(C_n)_{n\geq 0}$, respectively associated to the number of *cut-vertices*, *isthmuses* and *blocks* in a (uniform) random graph in $\mathcal{G}_n$, are all normally distributed with linear expectation and variance. A subgraph of a cubic planar graph is said to be *quasi-cubic* if at most two of its vertices have degree less than three. Next, we will look at the distributions of the sequences of random variables associated to the number of appearances of two special types of quasi-cubic subgraphs, namely the *cherries* and the *near-bricks*. Where a *cherry* is a planar graph in which all vertices have degree 3 except for one distinguished vertex of degree

1, and a *near-brick* as a graph obtained from a 3-connected cubic planar graph by removing one edge. We will also prove that both sequences are normally distributed. And finally, we will show that the number of triangles both in random arbitrary and 3-connected cubic planar graphs is distributed following a Gaussian limit law.

The second topic of this thesis is then developed in Chapter 5, where we will present the first combinatorial scheme for counting labelled 4-regular planar graphs through a complete recursive decomposition. More precisely, we will show that the exponential generating function of labelled 4-regular planar graphs can be computed effectively as the solution of a system of equations, from which the coefficients can be extracted. As a byproduct, we can also enumerate labelled 3-connected 4-regular planar graphs, and simple 4-regular rooted planar maps.

There are several references on the exhaustive *generation* of 4-regular planar graphs. Starting with a collection of basic graphs one shows how to generate all graphs in a certain class starting from the basic pieces and applying a sequence of local modifications. This was first done for the class of 4-regular planar graphs by Lehel [42], using as basis the graph of the octahedron. For 3-connected 4-regular planar graphs a similar generation scheme was shown by Boersma, Duijvestijn and Göbel [12]; by removing isomorphic duplicates they were able to compute the numbers of 3-connected 4-regular planar graphs up to 15 vertices. It is also the approach of the more recent work by Brinkmann, Greenberg, Greenhill, McKay, Thomas and Wollan [11] for generating planar quadrangulations of several types. The authors of [11] use several enumerative formulas to check the correctness of their generation procedure. However this does not include the class of 3-connected quadrangulations, which by duality correspond to 3-connected 4-regular planar graphs, a class for which no enumeration scheme was known until now. As we shall see, this class is the key to the enumeration of labelled 4-regular planar graphs.

Scientific publications.
This thesis is based on papers written by the author and co-authored with some subsets of Michael Drmota, Marc Noy and Juanjo Rué. All these papers have been published, submitted or in preparation:

- [59] Triangles in random cubic planar graphs (extended abstract).
  Electronic Notes in Discrete Mathematics 49 (2015) 383-391.
  With Juanjo Rué.

- [51] Random cubic planar graphs revisited (extended abstract).
  Electronic Notes in Discrete Mathematics 54 (2016) 211-216.
  With Marc Noy and Juanjo Rué.

- [52] Enumeration of labelled 4-regular planar graphs
  (extended abstract).
  Electronic Notes in Discrete Mathematics 61 (2017) 933-939.
  With Marc Noy and Juanjo Rué.

- [53] Enumeration of labelled 4-regular planar graphs.
  Accepted for publication in *Proceedings of the London Mathematical Society (2019).*
  With Marc Noy and Juanjo Rué.

- [54] Further results on random cubic planar graphs.
  Accepted for publication in *Random Structures and Algorithms (2019).*
  With Marc Noy and Juanjo Rué.

- [22] A unified approach to the enumeration of cubic planar maps
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  With Michael Drmota, Marc Noy and Juanjo Rué.

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Chapter 2

Preliminaries

2.1 Planar graphs

As notations in *Graph Theory* are not uniform throughout literature, we recall and fix here some basic definitions that we will use in the rest of this thesis. Our main reference for this part is the first chapter of Reinhard Diestel's book [19].

2.1.1 Basic definitions

Labelled graphs. A *labelled graph* G is defined by a pair $(V(G), E(G))$, where $V = V(G)$ is the vertex set of G and $E = E(G)$ is the edge set of G. We always consider finite graphs, and we use the set $[n] = \{1, 2, \ldots, n\}$ to denote the set of vertices of G. An *unlabelled graph* is a class of labelled graphs up to permutations of the labels of the vertex set.

An edge $e \in E$ is encoded by a pair of vertices $\{u, v\}$ (non-necessarily distinct) in V. In this case, u and v are said to be the *endpoints* of e. An edge with the same endpoints is called a *loop*.

A *simple graph* is a graph without loops. An edge is said to be *directed* when it is encoded by an ordered pair of vertices of V. A directed edge $e = (u, v) \in E$ of G can also be written as $e = uv$. A *multigraph* is a graph whose edge set is in fact a mutliset, thus allowing several repetitions of the same edge which will then be called a parallel edge. The number of repetitions of a parallel edge is called its *multiplicity*.

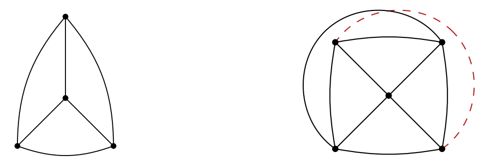


Figure 2.1: Two examples of complete graphs. One is cubic and planar, one is 4-regular and non-planar.

Connectivity. A graph G is *connected* if there exists a path between each pair of vertices. Every connected *maximal* subgraph of G is a *connected component* of G. A graph G is said to be *k-connected* if $|V| > k$ and if for every set $X \subset V$ with $|X| < k$, $G - X$ is connected. A set of vertices $\{v_1, v_2, \ldots, v_k\}$ which disconnects a k-connected graph G, which is not $(k+1)$-connected, is called a *k-cut* of G.

If $k = 1$, this set is also called a *cut vertex* of G. A *cut-edge* or *isthmus* is an edge e of G such that $G - e$ is disconnected. A *connected component* of G is a *maximal* connected subgraph in the sense that no other connected subgraph of G contains it. A *block* (resp. a *brick*) is then a maximal 2-connected (resp. 3-connected) subgraph of G. Even though edges are not 2-connected, we consider them as blocks when they are not contained in another block.

Regularity. The *degree* of a vertex v is denoted by $deg(v)$. A notable result, linking the sum of the degrees to the number of edges in a graph, is known as the *Handshaking Lemma* and is obtained by double counting

$$\sum_{v \in V(G)} deg(v) = 2|E(G)|.$$

A graph is said to be *k-regular* when all its vertices have degree k. In particular, a graph is called *cubic* when it is 3-regular. Two examples, one of a cubic graph and one of a 4-regular graph are given in Figure 2.1.

Interesting families of graphs. Let us introduce some specific families of graphs. The *cycle* with n vertices is denoted by C_n. A connected graph without cycles is called a *tree*. A vertex of degree 1 in tree is called a *leaf*. A *tree* with a distinguished vertex (resp. edge) is called a vertex-rooted (resp.

edge-rooted) tree, and the distinguished vertex (resp. edge) is called the root of the tree. The *complete graph* on n vertices is denoted by K_n. A graph is said to be *planar* when it can be drawn on the *sphere* without *edge-crossing*. In Figure 2.1 are pictured the complete graph on 4 vertices, K_4 which is planar, and the complete graph on 5 vertices, K_5 which is not: the dashed red edge is crossing another edge.

2.1.2 Planarity

Our main references for this part are the monograph of Mohar and Thomassen [45] and the book of Lando and Zvonkin [41].

Maps on surfaces. A *surface* is a compact (bounded and closed) connected 2-manifold (locally homeomorphic to a disk) without boundary. Let $\mathbb{S}$ be a surface without boundary. A *map* on $\mathbb{S}$ is a subdivision of $\mathbb{S}$ into 0-dimensional sets (vertices of the map), 1-dimensional contractible sets (edges of the map) and 2-dimensional contractible open sets (faces of the map). Maps are considered up to orientation preserving homeomorphisms of the underlying surface, preserving the combinatorial structure of the map (incidences between vertices, edges and faces).

Planar maps. *Planar maps* are maps on the 2-dimensional sphere $\mathbb{S}^2$ of $\mathbb{R}^3$. The fact that they are called planar comes from the fact that they are in bijection with maps on the plane $\mathbb{R}^2$ via the so-called stereographic projection. Alternatively, a planar map can be defined as a proper embedding of a connected planar multigraph on the oriented sphere, considered up to the homeomorphisms preserving the orientation. More precisely, vertices of the multigraph will be mapped to points of the sphere and edges xy are injectively mapped to arcs whose endpoints are the respective points of the sphere corresponding to x and y. Here and by abuse of language, we will refer to them as vertices and edges of the map. The *faces* of a map are the connected components of its complement, when the map is seen as a set of points, in the sphere. Observe that a given planar multigraph can admit several different proper embeddings, as illustrated by Figure 2.2. We will later see a necessary condition for a planar multigraph to admit a unique embedding on the sphere up to homeomorphism preserving the orientation.

An edge of a map has two *ends* (incidence with a vertex) and either one

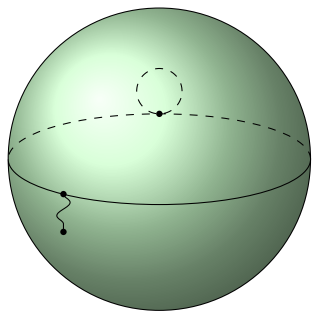


Figure 2.2: Two different planar maps drawn on the sphere and associated to the same planar multigraph.

Figure 2.3: Two equivalent rootings of a planar map. Left is rooted at the dotted edge, whose direction gives the root vertex in white and the root face on the right side. Right is the same map rooted at the corresponding corner.

or two sides (incidence with a face). A map is rooted if an end and a side of an edge are distinguished as the root-end and root-side respectively. Rooting of maps on oriented surfaces usually omits the choice of a root-side because the underlying surface is oriented and maps are considered up to orientation preserving homeomorphism. Our choice of a root-side is equivalent in the oriented case to the choice of an orientation of the surface. The vertex, edge and face defining these incidences are the root-vertex, root-edge and root-face, respectively. Rooted maps are considered up to homeomorphism preserving the root-end and root-side. A vertex is then said to be an *inner vertex* if it is not adjacent to the root face and all the non root faces are said to be *inner faces*. In the rest of this monograph, maps will always be planar and rooted.

A *corner* is an incidence between a vertex and a face. Hence, a vertex or a face of *degree* d defines d corners. Now a map is said to be a d-angulation when all its faces have degree d. In particular, a *triangulation* is a 3-angulation and a *quadrangulation* is a 4-angulation. Notice that a simple triangulation, i.e. with no multiple edge, is *maximal* in the sense that any added edge will create a crossing on the sphere. We also remark that the Handshaking lemma forces any triangulation to have an even number of vertices.

Connectivity and embeddings. A map is said to be k-connected, for any $k \geq 1$, if it has at least $k+1$ vertices and its underlying planar graph is k-connected. A famous result, concerning the number of embeddings of a 3-connected planar graphs on the sphere, implies the existence of a bijection between 3-connected planar graphs and 3-connected maps:

Theorem (Whitney [71])**.** *A 3-connected planar graph admits a unique embedding, up to homeomorphism preserving the orientation, on the sphere.*

Another fundamental result is an equation, known as *Euler's formula*, linking the number v of vertices of a planar map together with the number e of its edges and the number f of its faces, as follows:

$$v - e + f = 2.$$

A planar map is said to be simple when it is defined as a proper embedding of some connected simple planar graph. A corollary of Euler's formula states that a simple map cannot have more than $3n - 6$ edges. Together with the *Handshaking lemma*, this implies that there are no simple map or

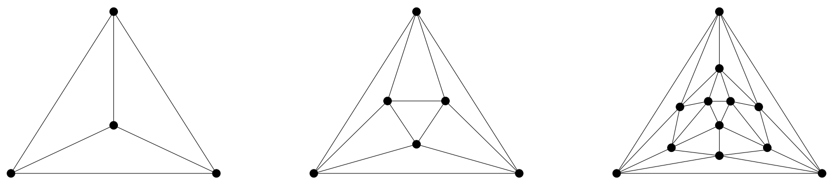


Figure 2.4: The smallest cubic, 4-regular and 5-regular planar graphs.

planar graph that is 6-regular or more. Simple planar graphs that are 1-regular are known as matchings and their properties are well understood, if not trivial. Similarly, simple planar graphs that are 2-regular are simply collections of cycles. Their asymptotic enumeration and properties are also well understood (see [25]). The most interesting and complex cases are then the three remaining classes of simple regular planar graphs, namely cubic, 4-regular and 5-regular, whose first respective elements are the graphs of the tetrahedron, of the octahedron and of the icosahedron, and are pictured in Figure 2.4.

Dual and medial. The dual map M^* of a map M on a surface without boundary is a map obtained by drawing the vertices of M^* in each face of M and edges of M^* across each edge of M. If the map M is rooted, the root edge of M^* corresponds to the root edge e of M; the root-end and root-side of M^* correspond respectively to the side and end of e which are not the root-side and root-end of M. An example of a dual map is pictured in Figure 2.5. It is immediate to show that $M^{**} = M$. Notice that the dual of a d-angulation is a d-regular map.

The *medial map* $\bar{M}$ of a map M on a surface without boundary is a map obtained as follows. One vertex of $\bar{M}$ is drawn on each edge of M. Then, two vertices of $\bar{M}$ are connected each time their corresponding edges share a corner of M. The edge of $\bar{M}$ associated to that corner is then drawn alongside it. An example of a medial map is pictured in Figure 2.5. Due to edges having two sides, notice that the medial of a map is always 4-regular.

2.2 Generating functions and enumeration

One of the most fundamental tools in enumeration is that of generating functions. In this section, we will briefly present some aspects of the exact

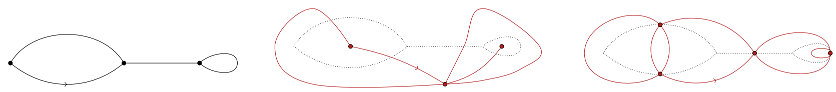


Figure 2.5: A map and its associated dual and medial maps.

enumeration of a combinatorial class. The two main references are the book from Flajolet and Sedgewick [25] and the monograph from Kauers and Paule [39] from which Sections 2.2.1 and 2.3.2 are directly inspired.

2.2.1 Formal power series

Sequences. An infinite sequence $(a_n)_{n\geq 0}$ in $\mathbb{C}$ is a map from $\mathbb{N}$ to $\mathbb{C}$. The set of all sequences in $\mathbb{C}$, denoted by $\mathbb{C}^{\mathbb{N}}$, forms a vector space over $\mathbb{C}$ if we define addition and scalar multiplication termwise:

$$(a_n)_{n\geq 0} + (b_n)_{n\geq 0} = (a_n + b_n)_{n\geq 0} \quad \text{and} \quad \alpha(a_n)_{n\geq 0} = (\alpha a_n)_{n\geq 0}, \text{ for } \alpha \in \mathbb{C}.$$

To a sequence $(a_n)_{n\geq 0}$ will be associated the following *formal power series with indeterminate* x:

$$a(x) = \sum_{n\geq 0} a_n x^n.$$

The set of all formal power series associated to sequences in $\mathbb{C}^{\mathbb{N}}$ and with indeterminate x is denoted by $\mathbb{C}[[x]]$. The n-th coefficient of the formal power series $a(x) = \sum_{n\geq 0} a_n x^n$ is denoted by $[x^n]a(x) = a_n$. Observe in particular that the constant term is given by $[a^0]a(x) = a(0)$. A *generating function* is a formal power series with non-negative coefficients, i.e. associated to a sequence in $\mathbb{N}$.

The ring of formal power series. Multiplication of two sequences is usually defined termwise (this would be the *Hadamard product*). This product together with the pairwise sum induces however a ring with divisors of zero. In the case of power series, we will instead use the *convolution product* (or *Cauchy product*):

$$(a_n)_{n\geq 0} \cdot (b_n)_{n\geq 0} = (c_n)_{n\geq 0}, \qquad \text{where} \qquad c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

And indeed, this time the following holds:

Theorem ([39] - Section 2)**.** *Together with the pairwise addition and the convolution product,* $\mathbb{C}[[x]]$ *forms a commutative ring. It is furthermore an integral domain.*

Differentiation and integration of a formal power series $a(x) = \sum_{n\geq 0} a_n x^n$ are respectively defined as follows:

$$D_x a(x) = \sum_{n\geq 0} a_{n+1}(n+1)x^n \qquad \text{and} \qquad \int_x a(x) = \sum_{n\geq 1} \frac{a_{n-1}}{n} x^n.$$

With respect to the first operation above, $\mathbb{C}[[x]]$ forms a *differential ring.* The following also holds:

Theorem ([39] - Section 2)**.**

- *Fundamental theorem of calculus I:* $D_x \int_x a(x) = a(x)$.
- *Fundamental theorem of calculus II:* $\int_x D_x a(x) = a(x) - [a^0]a(x)$.
- *Taylor's formula:* $[a^n]a(x) = \frac{1}{n!} D_x^n a(x)|_{=0}$.

Formal Laurent series. However, the set of formal power series does not form a field as not all element will admit a multiplicative inverse. In fact, only those associated to a sequence $(a_n)_{n\geq 0}$ for which $a_0 \neq 0$ admits one.

Notice now that it makes sense to consider $(a(x) - a(0))/x$, because the formal power series in the numerator only involves terms in $x, x^2, x^3, \ldots$ which are divisible by x. So that if we define the *order* $\mathrm{ord}(a(x))$, of a non-negative formal power series $a(x)$, as the smallest integer k such that $[a^k]a(x) \neq 0$, then the quotient $a(x)/b(x)$ of two formal power series $a(x)$ and $b(x)$ is itself a formal power series if and only if $\mathrm{ord}(b(x)) \leq \mathrm{ord}(a(x))$. This motivates the following definition.

A *formal Laurent series of order* k *and with indeterminate* x is a series of the form $x^k a(x)$ where $k \in \mathbb{Z}$ and $a(x) \in \mathbb{C}[[x]]$. It corresponds to a sequence with index set $\{k, k+1, k+2, \ldots\} \subset \mathbb{Z}$. Now, we have that the multiplicative inverse of $x^k a(x)$, where $a(0) \neq 0$ is simply $x^{-k}a(x)^{-1}$, where $a(x)^{-1}$ is the multiplicative inverse of $a(x)$ in $\mathbb{C}[[x]]$. The set of all such formal Laurent series is denoted by $\mathbb{C}((x))$. By construction, it is in fact a field, namely the *quotient field* of $\mathbb{C}[[x]]$.

2.2.2 The symbolic method

The *symbolic method* provides a dictionary to translate combinatorial relations between classes into algebraic relations between their associated generating functions. Using the terminology introduced by Flajolet and Sedgewick [25] in the context of *Analytic Combinatorics*, we present here the basic combinatorial constructions with their counterparts in terms of generating functions.

Combinatorial class. A *combinatorial class* is a pair $(\mathcal{A}, |\cdot|)$ where $\mathcal{A}$ is a set of objects and $|\cdot|$ is a mapping from $\mathcal{A}$ to $\mathbb{N}$. For every $a \in \mathcal{A}$, $|a|$ will be called the size of a, and for $n \in \mathbb{N}$, we define $a_n = |\mathcal{A}(n)|$, where $\mathcal{A}(n)$ is the set of objects of size n. We will restrict ourselves to the study of *admissible* combinatorial classes only, i.e. those for which the number of objects of size n is finite for every $n \in \mathbb{N}$. Let $(\mathcal{A}, |\cdot|)$ be a combinatorial class. The following power series

$$A(z) = \sum_{n \geq 0} a_n z^n = \sum_{a \in \mathcal{A}} z^{|a|},$$

is then called the *ordinary generating function* (OGF) associated to $(\mathcal{A}, |\cdot|)$.

The *neutral class* $\mathcal{E}$ is made of a single object of size 0, and its associated generating function is $E(z) = 1$. The *atomic class* $\mathcal{Z}$ is made of a single object of size 1, and its associated generating function is $Z(z) = z$. Let now $(\mathcal{B}, ||\cdot||)$ be another combinatorial class with ordinary generating function $B(z) = \sum_{n \geq 0} b_n z^n$. We write $B(z) \leq A(z)$ if and only if the inequality $b_n \leq a_n$ holds for every n. The union $\mathcal{A} \cup \mathcal{B}$ refers to the disjoint union of classes (and the corresponding induced size). The Cartesian product $\mathcal{A} \times \mathcal{B}$ is the set of pairs (a, b) where $a \in \mathcal{A}$, and $b \in \mathcal{B}$. The size of (a, b) is then the sum of the sizes of a and b. The sequence $\mathrm{Seq}(\mathcal{A})$ corresponds to the set $\mathcal{E} \cup \mathcal{A} \cup (\mathcal{A} \times \mathcal{A}) \cup (\mathcal{A} \times \mathcal{A} \times \mathcal{A}) \cup \ldots$ The size of an element $(a_1, a_2, \ldots, a_r)$ in $\mathrm{Seq}(\mathcal{A})$ is the sum of sizes of the elements a_i. The pointing operator over the class $\mathcal{A}$ works in the following way: for each element $a \in \mathcal{A}$, with $|a| = n$, the pointing operator distinguishes one of the n atoms that compounds a. Finally, the substitution of the class $\mathcal{B}$ in $\mathcal{A}$ substitutes each atom of every element of $\mathcal{A}$ by an element of $\mathcal{B}$.

Labelled combinatorial class. A combinatorial class is said to be *labelled* when to each element a in the class is attached $|a|$ different labels in the set $[|a|]$. When dealing with labelled structures, we introduce the following refinements. For an admissible labelled combinatorial class $(\mathcal{A}, |\cdot|)$ we define the *exponential generating function* (EGF) associated to $\mathcal{A}$ as the formal power series

$$A(z) = \sum_{n\geq 0} a_n \frac{z^n}{n!} = \sum_{a\in\mathcal{A}} \frac{z^{|a|}}{|a|!}.$$

Definition of combinatorial relations between labelled classes are quite different compared to ordinary classes. For instance, instead of considering the product of classes we consider the labelled product: let $(A, |\cdot|)$ and $(B, ||\cdot||)$ be labelled combinatorial classes, we define $\mathcal{A} * \mathcal{B}$ as the set of all possible labellings of the pairs of the form (a, b), with $a \in \mathcal{A}$ and $b \in \mathcal{B}$. This specification is translated in the language of exponential generating functions in the following way:

$$\sum_{(a,b)\in\mathcal{A}\times\mathcal{B}} \binom{|a| + ||b||}{|a|} \frac{z^{|a|+||b||}}{(|a| + ||b||)!} = \sum_{a\in\mathcal{A}} \frac{z^{|a|}}{|a|!} \sum_{b\in\mathcal{B}} \frac{z^{|b|}}{||b||!} = A(z) \cdot B(z).$$

All the other operations specified for ordinary generating functions can be rephrased in the exponential context using the labelled product. In Table 2.1 is depicted the translation between some basic combinatorial constructions and their associated generating functions, both for the ordinary and the exponential case. The particular constructions $\mathrm{Set}(\mathcal{A})$ and $\mathrm{Cyc}(\mathcal{A})$ are described for labelled classes only and refer to the classes made respectively of sets of elements of $\mathcal{A}$ and of cycles in which every vertex is substituted by an element of $\mathcal{A}$.

Parameters. A *parameter* of a combinatorial class $(\mathcal{A}, |\cdot|)$ is a function $\chi : \mathcal{A} \to \mathbb{N}$. In this context, the size function is called the main parameter. The consideration of additional parameters over the combinatorial classes gives rise to *multivariate generating functions.*

For example, the class of graphs with labelled vertices, where the number of vertices is the main parameter while the number of edges is the secondary parameter, will be associated to the bivariate generating function

$$A(x, y) = \sum_{n,k} a_{n,k} \frac{x^n}{n!} y^k,$$

Construction		OGF	EGF
Disjoint Union	$\mathcal{A} \cup \mathcal{B}$	$A(x) + B(x)$	$A(x) + B(x)$
Product	$\mathcal{A} \times \mathcal{B}$	$A(x) \cdot B(x)$	-
Labelled Product	$\mathcal{A} \star \mathcal{B}$	-	$A(x) \cdot B(x)$
Sequence	$\text{Seq}(\mathcal{A})$	$\frac{1}{1-A(x)}$	$\frac{1}{1-A(x)}$
Set	$\text{Set}(\mathcal{A})$	-	$\exp(A(x))$
Cycle	$\text{Cyc}(\mathcal{A})$	-	$\log \frac{1}{1-A(z)}$
Substitution	$\mathcal{A} \circ \mathcal{B}$	$A(B(x))$	$A(B(x))$
Pointing	$\mathcal{A}^\bullet$	$x\frac{\partial}{\partial x}A(x)$	$x\frac{\partial}{\partial x}A(x)$

Table 2.1: The dictionary of the symbolic method.

where $a_{n,k}$ is the number of labelled graphs with n vertices and k edges, and $\sum_{k\geq 0} a_{n,k} = a_n$. Setting a parameter y to one into $A(x, y)$ is formally equivalent to not considering it, i.e. $A(x, 1) = A(x)$.

2.2.3 Exact enumeration

Enumerating a combinatorial class $(\mathcal{A}, |\cdot|)$ is computing, for any $n \in \mathbb{N}$, the number $|\mathcal{A}(n)| = a_n$ of objects of size n. In terms of the associated generating function $A(z)$, this is translated into the computation of its n-th coefficient $[z^n]A(z)$ or $n![z^n]A(z)$, depending on whether $A(z)$ is ordinary or exponential.

One of the most well knonw result on enumeration is *Newton's binomial theorem*. On can rephrase it in terms of generating functions, as follows.

Theorem (Binomial formula)**.** *Let r be a any integer. Then*

$$(1 + z)^r = \sum_{n\geq 0} \binom{r}{n} z^n.$$

Bootstrapping. Using the symbolic method, some generating functions can be described implicitely as the solution of system of functional equations involving other generating functions. The following lemma guarantees the existence of a unique non-zero solution to some system of non-negative equations, under sufficient hypotheses.

Lemma 1. *Let* $y_1(z,u),\dots,y_m(z,u)$ *be power series satisfying the system of equations*

$$\begin{aligned} y_1 &= F_1(z,y_1,\dots,y_m,u),\\ y_2 &= F_2(z,y_1,\dots,y_m,u),\\ &\vdots\\ y_m &= F_m(z,y_1,\dots,y_m,u), \end{aligned}$$

where the F_i *are power series in the variables indicated. Assume that for each* i, F_i *has non-negative coefficients and is divisible by* z. *Assume also that there exists a solution* $\widehat{\mathbf{y}} = (y_1(z,u),\dots,y_m(z,u))$ *to the system which is not identically* 0 *for all* i. *Then this is the unique solution with non-negative coefficients.*

Moreover, the solution can be computed iteratively from the initial values $y_i = 0$ *up to any degree of* z.

Proof. We first recall that the order of a non-zero power series $A(z) = \sum a_n z^n$ is the minimum n such $a_n \neq 0$, and that a sequence $\{A_m(z)\}_{m\geq 0}$ is convergent in the ring of formal power series if the order of the $A_m(z)$ go to infinity.

Let $\mathbf{F} = (F_1,\dots,F_m)$. Start with the initial value $\mathbf{y}^{(0)}(z,u) = 0$ and let $\mathbf{y}^{(k+1)}(z,u) = \mathbf{F}(z,\mathbf{y}^{(k)}(z,u),u)$, where $\mathbf{y}^{(k)}(z,u) = (y_1^{(k)}(z,u),\dots,y_m^{(k)}(z,u))$. Since the F_i have non-negative coefficients, so do the $\mathbf{y}^{(k)}$. Since each F_i is divisible by z, the mapping $\mathbf{F}$ is a contraction, in the sense that the order of $y_i^{(k+1)}(z,u)$ is larger than the order of $y_i^{(k)}(z,u)$. Hence the solution is unique and is given by the limit of the $\mathbf{y}^{(k)}(z,u)$. The solution is non-zero since the F_i are non-zero. □

The former proof gives a procedure for computing iteratively the unique solution. Start with $y_i = 0$ for all i and compute the $y_i^{(k)}$ iteratively. Each $y_i^{(k)}(z,u)$ is a polynomial and, because of the hypothesis on the F_i, $y_{i+1}^{(k+1)}(z,u) = y_i^{(k)}(z,u) + M_i$, where M_i is a monomial of degree larger than the degree of $y_i^{(k)}(z,u)$. This can be iterated to any desired degree. Observe also that the variable u plays only the role of a parameter and that further parameters can be added without any change. We refer to Section 2.2.5 of [20] for further discussions.

Lagrange Inversion. In 1770, Lagrange [40] developed a method to relate the coefficients of the compositional inverse of an analytic function to the coefficients of the powers of the function itself. The inversion $z = \phi(u)$

consists in expressing u as a function of z and is solved by the so-called *Lagrange series*. The combinatorial version presented here is due to Bürmann [15] (see [25, Theorem A.2] for a proof).

Theorem (Lagrange-Bürmann inversion formula)**.** *Let $\phi(y) = \sum_{k\geq 0} \phi_k y^k$ be a power series of $\mathbb{C}[[y]]$ with $\phi_0 \neq 0$. And let H be an arbitrary function. Then the equation $u(z) = z\phi(u)$ admits a unique solution in $\mathbb{C}[[z]]$ whose coefficients are given by the Lagrange form*

$$[z^n]H(u(z)) = \frac{1}{n}[y^{n-1}](H'(y)\phi(y)^n).$$

2.3 Analytic combinatorics

The relationship between generating functions and combinatorial enumeration goes deeper than simply exact enumeration. Certain combinatorial classes admit an associated generating function that, when seen as a function of the complex variable, turns out to be analytic in a neighbourhood around the origin.

In this case, one can use the powerful tools from *Complex Analysis*, linking the expansion of the function near its dominant singularity to tight asymptotic estimates of its coefficients. This is aymptotic enumeration. Those estimates can in turn be used to compute the limiting probabilities of the random object associated to such a combinatorial class. Those *singular expansions* are in particular computed from the functional equations induced by the recursive decompositon of the combinatorial class, via the symbolic method. The main references for this section are the books from Flajolet and Sedgewick [24], Drmota [20], Walker [68] and Kauers and Paule [39].

2.3.1 Asymptotic enumeration

We say that two sequences of numbers $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are of the same *exponential order* if $\limsup |a_n|^{1/n} = \limsup |b_n|^{1/n}$. Under these assumptions, we write $a_n \bowtie b_n$. Sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are called *asymptotically equivalent*, and is denoted by $a_n \sim b_n$, if the limit of the quotient a_n/b_n exists and is equal to 1. This notation is also extended to generating functions, as we write $A(z) \sim B(z)$ when $[z^n]A(z) \sim [z^n]B(z)$.

When finite, the quantity lim sup $|a_n|^{1/n}$ is known as the *exponential growth* or the exponential order of the sequence $(a_n)_{n\geq 0}$. If R is the exponential growth of $(a_n)_{n\geq 0}$, then $a_n = \theta(n) \cdot R^n$, where lim sup $|\theta(n)|^{1/n} = 1$. The term $\theta(n)$ is called the *subexponential growth* of $(a_n)_{n\geq 0}$. Estimates of both the exponential and the subexponential growths of a sequence can be obtained by considering the dominant singularity of the associated generating function seen as an analytic function on a complex neighbourhood of the origin.

Analytic functions. A complex function $f(z)$ defined over a region Ω is said to be *analytic at a point* $z_0 \in \Omega$ if, for z in some open disc centred at z_0 and contained in Ω, it is representable by a convergent power series expansion

$$f(z) = \sum_{n\geq 0} c_n (z - z_0)^n.$$

A function is *analytic in a region* Ω if it is analytic at every point of Ω. An analytic function $f(z)$, defined over the interior region Ω determined by a simple closed curve γ, is said to be *analytically continuable* at $z_0 \in \gamma$ if there exists a an analytic function $f^*(z)$, defined over some set Ω^* containing z_0, such that $f^*(z) = f(z)$ in $\Omega^* \cap \Omega$.

Singularities and growth. A *singularity* of an analytic function $f(z)$ is a point z_0 on the boundary of its region of analycity for which f is not analytically continuable. Singularities of a function analytic at 0, which lie on the boundary of the disc of convergence, are called *dominant singularities.* In this case, a dominant singularity is a singularity with smallest modulus.

Theorem (Pringsheim [25, Theorem IV.6])**.** *A dominant singularity, if it exists, of a generating function with positive coefficients is always a positive real number.*

The location of a *dominant singularity* will give the exponential growth of the sequence, and the nature of this singularity the subexponential term. Indeed, the next theorem, stated by Cauchy in 1821 [16] then fully proved by Hadamard in 1892 [33], already relates in a case relevant to us the (positive real) dominant singularity with the exponential growth of an analytic function. A proof can be found in [25, Theorem IV.7].

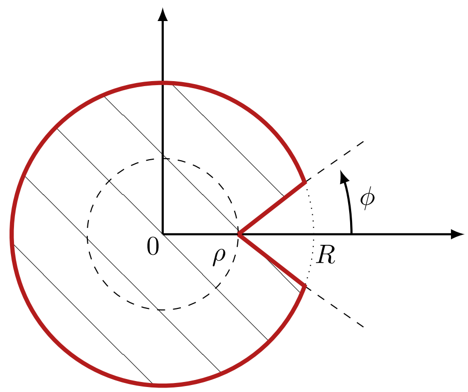


Figure 2.6: A typical Δ-domain at ρ.

Theorem (Exponential growth formula). *If the function $A(z)$ is analytic at 0 and one of its dominant singularities is the positive real number ρ, then*

$$[z^n]A(z) \bowtie \rho^{-n}.$$

To compute the subexponential term in the asymptotic growth of the coefficients of a generating function, we will need a more refined notion.

Singularity analysis. Given a complex number $\rho \neq 0$, a Δ-domain at ρ is an open set of the form

$$\Delta(R, \phi) = \{z \colon |z| < R, z \neq \rho, |\arg(z - \rho)| > \phi\}.$$

The typical shape of a Δ-domain is pictured in Figure 2.6.

Let now $A(z)$ and $B(z)$ be two generating functions with the same positive real number ρ as dominant singularity. We write

$$A(z) \underset{z \to \rho}{\sim} B(z) \text{ when } \lim_{z \to \rho} \frac{A(z)}{B(z)} = 1.$$

We obtain the asymptotic expansion of $[z^n]A(z)$ by transferring the behaviour of $A(z)$ around its dominant singularity from a simpler function $B(z)$, from which we know the analytic behaviour . This is the idea behind the so-called Transfer Theorems introduced by Flajolet and Odlyzko [24]. In this thesis, we will use a combination of Theorems VI.1 and VI.3 from [25].

Theorem (Sim-transfer)**.** *Assume $A(z)$ has a unique dominant singularity $\rho > 0$ and is analytic in a Δ-domain at ρ. If A satisfies, locally around ρ, the following estimate:*

$$A(z) \underset{z\to\rho}{\sim} c \cdot \left(1 - \frac{z}{\rho}\right)^{\alpha}$$

with $\alpha \in \mathbb{Q}_{>0} \setminus \mathbb{N}$, then the coefficients of $A(z)$ satisfy

$$[z^n]A(z) \underset{n\to\infty}{\sim} c \cdot \frac{n^{-\alpha-1}}{\Gamma(-\alpha)} \cdot \rho^{-n},$$

where Γ is the Gamma function: $\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}dt$.

All the singularities we will encounter are of square-root type, that is, the expansion of a function near a singularity ρ will be of the form

$$A(z) = A_0 + A_2X^2 + \cdots + A_{2k}X^{2k} + A_{2k+1}X^{2k+1} + O(X^{2k+2}),$$

where $X = \sqrt{1 - \frac{z}{\rho}}$ and $\alpha = 2k + 1$ is the smallest odd index i such that $A_i \neq 0$. Now, because ρ is a square-root singularity of $A(z)$, the even powers of X are analytic function at ρ, so all those that are even and asymptotically dominant (i.e. with a smaller power) compared to X^{α} cannot contribute to the asymptotic of $[z^n]A(z)$. This means that the dominant term is $A_{\alpha}X^{\alpha}$.

Multiple dominant singularities. If $A(z)$ has several dominant singularities coming from pure periodicities, then the contributions from each of them must be combined (see [25, IV.6.1]). In our case the periodicities will be due to the fact that cubic graphs have necessarily an even number of vertices and the corresponding generating functions will be even. We will locate the (unique) positive dominant singularity ρ and we will simply add the contributions from ρ and $-\rho$. In fact, to apply the Transfer Theorem in this case, $A(z)$ needs to be analytic in a *dented-domain* at $\pm\rho$.

Given a complex number $\rho \neq 0$, a *dented-domain* at $\pm\rho$ is an open set of the form

$$\Delta_2(R, \phi) = \{z\colon |z| < R, z \neq \pm\rho, |\arg(z \pm \rho)| > \phi\}.$$

The typical shape of a dented-domain is pictured in Figure 2.7.

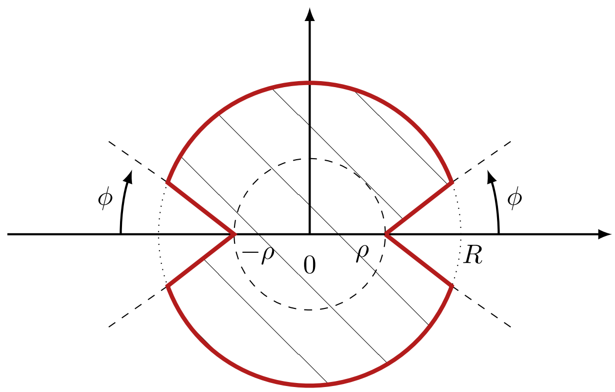


Figure 2.7: A typical dented-domain at $\pm\rho$.

Assume then that $A(z)$ is an even function and that its singular expansion $\sum A_i X^i$ around its dominant singularity is of square-root type, then by the Transfer Theorem, we have the following estimate

$$[z^n]A(z) = A_\alpha \cdot \frac{n^{-\alpha-1}}{\Gamma(\alpha)} \cdot (\rho^{-n} + (-\rho)^{-n}) \cdot (1 + o(n^{-1}),$$

where α is the smallest i with $A_i \neq 0$ in $\sum A_i X^i$.

2.3.2 Local expansions of certain implicit functions

In this thesis, every generating function will be defined implicitly via a system of equations or a single functional equations, thus reflecting the recursive nature of its associated combinatorial class. Furthermore, a direct application of the *implicit function theorem* would give us both a garanty of the analycity of the generating function and its a singular expansion, allowing us to use the Transfer theorem.

However, as we would like to compute the constants involved in the latter theorem, we would need a more constructive method to compute asymptotic expansions of implicit functions. We will now exhibit one such method when the function is algebraic or D-finite.

Algebraic generating functions. A function $f(x)$ is said to be *algebraic* of order $d \in \mathbb{N}$ when it satisfies a polynomial equation of the form:

$$P(f(x), x) := p_0(x) + p_1(x)f(x) + \ldots + p_d(x)f(x)^d = 0, \qquad (2.1)$$

where $p_k(x) \in \mathbb{C}[x]$ is a polynomial in x, for each $k \in \{0, \ldots, d\}$. In this case, we also say that $P(y, x)$ is an *annihilating polynomial* of $f(x)$. Formally, we think of $P(y, x)$ as an object in $\mathbb{C}[x][y] \simeq \mathbb{C}[x, y]$, so that when speaking about the *degree* of $P(y, x)$, we mean its degree with respect to y.

A *minimal polynomial* of $f(x)$ is an irreducible non identically zero polynomial of smallest degree among all the annihilating polynomials of $f(x)$. Note that in this definition, a minimal polynomial is not unique. This is because we would like to keep the nice following property: using the above notation, all polynomials $p_k(x)$ that arise throughout this thesis have integer coefficients.

Branch points. A bivariate polynomial $P(y, x)$ defines an *algebraic curve*, whose *critical points* are of particular interest to us, as they contain the singularities of the induced algebraic functions. In particular, a point (y, x) is said to be a *branch point* of the curve when it is a root of both $P(y, x)$ and $P_y(y, x)$, the derivative of $P(y, x)$ with respect to y. The x-coordinates of the branch points are then the roots of the *resultant* of $P(y, x)$ and $P_y(y, x)$ with respect to y. This is a polynomial in $\mathbb{C}[x]$ called the *discriminant* of $f(x)$.

A particular algebraic function need not have a singularity for everyone of those branch points, it corresponds in fact to just one branch of the algebraic curve, and x_0 being a critical point only means that some of the branches have a singularity at x_0 (see [39, Section 6.5]). For a given algebraic function $f(x)$ induced by a polynomial $P(y, x)$, branch points of $P(y, x)$ are hence called *potential singularities* of $f(x)$ (see [25, Section VII.7.1]). One has then to determine which of the potential singularities do actually belong to its associated branch. In particular, by Pringsheim's theorem the dominant singularity of an algebraic generating function $A(x)$, associated to a combinatorial class $\mathcal{A}$, will be the smallest positive root of one of the factors of the discriminant of $A(x)$. Throughout this thesis, the choice of the *good* factor will be done by an inspection into the combinatorial informations on $\mathcal{A}$.

Puiseux expansions. After finding the dominant singularity ρ of an algebraic generating function $f(x)$, one would like to compute its asymptotic expansion in a neighbourhood of ρ, namely its *singular expansion*. In some cases, roots of the minimal polynomial of $f(x)$ will be formal power series. But that is not always the case! To comprise all the solutions of any algebraic equation, one needs to introduce a more general notion, similarly to the formal Laurent series who were introduced to obtain a field. While formal Laurent series are obtained from power series by allowing terms x^n with negative integers n to appear in the series, we will now extend further and allow even rational numbers as exponents.

For that we first need to introduce a fundamental result due to Newton and later rediscovered by Puiseux. This version is taken from [39, Section 6] but a full constructive proof using the *Newton polygon* can be found in [68, Section III.3]

Theorem (Newton-Puiseux)**.** *If a polynomial $m(x, y) \in \mathbb{C}[x][y]$ is irreducible of degree d, then there exists a positive integer k and d distinct Laurent series $a_1(x), \dots, a_d(x)$ with $m(x^k, a_i(x)) = 0$ for $i = 1, \dots, d$.*

Let now r be a non-negative integer. After a formal substitution $x \to x^{1/r}$, the Laurent series $a_i(x)$ in the theorem become objects of the form $a_i(x^{1/r})$ which involve fractional exponents and satisfy $m(x, a_i(x^{1/r})) = 0$. They are called *Puiseux series of branching-type r with indeterminate x*. Observe that Puiseux series may involve fractional exponents, but not in an arbitrary fashion. The fractions appearing as exponents of a fixed Puiseux series must have a finite common denominator, here r. So that a typical Puiseux series of branching-type r has the form $x^{1/r}a(x)$, where $a(x) \in \mathbb{C}[[x]]$.

A corollary to the theorem of Newton-Puiseux is that the set of Puiseux series with indeterminate x form, together with the usual operations, an *algebraically closed field*, called the *fractionnal field* $\mathbb{C}(x)^*$ (for precise definitions and a proof, see [68, Theorem 3.1]). Another corollary wich will be central for us is the following (see [25, Theorem VII.7]):

Theorem (Newton-Puiseux singular expansion)**.** *Let $f(z)$ be an algebraic function. In a circular neighbourhood of a singularity ζ, slit along a ray emanating from ζ, $f(z)$ admits a Puiseux expansion that is locally convergent and of the form*

$$f(z) = \sum_{k \geq k_0} c_k \cdot (z - \zeta)^{k/r},$$

for a fixed determination of $(z - \zeta)^{1/r}$, where $k_0 \in \mathbb{Z}$ and r is a non-negative integer, called the branching type.

Notice that the singular expansions granted by the above theorem are of the exact type required to apply the theorem of *sim-transfer*.

Concerning computability of such an expansion, the constructive proof of the Newton-Puiseux theorem gives a polynomial-time algorithm that has been implemented in a several computer algebra systems. For this thesis, we use the function *puiseux* from the library *algcurves* of *Maple 2017*.

D-finite generating functions. A function $f(x)$ is said to be *D-finite*, or equivalently *holonomic*, of order $r \in \mathbb{N}$ if it satisfies a linear differential equation with polynomial coefficients:

$$q_0(x)f(x) + q_1(x)D_x f(x) + \ldots + q_r(x)D_x^r f(x) = 0. \tag{2.2}$$

The following facts about D-finite generating functions are well-known, we refer the reader to [62, Chapter 6] for details and proofs:

Theorem (Basic facts on D-finite generating functions)**.**

- *$A(x) = \sum_{n\geq 0} a_n x^n$ is D-finite if and only if $\{a_n\}_{n\geq 0}$ is P-*recursive*, that is, it satisfies a linear recurrence with polynomial coefficients.*
- *An algebraic function is D-finite.*
- *D-finite functions are closed under sums and products.*
- *If $A(x)$ is D-finite and $B(x)$ is algebraic, then $A(B(x))$ is D-finite.*
- *The derivative and the primitive of a D-finite function are D-finite.*

Furthermore, the next lemma states that a function is D-finite given that the derivative of its logarithm is algebraic.

Lemma 2. *If $f'(t)$ is an algebraic function of t, then $g(t) = e^{f(t)}$ is D-finite.*

The proof presented here is due to Mireille Bousquet-Mélou.

Proof. Start with the following, which is easily proved by induction on k: *If f' is algebraic, then each derivative $f^{(k)}$ is a rational function of f' and t.*

Using this observation we have

$$\begin{aligned} g &= 1 \cdot g \\ g' &= f'g = R_1(f', t)g \\ g'' &= f''g + (f')^2 g = R_2(f', t)g \\ &\vdots \\ g^{(k)} &= R_k(f', t)g, \end{aligned}$$

where R_k is a rational function, and we set $R_0 = 1$.

Since f' is algebraic, $\mathbb{Q}(f', t)$ is finite dimensional over $\mathbb{Q}(t)$, say of dimension k. Hence there are rational functions $S_i(t)$ such that

$$\sum_{i=0}^{k} S_i(t) R_i(f', t) = 0.$$

It follows that

$$S_0(t)g + S_1(t)g' + \cdots + S_k(t)g^{(k)} = 0.$$

This proves that g is D-finite. □

Note. The reciprocal holds but is harder to prove: if $f'(t)$ and $e^{f(t)}$ are D-finite, then $f'(t)$ is algebraic (see the last paragraph of [61]).

Transfer of singularities. Similarly to the algebraic case, solutions of linear differential equation with polynomial coefficients do not necesseraly admit a Puiseux expansion. There is a more general notion of series defined in [39, Section 7.3] and its computation, although more complex, also uses the Newton polygon. However, the number of solutions is not finite but is instead a vector space of dimension the order of the linear differential equation (see [39, Theorem 7.3]).

Thankfully, every D-finite generating functions that will be encountered in this thesis will either be an integral $\int_x f(x)$ or of the form $\exp \circ \int_x f(x)$ (as in Lemma 2), where $f(x)$ is algebraic. The next result (see [20, Section 2.2.4]) will guarantee us that in both cases, the D-finite generating functions will inherit the dominant singularity of $f(x)$, as well as its *singular behaviour*, that is they will have a similar expansion as $f(x)$ in a neighbourhood of the dominant singularity, namely a Puiseux expansion, even though the fractionnal exponent might change.

Theorem (Transfer of singularity)**.** *Suppose that the function $f(x)$ is analytic at zero with an expansion near its dominant singularity $\rho \in \mathbb{R}_{>0}$ of the form*

$$f(x) \underset{x\to\rho}{\sim} f_0 + f_2X^2 + \ldots + f_{2k}X^{2k} + f_{2k+1}X^{2k+1},$$

where $X = \sqrt{1 - x/\rho}$ and $k \in \mathbb{N}$.

Let now $H(x, y)$ be a function that is analytic at $(0, f(0))$ and such that $D_yH(0, f(0)) \neq 0$. Then $f_H(x) = H(x, f(x))$ admits the same kind of singular expansion as $f(x)$, that is

$$f_H(x) \underset{x\to\rho}{\sim} h_0 + h_2X^2 + \ldots + h_{2k}X^{2k} + h_{2k+1}X^{2k+1}.$$

Furthermore the derivative and the integral of $f(x)$ have singular expansions of the (respective) form:

$$D_xf(x) \underset{x\to\rho}{\sim} d_0 + d_2X^2 + \ldots + d_{2k}X^{2k} + d_{2k+2}X^{2k+2} + d_{2k+3}X^{2k+3},$$

$$\int_x f(x) \underset{x\to\rho}{\sim} i_0 + i_2X^2 + \ldots + i_{2k-2}X^{2k-2} + i_{2k-1}X^{2k-1}.$$

Regarding the computation of those Puiseux expansion, it will be done numerically by integrating the one of $f(x)$, computed via the Newton polygon, or by integrating it then taking the exponential. However, as we will see in Chapter 3, the subexponential term of the asymptotic estimate of the coefficients obtained in the second case will be undeterminated due to the integration step before taking the exponential.

To avoid having to integrate numerically, we will introduce in the next section a so-called *combinatorial integration* of certain classes of graphs. This is induced by a recursive decomposition of a connected graph in terms of its 2-connected and 3-connected components.

Bivariate singularity analysis. Suppose that we are now given a bivariate generating function $f(x, y) = \sum_{n,k\geq 0} f_{n,k}\, x^n y^k \in \mathbb{K}(x, y)$ that satisfies an irreducible trivariate polynomial implicit equation $P \in \mathbb{K}[X, Y, Z]$, i.e. $P(f(x, y), x, y) = 0$. To some extent, one can transfer the definitions from the univariate to the multivariate case. In particular, P will be called the minimal polynomial of $f(x, y)$.

When $f(x, y)$ is a function of complex variables, the groundbreaking monograph of Pemantle and Wilson [57] deals with the general analytic behaviour of multivariate generating functions. This is however beyond the scope of this thesis.

In our case, we will study the singular behaviour of $f(x, y)$ when y will be a postive real number in a small neighbourhood of 1. This has the advantage of reducing the bivariate analysis to the univariate case, and one can interpret it combinatorially as a small perturbation of the case $y = 1$ in which the second variable is not taken into account. Observe that if the generating function $f(x, y)$ has only non-negative coefficients, then it is continuous. So that if $f(x) = f(x, 1)$ is analytic in a neighbourhood of the origin, so is $f(x, y_0)$, for any positive real $y_0 \sim 1$. Now the discriminant $D(x, y)$ of P with respect to $f(x, y)$ will be called the *singularity curve* of $f(x, y)$. In $D(x, y)$, one can look at $x = x(y)$ as a function of y, so that when setting $y = 1$, we must recover the dominant singularity $\rho = \rho(1)$ of $f(x, 1) = f(x)$, as a positive root of the univariate polynomial $x(1)$. This also implies that when $f(x)$ is an even function that is analytic in a dented domain at $\pm\rho$, then so is $f(x, y_0)$, for any $y_0 \sim 1$.

Now for a fixed positive real number $y_0 \sim 1$, one can also apply Newton's algorithm to the bivariate polynomial $P(f(x, y_0), x, y_0)$ to compute the Puiseux expansion of $f(x, y_0)$ (which is now univariate) for x near $x(y_0)$. In this thesis, it will always be of the form:

$$f(x, y_0) = f_0(y_0) + f_2(y_0)X^2 + \cdots + f_{2k}(y_0)X^{2k} + f_{2k+1}(y_0)X^{2k+1} + O(X^{2k+2}),$$

where $X = \sqrt{1 - x/x(y_0)}$ and k is a positive odd integer. And we can apply the theorem of *sim-transfer* on this local expansion to obtain an asymptotic estimate on the coefficients of $f(x, y_0)$, as follows:

$$[x^n]f(x, y_0) = \frac{f_{2k+1}(y_0)}{\Gamma(-\alpha)} \cdot n^{-\alpha} \cdot \rho(y_0)^{-n}, \qquad \text{with } \alpha = \frac{k}{2} + 1.$$

2.3.3 Limiting distributions

We introduce here some of the basic definitions from probability theory as well as an important theorem that will be used in this thesis. INote that we jump directly to the notion of probability distribution without defining a probability space nor a random variable. For precise definitions, we send the reader to the monograph of Grimmett and Stirzaker [32].

Probability distribution. Let $(\Omega, \mathcal{P}(\Omega), \mathbf{p})$ be a probability space with a random variable $\mathbf{X}$. The quantity $F_{\mathbf{X}}(x) = \mathbf{p}[\{\mathbf{X} \leq x\}]$ is known as the *probability distribution function* of $\mathbf{X}$. Provided that its derivative exists, the probability distribution function has an interpretation in terms of $f_{\mathbf{X}}(x)$, the *density probability distribution* of $\mathbf{X}$

$$F_{\mathbf{X}}(x) = \int_{-\infty}^{x} f_{\mathbf{X}}(s) ds.$$

The *expectation* of $\mathbf{X}$ is then defined as

$$\mathbf{E}[\mathbf{X}] = \int_{-\infty}^{\infty} f_{\mathbf{X}}(s) ds,$$

and the *variance* of $\mathbf{X}$ is

$$\mathbf{Var}[\mathbf{X}] = \mathbf{E}[\mathbf{X}^2] - \mathbf{E}[\mathbf{X}]^2.$$

Gaussian distibution. Let $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}_{>0}$. Then a random variable $\mathbf{Z}$ is called *Gaussian* or *normal* with law $N(\mu, \sigma^2)$, if its probability distribution function is of the form

$$\mathbf{p}[\{\mathbf{Z} \leq x\}] = \Phi\left(\frac{x-\mu}{\sigma}\right), \qquad \text{with} \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt.$$

And we have $\mathbf{E}[\mathbf{Z}] = \mu$ and $\mathbf{Var}[\mathbf{Z}] = \sigma^2$.

Poisson distribution. Let $\lambda \in \mathbb{R}_{>0}$. Then a discrete random variable $\mathbf{Y}$ is called *Poisson* with law $Po(\lambda)$, if its probability distribution function is of the form

$$\mathbf{p}[\{\mathbf{Y} \leq \ell\}] = \sum_{k=0}^{\ell} \frac{\lambda^k}{k!} e^{-k}.$$

And we have $\mathbf{E}[\mathbf{Z}] = \mathbf{Var}[\mathbf{Z}] = \lambda$.

Probability generating function. Let $(\mathcal{A}, |\cdot|)$ a combinatorial class with an added parameter $\chi : \mathcal{A} \to \mathbb{N}$, so that its associated generating function is $A(z,u) = \sum_{n,k\geq 0} a_{n,k} z^n u^k$. One can then consider the sequence $\mathbf{X}_n$ of uniform random variables over $\mathcal{A}(n)$ defined for each $n \geq 0$ by the discrete probability density function:

$$\mathbf{p}[\{\mathbf{X}_n = k\}] = \frac{a_{n,k}}{a_n}.$$

This allows us to define the *probability generating function* of $\mathbf{X}_n$, for $n \geq 0$:

$$p_n(u) := \sum_{k \geq 0} \mathbf{p}[\{\mathbf{X}_n = k\}] u^k = \frac{[z^n]A(z,u)}{[z^n]A(z,1)}.$$

Convergence in distribution. Among the several modes of convergence of a sequence of random variables, we will only consider the *convergence in distribution.* We say that a sequence of random variables $(\mathbf{X}_n)_{n>0}$ tends in distribution to a random variable $\mathbf{X}$, and we denote it by $\mathbf{X}_n \xrightarrow{d} \mathbf{X}$, if the associated sequence of distribution probability functions $(F_{\mathbf{X}_n}(x))_{n>0}$ converges point wise to the distribution function $F_{\mathbf{X}}(x)$ of $\mathbf{X}$.

Asymptotic normality. In Chapter 4, we will mainly concern ourselves with sequences of random variable converging to normal distributions. The main method that will be used is due to Hwang [36]. He designed a way to deduce asymptotic normality from the singular expansions of generating functions. The following version can be found in [25, Theorem IX.8]:

Theorem (Quasi-powers [36])**.** *Let the $(\mathbf{X}_n)_{n>0}$ be a sequence of non-negative discrete random variables, each with probability density functions $p_n(u)$. Assume that, uniformly in a fixed complex neighbourhood of $u = 1$,*

$$p_n(u) = A(u) \cdot B(u)^n \left(1 + O\left(\frac{1}{n}\right)\right),$$

where $A(u)$ and $B(u)$ are both analytic at $u = 1$ and $A(1) = B(1) = 1$. Assume finally that $B(u)$ satisfies $B''(1) + B'(1) - B'(1)^2 \neq 0$.

Then after standardisation, the distribution of $\mathbf{X}_n$ is asymptotically Gaussian and the mean and variance satisfy

$$\mathbf{E}[\mathbf{X}_n] \sim \left(\frac{B'(1)}{B(1)}\right) n, \qquad \mathbf{Var}[\mathbf{X}_n] \sim \left(\frac{B''(1)}{B(1)} + \frac{B'(1)}{B(1)} - \left(\frac{B'(1)}{B(1)}\right)^2\right) n.$$

In the applications of the Quasi-Powers theorem throughout this thesis, we will always have $B(u) = \rho(1)/\rho(u)$, where $\rho(u)$ will be the singularity curve (as a function of u) of a bivariate generating function $f(z,u)$, that is $\rho(1)$ will be the dominant singularity of $f(z,1)$. The former expressions then become

$$\mathbf{E}[\mathbf{X}_n] \sim \left(-\frac{\rho'(1)}{\rho(1)}\right) n, \qquad \mathbf{Var}[\mathbf{X}_n] \sim \left(-\frac{\rho''(1)}{\rho(1)} - \frac{\rho'(1)}{\rho(1)} + \left(\frac{\rho'(1)}{\rho(1)}\right)^2\right) n.$$

2.4 Graphs decompositions

2.4.1 Root decompositions

As an illustration of how one can use some of the enumerative methods unified under the framework of *Analytic Combinatorics*, we will now study some classes of rooted planar maps following Tutte's seminal idea of *root decomposition*. In the rest of this thesis, all maps will be rooted and planar.

In 1968 [67], Tutte found a closed formula for the number m_n of maps with n edges:

$$m_n = \frac{2 \cdot 3^n}{n+2}\mathrm{Cat}(n),$$

where $\mathrm{Cat}(n)$ is the n^{th} Catalan number $\frac{1}{2n+1}\binom{2n}{n}$. To obtain that result, he recursively decomposed maps as follows: start with any map, remove its root edge and classify the resulting maps.

In 1962 following a similar recursive root decomposition, Tutte [64] also found closed formulas for the numbers of simple 3-connected triangulations.

Then 1965 [14] Brown found a root decomposition for *irreducible* quadrangulations, that are quadrangulations with at least 6 vertices and for whcih every 4-cycle defines a face.

3-connected triangulations. We would first like to point a slight difference in vocabulary between this thesis and [64]. For us, a simple triangulation is a triangulation without loop nor multiple edge, while for Tutte there are the triangulations in which every 3-cycle defines a face, i.e. no separating triangle. We will call the latter *irreducible triangulations*. In the rest of the thesis, we will refer as 3-connected (resp. 4-connected) simple triangulations those that are 3-connected resp. 4-connected) and without loop nor multiple edge.

For $n \geq 1$, let t_n be the number of 3-connected triangulations on $n+2$ vertices. So that $T(z) = \sum_{n\geq 1} t_n z^n$ is the (ordinary) generating function counting 3-connected triangulations, where the variable z marks the number of vertices minus two. Notice that we add here the single triangle, whose generating function is z, which is not 3-connected but will be counted by $T(z)$. Observe now that if one removes the root edge of a given 3-connected triangulation, which is not the single triangle, then the resulting map (re-rooted in some predefined canonical way) will have a root face of size four: it will be the triangulation of a square. This is because the root edge was adjacent to two triangles, the one defining the root face and another one. The two other edges of the latter triangle are now part of the boundary of the new root face. In general, the map resulting from the removal of the root edge of a triangulation of a k-gon is a triangulation of a $(k+1)$-gon.

Near triangulations. If we want to apply the root-decomposition method, it is then natural to introduce a new class of planar maps: the *near triangulations*. They are simple 3-connected planar maps in which every face has degree three but the root face, which has degree at least three, thus they include the 3-connected triangulations.

Conversely, let us now consider a near triangulation η, whose root edge (a, b) is adjacent to a k-gon, the root face, and to an *inner triangle* whose third vertex is c. Two cases arise: there are either $r > 0$ edges, different from $\{a, c\}$ and $\{b, c\}$, and directly connecting c to an *external vertex*, i.e. a vertex incident to the root face, or zero such edge (the case where c is directly incident with the root face is impossible as the map would then have a 2-vertex cut), as illustrated by Figure 2.8. Both cases can be obtained by considering a (finite) sequence of at least one near triangulations $\eta_1, \ldots, \eta_m$, with respective root edges $(a_1, b_1), \ldots, (a_m, b_m)$ and whose third unmarked external vertices are respectively $c_1, \ldots, c_m$, where (a_i, c_i) is an external edge of η_i ($i \in [m]$). Then by identifying together the following edges: (a_1, c_1) with (a_2, b_2), …, (a_{m-2}, c_{m-2}) with (a_{m-1}, b_{m-1}), and (a_{m-1}, c_{m-1}) with (a_m, b_m). And by finally adding a directed edge between c_m and b_m in the root face of the newly obtained map. We know obtained the near triangulation η, with $a = c_m$, $b = b_1$ and $c = a_1 = a_2 = \ldots = a_m$. Notice that in the second case, η_1 cannot have a root face of size three and that this all process excludes the single triangle.

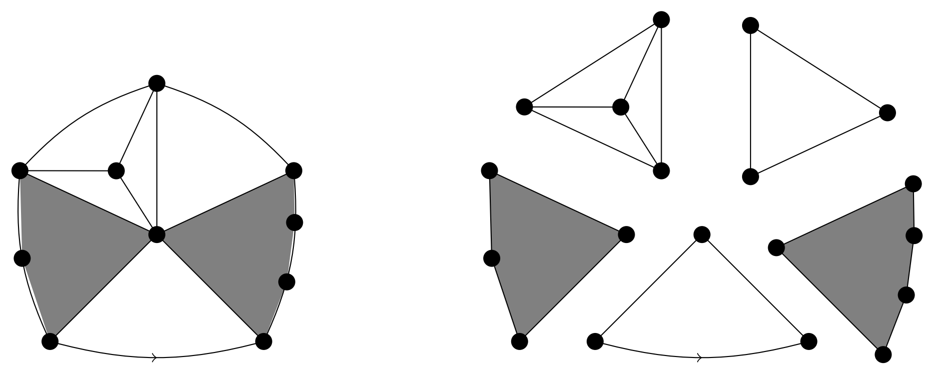


Figure 2.8: A near triangulation and its decomposition into two smaller near triangulations (the grayed areas) and two triangulations: K_3 and K_4.

A functional equation with a catalytic variable. As motivated above, we first introduce a new variable, u marking the number of external vertices minus three. The reason for the minus three is to avoid any over-counting when identifying edges of near triangulations. So that if for $n \geq 5$ and $k \geq 3$, $a_{n,k}$ is the number of near-triangulations with $n+2$ vertices and $k+3$ external vertices, so that $N(z,u) = \sum_{n,k\geq 0} a_{n,k} z^n u^k$ is the generating function of near triangulations, then the above combinatorial decompositon translates into the following functional equation:

$$N(z,u) = z + \frac{1}{u^2}\left(\frac{uN(z,u)}{1-uN(z,u)} - uT(z)\right). \tag{2.3}$$

The explanation is as follows. A near triangulation is either the single triangle or a non-empty sequence of near triangualtions. When the sequence has size one, then we must make sure that it is not a 3-connected triangulation as they have a triangular root face, thus the correcting term $-uT(z)$. The factors u and u^{-2} adjust the number of external vertices when identifying external edges of near triangulations of the sequence and when creating the new root edge, respectively.

Observe now that if we want to obtain $T(z)$ from Equation (2.3), we cannot just set $u = 0$. This behaviour is rather characteristic of a sort of equations that are said to have a *catalytic variable* (in our case u), as baptised by Zeilberger in [73].

The quadratic method. Let us first transform (2.3) into a polynomial equation in $N \equiv N(z,u)$ by multiplying both sides by $u(1-uN)$:

$$u^2N^2 + (uT + 1 - zu^2 - u)N + zu - T = 0, \tag{2.4}$$

where $T \equiv T(z)$. Observe again that setting $u = 0$ here would just give us $N(z,0) = T(z)$ which is true but not very interesting.

When the polynomial in N is quadratic, Tutte's trick is to *complete the square* so that one can eliminate the variable N, similarly to the classical resolution of quadratic equations. Notice that this amounts in fact to computing the discriminant of (2.4) with respect to N. Now, suppose that we already know $T(z)$ to be algebraic and with a minimal polynomial of genus zero. Then it will admit a rational parametrisation with an algebraic function. In [64], Tutte guessed such a parametrisation from the discriminant of (2.4) with respect to N:

$$T(z) = U(z)\left(1 - 2U(z)\right), \tag{2.5}$$

where $U(z)$ is an algebraic function defined by

$$U(z)(1-U(z))^3 - z = 0. \tag{2.6}$$

From there, one can obtain an annihilating polynomial of $T(z)$ by eliminating $U(z)$ from the system composed of Equations (2.5) and (2.6).

This method was later formalised by Brown in [13] and is known as the *quadratic method*. It has then been generalised to any polynomial with one catalytic variable in [8]. In our case, it works as follows. As we want a functional equation in T and z only, we would like to eliminate the variables N and u from (2.4). To do so, we iteratively compute the discriminant of (2.4), first with respect to N (thus obtaining a polynomial equation in T, z and u) then with respect to u.

Minimal polynomial of $T(z)$. In both cases, we end up with the following polynomial equation:

$$T^4 + 3T^3 + T^2(3 + 8z) + T(1 - 20z) + 16z^2 - z = 0, \tag{2.7}$$

This is in fact the minimal polynomial of $T(z)$ as we already removed a factor $-16z^3$ to make it irreducible. One of the solutions of (2.7) admits a unique local expansion near 0 in formal power series with non-negative coefficients. This is the generating function $T(z)$ whose first terms are

$$T(z) = z + z^2 + 3z^3 + 13z^4 + 68z^5 + 399z^6 + 2530z^7 + \cdots$$

Closed formula. Even though one does not really need Tutte's parametrisation to obtain the minimal polynomial of $T(z)$, it becomes useful if one wants to find a closed formula for the number of 3-connected triangulations. Indeed, Equation (2.6) has a so-called *Lagrangian form*, which means that one can apply Langrange-Bürmann inversion formula. So that with $\phi(y) = \frac{1}{(1-y)^3}$ and $H(y) = y$, we have

$$[z^n]U(z) = \frac{1}{n}[y^{n-1}]\left(\frac{1}{1-y}\right)^{3n} = \frac{3}{4n-1}\binom{4n-1}{n-1}, \tag{2.8}$$

where the last inequality is obtained by applying the binomial theorem with a negative entry on $(1-y)^{-3n}$. We can directly verify this as there is a unique formal power series solution of (2.6) with non-negative coefficients

$$U(z) = z + 3z^2 + 15z^3 + 91z^4 + 612z^5 + 4389z^6 + 32890z^7 + \cdots$$

Finally substituting $U(z)$ in (2.5) by the right hand-side of (2.8) gives us the closed formula for the number of 3-connected triangulations, as discovered by Tutte in [64]:

$$[z^n]T(z) = \frac{2(4n+1)!}{(n+1)!(3n+2)!}.$$

Asymptotic enumeration of $U(z)$. As we will need it in Chapter 3, we first compute both the local expansion of the generating function $U(z)$, together with an asymptotic estimate of its coefficients.

As it is irreducible, Equation (2.6) is in fact the minimal polynomial of $U(z)$. So that its discriminant with respect to $U(z)$,

$$z^2(256z - 27) = 0$$

has two solutions (up to multiplicity). As $U(z)$ has only non-negative coefficients, its dominant singularity is the unique solution which is a positive (non zero) real number, namely $\tau = 27/256$.

Now if we compute the possible Puiseux expansions of $U(z)$ for z near τ, we obtain the three following truncations, setting $Z = \sqrt{1 - z/\tau}$:

$$\frac{5 + i\sqrt{2}}{4} - \frac{8 + 7i\sqrt{2}}{96} Z^2 - \frac{1184 + 1117i\sqrt{2}}{41472} Z^4 + O(Z^6),$$
$$\frac{5 - i\sqrt{2}}{4} - \frac{8 - 7i\sqrt{2}}{96} Z^2 - \frac{1184 - 1117i\sqrt{2}}{41472} Z^4 + O(Z^6),$$
$$\frac{1}{4} - \frac{\sqrt{6}}{8} Z + \frac{1}{12} Z^2 - \frac{31\sqrt{6}}{1728} Z^3 + O(Z^4).$$

Observe that the first two are conjugate from one another but that both are analytic at $z = \tau$. So the only remaining candidate for the local expansion of $U(z)$ is the third one, for which only the first two summands suffice:

$$U(z) \underset{z \to \tau}{\sim} \frac{1}{4} - \frac{\sqrt{6}}{8} \sqrt{1 - \frac{z}{\tau}}. \tag{2.9}$$

Applying finally the theorem of sim-transfer on (2.9), we obtain the following asymptotic estimate:

$$[z^n]U(z) \underset{n \to \infty}{\sim} \frac{\sqrt{3}}{8\sqrt{2\pi}} \cdot n^{-3/2} \cdot \left(\frac{256}{27}\right)^n.$$

Asymptotic enumeration of $T(z)$. By transfer of singularity, Equation (2.5) tells us that $T(z)$ has the same dominant singularity has $U(z)$, namely $\tau = 27/256$ (this can be verified by computing the discriminant of (2.7), the minimal polynomial of $T(z)$, with respect to T). And from the polynomial (2.7), one can compute the first terms of the only Puiseux expansion of $T(z)$ near τ that is not analytic at τ. So that:

$$T(z) \underset{z \to \tau}{\sim} \frac{1}{8} - \frac{3}{16} \left(1 - \frac{z}{\tau}\right)^2 + \frac{\sqrt{6}}{24} \left(1 - \frac{z}{\tau}\right)^{3/2}. \tag{2.10}$$

And applying the theorem of *sim-transfer* on (2.10), gives the following asymptotic estimate:

$$t_n = [z^n]T(z) \underset{n \to \infty}{\sim} \frac{\sqrt{3}}{16\sqrt{2\pi}} \cdot n^{-5/2} \cdot \left(\frac{256}{27}\right)^n.$$

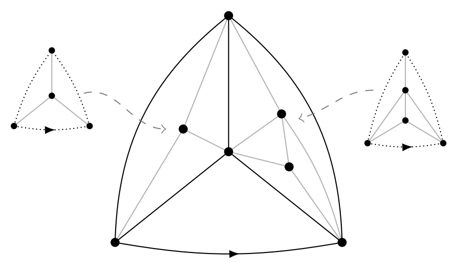


Figure 2.9: Two 3-connected triangulations pasted on the faces of an irreducible triangulation, here K_4.

2.4.2 Composition schemes

In [64], Tutte also used his results on 3-connected triangulations to study the generating function of simple 4-connected triangulations via a *composition scheme*. Thus leading to a closed formula for the number of simple 4-connected triangulations.

Similarly in 1968, Mullin and Schellenberg [47] found another composition scheme relating the family of irreducible quadrangualtions with that of simple quadrangulations.

Irreducible triangulations. Whitney showed in [70] that a triangulation is irreducible if and only if it is either 4-connected or K_4, the graph of the tetrahedron, or the single triangle K_3.

In [64], Tutte remarked that any 3-connected triangulation can be obtained recursively by replacing each inner (non root) face of an irreducible triangulation, that is not the single triangle, by some 3-connected triangulation (see Figure 2.9). Notice that because the single triangle is counted as a 3-connected triangulation, substituting a triangular face with nothing is like substituting it with a triangle. As irreducible quadrangulations are composed of the single triangle, the tetrahedron and the 4-connected triangulations, this gives us a way to access the generating function of the latter from that of 3-connected triangulations.

4-connected triangulations. For $n \geq 4$, let f_n be the number of 4-connected triangulations with $n + 2$ vertices. So that $T_4(y) = \sum_{n \geq 4} f_n y^n$

is the generating function counting 4-connected triangulations, where the variable y also marks the number of vertices minus two.

The above discussion can be interpreted in terms of the following functional equation:

$$T(z) = z + z^2(z^{-1}T(z))^3 + \frac{T_4(z(z^{-1}T(z))^2)}{z^{-1}T(z)}. \tag{2.11}$$

It is explained as follows. Substituting a triangular face by any 3-connected triangulation Γ is encoded by the generating function $z^{-1}T(z)$, as we want the three external vertices of Γ to be unmarked by the variable z (two originally and one more from the factor z^{-1}). The right hand-side is composed of the single triangle whose unique inner face is not substituted, the tetrahedron whose three inner faces are substituted, and of any 4-connected triangulation whose inner faces are substituted. The latter substitution is encoded by composition of the variable y in $T_4(y)$, which marks vertices minus two, with the generating function $z(z^{-1}T(z))^2$. The square is there to substitute faces instead of vertices, as Euler's formula tells us that a simple triangulation with $n+2$ vertices has $2n$ faces. And the factors z (or z^2 for the tetrahedron) exist to keep track of the original number of vertices. Finally, the division by $z^{-1}T(z)$ is there to ensure that we do not substitute the root face.

Minimal polynomial of $T_4(y)$. One can then rewrite (2.11) into two polynomial equations:

$$\begin{aligned} &T^4 - zT^2 + z^2T + z^2T_4 = 0,\\ &T^2 - zy = 0, \end{aligned}$$

where $T_4 \equiv T_4(y)$ and $T \equiv T(z)$. So that together with Equation (2.7), they form a polynomial system from which one can eliminate the two variables T and z. Thus resulting in the following single polynomial equation:

$$\begin{aligned} &(y^2 + 8y + 16)T_4^3 + (3y^4 + 21y^3 + 21y^2 - 28y)T_4^2\\ &+ (3y^6 + 18y^5 - 3y^4 - 26y^3 + 11y^2 - y)T_4\\ &+ y^8 + 5y^7 - 8y^6 + y^5 = 0, \end{aligned} \tag{2.12}$$

which is irreducible and is hence the minimal polynomial of $T_4(y)$. And the unique solution of Equation (2.12) that admits a formal power series expansion at $y = 0$ with non-negative coefficients is:

$$T_4(z) = y^4 + 3y^5 + 12y^6 + 52y^7 + 241y^8 + 1173y^9 + 5929y^{10} + \cdots$$

Rational parametrisation of $T_4(z)$. Alternatively, after observing that Equation (2.12) has genus zero, one could substitute T by the right hand-side of Equation (2.5), as well as z by $U(z)(1-U(z))^3$ (see (2.6)) into (2.11) to obtain, after some manipulations, the following rational parametrisation

$$T_4(y) = y + \frac{V(y)(V(y)-1)}{(1+V(y))^2} - y^2, \tag{2.13}$$

where $V(y)$ is an algebraic function defined by

$$V(y)(1-V(y))^2 - y = 0. \tag{2.14}$$

Similarly to above, one could exploit the fact that (2.14) admits a *Lagrangian form* to obtain a closed, although quite involve, formula for the coefficients of $T_4(z)$ (see [64, Section 7]).

Asymptotic enumeration of $T_4(z)$. Now taking the discriminant of (2.12) with respect to T_4 gives

$$-y^3(27y-4)^3 = 0. \tag{2.15}$$

From there, one can directly deduce that the dominant singularity of $T_4(y)$ is $\varsigma = 4/27$. And the only solution of (2.12) that admits a Puiseux expansion, for $y \to \varsigma$, that is not analytic at ς is given by:

$$T_4(y) \underset{y\to\varsigma}{\sim} \frac{7}{5832} + \frac{245}{23328}\left(1-\frac{y}{\varsigma}\right) + \frac{\sqrt{3}}{96}\left(1-\frac{y}{\varsigma}\right)^{3/2}. \tag{2.16}$$

So that an application of the theorem of *sim-transfer* on (2.16), gives the following asymptotic estimate for the number of 4-connected triangulations:

$$f_n = [y^n]T_4(y) \underset{n\to\infty}{\sim} \frac{\sqrt{3}}{128\sqrt{\pi}} \cdot n^{-5/2} \cdot \left(\frac{27}{4}\right)^n.$$

Critical composition scheme. Let us now observe a bit the *composition scheme equation* (4.14) relating $T(z)$ with $T_4(z)$. In particular, notice that in the composition $T_4(z^{-1}T(z)^2)$ it holds that

$$\tau^{-1}T(\tau)^2 = \varsigma, \tag{2.17}$$

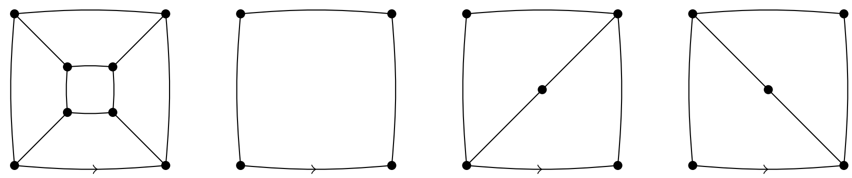


Figure 2.10: From left to right. the unique irreducible quadrangulation with five inner faces: the graph of the cube, the single quadrangle, then the two symmetric ones counted by $2z^2$.

where τ and ς are the respective dominant singularities of $T(z)$ and $T_4(y)$. Indeed, as proven in (2.10) $T(\tau) = 1/8$ so that $\frac{256}{27\cdot 64} = \frac{4}{27}$. In this case, we call the above composition scheme *critical*. Equation (2.17) gives us in fact a mean to directly compute ς, the dominant singularity of $T_4(y)$, given that we know the dominant singularity $T(z)$ as well as its singular expansion. There is a whole family of combinatorial classes for which this type of critical composition holds, as such it defines a combinatorial characterisation that is universal. This is typically the case for planar-like families of graphs.

If, in Equation (2.17), it were instead that $\tau^{-1}T(\tau)^2 > \varsigma$, the composition scheme would be called *sub-critical*. Notable classes of connected graphs for which the latter holds are trees, cactus graphs, outerplanar graphs or series-parallel graphs.

Irreducible quadrangulations. A quadrangulation is said to be *irreducible* if it has no separating quadrangle, i.e. if every 4-cycle defines a face. We denote by $\mathcal{S}$ the set of all irreducible quadrangulations with at least five inner faces. For any $n \geq 6$, let s_n be the number of irreducible quadrangulations with n inner faces. So that, the generating function $S(t) = \sum_{n\geq 0} s_n t^n$ counts irreducible quadrangulations (with at least five inner faces), where the variable t marks the number of inner (non root) faces. Notice that by considering only the irreducible quadrangulations with at least five inner faces, we exclude the single quadrangle, whose generating function is z, and the two symmetric quadrangulations counted by $2z^2$ (see Figure 2.10). The reason will be made more clear when decompsing simple quadrangulations. In the rest of this thesis, an irreducible quadrangulation will always have at least five inner faces.

Bijection with 3-connected maps. The generating function $S(t)$ is well-known, since $\mathcal{S}$ is in bijection with rooted 3-connected planar maps counted by number of edges (see [65] or [47], and [3]). This bijection is obtained as follows. Start from a 3-connected map and take its medial, it is a 4-regular map, then take the dual to obtain an irreducible quadrangulation with at least five inner faces. Now setting $x = 1$ in the system for 3-connected maps, composed of Equations (9) and (10) in [3], gives us the following rational parametrisation of $S(t)$:

$$S(t) = \frac{2t}{1+t} - t - \frac{W(t)^2}{(1+2W(t))^3 t}, \tag{2.18}$$

where the algebraic generating function $W(t)$ is defined by

$$W(t) = (1 + W(t))^2 t. \tag{2.19}$$

Notice that $W(t) = \sum_{n \geq 0} C_n t^n$ is the generating function of the Catalan numbers.

Minimal polynomial of $S(t)$. Eliminating $W(t)$ from the system composed of Equations (2.18) and (2.19) gives an annihilating polynomial of $S(t)$. The only factor of the resulting annihilating polynomial whose root, expressed as a formal power series of t near zero, has non-negative coefficients and is

$$t^5 + 4t^7 + 6t^8 + 24t^9 + 66t^{10} + 214t^{11} + 676t^{12} + \cdots$$

is in fact the minimal polynomial of $S(t)$ and is given by

$$\begin{aligned} &t^7 + 4t^6 - t^5 + (2t^6 + 12t^5 + 20t^4 + 10t^3 - 5t^2 - 4t + 1)S \\ &+ (t^5 + 8t^4 + 25t^3 + 38t^2 + 28t + 8)S^2 = 0. \end{aligned} \tag{2.20}$$

Asymptotic enumeration of $S(t)$. The discriminant of (2.20) with respect to S is

$$(4z - 1)^3 (z + 1)^4 = 0,$$

whose unique positive root 1/4 is the dominant singularity of $S(t)$. And the unique Puiseux expansion near 1/4 computed from (2.20) and that is not analytic at 1/4 is

$$S(t) \underset{t \to \frac{1}{4}}{\sim} \frac{1}{540} - \frac{167}{8100}(1 - 4t) + \frac{32}{729}(1 - 4t)^{3/2}. \tag{2.21}$$

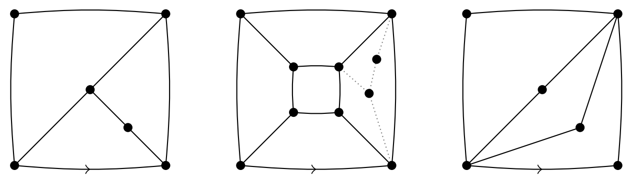


Figure 2.11: A typical simple quadrangulation (left) pasted in the inner face of an irreducible quadrangulation (middle), thus depicting a quadrangulation in $\mathcal{R}$. Right is the quadrangulation κ in which either one of the two inner faces f_1 or f_2 has been substituted by the quadrangulation κ.

So that an application of the theorem of *sim-transfer* on (2.21) gives the following asymptotic estimate for the number of irreducible quadrangualtions

$$s_n = [t^n]S(t) \underset{n\to\infty}{\sim} \frac{8}{243\sqrt{\pi}} \cdot n^{-5/2} \cdot 4^n.$$

Simple quadrangulations. Let $\mathcal{Q}$ be the class of all simple quadrangulations and for $n \geq 1$, let q_n be the number of simple quadrangulations with n inner faces. Its associated generating function will be denoted by $Q(z) = \sum_{n\geq 1} q_n z^n$.

In [47], the authors showed that simple quadrangulations can be decomposed via a composition scheme from the irreducible ones: take an irreducible quadrangulation and replace its inner faces by a simple quadrangulation. Let then $\mathcal{R}$ be the sub-class of $\mathcal{Q}$ containing all simple quadrangulations obtained from $\mathcal{S}$ by replacing each internal face with a quadrangulation in $\mathcal{Q}$, all illustrated by the two leftmost pictures of Figure 2.11. Its associated generating function will be denoted by $R(z) = S(Q(z))$, where the variable z marks inner faces.

Quadrangulations with a diagonal. Now the reasons we excluded the single quadrangle and the quadrangulations counted by $2z^2$ from the generating function $S(t)$ counting irreducible quadrangulations can be made clear. First, it would be meaningless to replace the inner face of the single quadrangle. Secondly, let us define the quadrangulation κ as the third quadrangulation from the left of Figure 2.10, whose generating function is z^2. κ has two inner faces, f_1 the one incident with the root edge and f_2. Notice then that if one would replace f_1 by the simple quadrangulation κ, then the resulting simple quadrangulation would be exaclty the same as the one obtained by instead having replaced f_2 by κ (see the rightmost picture of Figure 2.11).

This would break the unicity of the decomposition. Hence we need to treat this case separately and introduce the following new sub-class of simple quadrangulations: $\mathcal{N}$ is the class of simple quadrangulations containing a *diagonal* incident with the root vertex, where a diagonal is a path of length two joining two vertices incident with the root face and passing through an inner vertex. By symmetry they are in bijection with simple quadrangulations containing a diagonal not incident with the root vertex. The smallest quadrangulation in $\mathcal{N}$ is κ, as defined above, and its symmetric is the rightmost quadrangulation of Figure 2.10. The generating function associated with $\mathcal{N}$ will be denoted by $N(z)$, where the variable z marks inner faces. Observe that any quadrangulation with a diagonal, other than κ or its symmetric, admits a 4-cycle that does not define a face.

Recursive composition scheme. In [47], the authors showed that the following system of equations holds:

$$\begin{aligned} Q(z) &= z + 2N(z) + S(Q(z)), \\ N(z) &= Q(z) \cdot (z + N(z) + S(Q(z))). \end{aligned} \tag{2.22}$$

Indeed, a simple quadrangulation is either the single quadrangle, a quadrangulation in $\mathcal{R}$, or the quadrangulations containing a diagonal (where the factor 2 stands for the symmetric ones). Those classes form in fact a partition of $\mathcal{Q}$. This justifies the first equation. Now for the second equation, notice that a quadrangulation with a diagonal adjacent to the root vertex is obtained by substituting the two inner faces of κ as follows: f_2 can be substituted by any simple quadrangulation, but f_1 can only be substituted by simple quadrangulations not in $\mathcal{N}$ (the class of symmetrics of $\mathcal{N}$ can be decomposed similarly by considering the symmetric of κ as based quadrangulation).

Minimal polynomial of $Q(z)$. We first rewrite (2.22) as a polynomial system of equations, where $Q \equiv Q(z)$, $S \equiv S(t)$ and $N \equiv N(z)$, as follows:

$$\begin{aligned} &Q - t = 0, \\ &z + 2N + S - Q = 0, \\ &(z + N + S)Q - N = 0. \end{aligned} \tag{2.23}$$

So that from the system composed of (2.23) and the minimal polynomial (2.20) of $S(t)$, one can eliminate the variables S, t, and N to obtain the following annihilating polynomial of $Q(z)$:

$$z^2Q^3 + (6z^2 + 2z)Q^2 + (12z^2 - 10z + 1)Q + 8z^2 - z = 0, \tag{2.24}$$

which is irreducible and hence the minimal polynomial of $Q(z)$.

Asymptotic enumeration of $Q(z)$. The discriminant of (2.24) is

$$-z^3(27z - 4)^3,$$

whose unique positive root is $\varsigma = 4/27$ (the same as 4-connected triangulations). And the unique Puiseux expansion near ς computed from (2.24) and that is not analytic at ς is given by

$$Q(z) \underset{z \to \varsigma}{\sim} \frac{1}{4} - \frac{3}{4}\left(1 - \frac{z}{\varsigma}\right) + \frac{2\sqrt{3}}{3}\left(1 - \frac{z}{\varsigma}\right)^{3/2}. \tag{2.25}$$

So that an application of the theorem of *sim-transfer* on (2.25) gives us the following asymptotic estimate on the number of simple quadrangulations with n inner faces:

$$q_n = [z^n]Q(z) \underset{n \to \infty}{\sim} \frac{\sqrt{3}}{2\sqrt{\pi}} \cdot n^{-5/2} \cdot \left(\frac{27}{4}\right)^n.$$

Bijective methods. Equations (2.5), (2.13) and (2.18) can however be derived by some more direct means. Indeed, one can find some bjective correspondance between classes of maps and some simpler sets whose enumeration is known. Cori and Vauquelin [18] first found such a bijection between general planar maps and some tree structure called *well labelled trees*, later rediscovered by Arquès [1]. But the real development of this method came with the thesis of Schaeffer [60], who was able to re-obtain most of Tutte's results through bijections. Finally, Bouttier, Di Francesco and Guitter [10] found another bijection between classes of maps and so-called *mobiles*, which allows one to clearly control the degrees of the vertices (see also [5] for a generalisation).

Two other noteworthy advantages of the bijective methods are to provide both some powerful algorithmic tools to generate random planar graphs, via *Boltzmann samplers* [26], and some random models to study the scaling limit of random planar maps [9].

2.4.3 Connectivity decompositions

Let $\mathcal{G}$ be a family of (simple) labelled graphs, with its associated generating function $G(x) = \sum_{n\geq 0} g_n \frac{x^n}{n!}$, where the variable x encodes vertices.

Component stable classes. A graph $\gamma \in \mathcal{G}$ can always be considered as the disjoint union of its connected components. Notice then that the connected components of $\gamma \in \mathcal{G}$ can be formed from any connected graph in $\mathcal{G}$. So that if we define the subclass $\mathcal{G}_1$ as containing every connected graphs in $\mathcal{G}$, then the symbolic method translates the above relationship into a $\mathcal{G} = Set(\mathcal{G}_1)$. This allows us to derive the generating function $G_1(x) = \sum_{n\geq 0} c_n \frac{x^n}{n!}$, associated with the class $\mathcal{G}_1$, from $G(x)$ as follows

$$G(x) = e^{G_1(x)}. \tag{2.26}$$

This way, we can always derive the subclass $\mathcal{G}_1$ of connected graphs of a given labelled graph class $\mathcal{G}$. Furthermore, a graph class $\mathcal{G}$ defined by a property P is said to be *component stable* when $\mathcal{G}_1$, the class of connected graph in $\mathcal{G}$, is also the class of connected graphs having property P. Classes that are component stables include for example the class of cycle-free graphs (i.e. forests), or that of planar graphs.

We will see in the rest of this section that similar derivations for the subclasses of 2-connected or 3-connected graphs, athough more involve, can be expressed using the symbolic method.

Rooted connected graphs. Let $\mathcal{G}$ be a component stable class of graphs and let $\gamma \in \mathcal{G}_1$ be a connected graph in $\mathcal{G}_1$. If γ has n vertices, then there are n ways to root γ at a vertex v (when we root at a vertex, we keep its label). As explained before, we denote by $\mathcal{G}_1^\bullet$ the class of (vertex-)rooted graphs in $\mathcal{G}_1$. And the associated generating function $G_1^\bullet(x)$ can be obtained as follows

$$G_1^\bullet(x) = xD_xG_1(x) = xG_1'(x) = \sum_{n\geq 1} nc_n \frac{x^n}{n!}. \tag{2.27}$$

Block-decomposition. Let $\mathcal{G}_1^\bullet$ be the subclass of rooted connected graphs of a component stable class of labelled graphs $\mathcal{G}$, and let $\gamma \in \mathcal{G}_1^\bullet$. Then γ can be decomposed along its root vertex v as follows.

Suppose that v is not a cut vertex of γ. Then γ is simply a block β, i.e. a maximal 2-connected component, rooted at v in which every other vertex u is substituted by a connected graph in $\mathcal{G}_1^\bullet$ (and u is a cut vertex of β) whose root vertex is identified with u (we consider here the graph reduced to a single vertex as connected). Now if v is a cut vertex itself, then γ is decomposed as a set (and not as a sequence, as graphs are not embedded) of rooted blocks for which: each root vertex is identified with v (so that its label was forgotten) and each other vertex is substituted by graphs in $\mathcal{G}_1^\bullet$ as described above. Observe that such a rooted block with substituted vertices can also be a single rooted edge with substituted vertex.

Notice the, that this decomposition can be carried recursively on the rooted blocks, whose vertices are substituted, adjacent to the root vertex, so that what remains at the end of the recursion is a set of rooted blocks or edges, see Figure 2.12. In [34], it can be further observed that this terminal set has a particular tree-like structure: consider a tree whose vertices are the terminal rooted blocks and rooted edges, and where two vertices are adjacent when their corresponding rooted blocks or edges have their root vertices identified together in the recursion. This is indeed a tree as otherwise any cycle with vertices $v_1, \ldots, v_t$, respectively associated to the rooted blocks or edges $b_1, \ldots, b_t$ would induce a bigger block in γ containing all the blocks $b_1, \ldots, b_t$, thus contradicting the maximality of each b_i (for $i \in [t]$). Such a structure is called a *block-decomposition* of the rooted connected graph γ.

Rooted 2-connected graphs. Let us now denote by $\mathcal{G}_2$ the subclass of all the 2-connected graphs in $\mathcal{G}$, with its associated generating function $G_2(x)$.

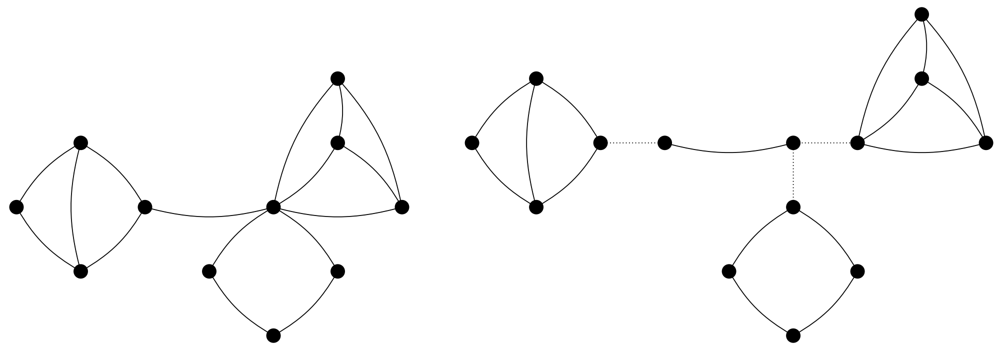


Figure 2.12: An iterated block-decomposition of a connected graph.

A class of graphs $\mathcal{G}$, defined by a property P, is said to be *block stable* when $\mathcal{G}_2$ is the class of 2-connected graphs verifying property P. Suppose now that $\mathcal{G}$ is both component and block stable. Then applying the symbolic method to what was said above, the generating function $G_2^\bullet(x) = xG_2'(x)$, associated to the class of rooted 2-connected graphs in $\mathcal{G}$, can be derived from that of $\mathcal{G}_1^\bullet$ as follows (see [34] for a complete proof and further details):

$$G_1^\bullet(x) = xe^{G_2'(G_1^\bullet(x))}. \tag{2.28}$$

Notice that Equation (2.28) is a composition scheme. There seems to be a universal behaviour in the asymptotic enumeration of a labelled graph class for which the scheme is critical and another one for when it is sub-critical. In the first case, for example the family of planar graphs, the generating function counting the connected graphs in the class will admit a (singular) Puiseux expansion of type 5/2. While in the second case, it will be of type 3/2. Known classes for which the scheme is sub-critical are forests, cactus graphs, outerplanar graphs and series-parallel graphs.

Enumeration of some labelled sub-critical graph classes. Such decomposition scheme can alternatively be used to enumerate a family of graphs that is component and block stable, when the generating function counting the 2-connected graphs of the family is known. This was for example the methods followed by Bodirsky, Giménez, Kang and Noy to enumerate both outerplanar and series-parallel labelled graphs:

Theorem ([6])**.** *Let ℓ_n and s_n respectively be the numbers of labelled outerplanar and of series parallel graphs on $n \in \mathbb{N}$ vertices. The the following two estimates hold:*

$$\ell_n \underset{n\to\infty}{\sim} \ell \cdot n^{-5/2} \cdot \lambda^{-n} \cdot n!, \qquad s_n \underset{n\to\infty}{\sim} s \cdot n^{-5/2} \cdot \sigma^{-n} \cdot n!,$$

where $\ell \approx 0.0182016$, $\lambda^{-1} \approx 7.321$, $s \approx 0.10131 \cdot 10^{-2}$ and $\sigma^{-1} \approx 7.812267$.

As the 2-connected blocks of trees are simply rooted edges, they form a nice example of a block decomposition. We will briefly present it next, following the monograph of Moon [46], but extended to forests.

Enumeration of labelled forests. Labelled graph classes that are component and block stable, and without 3-connected components, such as forests can be decomposed following a connectivity-decomposition scheme if the subclass 2-connected components is known. In the case of labelled forests, the 2-connected forests are simply edges. So that the (exponential) generating function counting the number of 2-connected forests by vertices is $x^2/2$.

Consider now the subclass of connected labelled forests, i.e. labelled trees. Applying the block decomposition on the class of trees rooted at a vertex gives us the following functional equation for the (exponential) generating function $T(x)$ counting trees by vertices:

$$xT'(x) = xe^{xT'(x)}. \tag{2.29}$$

Observe that Equation (2.29) has a *Lagrangian form* $u(x) = x\phi(u(x))$ with $u(x) = xT'(x)$, $H(y) = y$ and $\phi(y) = \exp(y)$. So that the *Lagrange-Bürmann inversion theorem* gives

$$[x^n]xT'(x) = \frac{1}{n}[y^{n-1}]\sum_{k\geq 0}\frac{(ny)^k}{k!} = \frac{n^{n-2}}{(n-1)!} = \frac{n^{n-1}}{n!}.$$

Notice now that Equation (2.27), applied to the case of trees, can be reintepreted into the following relationship between the number of rooted labelled trees on n vertices and that of (unrooted) labelled trees on n vertices:

$$n![x^n]T(x) = \frac{1}{n}n![x^n]xT'(x) = n^{n-2},$$

which is known as *Cayley's formula*.

Now using Equation (2.26), we can access the number f_n of forests with $n \geq 1$ vertices. For that, we will first compute the number $f_{n,k}$ of such forests but composed of $k \geq 1$ disjoint trees. It is given by

$$f_{n,k} = n![x^n]\frac{T(x)^k}{k!} = \binom{n-1}{k-1} n^{n-k}.$$

So that:

$$f_n = \sum_{k=1}^{n} \binom{n-1}{k-1} n^{n-k}.$$

Network-decomposition. Similarly to the decompositon of a connected graph, rooted at a vertex, along its cut vertices, we will now see how to decompose a 2-connected graph $\beta \in \mathcal{G}_2$, rooted at a pair of vertices $\{s, t\}$, along its 2-vertex cuts.

This was one of the motivations for Trakhtenbrot [63] and independently Tutte [66] to introduce the notion of *networks*: a network is a graph with two distinguished vertices, called *poles*, such that the graph obtained by adding an edge between the two poles (if they were not already adjacent) is 2-connected. The set of networks can be decomposed following a partition (see [63] or [29]) into the three following subsets, plus the single edge:

- The *series networks* are the sequences of $k \geq 2$ non-series networks, where the second pole of the i-th network is identified with the first pole of the $(i+1)$-th network. The poles of the resulting network are then the first pole of the first network and the second pole of the last network in the sequence. Notice that one can make this decomposition unique by observing that a series network is the composition of a network which is not series with an arbitrary network.

- The *parallel networks* are those obtained by gluing two or more non-parallel networks, none of them containing the root edge, along the common poles.

- The *h-networks* are those obtained as follows. Consider a 3-connected graph, the *core*, rooted at a directed edge e (it is in fact itself an h-network). And replace each edge $ab \neq e$ of the core by a network with ditinguished vertices (s, t). The replacement is done by first removing the edge ab from the core and then by identifying the vertices a with s and b with t.

So that when decomposing β along $\{s,t\}$, two cases can arise: either $\{s,t\}$ is or is not a 2-cut of β. In the first case, β forms either a series or a parallel network. While in the second case, β forms an h-network. Tutte proved in [66] that by applying such a decomposition recursively, one ends up with a tree-like structure, whose vertices are associated to the networks induced by the decomposition, and two vertices of the tree form an edge of the tree when their two associated networks are composed together in a bigger network. Notice then that the leaves of this tree are associated to the cores of the h-networks.

Rooted 3-connected graphs. Looking at the above network-decomposition, one can see that rooted 3-connected graphs are the essential bricks of the decomposition of rooted 2-connected graphs. This decomposition can indeed be translated by Walsh [69] into a system of functional equations between exponential generating functions.

To that end, let $\mathcal{G}$ be a class of graphs that is both component and block stable. And let us denote by $\mathcal{G}_3$ the subclass of all the 3-connected graphs in $\mathcal{G}$, with its associated generating function $G_3(x)$. A class of graphs $\mathcal{G}$, defined by a property P, is said to be *brick stable* when $\mathcal{G}_3$ is the class of 3-connected graphs verifying property P.

Let finally $\vec{G}_3(x,y) = \frac{2y}{x^2} \cdot \frac{\partial}{\partial y} G_3(x,y)$ be the bivariate exponential generating function counting the graphs in $\mathcal{G}_3$ that are rooted at a directed edge whose two endpoints are distinguished, and where the variables x marks vertices and y marks edges. So that if $N(x,y)$, $S(x,y)$, $P(x,y)$ and $H(x,y)$ respectively are the bivariate exponential generating functions of arbitrary, series, parallel and h-networks, then the following system of equations holds, as shown in [69] (see also [29]):

$$
\begin{aligned}
N(x) &= \frac{2y}{x^2} \cdot \frac{\partial}{\partial y} G_2(x,y), \\
N(x,y) &= 1 + S(x,y) + P(x,y) + H(x,y), \\
S(x,y) &= N(x,y) \cdot x \cdot (N(x,y) - S(x,y)), \\
P(x,y) &= \exp_{\geq 2}(1 + N(x,y) + S(x,y)), \\
H(x,y) &= \vec{G}_3(x, N(x,y)),
\end{aligned}
\tag{2.30}
$$

where $\exp_{\geq 2}(z) = \exp(z) - 1 - z$. The first equation translates the fact that a network is a 2-connected graph rooted at a directed edge (the factor 2 encodes the two possible directions) whose two endpoints are distinguished.

Suppose now that a graph class $\mathcal{G}$ is component, block and brick stable. Then the above three connectivity decomposition schemes give us a way to decompose any graph in $\mathcal{G}$ with graphs in $\mathcal{G}_3$ that are rooted at a directed edge (whose endpoints are distinguished). So that when the generating function of the 3-connected (labelled) graphs of a class is known, then the whole class can be known (see [30] for a general framework).

Further decompositions of 3-connected graphs into their 4-connected components and so forth seem to however be more involved. Indeed, as shown in [35], any similar decomposition along the k-cuts of a k-connected graph is not unique for $k > 3$.

Enumeration of labelled planar graphs. An example of such a graph class is that of planar graphs as one can check that a graph is planar if and only if all its connected, 2-connected and 3-connected components are planar. Notice now that by Withney's theorem the family of 3-connected planar graphs rooted at directed edge is in bijection with the family of 3-connected rooted planar maps. And as mentionned before, the latter family was enumerated by Tutte in [65]. So that one can relate the family of 3-connected rooted planar maps to that of (labelled) 2-connected planar graphs, then to that of connected planar graphs, to finally obtain the family of arbitrary planar graphs.

This is the scheme first adapted by Bender, Gao and Wormald in [3], where they enumerated the family of 2-connected planar graphs:

Theorem ([3])**.** *Let b_n be the number of 2-connected labelled planar graphs on $n \in \mathbb{N}$ vertices. The the following estimate holds:*

$$b_n \underset{n\to\infty}{\sim} b \cdot n^{-7/2} \cdot \beta^{-n} \cdot n!,$$

where $b \approx 0.37042 \cdot 10^{-5}$ and $\beta^{-1} \approx 26.18412$.

This scheme was then continued by Giménez and Noy in their breakthrough paper [28], in which they were able to obtain the asymptotic enumeration of the family of connected and arbitray labelled planar graphs:

Theorem ([28]). *Let c_n and p_n respectively be the numbers of connected and arbitrary labelled planar graphs on $n \in \mathbb{N}$ vertices. The the following two estimates hold:*

$$c_n \underset{n\to\infty}{\sim} c \cdot n^{-7/2} \cdot \rho^{-n} \cdot n!, \qquad p_n \underset{n\to\infty}{\sim} p \cdot n^{-7/2} \cdot \rho^{-n} \cdot n!,$$

where $c \approx 0.41043 \cdot 10^{-5}$, $p \approx 0.42609 \cdot 10^{-5}$ and $\rho^{-1} \approx 27.22688$.

The computation of the constant p in the above theorem was obtain after a rather involved algebraic integration of the generating function counting rooted connected planar graphs.

Combinatorial integration. In [17] however, the authors were able to obtain the same integration solely using a combinatorial decomposition. They showed that one can relate (under certain conditions) the generating function of a rooted family of graphs to its unrooted counterpart via a functional equation.

The idea comes from the book of Bergeron, Labelle and Leroux [4], in which the authors generalised a famous relation between rooted and unrooted trees, namely the *Dissimilarity Characteristic Theorem* developed by Otter in [55] to prove his famous formula, to generating functions (we follow the formulation from [17]):

Theorem (Dissymetry for trees [4]). *Let $\mathcal{T}$ and $\mathcal{T}_\bullet$ be the class of trees and vertex-rooted trees respectively. Let also $\mathcal{T}_{\bullet-\bullet}$ and $\mathcal{T}_{\bullet\to\bullet}$ be the classes of trees respectively rooted at an edge and at a directed edge. Then the following holds*

$$\mathcal{T} \cup \mathcal{T}_{\bullet\to\bullet} \simeq \mathcal{T}_\bullet \cup \mathcal{T}_{\bullet-\bullet},$$

where $\simeq$ is a bijection preserving the number of nodes.

Tree-decomposable classes of graphs. The authors of [17] then generalised the *dissymetry theorem for trees* to any class of graphs that can be decomposed in a tree-like manner, as for example in a block or a network-decomposition.

A class of graphs $\mathcal{A}$ is said to be *tree-decomposable* if one can associate to each graph $\gamma \in \mathcal{A}$ a tree $\tau(\gamma)$ whose nodes are distinguishable in some way (e.g., using the labels on the vertices of γ). Let now $\mathcal{A}_\bullet$ denote the class of graphs in $\mathcal{A}$ where a node of $\tau(\gamma)$ is distinguished. Similarly, $\mathcal{A}_{\bullet-\bullet}$ will be the class of graphs in $\mathcal{A}$ where an edge of $\tau(\gamma)$ is distinguished, and $\mathcal{A}_{\bullet\to\bullet}$ those were an edge $\tau(\gamma)$ is distinguished and given a direction. So that:

Theorem (Dissymetry for tree-decomposable classes [17]). *Let $\mathcal{A}$ be a tree-decomposable class. Then*

$$\mathcal{A} \cup \mathcal{A}_{\bullet \to \bullet} \simeq \mathcal{A}_{\bullet} \cup \mathcal{A}_{\bullet - \bullet},$$

where $\simeq$ is a bijection preserving the number of nodes.

In particular, if one associate the generating function $A(x)$ (resp. $A^{\bullet}(x)$, $A^{\bullet-\bullet}(x)$ and $A^{\bullet\to\bullet}(x)$) to the graph class $\mathcal{A}$ (resp. $\mathcal{A}_{\bullet}$, $\mathcal{A}_{\bullet-\bullet}$ and $\mathcal{A}_{\bullet\to\bullet}$), then the following equality holds:

$$A(x) = A^{\bullet}(x) + A^{\bullet-\bullet}(x) - A^{\bullet\to\bullet}(x).$$

Note that the above *dissymetry theorem* still holds for unlabelled combinatorial graph classes, as long as they admit a tree-like decomposition.

The method used in [17] to derive the generating function counting labelled (unrooted) connected planar graphs, was to root the tree associated to the network-decomposition of 2-connected planar graphs graphs at either a vertex (i.e. a network of the decomposition), an edge (i.e. two networks associated in series, parallel or h-composition) or a directed edge. Each such rooting can then be translated into functional equations involving the different generating functions counting networks. The main difficulty in this type of combinatorial integration scheme always seems to obtain the generating function of unrooted 3-connected graphs in the family. For planar graphs, it can be for example done by integrating (algebraically) the rational parametrisation of irreducible quadrangulations (and hence 3-connected rooted planar maps) given in Equation (2.18).

Algebraic connectivity decompositions. One can similarly decompose rooted planar maps into 2-connected and 3-connected components. This was in fact the main method used by Tutte in [65] to obtain the generating function counting 3-connected planar maps. In this case, the functional equations involved become algebraic (the sets become sequences so that $\exp(x)$ becomes $(1-x)^{-1}$). It is in general not the case for classes of graphs.

In [7] however, Bodirsky, Kang, Löffler and McDiarmid adapted the network-decomposition to cubic planar planar graphs. As we will see next, the sets involved in this decomposition have size at most three, so that the functional equations become algebraic. It seems to always be the case when the graphs in the class have bounded degrees.

Chapter 3

Enumeration of cubic planar graphs

3.1 Introduction

This chapter is about the asymptotic enumeration of the family of labelled cubic planar graphs, counted by vertices. To that end, we will follow the cubic network-decomposition scheme of vertex-rooted connected planar graphs developed by Bodirsky, Kang, Löffler and McDiarmid in [7]. As mentionned before, they adapted the classical network-decomposition scheme of [69] to the particular case of vertex-rooted cubic graphs, where the generating functions involved are algebraic. Computing then each singular expansions of the associated exponential generating functions, we extend their scheme and recover their main results: the asymptotic enumerations of 3-connected, 2-connected and connected cubic planar graphs, but this time with complete asymptotic estimates (including the multiplicative constants).

The first result concerns the full asymptotic enumeration of the class of 3-connected cubic planar graphs:

Theorem 3. *The number t_n of 3-connected cubic planar graphs on $n \in \mathbb{N}$ vertices is asymptotically*

$$t_n \underset{n\to\infty}{\sim} t \cdot n^{-7/2} \cdot \gamma_t^n \cdot n!,$$

with $t \approx 0.0407168760$ and $\gamma_t = \rho_t^{-1} \approx 3.0792014357$, where $\rho_t = \frac{3\sqrt{3}}{16} \approx 0.3247595264$.

The second result is then the (full) asymptotic enumeration of 2-connected cubic planar graphs:

Theorem 4. *The number b_n of 2-connected cubic planar graphs on $n \in \mathbb{N}$ vertices is asymptotically*

$$b_n \underset{n\to\infty}{\sim} b \cdot n^{-7/2} \cdot \gamma_b^n \cdot n!,$$

with $b \approx 0.0592436837$, $\gamma_b = \rho_b^{-1} \approx 3.1296662937$ and $\rho_b \approx 0.3195228840$ is the smallest positive solution of

$$54x^6 + 324x^4 - 4265x^2 + 432 = 0.$$

And we finally obtain the asymptotic enumeration of the family of connected cubic planar graphs:

Theorem 5. *The number c_n of connected cubic planar graphs on $n \in \mathbb{N}$ vertices is asymptotically*

$$c_n \underset{n\to\infty}{\sim} c \cdot n^{-7/2} \cdot \gamma^n \cdot n!,$$

with $c \approx 0.0609730610$, $\gamma = \rho^{-1} \approx 3.1325905979$, and $\rho \approx 0.3192246062$ is the smallest positive solution of

$$729x^{12} + 17496x^{10} + 148716x^8 + 513216x^6 - 7293760x^4 + 279936x^2 + 46656 = 0.$$

Furthermore, the cubic network-decomposition of [7] gives a natural tree-decomposition of the class of connected cubic planar graphs. One can in particular compute this way the constant coefficient of the singular expansion of the generating function counting connected cubic planar graphs. So that we can now give a complete asynptotic estimate of the family of labelled cubic planar graphs:

Theorem 6. *The number g_n of cubic planar graphs on $n \in \mathbb{N}$ vertices is asymptotically*

$$g_n \underset{n\to\infty}{\sim} g \cdot n^{-7/2} \cdot \gamma^n \cdot n!,$$

with $g \approx 0.0610098696$ and γ is as in Theorem 5.

The previous theorems were stated in [7, Theorem 2] but without computations of the multiplicative constants involved in the asymptotic estimates. Our first goal is to provide a proof of these estimates. In particular, the one for arbitrary cubic planar graphs requires an application of the *dissymetry theorem for tree-decomposable classes* that only appeared one year later in [17]. We also remark that the value of $\gamma \approx 3.132595$ given in [7] is only correct up to the fifth decimal place, and the value for $\gamma_b \approx 3.129684$ given there is correct only up to the fourth decimal place.

Our next result is an estimate on the number of cubic planar *multigraphs*. This class of graphs is instrumental in the study of the phase transition of the Erdős-Rényi random graph [37, 38, 50]. In these references cubic multigraphs are equipped with a weight that depends on the number of loops and multiple edges. Here we count unweighted cubic multigraphs, which is a result interesting by itself:

Theorem 7. *The numbers m_n and m'_n of cubic planar multigraphs and cubic planar connected multigraphs on $n \in \mathbb{N}$ vertices are asymptotically*

$$m_n \underset{n\to\infty}{\sim} m \cdot n^{-7/2} \cdot \gamma_m^n \cdot n! \qquad \text{and} \qquad m'_n \underset{n\to\infty}{\sim} m' \cdot n^{-7/2} \cdot \gamma_m^n \cdot n!,$$

with $m \approx 0.2247427548$, $m' \approx 0.2094103951$ and $\gamma_m = \rho_m^{-1} \approx 3.9855373662$, where $\rho_m \approx 0.2509071947$ is the smallest positive root of the equation

$$729x^{12} - 17496x^{10} + 148716x^8 - 513216x^6 - 7293760x^4 - 279936x^2 + 46656 = 0.$$

The exponential growth of the number of cubic planar multigraphs γ_m is also computed in [23] but in the more general setting of cubic multigraphs on an orientable surface of fixed genus.

We finally reprove two results of [27], on the enumeration of simple cubic maps, in a different way using a connectivity decomposition adapted to maps:

Theorem 8. *Let s_n be the number of simple cubic planar maps on $n \in \mathbb{N}$ vertices. Then the following estimate holds:*

$$s_n \underset{n\to\infty}{\approx} s \cdot n^{-\frac{5}{2}} \sigma^n,$$

with $s \approx 0.9367499783$ and $\sigma = \rho_s^{-1} \approx 3.2231120230$, where $\rho_s \approx 0.3102591511$ is the smallest positive solution of

$$27z^{12} + 216z^{10} + 171z^8 - 208z^6 - 339z^4 + 24z^2 + 1 = 0. \qquad (3.1)$$

By setting $x^2 = z^3$ in (3.1), we can recover the polynomial obtained in [27, Corollary 1].

Theorem 9. *Let s'_n be the number of 2-connected simple cubic planar maps on $n \in \mathbb{N}$ vertices. Then the following estimate holds:*

$$s'_n \underset{n\to\infty}{\sim} s' \cdot n^{-\frac{5}{2}} \cdot \beta^n,$$

with $s' \approx 0.6336562882$ *and* $\beta = \rho_s'^{-1} = \frac{2}{\sqrt{6\sqrt{3}-10}} \approx 3.1931414661$, *where* $\rho'_s = \frac{\sqrt{6\sqrt{3}-10}}{2} \approx 0.3131712173$ *is the smallest positive solution of*

$$2x^4 + 10x^2 - 1 = 0.$$

The plan of this chapter is the following: after briefly introducing the method in [7], we will discuss the generating functions of 3-connected, 2-connected then connected labelled cubic planar graphs and estimate their coefficients. Then, applying the *dissymetry theorem for tree-decomposable classes* of [17], we will estimate the number of labelled cubic planar graphs on n vertices. Finally, using a similar method, we will discuss and estimate the coefficients of the exponential generating functions counting cubic planar multigraphs and of the two ordinary generating functions respectively counting arbitrary and 2-connected simple cubic planar maps.

3.2 Preliminaries

3.2.1 Cubic network-decomposition

We follow the definitions from [7] but deviate slightly from the notation there.

Cubic networks. A *cubic network* is a connected cubic multigraph μ with an ordered pair of adjacent vertices (s, t), such that the graph obtained by removing the edge st is simple. Observe that st can be a loop. The oriented edge st is called the *root* of the network, and s and t are the *poles*.

Let μ be a network with root st and $e = uv$ be an edge of another network η. The *replacement*, or *substitution* of e with μ is the network obtained by the following operations. Subdivide e twice transforming it in to the path $uu'v'v$, remove the edge $u'v'$, and respectively identify u' with s and v' with t.

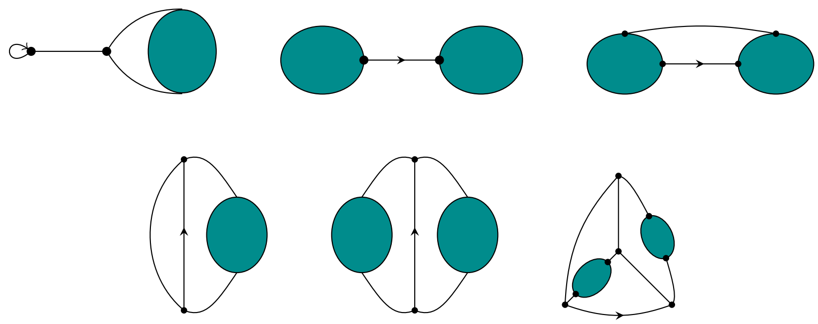


Figure 3.1: The five different types of cubic networks. Top, from left to right, are a loop, an isthmus and a series cubic network. Bottom, from left to right, are the two types of parallel cubic networks and an h-composition.

Notice that if η and μ are cubic and planar, so is the resulting network. A cut vertex in a cubic graph is necessarily incident with one or three isthmuses. For each cut vertex u incident with exactly one isthmus e, one can remove the component containing e and erase the resulting vertex of degree two, resulting in a cubic graph. We call this operation *suppressing* the cut vertex u. A *3-connected core* is in particular a 3-connected network in which no edge has been substituted. Notice that 3-connected networks are simple graphs.

Classification of cubic networks. Networks can be classified by the different types of graphs obtained after the removal of the root. They fall into five classes, as shown in [7] . Some examples of each different class is pictured in Figure 3.1. Let μ be a network with root st. Then it belongs to one and only one of the following classes (see [17] for a detailed exposition).

- $\mathcal{L}$ (Loop). The root is a loop.
- $\mathcal{I}$ (Isthmus). The root is an isthmus.
- $\mathcal{S}$ (Series). $\mu - st$ is connected but is not 2-connected.
- $\mathcal{P}$ (Parallel). $\mu - st$ is 2-connected and $\mu - \{u, v\}$ is not connected.
- $\mathcal{H}$ (3-connected). μ is obtained from a 3-connected core by possibly substituting each non-root edge with a network.

Recursive decomposion. To decompose networks, we will need to introduce the class $\mathcal{C}$ of connected cubic networks, that are the cubic networks μ with root st, such that $\mu - st$ is connected. Notice that every (cubic) network is connected apart for the isthmus networks. So that if we now let $L(x)$, $I(x)$, $S(x)$, $P(x)$, $H(x)$ and $C(x)$ be the exponential generating functions respectively counting cubic loop, isthmus, series, parallel, 3-connected and connected networks, where the variable x marks vertices, then the following equation holds:

$$C(x) = L(x) + S(x) + P(x) + H(x).$$

We now present a recursive decomposition of networks that will link the exponential generating function $\vec{G}_3(x, y)$ counting 3-connected cores (i.e. 3-connected cubic planar graphs rooted at a directed edge), where the variables x and y respectively mark vertices and edges, to that of connected networks. The next lemma was first proven in [7, Section 3], and for the sake of completeness we offer a proof here.

Lemma 10. *The following system of equations holds:*

$$\begin{aligned} L &= \frac{x^2}{2}(I + C - L), & I &= \frac{L^2}{x^2}, \\ S &= (C - S)C, & P &= x^2\left(C + \frac{C^2}{2}\right), \\ C &= L + S + P + H, & H &= \frac{\vec{G}_3(x, 1 + C)}{1 + C}, \end{aligned} \tag{3.2}$$

where $L \equiv L(x)$, $I \equiv I(x)$, $S \equiv S(x)$, $P \equiv P(x)$, $H \equiv H(x)$, $C \equiv C(x)$.

Proof. Equation for $L(x)$. Take any cubic network η and subdivide its root with a new vertex v. Create finally another new vertex u and add both the edge $\{u, v\}$ and the directed loop uu. Notice now that the resulting graph is a network if and only if η was not rooted at a loop. This means that η belongs to the family of cubic networks counted by $I + C - L$. The factor $1/2$ in the equation for $L(x)$ then comes from the fact that the direction of the loop uu does not matter. Alternatively, consider the *double-loop*, i.e. the cubic multigraph with two vertices, both adjacent to one loop and to one another, and rooted at a loop, and substitute the non-rooted loop by any cubic network not rooted at a loop.

Equation for $I(x)$. Consider two ordered loop cubic networks (η_1, η_2) from which the root vertex has been removed and notice that each of the resulting graphs has now a vertex of degree two. Connect then those two vertices by a directed edge, whose direction follows the order (η_1, η_2). Observe that this directed edge is now an isthmus.

Equation for $C(x)$. By definition, the class $\mathcal{C}$ is in bijection with the class $\mathcal{L} \cup \mathcal{S} \cup \mathcal{P} \cup \mathcal{H}$. We exclude the class $\mathcal{I}$ since removing the root edge disconnects the graph.

Equation for $S(x)$. Consider the two connected cubic networks η_1 and η_2, with respective root edges $s_1 t_1$ and $s_2 t_2$. Now remove both their root edges and add the edge $\{t_1, s_2\}$ and the directed edge $s_1 t_2$. By minimality of the decomposition, one of the two cubic networks cannot be series.

Equation for $P(x)$. Consider the *3-bond*, i.e. the multigraph with two vertices connected by a triple edge, one of which is directed. Now, if we substitute at least one of the non-root edges by a connected cubic network, then the resulting graph becomes itself a cubic network. It is in fact a parallel cubic network. Observe the absence of a factor 2 in the equation for $P(x)$, as the direction of any directed edge of the 3-bond does not matter.

Equation for $H(x)$. Consider a 3-connected core and possibly replace each non-root edge by any network. It is thus translated in terms of generating functions by substituting the variable y of $\vec{G}_3(x, y)$, by $1 + C(x)$. The division ensures that the root edge is not replaced. □

Asymptotic enumeration. Recall that all the functions involved are even, in agreement with the fact that a cubic graph has an even number of vertices. Following [7] and by algebraic elimination, the system (3.2) can be written as a single polynomial equation $\Phi(x, C)$ satisfied by C. It is then claimed in [7] that the smallest positive root $\rho_c \approx 0.319224$ of the discriminant of Φ with respect to C provides the dominant singularity of C. The analysis is here slightly incomplete since one must guarantee that this is indeed a singularity of the generating function $C(x)$. Then a singular expansion of $C(x)$ near ρ_c of square-root type is deduced as

$$C(x) = C_0 + C_2 X^2 + C_3 X^3 + O(X^4), \qquad \text{where } X = \sqrt{1 - \frac{x}{\rho_c}}.$$

The coefficients C_i are however not computed in [7]. Using a transfer-type theorem, an estimate for the coefficients of $C(x)$ is finally derived. This implies a corresponding estimate for the coefficients of G_1. We provide in section 3.5 both a complete proof of such an estimate and compute every constants involved.

3.2.2 An analytic lemma

Lemma 11. *Let $A(x)$ be an even algebraic exponential power series with positive coefficients which satisfies both a polynomial equation $P(A(x), x)$ and a functional equation of the form*

$$F(A(x), x) = f(A(x), x) + h(T(x^2(1 + A(x))^3), x),$$

where $T(z)$ is the generating function of triangulations as in (2.5), and $f(y, x)$ and $h(y, x)$ are bivariate polynomials. Let also $\Delta(x)$ be the discriminant of $P(A(x), x)$ with respect to $A(x)$ such that there exists at least two factors of $\Delta(x)$ giving rise to positive (real) roots. And assume the following conditions:

1. *x_0 is a positive solution of the equation $x^2(1 + A(x))^3 = \tau$.*
2. *For all $y \in \mathbb{R}$, with $|y| \leq x_0$, it holds that $F_A(A(y), y) \neq 0$, where F_A is the derivative of $F(A, x)$ with respect to A.*
3. *x_0 is the smallest positive root of some factor of $\Delta(x)$.*

Then x_0 and $-x_0$ are the only two dominant singularities of $A(x)$, and the singular expansions at x_0 is of the form

$$A(x) \underset{x \to x_0}{\sim} A_0 + A_2\left(1 - \frac{x}{x_0}\right) + A_3\left(1 - \frac{x}{x_0}\right)^{3/2},$$

where A_0, A_2, A_3 and x_0 are non-nul and computable algebraic numbers. Furthermore, the following asymptotic estimate holds, for n even:

$$n![x^n]A(x) \underset{n \to \infty}{\sim} \frac{3A_3}{2\sqrt{\pi}} \cdot n^{-5/2} \cdot x_0^{-n} \cdot n!.$$

Proof. When $\Delta(x)$ admits several factors each with some (smallest) positive roots, one wants to consider the combinatorial equation $F(A(x), x)$ in order to decide which factor gives rise to the dominant singularities of $A(x)$, the combinatorial branch of $P(A(x), x)$. Any solution of $x^2(1 + A(x))^3 = \tau$, where τ is the dominant singularity of $T(z)$, is a singularity of $A(x)$. But also any solution of $F_A(A(x), x) = 0$.

First observe that conditions 1. and 2. ensure that the composition scheme $T(x^2(1+A(x))^3)$ is critical, i.e. that there is no positive solution of $F_A(A(x), x) = 0$ that is smaller than x_0, the smallest positive solution of $x^2(1+A(x))^3 = \tau$. Given that $A(x)$ is an even function, the above implies that $\pm x_0$ are the dominant singularities of $A(x)$.

Now condition 3. ensures that $A(x)$ is analytic in a disk of radius x_0 centered at the origin after slicing the two rays $[x_0, +\infty]$ and $[-x_0, -\infty]$. Furthermore, a standard compactness argument (see the last part of the proof of Theorem 2.19 in [20]) shows that $A(x)$ is analytic in a dented domain at $\pm x_0$. And using the Newton-Puiseux theorem of *singular expansion*, we obtain

$$A(x) \underset{x \to x_0}{\sim} A_0 + A_2\left(1 - \frac{x}{x_0}\right) + A_3\left(1 - \frac{x}{x_0}\right)^{3/2}.$$

Indeed, first notice that due to $A(x)$ having non negative coefficients, $x_0 > 0$ implies that $A_0 = A(x_0) > 0$. By then differentiating $x^2(1 + A(x))^3$ with respect to x and substituting $x = x_0$, we deduce both from the critical composition scheme $x_0^2(1 + A(x_0))^3 = \tau$ and from $A(x_0) \neq 0$, that $|A'(x_0)| = |2(1 + A_0)^2/(3x_0)| < +\infty$. Thus showing that $A_1 = 0$. Using finally the parametrisation of $T(z)$ by the algebraic generating function $U(z)$ in (2.5), observe that $T'(z) = (1 - U(z))^{-2}$, which implies that the derivative $F_x(A(x), x)$ contains the term $U(x^2(1 + A(x))^3)$, so that the second derivative is in terms of $U'(z)$. It follows that the expression for $A''(x_0)$ contains the term $U'(x_0^2(1 + A_0(x_0))^3) = U'(\tau)$, which is infinite because of (2.9). So that $|A''(x_0)| = +\infty$, which implies that $A_3 \neq 0$. The coefficients A_i are algebraic numbers since $A(x)$ is an algebraic function and so are $\pm x_0$ by the first condition.

We can now apply the theorem of *sim-transfer* to obtain the estimate for $n \in \mathbb{N}$ even as claimed, using $\Gamma(-3/2) = 4\sqrt{\pi}/3$. Notice that the contributions from x_0 and $-x_0$ are added, so that the multiplicative constant is $2A_3/\Gamma(-3/2)$. □

Notes. In all our applications $f(A(x), x)$ will be either a rational or a quadratic function. Furthermore, when computing Puiseux expansions from the minimal polynomial of $A(x)$, it may be that several solutions appear due to the different branches at a given point. In all our proofs we find a single expansion containing a non-zero term A_3X^3, which has to correspond to the branch of the combinatorial solution due to the above considerations.

3.3 3-connected cubic planar graphs

The main goal of this section is to estimate the number of labelled 3-connected cubic planar graphs. As a byproduct, we will settle the algebraicity of some associated generating functions. In particular, the one counting the 3-connected cores is central for the network-decomposition of both connected and 2-connected cubic planar graphs, of cubic planar multigraphs and finally in the root-decomposition of simple cubic maps.

3.3.1 3-connected cores are algebraic

Recall that $\vec{G}_3(x,y)$ is the exponential generating function counting labelled 3-connected cubic graphs rooted at a directed edge, where the variables x and y respectively mark the number of vertices and edges. We will first derive the minimal polynomial of $\vec{G}_3(x,y)$.

The duality argument mentioned before shows that $\vec{G}_3(x,y)$ can be obtained from the generating function $T(z)$ of rooted (unlabelled) triangulations, where the variable z counts the number of vertices minus two. The relation is

$$2 \cdot \vec{G}_3(x,y) = T(x^2y^3) - x^2y^3. \tag{3.3}$$

The subtracted term x^2y^3 corresponds to the triangulation consisting of a single triangle and the factor 2 to the two possible choices of a root face from a root edge when drawing a 3-connected planar graph on the sphere.

Let us now consider the algebraic system of equations composed of Equations (2.5), (2.6) and (3.3), together with the equation $z = x^2y^3$ encoding the change of variable in (3.3). This gives:

$$\begin{aligned} &(1-U)^3U - z = 0, \qquad &&(1-2U)U - T = 0, \\ &T - z - \vec{G}_3 = 0, &&x^2y^3 - z = 0, \end{aligned} \tag{3.4}$$

where $\vec{G}_3 \equiv \vec{G}_3(x,y)$. From (3.4), one can eliminate the variables z, U and

T to obtain an annihilating polynomial of $\vec{G}_3 \equiv \vec{G}_3(x,y)$:

$$\begin{aligned} &16\vec{G}_3^4 + (32x^2y^3+24)\vec{G}_3^3 + (24x^4y^6+68x^2y^3+12)\vec{G}_3^2 \\ &+(8x^6y^9+50x^4y^6-28x^2y^3+2)\vec{G}_3 + x^8y^{12}+11x^6y^9-x^4y^6 = 0. \end{aligned} \tag{3.5}$$

This polynomial turns out to be irreducible and is hence the minimal polynomial of $\vec{G}_3(x,y)$, thus proving its algebraicity.

3.3.2 Enumeration of 3-connected cubic planar graphs

Let $G_3(x,y) = \sum_{n\geq 0} t_{n,k}\frac{x^n}{n!}y^k$ be the exponential generating function counting (unrooted) labelled 3-connected cubic planar graphs. The next statement won't be used to compute the asymptotic estimate of the coefficients $t_n = t_{n,1}$. The holonomic functional equation satisfied by $G_3(x,y)$ will however be of much use later on, when we will apply the *dissymetry theorem for tree-decomposable classes*.

Proposition 12. *The exponential generating function $G_3(x,y)$ is D-finite.*

Proof. We have $\vec{G}_3(x,y) = 2yD_yG_3(x,y)$, so that

$$G_3(x,y) = \frac{1}{2}\int_y \frac{M(x,y)}{y} = \frac{1}{4}\int_y \frac{T(x^2y^3)-x^2y^3}{y},$$

where the $1/2y$ of the first equality arises from the label of the root-edge and its orientation, and the second equality from Equation (3.3).

We now apply the change of variables $x^2y^3 = z$ and are left with the integral $\frac{1}{12}\int T(z)/z\,dz$. We make the further change $w = U(z)$, and using Equations (2.5) and (2.6), we get

$$\begin{aligned} G_3(x,y) &= -\frac{1}{12}\left(z - \int_z \frac{T(z)}{z}\right) = -\frac{1}{12}\left(z - \int_w \frac{(1-2w)(1-4w)}{1-w}\right) \\ &= -\frac{1}{12}\left(4w^2+2w+3\log(1-w)+z\right). \end{aligned}$$

This gives us a functional equation relating $G_3(x,y)$ with the (exponential with respect to the variable x) generating function $U(x^2y^3)$:

$$12G_3(x,y) + 4U(x^2y^3)^2 + 2U(x^2y^3) + x^2y^3 + 3\log(1-U(x^2y^3)) = 0. \tag{3.6}$$

Since $U(z)$ is algebraic (see (2.6)) so is $U(x^2y^3)$, and the function $\log(U(x^2y^3))$ is D-finite. Then from (3.6) one can directly see that so is $G_3(x,y)$. □

Singularity analysis. We will now estimate the coefficients of the exponential generating function $G_3(x) = G_3(x,1) = \sum_{n\geq 0} t_n x^n/n!$. Let us first consider the univariate version of Equation (3.6):

$$G_3(x) = \frac{1}{12}(4W(x)^2 + 2W(x) + x^2 + 3\log(1-W(x))), \qquad (3.7)$$

where $W(x) = U(x^2)$ is an exponential generating function. By setting $z = x^2$ and $U(z) = W(x)$ in the polynomial equation (2.6), we directly obtain the minimal polynomial $Q(W(x), x)$ of $W(x)$. So that $W(x)$ is algebraic and its dominant singularity, the unique positive root of the discriminant of Q with respect to W, is:

$$\rho_t = \sqrt{\tau} = \frac{3\sqrt{3}}{16} \approx 0.3247595264.$$

And $W(x)$ admits a local Puiseux expansion near ρ_t of the from:

$$W(x) = \frac{1}{4} - \frac{\sqrt{3}}{4}X + \frac{1}{6}X^2 - \frac{\sqrt{3}}{108}X^3 + \frac{5}{162}X^4 - \frac{11\sqrt{3}}{1944}X^5 + O(X^6), \quad (3.8)$$

with $X = \sqrt{1 - x/\rho_t}$.

Notice now that the only possible source of singularities of $G_3(x)$ arises from Equation (3.7), in particular from $W(x)$. So that $\pm\rho_t$ are also the dominant singularities of $G_3(x)$, as $G_3(x)$ is an even function. Using the theorem on the *transfer of singularity* for the logarithm in Equation (3.7), the type of expansion of $W(x)$ is transfered to $G_3(x)$, i.e. it admits a local Pusieux expansion near ρ_t of branching-type 2.

Proof of Theorem 3. To compute the coefficients of the singular expansions of $G_3(x)$, we simply substitute $W(x)$ by its local expansion (3.8) in the right hand-side of Equation (3.7). So that:

$$G_3(x) \underset{x\to\rho_t}{\sim} \frac{\log 2}{2} - \frac{73}{1024} - \frac{\log 3}{4} - \frac{5}{1536}X^2 + \frac{37}{3072}X^4 - \frac{\sqrt{3}}{90}X^5,$$

with $X = \sqrt{1 - x/\rho_t}$. Notice that the first odd index of the above expansion is five, this justifies our need to compute the coefficients of the expansion of $W(z)$ up to the fifth index in (3.8).

Now, because of the expansion in powers of X, $G_3(x)$ is analytic in a disk of radius ρ_t centered at the origin after slicing the rays $[\rho_t, +\infty]$ and $[-\rho_t, -\infty]$. The compactness argument mentionned in the proof of Lemma 11 then shows that $G_3(x)$ is analytic in a dented domain at both $\pm\rho_t$. So that an application of the theorem of *sim-transfer* gives us the following estimate on t_n (for $n \in \mathbb{N}$ even):

$$t_n = n![x^n]G_3(x) \underset{n\to\infty}{\sim} \frac{\sqrt{3}}{24\sqrt{\pi}} \cdot n^{-7/2} \cdot \left(\frac{16\sqrt{3}}{9}\right)^n \cdot n!.$$

This concludes the proof.

3.4 2-connected cubic planar graphs

The main goal of this section is to asymptotically estimate the number of labelled 2-connected cubic planar graphs. It will be done by studying the coefficients of the exponential generating function $G_2(x) = \sum_{n\geq 0} b_n x^n/n!$ counting labelled 2-connected cubic planar graphs, where the variable x will mark the number of vertices. As a byproduct, we will settle some useful algebraic properties of some of the associated generating functions.

3.4.1 2-connected cubic networks are algebraic

We define a *2-connected cubic network* as a 2-connected cubic planar multigraph β, rooted at a directed edge e, and such that $\beta - e$ is a simple connected graph. We denote their exponentail generating function by $B \equiv B(x)$.

Notice that one can directly obtain 2-connected cubic networks from cubic networks by simply discarding the two classes $\mathcal{L}$ and $\mathcal{I}$: these are the only ones that produce cut-vertices. So that restricting Lemma 10 to 2-connected cubic networks gives us directly

$$S = (B - S)B, \qquad B = S + P + H,$$
$$P = x^2\left(B + \frac{B^2}{2}\right), \qquad H = \frac{\vec{G}_3(x, 1+B)}{1+B},$$

where the generating functions $S \equiv S(x)$, $P \equiv P(x)$ and $H \equiv H(x)$ have the same meaning as before, except that they are now restricted to 2-connected cubic networks. Using Equation (3.3), relating $\vec{G}_3(x,y)$ with $T \equiv T(z)$, we now rewrite it as the following system of five polynomial equations:

$$\begin{array}{ll} B^2-(1+B)S=0, & S+P+H-B=0, \\ x^2(2B+B^2)-2P=0, & x^2(1+B)^3-z=0, \\ T-x^2(1+B)^3-2(1+B)H=0, & \end{array} \tag{3.9}$$

where the equation $x^2(1+B)^3 - z = 0$ encodes the compositon scheme $T(x^2(1+B)^3)$ with the generating function $T(z)$.

From (3.9) one can first eliminate the variables S, P and H to obtain after a simple algebraic manipulation a functional equation relating $B(x)$ with $T(z)$:

$$2B + x^2(1+B) - T(x^2(1+B)^3) = 0. \tag{3.10}$$

And using the minimal polynomial of $T(z)$, together with $x^2(1+B)^3 - z = 0$, one can now eliminate both z and T to obtain the following annihilating polynomial of $B(x)$:

$$\begin{array}{l} 16x^4B^6 + B^5(8x^6+128x^4+32x^2) + B^4(x^8+48x^6+372x^4 \\ +88x^2+16) + B^3(4x^8+107x^6+498x^4+43x^2+24) \\ +B^2(6x^8+113x^6+311x^4-43x^2+12) \\ +B(4x^8+57x^6+72x^4-30x^2+2) + x^8+11x^6-x^4 = 0. \end{array} \tag{3.11}$$

Notice that the above polynomial is irreducible, so that it is in fact the minimal polynomial of $B(x)$.

3.4.2 Enumeration of 2-connected cubic planar graphs

Proposition 13. *The sequence $\{b_n\}_{n\geq 0}$ is P-recursive, that is, it satisfies a linear recurrence with polynomial coefficients.*

Proof. Notice first that the following relation between the exponential generating function $\vec{G}_2(x)$, counting 2-connected cubic planar graphs rooted at a directed edge, and $B(x)$ holds:

$$\vec{G}_2(x) = B(x) - x^2B(x). \tag{3.12}$$

The reason is that from the cubic networks encoded by $B(x)$ we have to exclude the parallel networks with a double edge, that correspond to $x^2B(x)$.

Observe that for a given labelled graph on n vertices, one has n possible ways of rooting it at a vertex. So that the n-th coefficient (for n even) of the exponential generating function $xG_2'(x) = G_2^\bullet(x)$, counting labelled 2-connected cubic planar graphs rooted at a vertex, is equal to $nb_n/n!$, where $b_n/n!$ is the n-th coefficient of $G_2(x)$. And by double counting we have

$$G_2^\bullet(x) = \frac{\vec{G}_2(x)}{3} = \frac{B(x)(1-x^2)}{3}. \tag{3.13}$$

And finally using that $xG_2'(x) = G_2^\bullet(x)$ in (3.13), we get

$$G_2(x) = \int \frac{B(x) - x^2 B(x)}{3x} dx.$$

Since $B(x)$ is algebraic and divisible by x, the above equation directly implies that $G_2(x)$ is D-finite, i.e. the sequence $\{b_n\}_n$ is P-recursive. □

Singularity analysis. Rewritting Equation (3.13) gives us the following algebraic equation between $G_2^\bullet(x)$ and $B(x)$:

$$(1-x^2)B(x) - 3G_2^\bullet(x) = 0. \tag{3.14}$$

Thus by the theorem of *transfer of singularities*, $G_2^\bullet(x)$ will have the same dominant singularities and singular behaviour as that of $B(x)$. So in order to estimate the coefficients nb_n, as $n \to \infty$, it is enough to study the analytic behaviour of $B(x)$.

We compute the discriminant of (3.11) with respect to B and consider the following irreducible factor:

$$54x^6 + 324x^4 - 4265x^2 + 432 = 0, \tag{3.15}$$

whose smallest positive root is given by $\rho_b \approx 0.3195228840$. We claim that ρ_b is indeed the dominant singularity of the generating function $B(x)$. To prove it, we will now verify the conditions of Lemma 11 with the functional equation (3.10):

$$F(B(x), x) := B(x) = \frac{T(x^2(1+B(x))^3)}{2} - \frac{x^2(1+B(x))}{2},$$

Now, to show that the composition scheme

$$x^2(1+B(x))^3 = \tau. \tag{3.16}$$

is indeed critical with $x = \rho_b$, it is sufficient to prove that $R(B(x), x)$, the right hand-side of $F_B(B(x), x)$, does not equal one for $|x| \leq \rho_b$. A simple computation using Equations (2.5) and (2.6) gives $T'(z) = (1-U(z))^{-2}$, and we arrive at

$$R(B(x), x) = \frac{3x^2(1+B(x))^2}{2(1-U(x^2(1+B(x))^3))^2} - \frac{x^2}{2}.$$

Substituting $x = \rho_b$ in Equation (3.16) and since $B(x)$ has non-negative coefficients, we obtain

$$|B(x)| \leq |B(\rho_b)| = \left| \sqrt[3]{\frac{\tau}{\rho_b^2}} - 1 \right| \approx 0.0108963334.$$

And since also $U(z)$ and $T(z)$ both have non-negative coefficients, we get

$$\begin{aligned} |U(x^2(1+B(x))^3)| &\leq |U(\rho_b^2(1+B(\rho_b))^3)| = |U(\tau)| = 1/4, \\ |T(x^2(1+B(x)))^3)| &\leq |T(\rho_b^2(1+B(\rho_b)))^3)| = |T(\tau)| = 1/8. \end{aligned}$$

It is now a simple matter to check the following inequalities:

$$|R_B(B(x), x)| \leq |R_B(B(\rho_B), \rho_B)| \leq \left| \frac{3\rho_b^2(1+B(\rho_b))^2}{2(1-1/4)^2} \right| - \left| \frac{\rho_b^2}{2} \right| < 0.227.$$

Consider finally the resultant, with respect to B, between the two polynomials (3.11) and (3.16). And observe that (3.15) is its unique factor which is also a factor of the discriminant of (3.11). So that ρ_b is a positive root of (3.16). And we can now apply Lemma 11 to obtain the singular expansion of $B(x)$ near ρ_b:

$$B(x) \underset{x \to \rho_b}{\sim} B_0 + B_2 \left(1 - \frac{x}{\rho_b}\right) + B_3 \left(1 - \frac{x}{\rho_b}\right)^{3/2}, \tag{3.17}$$

where $B_0 = B(\rho_b)$, $B_2 \approx -0.1090702692$ and $B_3 \approx 0.2338926294$.

Proof of Theorem 4. Taking the resultant with respect to B of the minimal polynomial of $B(x)$ (3.11) together with the polynomial equation (3.14) gives us a polynomial equation of degree six: $P(G_2^\bullet(x), x)$ that turns out to be irreducible. Thus $P(G_2^\bullet(x), x)$ is the minimal polynomial of $G_2^\bullet(x)$.

As mentionned before, $G_2^\bullet(x)$ admits the same dominant singularity ρ_b and the same singular behaviour near ρ_b as $B(x)$. So that computing the singular Puiseux expansion of $G_2^\bullet(x)$ from $P(G_2^\bullet(x), x)$ gives:

$$G_2^\bullet(x) \underset{x\to\rho_b}{\sim} G_{2,0}^\bullet + G_{2,2}^\bullet\left(1-\frac{x}{\rho_b}\right) + G_{2,3}^\bullet\left(1-\frac{x}{\rho_b}\right)^{3/2},$$

where $G_{2,0}^\bullet \approx 0.0032612912$, $G_{2,2}^\bullet \approx -0.0319032781$ and $G_{2,3}^\bullet \approx 0.0700044636$.

And using (3.4.2), an application of the theorem of *sim-transfer* to $G_2^\bullet(x)$ gives the following asymptotic estimate for n even:

$$nb_n = n![x^n]G_2^\bullet(x) \sim b \cdot n^{-5/2} \cdot \rho_b^{-n} \cdot n!,$$

where

$$b = \frac{3}{2\sqrt{\pi}} G_{2,3}^\bullet \approx 0.0592436837.$$

From there the estimate on b_n follows.

3.5 Connected cubic planar graphs

In this section, similarly to the previous one, our main goal is to estimate the number c_n of labelled connected cubic planar graphs with $n \in \mathbb{N}$ vertices, by studying the coefficients of the exponential generating function $G_1(x) = \sum_{n\geq 0} c_n x^n/n!$, where the variable x marks the number of vertices.

To do so, we will first exhibit the minimal polynomial of the exponential generating function counting cubic networks, then studying its singular behaviour. We will then transfer this behaviour to the generating function counting connected cubic planar graphs rooted at a vertex and conclude.

3.5.1 Cubic networks are algebraic

Using Equation (3.3), and the composition scheme $T(x^2(1+C)^3)$, the system (3.2) can be rewritten as a system of seven polynomial equations:

$$\begin{array}{ll} x^2(I+C-L) - 2L = 0, & L^2 - x^2 I = 0, \\ L + S + P + H - C = 0, & C^2 - (1+C)S = 0, \\ x^2(2C + C^2) - 2P = 0, & x^2(1+C)^3 - z = 0, \\ T - x^2(1+C)^3 - 2(1+C)H = 0. & \end{array} \tag{3.18}$$

One can first eliminate the variables L, I, S, P, H and z from the five remaining equations of (3.18), to obtain after a simple simple manipulation a functional equation relating $C(x)$ with $T(x^2(1+C(x))^3)$:

$$C = \frac{T(z)}{2}\left(1 + \frac{T(z)^2}{4}\right) - \frac{x^2(1+C)^2}{2}\left(1 - C + \frac{x^2}{2}\right) - \frac{C^2}{2}, \qquad (3.19)$$

where $z = x^2(1+C(x))^3$.

Then using the minimal polynomial (2.7) of $T(z)$, one can eliminate $T(z)$ to obtain an annihilating polynomial of $C(x)$:

$$\begin{aligned}
&4096x^6C^9 + C^8(x^{16} + 32x^{14} + 400x^{12} + 2432x^{10} + 7520x^8\\
&+48640x^6 + 7424x^4 + 2048x^2 + 256) + C^7(8x^{16} + 256x^{14}\\
&+3200x^{12} + 19306x^{10} + 57760x^8 + 230864x^6 + 49792x^4\\
&+13984x^2 + 2048) + C^6(28x^{16} + 896x^{14} + 11193x^{12}\\
&+66878x^{10} + 192332x^8 + 593264x^6 137977x^4 + 37856x^2\\
&+6720) + C^5(56x^{16} + 1792x^{14} + 22358x^{12} + 132034x^{10}\\
&+362152x^8 + 919646x^6 + 201670x^4 + 50008x^2 + 11648)\\
&+C^4(70x^{16} + 2240x^{14} + 27895x^{12} + 162470x^{10} + 420995x^8\\
&+891046x^6 + 162815x^4 + 29240x^2 + 11440)\\
&+C^3(56x^{16} + 1792x^{14} + 22260x^{12} + 127582x^{10} + 308620x^8\\
&+532540x^6 + 67124x^4 - 416x^2 + 6336)\\
&+C^2(28x^{16} + 896x^{14} + 11095x^{12} + 62426x^{10} + 138830x^8\\
&+182524x^6 + 9274x^4 - 8232x^2 + 1836) + C(8x^{16} + 256x^{14}\\
&+3158x^{12} + 17398x^{10} + 34852x^8 + 29174x^6 - 1204x^4\\
&-2664x^2 + 216) + x^{16} + 32x^{14} + 393x^{12}\\
&+2114x^{10} + 3707x^8 + 846x^6 - 108x^4 = 0,
\end{aligned} \qquad (3.20)$$

Which is irreducible and is hence the minimal polynomial of $C(x)$.

3.5.2 Enumeration of connected cubic planar graphs

Proposition 14. *The sequence $\{c_n\}_{n\geq 0}$ is P-recursive, that is, it satisfies a linear recurrence with polynomial coefficients.*

Proof. As shown in [7], the exponential generating function counting connected cubic planar graphs rooted at a vertex $G_1^\bullet(x)$ can be expressed in terms of cubic networks as

$$3G_1^\bullet(x) = C(x) + I(x) - L(x) - x^2C(x) - L(x)^2. \qquad (3.21)$$

The factor 3 comes from double counting since at every root vertex v we have three possible choices for a root edge having v as a tail. The terms $C(x) + I(x)$ encodes all the cubic networks, from which one has to subtract those which are not simple. The latter those for which the root edge is a loop, i.e. those in $\mathcal{L}$, and those where the root edge is a double edge which are encoded by the two generating functions $x^2C(x)$ (parallel-composition of a network with an edge) and $L(x)^2$ (two loop networks in series).

From the knowledge of $G_1^\bullet(x)$ (recall that $G_1^\bullet(x) = xG_1'(x)$), one can then directly compute the generating function of connected cubic planar graphs:

$$G_1(x) = \int \frac{G_1^\bullet(x)}{x} dx. \tag{3.22}$$

Now, plugging (3.21) into the above equation gives

$$G_1(x) = \int \frac{C(x) + I(x) - L(x) - x^2C(x) - L(x)^2}{3x} dx. \tag{3.23}$$

Similarly to the proof of the algebraicity of $C(x)$, one can eliminate z, $T(z)$, H, S, P, I and C from the system (3.18) to obtain an annihilating polynomial of $L(x)$, thus implying that it is algebraic. The algebraicity of $I(x)$ follows from the same elimination argument. And since $C(x)$ is algebraic and divisible by x, Equation (3.23) directly implies that $G_1(x)$ is D-finite. □

Singularity analysis. Similarly to the previous section, applying the theorem of *transfer of singularities* to (3.21) implies that $G_1^\bullet(x)$ shares the dominant singularity ρ and the singular behaviour of $C(x)$. So that to estimate the coefficients nc_n, as $n \to \infty$, it is enough to study the singular behaviour of $C(x)$.

To that end, let us compute the discriminant of (3.20) with respect to C and consider the following irreducible factor:

$$\begin{aligned} &729x^{12} + 17496x^{10} + 148716x^8 + 513216x^6 - 7293760x^4 \\ &+279936x^2 + 46656 = 0, \end{aligned} \tag{3.24}$$

whose smallest positive root is given by $\rho \approx 0.3192246062$. We claim that ρ is indeed the dominant singularity of the generating function $C(x)$. To prove it, we will now verify the conditions of Lemma 11 with the functional equation

(3.19). Consider then $R(C(x), x)$, the right hand-side of the derivative of Equation (3.19) with respect to $C(x)$:

$$\frac{x^2(1+C)^2}{4}\left(\frac{6+3T(z)}{(1-U(z))^2}-x^2\right)+\frac{2x^2(3C^2+2C-1)}{4}-C,$$

where $z = x^2(1+C(x))^3$ and where we again used that $T'(z) = (1-U(z))^{-2}$. Similarly to the previous section, we will prove the criticality of the composition scheme

$$x^2(1+C(x))^3 = \tau \tag{3.25}$$

by showing that $|R(C(x), x)| \neq 1$ for any $x \in \mathbb{R}$ such that $|x| \leq \rho$. Indeed, similarly to before, since $C(x)$, $U(z)$ and $T(z)$ all have non-negative coefficients, it holds that

$$\begin{aligned}
&|U(x^2(1+C(x))^3)| \leq |U(\rho^2(1+C(\rho))^3)| = |U(\tau)| = 1/4,\\
&|T(x^2(1+C(x))^3)| \leq |T(\rho^2(1+C(\rho))^3)| = |T(\tau)| = 1/8,\\
&\text{and } |C(x)| \leq |C(\rho)| = \left|\sqrt[3]{\frac{\tau}{\rho^2}}-1\right| \approx 0.0115259444.
\end{aligned}$$

So that it is a simple matter to check the following inequalities:

$$\begin{aligned}
&|R(C(x), x)| \leq |R(C(\rho), \rho)|\\
&\leq \left|\frac{\rho^2(1+C(\rho))^2}{4}\left(\frac{34}{3}-\rho^2\right)+\frac{2\rho^2(3C(\rho)^2+2C(\rho)-1)}{4}-C(\rho)\right|\\
&< 0.229.
\end{aligned}$$

As before, one can check that (3.24) is the only factor of both the discriminant of (3.20) with respect to C, and of the resultant of (3.20) and (3.25) with respect to C. So that ρ is a positive root of (3.25). And we can now apply Lemma 11 to obtain the singular expansion of $C(x)$ near ρ:

$$C(x) = C_0 + C_2X^2 + C_3X^3 + C_4X^4 + C_5X^5 + O(X^6), \tag{3.26}$$

where $X = \sqrt{1-x/\rho}$, $C_0 = C(\rho) \approx 0.0115259444$ and:

$$\begin{aligned}
&C_2 \approx -0.1182076128, && C_4 \approx -0.2260760661,\\
&C_3 \approx 0.2542672141, && C_5 \approx 0.0725519854.
\end{aligned}$$

Remark that we compute here the expansion up to the fifth power of X as it will be needed for the next section.

Proof of Theorem 5. The resultant with respect to C of (3.20) and (3.21) is an irreducible polynomial equation of degree nine: the minimal polynomial of $G_1^\bullet(x)$. As mentionned before, $G_1^\bullet(x)$ admits the same dominant singularity ρ and the same singular behaviour near ρ as $C(x)$. So that computing the singular Puiseux expansion of $G_1^\bullet(x)$ from its minimal polynomial gives:

$$G_1^\bullet(x) \underset{x\to\rho}{\sim} G_{1,0}^\bullet + G_{1,2}^\bullet\left(1-\frac{x}{\rho}\right) + G_{1,3}^\bullet\left(1-\frac{x}{\rho}\right)^{3/2}, \qquad (3.27)$$

where $G_{1,0}^\bullet \approx 0.0032650685$, $G_{1,2}^\bullet \approx -0.0323585164$ and $G_{1,3}^\bullet \approx 0.0720479578$.

And using (3.27), an application of the theorem of *sim-transfer* to $G_1^\bullet(x)$ gives the following asymptotic estimate for n even:

$$nc_n = n![x^n]G_1^\bullet(x) \sim c \cdot n^{-5/2} \cdot \rho^{-n} \cdot n!, \qquad (3.28)$$

where

$$c = \frac{3}{2\sqrt{\pi}} G_{1,3}^\bullet \approx 0.0609730610.$$

The estimate on c_n follows.

3.6 Cubic planar graphs

The main goal of this section is to estimate the number g_n of labelled cubic planar graphs on $n \in \mathbb{N}$ vertices. We do so following the same procedure as before, by looking at the singular expansion of the generating function $G(x) = \sum_{n\geq 0} g_n x^n/n!$, counting cubic planar graphs, where the variable x marks the number of vertices, but with a twist. Let us first prove the following statement:

Proposition 15. *The sequence $\{g_n\}_{n\geq 0}$ is P-recursive, that is, it satisfies a linear recurrence with polynomial coefficients.*

Proof. Using Equation (3.22) and the decomposition of a grqph into a set of its connected components, one directly gets:

$$G(x) = \exp(G_1(x)) = \int_x \frac{G_1^\bullet(x)}{x}, \qquad (3.29)$$

The statement is then a direct consequence of Lemma 2. □

Applying then the theorem of *transfer of singularity* to (3.29) implies that both $G_1(x)$ and $G(x)$ inherit from $G_1(x)$ the fact that their expansion near their dominant singularity ρ is Puiseux of branching-type 2. Plugging now (3.27) into Equation (3.22) gives us the singular expansion of $G_1(x)$:

$$G_1(x) = G_{1,0} + G_{1,2}X^2 + G_{1,4}X^4 + G_{1,5}X^5 + O(X^6),$$

where $G_{1,3} = 0$ follows by integrating (3.27) with respect to x together with the fact that $G^{\bullet}_{1,1} = 0$. Taking then the exponential will give us the singular expansion of $G(x)$:

$$G(x) = G_0 + G_2X^2 + G_4X^4 + G_5X^5 + O(X^6),$$

where

$$\begin{array}{ll} G_0 = e^{G_{1,0}}, & G_2 = e^{G_{1,0}}G_{1,2}, \\ G_4 = e^{G_{1,0}}\left(\frac{1}{2}G_{1,2}^2 + G_{1,4}\right), & G_5 = e^{G_{1,0}}G_{1,5}. \end{array} \tag{3.30}$$

Observe however that after integrating the singular expansion of $G^{\bullet}_1(x)$, the resulting constant term $G_{1,0}$ becomes indeterminate. In order to prove Theorem 6 we need an expression of $G_1(x)$ in terms of the generating functions of networks. Given Equation (3.23), all that would be necessary is to integrate $C(x)$, which is an algebraic function, but we have not been able to solve this integration problem. This may be due to the fact that the algebraic equation defining $C(x)$ has genus 20; in a similar situation when integrating the generating function of general planar networks, the corresponding curve has genus 0 (see [28, page 319]) and determines a rational curve. Instead, we use the theorem of *dissymetry for tre-decomposable classes*. This approach is more combinatorial and has the additional advantage of giving an alternative proof to Proposition 15.

As we shall also see in the next chapter, the constant $G_{1,0}$ has another important use when one wants to compute the probability of connectivity of a uniformly at random cubic planar graph.

3.6.1 Combinatorial integration

The key tool is to associate to a cubic planar graph γ, a canonical tree $\tau(\gamma)$ and to encode its different rootings using the generating functions of cubic networks introduced before. We follow the development and terminology from [17], adapted to our situation, where the main novelty is that, due to their bounded degree, there is a finite number of cases to encode cut-vertices using cubic networks.

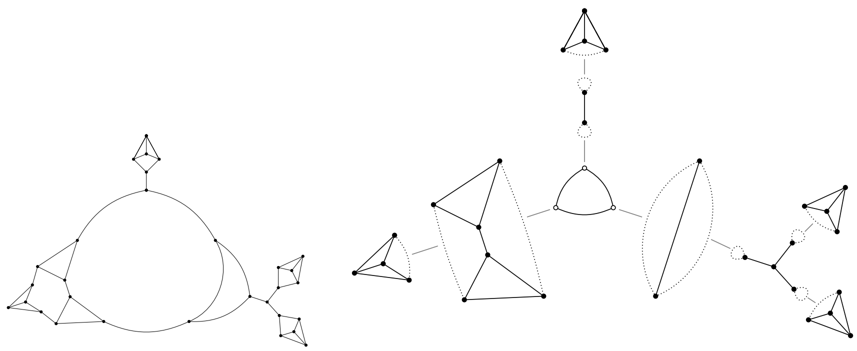


Figure 3.2: A cubic planar graph γ and its tree-decomposition $\tau(\gamma)$.

Connected cubic graphs are tree-decomposable. Let $\gamma \in \mathcal{G}_1$ be a labelled connected cubic planar graph. Similar to the recursive decomposition of γ in terms of cubic networks, as in [7, Section 3], we associate to γ a tree $\tau(\gamma)$ as follows. Its nodes will be of four different types $\mathcal{L}$ (loop), $\mathcal{R}$ (ring), $\mathcal{M}$ (multiple edge) and $\mathcal{T}$ (tri-connected) each corresponding to some cubic multigraph: $\mathcal{L}$ is associated to the cubic multigraph on two vertices with two loops, $\mathcal{R}$ with any simple cycle, $\mathcal{M}$ to the cubic parallel edge, i.e. the graph of the *3-bond*, and finally $\mathcal{T}$ is associated to any 3-connected cubic planar graph. An edge between two nodes of $\tau(\gamma)$ will correspond to compositions similar to those found in the cubic network-decomposition, but now it will be between the associated cubic multigraphs. An example of such decomposition is depicted in Figure 3.2.

The trick is to express each edge e or node v of $\tau(\gamma)$ while $\tau(\gamma)$ is itself rooted at e or at v. This will allow us to use the different cubic networks arising while decomposing the graph γ rooted at a directed edge. From there, the theorem of *dissymetry for trees* will allow us to recover the exponential generating function $t(x)$ counting all the (unrooted) trees $\tau(\gamma)$, for every $\gamma \in \mathcal{G}_1$, in terms of the different generating functions counting cubic networks. So that the bijection associating such a tree to each graph in $\mathcal{G}_1$ implies that $t(x) = G_1(x)$.

Next, we will detail the different compositions. We use here the same notation as introduced in [17], with the difference of the new category of $\mathcal{L}$-nodes (and any edge adjacent to an $\mathcal{L}$-node).

Rooting at a vertex. An $\mathcal{R}$-node is a cycle of length at least 3 in which we replace every vertex with a connected cubic network that is not series, i.e. counted by the generating function $C - S$. Notice that, by maximality of the series construction, two $\mathcal{R}$-nodes cannot be adjacent in the tree. The generating function counting trees where an $\mathcal{R}$-node is distinguished is:

$$\begin{aligned} t_{\mathcal{R}} \equiv t_{\mathcal{R}}(x) &= \mathrm{Cycl}_{\geq 3}(C(x) - S(x)) \\ &= \frac{1}{2}\left(\log \frac{1}{1-(C(x)-S(x))} - (C(x)-S(x)) - \frac{(C(x)-S(x))^2}{2}\right). \end{aligned}$$

An $\mathcal{M}$-node is the cubic parallel edge in which at least two (so that the resulting graph is simple) of the edges are substituted by a connected cubic network. The generating function counting trees where a $\mathcal{M}$-node is distinguished is:

$$t_{\mathcal{M}} \equiv t_{\mathcal{M}}(x) = \frac{x^2}{2}\left(\frac{C(x)^2}{2} + \frac{C(x)^3}{6}\right).$$

A $\mathcal{T}$-node encodes a 3-connected cubic planar graph, the *core*, in which every edge is possibly substituted by a connected cubic network. The generating function counting trees where a $\mathcal{T}$-node is distinguished is given by

$$t_{\mathcal{T}} \equiv t_{\mathcal{T}}(x) = G_3(x, 1 + C(x)).$$

Notice that the leaves of this tree-decomposition are always $\mathcal{T}$-nodes. This is the combinatorial translation of the fact discussed earlier that 3-connected graphs can be considered as the building blocks of all graphs.

Finally, a $\mathcal{L}$-node encodes a cut-vertex of the graph γ, which separates the graph into either two or three connected components. The first case, counted by the generating function $L(x)(D(x)-L(x))/2$, is obtained by replacing the root of a loop cubic network with a connected cubic network not rooted at a loop (it cannot be another loop cubic network as otherwise a double-edge would be created). The second case, counted by the generating function $L(x)^3/(6x^2)$, is obtained by pasting together three loop cubic networks. Both cases are depicted in Figure 3.3. The generating function counting trees where a $\mathcal{L}$-node is distinguished is given by

$$t_{\mathcal{L}} \equiv t_{\mathcal{L}}(x) = \frac{L(x)(C(x) - L(x))}{2} + \frac{L(x)^3}{6x^2}.$$

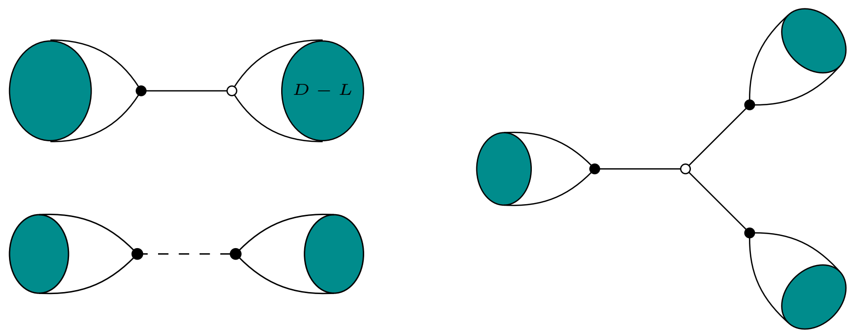


Figure 3.3: The $\mathcal{L}$-nodes of the tree-decomposition for graphs.

Rooting at an edge. The adjacency between a $\mathcal{R}$-node of the tree $\tau(\gamma)$ and any other node can be described as the Cartesian product of a series-network with another network. Recall that by maximality, an edge between two R-nodes is impossible.

Then, an adjacency between any node and a $\mathcal{M}$-node (resp. $\mathcal{T}$-node, resp. $\mathcal{L}$-node) will be described by taking the Cartesian product of a parallel-network (resp. H-network, resp. loop-network) with another network. We must then divide by two to take into account the symmetry inherent to the undirectedness of the edge.

The novelty here compared to [17] is the edge between two $\mathcal{L}$-nodes (see Figure 3.3). The associated generating functions are listed below:

$$t_{\mathcal{R}-\mathcal{M}} \equiv t_{\mathcal{R}-\mathcal{M}}(x) = \tfrac{1}{2}S(x)P(x), \quad t_{\mathcal{R}-\mathcal{T}} \equiv t_{\mathcal{R}-\mathcal{T}}(x) = \tfrac{1}{2}S(x)H(x),$$

$$t_{\mathcal{R}-\mathcal{L}} \equiv t_{\mathcal{R}-\mathcal{L}}(x) = \tfrac{1}{2}S(x)L(x), \quad t_{\mathcal{M}-\mathcal{T}} \equiv t_{\mathcal{M}-\mathcal{T}}(x) = \tfrac{1}{2}P(x)H(x),$$

$$t_{\mathcal{M}-\mathcal{L}} \equiv t_{\mathcal{M}-\mathcal{L}}(x) = \tfrac{1}{2}P(x)L(x), \quad t_{\mathcal{M}-\mathcal{M}} \equiv t_{\mathcal{M}-\mathcal{M}}(x) = \tfrac{1}{4}P(x)^2,$$

$$t_{\mathcal{T}-\mathcal{L}} \equiv t_{\mathcal{T}-\mathcal{L}}(x) = \tfrac{1}{2}H(x)L(x), \quad t_{\mathcal{T}-\mathcal{T}} \equiv t_{\mathcal{T}-\mathcal{T}}(x) = \tfrac{1}{4}H(x)^2,$$

$$t_{\mathcal{L}-\mathcal{L}} \equiv t_{\mathcal{L}-\mathcal{L}}(x) = \frac{1}{2}\frac{L(x)^2}{x^2}.$$

For instance, the generating function $t_{\mathcal{R}-\mathcal{M}}$ will count every tree-decompositon rooted at an undirected edge between a $\mathcal{R}$-node and a $\mathcal{M}$-node. It corresponds to an unordered pair of a series and parallel cubic networks, and similarly for the remaining expressions. The case when the root of the tree is an undirected the edge between two $\mathcal{L}$-nodes is illustrated as a dashed edge in the bottom left picture of Figure 3.3.

Rooting at an oriented edge. If $\mathcal{A}$ and $\mathcal{B}$ are two nodes of different types, then $t_{\mathcal{A}\to\mathcal{B}}(x) = t_{\mathcal{B}\to\mathcal{A}}(x)$ and $t_{\mathcal{A}\to\mathcal{B}}(x) = t_{\mathcal{A}-\mathcal{B}}(x)$, because there are no symmetries. When $\mathcal{A} = \mathcal{B}$ we have $t_{\mathcal{A}\to\mathcal{A}}(x) = 2t_{\mathcal{A}-\mathcal{A}}(x)$ because there are two possible orientations, hence

$$\begin{aligned} t_{\mathcal{M}\to\mathcal{M}} &\equiv t_{\mathcal{M}\to\mathcal{M}}(x) = \frac{1}{2}P(x)^2, \\ t_{\mathcal{T}\to\mathcal{T}} &\equiv t_{\mathcal{T}\to\mathcal{T}}(x) = \frac{1}{2}H(x)^2, \\ t_{\mathcal{L}\to\mathcal{L}} &\equiv t_{\mathcal{L}\to\mathcal{L}}(x) = \frac{L(x)^2}{x^2}. \end{aligned}$$

A dissymetry theorem. A theorem of *dissymetry* for $\mathcal{G}_1$ can be obtained by applying the theorem of *dissymetry for trees* on the class τ of all the trees $\tau(\gamma)$ associated to the graphs $\gamma \in \mathcal{G}_1$ and then using the bijection $\mathcal{G}_1 \simeq \tau$ explained above. So that the following combinatorial equation between labelled classes:

$$\tau \cup \tau_{\bullet\to\bullet} \simeq \tau_{\bullet} \cup \tau_{\bullet-\bullet}.$$

directly translates via the symbolic method into a functional equation between their associated exponential generating functions:

$$\begin{aligned} G_1(x) = t(x) = \;& t_{\mathcal{R}} + t_{\mathcal{M}} + t_{\mathcal{T}} + t_{\mathcal{L}} + t_{\mathcal{M}-\mathcal{M}} + t_{\mathcal{T}-\mathcal{T}} + t_{\mathcal{L}-\mathcal{L}} \\ & + t_{\mathcal{R}-\mathcal{M}} + t_{\mathcal{R}-\mathcal{T}} + t_{\mathcal{R}-\mathcal{L}} + t_{\mathcal{M}-\mathcal{T}} + t_{\mathcal{M}-\mathcal{L}} + t_{\mathcal{T}-\mathcal{L}} \\ & -2(t_{\mathcal{R}\to\mathcal{M}} + t_{\mathcal{R}\to\mathcal{T}} + t_{\mathcal{R}\to\mathcal{L}} + t_{\mathcal{M}\to\mathcal{T}} + t_{\mathcal{M}\to\mathcal{L}} + t_{\mathcal{T}\to\mathcal{L}}) \\ & -(t_{\mathcal{M}\to\mathcal{M}} + t_{\mathcal{T}\to\mathcal{T}} + t_{\mathcal{L}\to\mathcal{L}}). \end{aligned}$$

Using now the above functional equations relating the different generating functions of rooted trees into those counting the different classes of cubic networks, we obtain after some simplifications a functional equation relating

$G_1(x)$ and the cubic networks:

$$\begin{aligned} G_1(x) = \ & \tfrac{x^2}{2}\left(\tfrac{C^2}{2}+\tfrac{C^3}{6}\right)+G_3(x,1+C)+\tfrac{L^3}{6x^2} \\ & -\tfrac{1}{2}\left(\log(1-C+S)+(C-S)+\tfrac{(C-S)^2}{2}\right) \\ & -\tfrac{1}{2}\left(SP+SH+HP+\tfrac{P^2+H^2}{2}+\tfrac{L^2}{x^2}\right). \end{aligned} \tag{3.31}$$

An alternative proof of Proposition 15. We will use the expression for $G_1(x)$ obtained in (3.31) to show that $G(x) = e^{G_1(x)}$ is D-finite. First, similarly to $C(x)$, one can prove using the system (3.4) that all the involved generating functions counting cubic networks are algebraic. Now, since the exponential is clearly a D-finite function, so are the exponentials of $C(x)$, $L(x)$, $P(x)$, $S(x)$ and $H(x)$, and the same is true for the exponential of $-\log(1-C(x)+S(x))/2$, since the logarithm is canceled.

Finally, we know from Proposition 12, that the generating function $G_3(x, y)$ is D-finite. But this implies that the composition $\exp(G_3(x, 1+C))$ also is. This concludes the proof.

3.6.2 Enumeration of cubic planar graphs

Singularity analysis. As mentionned above, the theorem of *transfer of singularity* applied to Equation (3.29) tells us that $G(x)$ shares the same analytic behaviour as $C(x)$ near the positive dominant singularity ρ. It will be a Puiseux expansion of branching-type 2. To compute its coefficients, we will first compute those of the singular expansion of $G_1(x)$ by substituting each generating function in right hand-side of Equation (3.31) by its own singular expansion. Note that the expansion obtained this way will have its smallest odd-indexed summand equal to five. So that, even though each expansion involved in the right hand-side of Equation (3.31) will have its smallest odd-indexed summand equal to three, we will compute those expansions up to the fifth index.

To that end, observe that the same theorem applied to the system (3.4) implies that each of the generating functions $L(x)$, $S(x)$, $P(x)$ and $H(x)$ also share the same singular behaviour as $C(x)$. From this system one can then compute the minimal polynomial of each one of those generating functions (all of degree nine) and this way their respective Puiseux expansions near ρ.

By Lemma 11 they will be of the form:

$$\begin{aligned}
L(x) &= L_0 + L_2X^2 + L_3X^3 + L_4X^4 + L_5X^5 + O(X^6),\\
S(x) &= S_0 + S_2X^2 + S_3X^3 + S_4X^4 + S_5X^5 + O(X^6),\\
P(x) &= P_0 + P_2X^2 + P_3X^3 + P_4X^4 + P_5X^5 + O(X^6),\\
H(x) &= H_0 + H_2X^2 + H_3X^3 + H_4X^4 + H_5X^5 + O(X^6),
\end{aligned} \tag{3.32}$$

where $X = \sqrt{1 - x/\rho}$ and the computed constants are given by:

$$\begin{aligned}
&L_0 \approx 0.0005589485, L_2 \approx -0.0067979489, L_3 \approx 0.0123339214,\\
&L_4 \approx 0.0003960045, L_5 \approx -0.0200317540,\\
&S_0 \approx 0.0001313336, S_2 \approx -0.0026785117, S_3 \approx 0.0057615385,\\
&S_4 \approx 0.0083780652, S_5 \approx -0.0564371027,\\
&P_0 \approx 0.0011813127, P_2 \approx -0.0145473353, P_3 \approx 0.0262095830,\\
&P_4 \approx 0.0029590186, P_5 \approx -0.0480034574,\\
&H_0 \approx 0.0096543495, H_2 \approx -0.0941838168, H_3 \approx 0.2099621712,\\
&H_4 \approx 0.2378091544, H_5 \approx -0.19702429942.
\end{aligned}$$

Now the last remaining generating function is the one counting the trees rooted at a $\mathcal{T}$-node, i.e. $t_{\mathcal{T}}(x) = G_3(x, 1 + C(x))$. In order to compute its singular expansion, let us first consider the exponential generating function $Z(x) = U(x^2(1+C(x))^3)$. Eliminating the variables C and z from the system composed of Equations (2.6), (3.20) and $x^2(1 + C)^3 = z$, one can obtain the minimal polynomial of $Z(x)$ (it is also of degree nine). It is obvious from its definition, that the only source of singularity of $Z(x)$ comes from the composition scheme $x^2(1 + C)^3 = z$. It is now easy to check that ρ is the only positive root of both the composition scheme and the discriminant of the minimal polynomial of $Z(x)$ with respect to Z. And by arguments similar to the proof of Lemma 11, one can prove that $Z(x)$ admits a Puiseux expansion of branching-type 2 near ρ and whose coefficients are computed using its minimal polynomial as follows:

$$Z(x) = \frac{1}{4} + Z_1X + Z_2X^2 + Z_3X^3 + Z_4X^4 + Z_5X^5 + O(X^6), \tag{3.33}$$

where X is as above and

$$\begin{aligned}
&Z_1 \approx -0.4694327078, \quad Z_2 \approx 0.2711831468, \quad Z_3 \approx -0.1081595822,\\
&Z_4 \approx 0.0160663764, \quad Z_5 \approx 0.0862159165.
\end{aligned}$$

Notice that here, $Z_1 \neq 0$. Similarly to Lemma 11, this is because $Z'(\rho)$ contains the term $U'(\tau) = \infty$.

So that substituting $y = 1+C(x)$ into (3.6), then $Z(x) = U(x^2(1+C(x))^3)$ and $C(x)$ by their respective Puiseux expansions near ρ, directly gives us the Puiseux expansion of $t_{\mathcal{T}}(x)$ near ρ:

$$t_{\mathcal{T}}(x) = G_3(x, 1+C(x)) = t_0 + t_2X^2 + t_3X^3 + t_4X^4 + t_5X^5 + O(X^6), \quad (3.34)$$

where X is as above and

$$t_0 \approx 0.0006314556, \qquad t_2 \approx -0.0038258171, \qquad t_3 \approx 0.0012273923,$$
$$t_4 \approx 0.0161328302, \qquad t_5 \approx -0.0404428904.$$

The next step is to substitute each generating function of the right hand-side of Equation (3.31) by their respective Puiseux expansions near ρ. This gives us the singular expansion of $G_1(x)$ near ρ:

$$G_1(x) \underset{x\to\rho}{\sim} G_{1,0} + G_{1,2}X^2 + G_{1,4}X^4 + G_{1,5}X^5, \quad (3.35)$$

where we can finally compute every coefficient, as follows:

$$G_{1,0} \approx 0.0006035047, \qquad G_{1,2} \approx -0.0032650685,$$
$$G_{1,4} \approx 0.0145467239, \qquad G_{1,5} \approx -0.0288191831.$$

Proof of Theorem 6. By substituting $G_1(x)$ with its local expansion (3.35) into Equation (3.22), one can compute the singular expansion of $G(x)$ near ρ:

$$G(x) \underset{x\to\rho}{\sim} G_0 + G_2X^2 + G_4X^4 + G_5X^5, \quad (3.36)$$

where

$$G_0 \approx 1.0006036868, \qquad G_2 \approx -0.0032670396,$$
$$G_4 \approx 0.0145608392, \qquad G_5 \approx -0.0288365809.$$

Now, because of the expansion in powers of X, $G(x)$ is analytic in a disk of radius ρ centered at the origin after slicing the rays $[\rho, +\infty]$ and $[-\rho, -\infty]$. The compactness argument mentionned in the proof of Lemma 11 then shows that $G(x)$ is analytic in a dented domain at both $\pm\rho$. So that an application of the theorem of *sim-transfer* gives us the following estimate on g_n (for $n \in \mathbb{N}$ even):

$$g_n = n![x^n]G(x) \underset{n\to\infty}{\sim} g \cdot n^{-7/2} \cdot \rho^{-n} \cdot n!, \tag{3.37}$$

where

$$g = -\frac{15}{4\sqrt{\pi}} G_5 \approx 0.0610098696.$$

This concludes the proof.

3.7 Cubic planar multigraphs

This section discusses an asymptotic estimate for the number m_n of labelled cubic planar multigraphs on n vertices. To that end, we study the exponential generating function $M(x) = \sum_{n\geq 0} m_n x^n/n!$.

As in the case of simple graphs, we will first decompose connected cubic planar multigraphs by rooting them at an edge, using so-called *cubic multi-networks*. This time however, due to the presence of loops and multiple edges, there seems to be no direct way to describe the family of vertex-rooted connected planar multigraphs from the edge-rooted ones by integration and differentiation. Hence the apparent necessity of the dissymetry theorem to get a combinatorial description of the class of (unrooted) connected cubic planar multigraphs. This is interesting in itself as it is the first example that we could find in the literature where this theorem appears necessary.

3.7.1 Cubic multi-networks

A cubic multi-network is a labelled connected cubic planar multigraph rooted at a directed edge. So that, using similar notation as before, $\vec{M}_1(x)$ will be the generating function counting cubic multi-networks, where the variable x marks the number of vertices. For the sake of readability, we will use the notation $D \equiv D(x) = \vec{M}_1(x)$ for this generating function.

Proposition 16. *The generating function $D(x)$ is algebraic.*

Proof. We define the other families of cubic multi-networks as in the case of simple graphs and use the same letters $L(x)$, $I(x)$, $S(x)$, $P(x)$ and $H(x)$ for

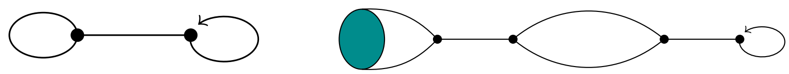


Figure 3.4: Left is the double-loop. Right is a loop cubic multi-network counted by the generating function $x^2 L$.

their associated generating functions. Then the following system of functional equations holds:

$$\begin{aligned} L &= x^2(1+L) + \frac{x^2}{2}(I+D-L), & I &= \frac{L^2}{x^2}, \\ S &= D(D-S), & P &= x^2\left(1+D+\tfrac{D^2}{2}\right), \\ D &= L+S+P+H, & H &= \frac{\vec{G}_3(x,1+D)}{1+D}. \end{aligned} \tag{3.38}$$

Observe the differences with the simple case: the first loop and parallel-compositions are now respectively the double-loop and the 3-bond, both counted by the generating function x^2. Then the case where the non-rooted edge of the double-loop is substituted by a cubic multi-network rooted at a loop is now valid, see Figure 3.4. Using Equation (3.3), we then rewritte (3.38) into a polynomial system of equations:

$$\begin{aligned} &L+S+P+H-D=0, \\ &2x^2(1+L)+x^2(I+D-L)-2L=0, \\ &L^2-x^2 I=0, \\ &D^2-(1+D)S=0, \\ &x^2(2+2D+D^2)-2P=0, \\ &T(z)-x^2(1+D)^3-2(1+D)H=0, \\ &x^2(1+D)^3-z=0. \end{aligned} \tag{3.39}$$

Together with the minimal polynomial (2.7) of $T(z)$, one can then eliminate, from the algebraic system (3.39), the variables $T(z)$, z, L, I, S, P and H to obtain the following annihiliting polynomial of $D(x)$:

$$\begin{aligned}
&4096x^6D^9 + (x^{16} - 32x^{14} + 400x^{12} - 2432x^{10} + 7520x^8 \\
&+25088x^6 + 7424x^4 - 2048x^2 + 256)D^8 + (8x^{16} - 256x^{14} \\
&+3200x^{12} - 19606x^{10} + 62560x^8 + 42448x^6 + 68992x^4 \\
&-18784x^2 + 2048)D^7 + (28x^{16} - 896x^{14} + 11193x^{12} - 68978x^{10} \\
&+225932x^8 - 56336x^6 + 272377x^4 - 71456x^2 + 6720)D^6 \\
&+(56x^{16} - 1792x^{14} + 22358x^{12} - 138334x^{10} + 462952x^8 \\
&-340130x^6 + 597478x^4 - 147112x^2 + 11648)D^5 + (70x^{16} \\
&-2240x^{14} + 27895x^{12} - 172970x^{10} + 588995x^8 - 610234x^6 \\
&+797855x^4 - 178760x^2 + 11440)D^4 + (56x^{16} - 1792x^{14} \\
&+22260x^{12} - 138082x^{10} + 476620x^8 - 591172x^6 + 665204x^4 \\
&-129952x^2 + 6336)D^3 + (28x^{16} - 896x^{14} + 11095x^{12} - 68726x^{10} \\
&+239630x^8 - 331972x^6 + 338554x^4 - 53592x^2 + 1836)D^2 \\
&+(8x^{16} - 256x^{14} + 3158x^{12} - 19498x^{10} + 68452x^8 - 102026x^6 \\
&+96236x^4 - 10440x^2 + 216)D + x^{16} - 32x^{14} + 393x^{12} \\
&-2414x^{10} + 8507x^8 - 13330x^6 + 11700x^4 - 432x^2 = 0.
\end{aligned} \tag{3.40}$$

This polynomial is irreducible and is in fact the minimal polynomial of $D(x)$. Thus concluding the proof. □

To be able to later locate the dominant singularities of $D(x)$, we will need an equation with a composition scheme in the style of Lemma 11. To that end, let us only elimnate z, L, I, S, P and H from the system (3.39), which after a simple manipulation gives:

$$D = \frac{x^2(1+D)^2}{2}\left(3 + D - \frac{1}{8}\right) + \frac{T(z)}{2} + \frac{T(z)^2}{8} - \frac{D^2}{2}, \tag{3.41}$$

where $z = x^2(1 + D(x))^3$.

3.7.2 Combinatorial integration

Let $\mathcal{M}_1$ be the class of (unrooted) labelled connected cubic planar multigraphs, with its associated generating function $M_1(x) = \sum_{n\geq 0} m'_n \frac{x^n}{n!}$. As discussed above, in order to access $M_1(x)$, we will first need to prove that connected cubic planar multigraphs are tree-decomposable. And then apply a theorem of *dissymetry* on the induced tree-decomposition.

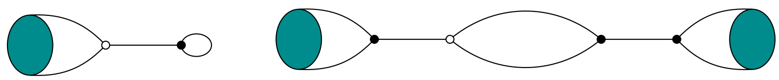


Figure 3.5: In white, the two new types of cut-vertices of a multigraph.

Connected cubic planar multigraphs are tree-decomposable. Similarly to the case of simple graphs, to any a connected cubic planar multigraph $\gamma \in \mathcal{M}_1$, we associate a unique tree $\tau(\gamma)$. And we define the class τ of all such trees, where $t(x)$ is the associated exponential generating function. So that the bijection $M_1 \simeq \tau$ holds.

We will then consider the different generating functions of those trees rooted at a vertex, at an edge, or at a directed edge, and use the exact same letters as before for both the combinatorial classes of rooted trees and for their associated generating functions.

Those generating functions can also be described in terms of multi-networks only. In fact, they will be exactly as in the simple graphs case, changing $C(x)$ to $D(x)$, but with some slight differences, reflecting the fact that we now deal with multigraphs, which we present next.

Rooting at a $\mathcal{M}$-node. The number of edges of the 3-bond graph that can now be substituted by a connected cubic multi-network, is either zero, one, two or three. This gives the new generating function:

$$t_{\mathcal{M}}(x) = \frac{x^2}{2}\left(1 + D + \frac{D^2}{2!} + \frac{D^3}{3!}\right).$$

Rooting at a $\mathcal{L}$-node. The corresponding cut-vertices of the graph can now be of two additional types: the top picture of Figure 3.5 corresponds to the new case where a cut-vertex is adjacent to another cut-vertex, itself incident with a loop. This is counted by the generating function $L(x)$. The bottom picture of Figure 3.5 corresponds to the case where a cut-vertex is incident with a double-edge. Notice that now, because of the rooting, there is no symmetry, so that it is encoded by the generating function $x^2L(x)$. Altogether, this gives the new generating function

$$t_{\mathcal{L}}(x) = L + L^2 + \frac{L(D-L)}{2} + \frac{L^3}{6x^2}$$

Rooting at an edge between two $\mathcal{L}$-nodes. We need here to add the new case in which two cut-vertices are connected by a double edge, illustrated by the multigraph on the right of Figure 3.5. This is counted by the generating function $L^2/2$, where the factor $1/2$ encodes the symmetry induced by exchanging both sides of the double edge.

The directed case is then obtained by giving a direction to one edge of the double edge from the undirected case, thus breaking the symmetry. So that the new generating functions for the undirected and the directed cases are respectively extended to:

$$t_{\mathcal{L}-\mathcal{L}}(x) = \frac{L^2}{2x^2} + \frac{L^2}{2} \qquad \text{and} \qquad t_{\mathcal{L}\to\mathcal{L}}(x) = 2 \cdot t_{\mathcal{L}-\mathcal{L}}(x) = \frac{L^2}{x^2} + L^2.$$

Another dissymetry theorem. As in the simple graphs case, we obtain a theorem of *dissymetry* for the class $\mathcal{M}_1$ by applying the theorem of *dissymetry for trees* on the class τ and using the bijection $\mathcal{M}_1 \simeq \tau$:

$$\mathcal{M}_1 \cup \tau_{\bullet\to\bullet} \simeq \tau_{\bullet} \cup \tau_{\bullet-\bullet}.$$

This directly translates via the symbolic method into a functional equation between their associated exponential generating functions:

$$\begin{aligned} M_1(x) \;&= x^2\left(\tfrac{1}{2} + \tfrac{D}{2} + \tfrac{D^2}{4} + \tfrac{D^3}{12}\right) + G_3(x, 1+D) + \tfrac{L^3}{6x^2} + \tfrac{L^2}{2} + L \\ &-\tfrac{1}{2}\left(\log(1-D+S) + D - S + \tfrac{(D-S)^2}{2}\right) \\ &-\tfrac{1}{2}\left(PS + HS + HP + \tfrac{P^2+H^2}{2} + \tfrac{L^2}{x^2}\right). \end{aligned} \tag{3.42}$$

Proposition 17. *The sequence $\{m_n\}_{n\geq 0}$ is P-recursive, that is, it satisfies a linear recurrence with polynomial coefficients.*

Proof. The claim follows from a direct adaptation of the second proof of Proposition 15, presented in the last paragraph of Subsection 3.6.1, to Equation 3.42, using Propositions 12 and 16. □

3.7.3 Enumeration of cubic planar multigraphs

Singularity analysis. The theorem of *transfer of singularities* applied to (3.42) shows that one can derive the singular behaviour of $M_1(x)$ from that of $D(x)$. To that end, let us compute the discriminant of (3.40) with respect to D and consider the following irreducible factor:

$$\begin{array}{l} 729x^{12} - 17496x^{10} + 148716x^8 - 513216x^6 - 7293760x^4 \\ -279936x^2 + 46656 = 0, \end{array} \tag{3.43}$$

whose smallest positive root is given by $\rho_m \approx 0.2509071947$. Notice that the coefficients are exactly those of (3.24), up to some changes of signs. We have no satisfactory interpretation of this similarity between the simple and the multigraphs cases.

Let us now check that ρ_m is the dominant singularity of $D(x)$ by verifying the conditions of Lemma 11 with the functional equation (3.41). Let as before $R(D(x), x)$ be the right hand-side of the derivative of Equation (3.41) with respect to D:

$$\frac{x^2(1+D)^2}{4}\left(\frac{6+3T(z)}{(1-U(z))^2} - x^2\right) + \frac{x^2(3D^2+10D-7)}{2} - D,$$

where $z = x^2(1+D(x))^3$ and where we again used that $T'(z) = (1-U(z))^{-2}$. To prove the criticality of the composition scheme

$$x^2(1+D(x))^3 = \tau, \tag{3.44}$$

it is a simple matter to check that for any $x \in \mathbb{R}$ such that $|x| \leq \rho_m$:

$$|R(D(x), x)| \leq |R(D(\rho_m), \rho_m)| < 0.3432,$$

where we used that $|D(x)| \leq |D(\rho_m)| = \left|\sqrt[3]{\frac{\tau}{\rho_m^2}} - 1\right| \approx 0.1876793068$.

As before, one can check that (3.43) is the only factor of both the discriminant of (3.40) with respect to D, and of the resultant of (3.40) and (3.44) with respect to D. And we can now apply Lemma 11 to obtain the singular expansion of $D(x)$ near ρ_m:

$$D(x) = D_0 + D_2X^2 + D_3X^3 + D_4X^4 + D_5X^5 + O(X^6), \tag{3.45}$$

where $X = \sqrt{1 - x/\rho_m}$, $D_0 = D(\rho_m) \approx 0.1876793068$ and:

$$\begin{array}{ll} D_2 \approx -0.7326085490, & D_4 \approx 0.35908310, \\ D_3 \approx 0.6259341848, & D_5 \approx -0.9183087339. \end{array}$$

Remark that similarly to the simple case, we need to compute the expansion up to the fifth power of X.

Singular expansion of connected cubic planar multigraphs. The same arguments as in the previous section applied to the system (3.39) shows that the exponential generating functions $L(x)$, $S(x)$, $P(x)$ and $H(x)$ counting the associated families of cubic multi-networks are algebraic and one can compute their respective minimal polynomials in a similar manner as that of $D(x)$. From there, we compute their respective Puiseux expansions near ρ_m:

$$\begin{aligned} L(x) &= L_0 + L_2X^2 + L_3X^3 + L_4X^4 + L_5X^5 + O(X^6), \\ S(x) &= S_0 + S_2X^2 + S_3X^3 + S_4X^4 + S_5X^5 + O(X^6), \\ P(x) &= P_0 + P_2X^2 + P_3X^3 + P_4X^4 + P_5X^5 + O(X^6), \\ H(x) &= H_0 + H_2X^2 + H_3X^3 + H_4X^4 + H_5X^5 + O(X^6), \end{aligned} \quad (3.46)$$

where $X = \sqrt{1 - x/\rho_m}$ and the computed constants are given by:

$$\begin{aligned} &L_0 \approx 0.0739210283, L_2 \approx -0.1849294497, L_3 \approx 0.022023947, \\ &L_4 \approx 0.1758931558, L_5 \approx -0.0824618341, \\ &S_0 \approx 0.0296574353, S_2 \approx -0.2132424445, S_3 \approx 0.1821924353, \\ &S_4 \approx 0.4248853189, S_5 \approx -0.8147303774, \\ &P_0 \approx 0.0758784006, P_2 \approx -0.2065336949, P_3 \approx 0.0468008876, \\ &P_4 \approx 0.2291750107, P_5 \approx -0.1911320863, \\ &H_0 \approx 0.0082224427, H_2 \approx -0.1279029599, H_3 \approx 0.3749169147, \\ &H_4 \approx 0.4708703797, H_5 \approx -0.1700155640. \end{aligned}$$

This time, the exponential generating function counting the decomposition-trees rooted at a $\mathcal{T}$-node is given by $t_{\mathcal{T}}(x) = G_3(x, 1+D(x))$. The exact same method as in the previous section (changing $C(x)$ to $D(x)$ and the associated minimal polynomials) allows us to compute its singular expansion near ρ_m:

$$t_{\mathcal{T}}(x) = G_3(x, 1 + D(x)) = t_0 + t_2X^2 + t_3X^3 + t_4X^4 + t_5X^5 + O(X^6), \quad (3.47)$$

where X is as above and

$$\begin{aligned} &t_0 \approx 0.0006314556, \quad &&t_2 \approx -0.0062671242, \quad &&t_3 \approx 0.0025733540, \\ &t_4 \approx 0.0495962882, \quad &&t_5 \approx -0.1427835242. \end{aligned}$$

The final step is to substitute each generating function of the right hand-side of Equation (3.42) by their respective Puiseux expansions near ρ_m. This gives us the singular expansion of $M_1(x)$ near ρ_m:

$$M_1(x) \underset{x\to\rho}{\sim} M_{1,0} + M_{1,2}X^2 + M_{1,4}X^4 + M_{1,5}X^5, \quad (3.48)$$

where $X = \sqrt{1 - x/\rho_m}$ and:

$$M_{1,0} \approx 0.0706604969, \qquad M_{1,2} \approx -0.1638622489,$$
$$M_{1,4} \approx 0.1725921179, \qquad M_{1,5} \approx -0.0989787363.$$

By contradiction, let us illustrate here why is $M_{1,3} = 0$. Indeed, otherwise and by the theorem of *sim-transfernsfer*, the ratio between the number of connected cubic planar multigraphs with n vertices and the number of connected networks with n vertices will be constant as n goes to infinity. Let us now define a *bad edge* as either a double edge or a loop. Observe that for $n \geq 4$, a vertex of a connected cubic planar multigraph can be adjacent to at most one bad edge. Hence, every vertex is adjacent to at least one simple edge, and because a simple edge is shared by two vertices, a connected cubic planar multigraph admits at least $n/2$ simple edges, for $n \geq 4$. Notice now that every time a simple edge of a connected cubic planar multigraph is distinguished and directed, we get a different connected network, i.e. for $n \geq 4$ the number of connected networks with n vertices is at least $n/2$ times greater than the number of connected cubic planar multigraphs with n vertices. Which contradicts the hypothesis that their ratio converges as n goes to infinity.

Proof of Theorem 7. The expansion (3.48) in powers of $X = \sqrt{1 - x/\rho_m}$ implies that $M_1(x)$ is analytic in a disk of radius ρ_m centered at the origin after slicing the rays $[\rho_m, +\infty]$ and $[-\rho_m, -\infty]$. The compactness argument mentionned in the proof of Lemma 11 then shows that it is analytic in a dented domain at both $\pm\rho_m$. So that an application of the theorem of *sim-transfer* gives the following estimate on m'_n (for n even):

$$m'_n = n![x^n]M_1(x) \underset{n\to\infty}{\sim} m' \cdot n^{-7/2} \cdot \rho_m^{-n} \cdot n!, \tag{3.49}$$

where

$$m' = -\frac{15}{4\sqrt{\pi}} M_{1,5} \approx 0.2094103951.$$

Now from the knowledge of $M_1(x)$, one can recover the generating function $M(x)$ of labelled cubic planar multigraphs as follows

$$M(x) = \exp\Big(M_1(x)\Big). \tag{3.50}$$

By the theorem of *transfer of singularity*, ρ_m is the dominant singularity of $M(x)$ and its singular expansion is given by:

$$M(x) \underset{x \to \rho_m}{\sim} M_0 + M_2 X^2 + M_4 X^4 + M_5 X^5, \tag{3.51}$$

where $X = \sqrt{1 - x/\rho_m}$ and:

$$\begin{aligned} M_0 &\approx 1.0732168035, & M_2 &\approx -0.1758597189, \\ M_4 &\approx 0.1996371456, & M_5 &\approx -0.1062256430. \end{aligned}$$

The same arguments as above on the expansion (3.51) show that $M(x)$ is analytic in a dented domain at both $\pm\rho_m$. So that an application of the theorem of *sim-transfer* gives us the following estimate on m_n (for $n \in \mathbb{N}$ even):

$$m_n = n![x^n]M(x) \sim m \cdot n^{-7/2} \cdot \rho_m^{-n} \cdot n!, \tag{3.52}$$

where

$$m = -\frac{15}{4\sqrt{\pi}} M_5 \approx 0.2247427548.$$

This concludes the proof.

3.8 Simple cubic planar maps

In this section, we will show that the connectivity-decomposition method developed so far allows one to also enumerate both arbitrary and 2-connected simple cubic planar maps. We will first need to adapt this decompositon method to the context of maps, where so-called *near-simple* maps will play an analogous role as that of netwroks in graphs.

3.8.1 Enumeration of simple cubic maps

We will enumerate here the class $\mathcal{C}$ of simple rooted cubic planar maps, by computing an asymptotic estimate of the number s_n of simple rooted cubic planar maps with n vertices, as n goes to infinity. We set $C(x) = \sum_{n \geq 0} s_n x^n$ to be the associated ordinary generating function (recall that here we only consider maps with unlabelled vertices).

Near-simple cubic maps. A cubic map $N \in \mathcal{C}$ with root edge st is said to be *near-simple* when $N - st$ is connected and simple. Let $\bar{\mathcal{C}} \subseteq \mathcal{C}$ be the class of near-simple maps in $\mathcal{C}$, with its associated ordinary generating function $\bar{C} \equiv \bar{C}(x)$. Analogue to the different classes of networks, we now define some subclasses of $\bar{\mathcal{C}}$:

- $N \in \mathcal{L}$ is a *loop map* if st is a loop,
- $N \in \mathcal{S}$ is a *series map* if $N - st$ admits a bridge,
- $N \in \mathcal{P}$ is a *parallel map* if $N - st$ is 2-connected but $\{s, t\}$ is a 2-cut,
- $N \in \mathcal{H}$ is an *h-map* if it can be obtained by possibly substituting the edges of some 3-connected cubic map by maps in $\bar{\mathcal{C}}$.

To each of those classes is respectively associated the ordinary generating functions $L \equiv L(x)$, $I \equiv I(x)$, $S \equiv S(x)$, $P \equiv P(x)$ and $H \equiv H(x)$. This allows us to derive a system of equations relating $C(x)$ with the ordinary generating function $T(z)$, counting 3-connected cubic planar maps.

Lemma 18. *The following system of equations holds:*

$$\begin{aligned} &\bar{C} = L + S + P + H, && L = 2x^2\left(I + \bar{C} - L\right), \\ &I = \frac{L^2}{4x^2}, && P = x^2(2\bar{C} + \bar{C}^2), \\ &S = \bar{C}(\bar{C} - S), && H = \frac{T(x^2(1+\bar{C})^3) - x^2(1+\bar{C})^3}{1+\bar{C}}. \end{aligned} \tag{3.53}$$

Proof. It suffices to adaptate the proof of Lemma 10 to the case of maps. The only differences are the absence of symmetries in the loop, the isthmus, the parallel and the h-compositions. □

Proposition 19. *The generating function $C(x)$ is algebraic.*

Proof. We first rewrite (3.53) into a system of polynomial equations:

$$\begin{aligned} &\bar{C} - L - L^2 - 2x^2\bar{C} + I - C = 0, \\ &2x^2(I + \bar{C}) + (2x^2 - 1)L = 0, \\ &L + S + P + H - \bar{C} = 0, \\ &L^2 - 4x^2 I = 0, \\ &x^2(2\bar{C} + \bar{C}^2) - P = 0, \\ &\bar{C} - (1 + \bar{C})S = 0, \\ &T(z) - x^2(1 + \bar{C})^3 - (1 + \bar{C})H = 0, \\ &x^2(1 + \bar{C})^3 - z = 0. \end{aligned} \tag{3.54}$$

Similarly to the graph case, in order to obtain a simple map in $\mathcal{C}$ one needs to remove the loop maps and the maps rooted at a double edge from the class $\bar{\mathcal{C}}$, and then to add the maps rooted at an isthmus. This gives the following polynomial equation relating $C(x)$ with the ordinary generating functions of near-simple cubic maps:

$$C(x) = \bar{C}(x) - L(x) - L^2(x) - 2x^2\bar{C}(x) + I(x), \tag{3.55}$$

where the terms $L^2(x)$ and $2x^2\bar{C}(x)$ encode maps rooted at a double edge, respectively obtained from the series composition of two loop maps and the two possible parallel compositions of a map in $\bar{\mathcal{C}}$ with an edge.

So that by eliminating variables I, L, S, P, H, z and T from the system (3.54), together with Equations (2.7) and (3.55), we obtain an annhilating polynomial for $C(x)$. Furthermore, one can factorise this annihilating polynomial and choose the "right" factor by checking the first terms of the Taylor expansions of its roots, for x near the origin. This factor will then be the minimal polynomial of $C \equiv C(x)$:

$$\begin{aligned}
&64x^{10}C^4 + (912x^{14} + 640x^{12} + 384x^{10} + 3328x^8 + 2864x^6)C^3\\
&-(1743x^{18} + 13968x^{16} + 13344x^{14} - 52888x^{12} - 116934x^{10}\\
&-71248x^8 - 4064x^6 + 3768x^4 - 41x^2)C^2 + (784x^{22} + 13524x^{20}\\
&+29478x^{18} - 51033x^{16} - 194686x^{14} - 166400x^{12} - 5454x^{10}\\
&+43746x^8 + 4030x^6 - 5652x^4 + 904x^2 - 41)C - 1568x^{24}\\
&-17724x^{22} + 3788x^{20} + 90950x^{18} + 85609x^{16} - 10833x^{14}\\
&-29549x^{12} + 1572x^{10} + 3719x^8 - 781x^6 + 41x^4 = 0.
\end{aligned} \tag{3.56}$$

□

Singularity analysis. As before, the theorem of *transfer of singularity* applied to Equation 3.55 implies that to study the singular behaviour of $C(x)$ it is enough to study that of $\bar{C}(x)$.

To do the latter, we derive an equation verifying the hypotheses of Lemma 11 from the system (3.54): after eliminating the variables I, L, S, P, H and z, we obtain a single polynomial equation with a composition scheme, relating $\bar{C} \equiv \bar{C}(x)$ with $T(z)$. It is given by:

$$\begin{aligned}
\bar{C} = &\frac{T(z)^2}{2} + (x^2(1+\bar{C}) + 1)T(z) + 2x^2\bar{C}^3\\
&+ \frac{1}{2}(1-x^2)(3x^2-1)\bar{C}^2 - x^2\left(1 + (3x^2+1)\bar{C} + \frac{3x^2}{2}\right),
\end{aligned} \tag{3.57}$$

where $z = x^2(1+\bar{C})^3$ is the composition scheme. And using (2.7), the minimal polynomial of $T(z)$, one can eliminate $T(z)$ from (3.57) to obtain the following annihilating polynomial of $\bar{C} \equiv \bar{C}(x)$:

$$\begin{aligned}&256x^{12}\bar{C}^4 + 32x^6\bar{C}^3(9x^8+80x^6+206x^4+272x^2+161)\\&+\bar{C}^2(81x^{16}+1728x^{14}+7260x^{12}+16032x^{10}+17366x^8\\&+8832x^6-548x^4+32x^2+1)+\bar{C}(162x^{16}+2538x^{14}+7432x^{12}\\&+12324x^{10}+9324x^8+2970x^6-1240x^4+56x^2+2)+81x^{16}\\&+1098x^{14}+2449x^{12}+2668x^{10}+761x^8-70x^6-2x^4=0.\end{aligned} \tag{3.58}$$

This polynomial turns out to be irreducible and is hence the minimal polynomial of $\bar{C}(x)$.

Next, we consider the following factor of the discriminant of (3.58) with respect to $\bar{C}$:

$$\begin{aligned}&729x^{12}+17496x^{10}+148716x^8+513216x^6\\&-7293760x^4+279936x^2+46656=0,\end{aligned} \tag{3.59}$$

whose smallest positive root is given by $\rho_s \approx 0.3192246062$. This is the unique such factor that is also a factor of the polynomial obtained by taking the resultant, with respect to $\bar{C}$, of (3.58) with the composition scheme:

$$x^2(1+\bar{C}(x))^3 = \tau. \tag{3.60}$$

To finally apply Lemma 11, we need to prove that the above scheme is critical. To that end, let us compute the derivative of the right hand-side of (3.57) with respect to $\bar{C}$:

$$\begin{aligned}R(\bar{C}(x),x) :=\ & 3x^2(1+\bar{C})\left(x^2(2\bar{C}+\bar{C}^2)+\frac{T(z)-1}{(1-U(z))^2}\right)\\&+x^2T(z)-x^2(1-4\bar{C}-6\bar{C}^2)-\bar{C},\end{aligned}$$

where $z = x^2(1+\bar{C}(x))^3$, and where we again used that $T'(z) = (1-U(z))^{-2}$. Using the fact that both $\bar{B}(x)$ and $U(z)$ have non-negative coefficients, it is a simple matter to check that for any $x \in \mathbb{R}$ such that $|x| \le \rho_s$:

$$|R(\bar{C}(x),x)| \le |R(\bar{C}(\rho_s),\rho_s)| < 0.555,$$

where we used that $|\bar{C}(x)| \le |\bar{C}(\rho_s)| = \left|\sqrt[3]{\frac{\tau}{\rho_s^2}}-1\right| \approx 0.0309197658$. So that by Lemma 11, $\pm\rho_s$ are the two dominant singularities of both $\bar{C}(x)$ and $C(x)$, and their respective expansions near ρ_s are Puiseux series of branching type 3/2. This also implies that $\bar{C}(x)$ is analytic in a dented domain at $\pm\rho_s$.

Proof of Theorem 8. Using (3.56), we can compute the Puiseux expansion of $C(x)$ near ρ_s:

$$C(x) \underset{x\to\rho_s}{\sim} C_0 + C_2\left(1-\frac{x}{\rho_s}\right) + C_3\left(1-\frac{x}{\rho_s}\right)^{3/2}, \tag{3.61}$$

where $C_0 \approx 0.0200048103$, $C_2 \approx -0.296721268$, and $C_3 \approx 1.1068974043$.

The theorem of *transfer of singularity* on (3.55) tells us that $C(x)$ is also analytic in a dented domain at $\pm\rho_s$. So that the theorem of *sim-transfer*, applied on (3.61), implies the following estimate on s_n (for n even):

$$s_n = [x^n]C(x) \underset{n\to\infty}{\sim} s \cdot n^{-5/2} \cdot \rho_s^{-n}, \tag{3.62}$$

where $s = \dfrac{3}{2\sqrt{\pi}} \cdot C_3 \approx 0.9367499783$.

3.8.2 Enumeration of 2-connected simple cubic maps

We now enumerate the class $\mathcal{B}$ of 2-connected simple cubic planar maps. In particular, we will give an asymptotic estimate of s'_n, the number of 2-connected simple cubic planar maps on n vertices, by studying the associated ordinary generating function $B(x) = \sum_{n\geq 0} s'_n x^n$.

Proposition 20. *The ordinary generating function $B(x)$ is algebraic.*

Proof. Similarly to the case of arbitray cubic maps, we define the notion of *near-simple* 2-connected cubic planar maps: let $\bar{\mathcal{B}}$ be the class of 2-connected cubic planar maps N, rooted at the edge st, such that $N - st$ is simple and connected. Notice that, because it is 2-connected, a map in $\bar{\mathcal{B}}$ cannot have loops nor isthmuses. The subclasses $\mathcal{S}$, $\mathcal{P}$ and $\mathcal{H}$ of $\bar{\mathcal{B}}$ are then defined as their connected counterparts. Thus, from a direct adaptation of the proof of Lemma 18, one can see that the following polynomial system of equations holds:

$$\begin{array}{l} S + P + H - \bar{B} = 0, \\ x^2(2\bar{B} + \bar{B}^2) - P = 0, \\ \bar{B}^2 - S(1+\bar{B}) = 0, \\ T(z) - x^2(1+\bar{B})^3 - (1+\bar{B})H = 0, \\ x^2(1+\bar{B})^3 - z = 0. \end{array} \tag{3.63}$$

Notice now that the class $\mathcal{B}$ can be obtained by removing from the class $\bar{\mathcal{B}}$ the maps rooted at a double edge coming from the parallel compositions.

This is directly translated into the following relation between the associated generating functions $B(x)$ and $\bar{B}(x)$:

$$B(x) = (1 - 2x^2)\bar{B}(x). \tag{3.64}$$

So that using (3.64) together with (2.7), one can eliminate variables S, P, H, z, $T(z)$ and $\bar{B}$ from the system (3.63) to obtain an annihilating polynomial of $B(x)$. And after factorisation, one can compute the (irreducible) "right" factor of this annihilating polynomial. It is obtained as in the connected case, and is the minimal polynomial of $B \equiv B(x)$:

$$\begin{aligned} &16x^4B^3 - (16x^8 + 120x^6 - 48x^4 - 8x^2)B^2 \\ &+ (4x^{12} + 76x^{10} + 121x^8 - 244x^6 + 118x^4 - 20x^2 + 1)B \\ &- 8x^{14} - 76x^{12} + 134x^{10} - 77x^8 + 17x^6 - x^4 = 0. \end{aligned} \tag{3.65}$$

Thus proving that $B(x)$ is algebraic. □

Singularity analysis. Again by applying the theorem of *transfer of singularity* on Equation 3.64, one can see that $B(x)$ and $\bar{B}(x)$ both share the same dominant singularities and singular behaviour.

We will now study that of $\bar{B}(x)$ by deriving an equation with a composition scheme and show that it satisfies Lemma 11. After eliminating variables S, P and H from the system (3.63), we obtain the following single polynomial equation relating $\bar{B}(x)$ with $T(z)$:

$$\bar{B}(x) = T(z) - x^2(1 + \bar{B}), \tag{3.66}$$

where $z = x^2(1 + \bar{B})^3$ is the composition scheme. Using (2.7), we can now eliminate $T(z)$ from (3.66) to obtain an annhilating polynomial of $\bar{B} \equiv \bar{B}(x)$:

$$\begin{aligned} &16x^4\bar{B}^3 + 8x^2\bar{B}^2(x^4 + 8x^2 + 1) \\ &+\bar{B}(x^8 + 20x^6 + 50x^4 - 16x^2 + 1) + x^8 + 11x^6 - x^4 = 0. \end{aligned} \tag{3.67}$$

One can check that this polynomial is irreducible and is hence the minimal polynomial of $\bar{B}(x)$. Let us now consider the following factor of the discriminant of (3.67) with respect to $\bar{B}$:

$$2x^4 + 10x^2 - 1 = 0, \tag{3.68}$$

whose smallest positive root is:

$$\rho'_s = \frac{\sqrt{6\sqrt{3} - 10}}{2} \approx 0.3131712173.$$

We will now prove that the composition scheme is critical, i.e. that the following equation holds:

$$x^2(1+\bar{B}(x))^3 = \tau. \tag{3.69}$$

Let us first compute the derivative of the right hand-side of (3.66) with respect to $\bar{B}$:

$$R(\bar{B}(x), x) := \frac{3x^2(1+\bar{B}(x))^2}{(1-U(z))^2} - x^2,$$

where $z = x^2(1+\bar{B}(x))^3$ and $T'(z) = (1-U(z))^{-2}$. Using then that both $\bar{B}(x)$ and $U(z)$ have non-negative coefficients and that

$$\bar{B}(\rho'_s) = \frac{3\sqrt{3}-5}{8} \approx 0.0245190528,$$

it is a simple matter to check that the following inequality holds for any $x \in \mathbb{R}$ such that $|x| \leq \rho'_s$:

$$|R(\bar{B}(x), x)| \leq |R(\bar{B}(\rho'_s), \rho'_s)| < 0.451.$$

One can finally verify that (3.68) is the unique factor of the discriminant, with respect to $\bar{B}$, of (3.67) that is also a factor of the resultant, with respect to $\bar{B}$, of (3.67) with (3.69). And Lemma 11 implies that $\pm\rho'_s$ are the two dominant singularities of $\bar{B}(x)$. It also implies that $\bar{B}(x)$ both admits a Puiseux series expansion near ρ'_s of branching type 3/2, and is analytic in a dented domain at $\pm\rho'_s$.

Proof of Theorem 9. As mentionned nefore, the theorem of *transfer of singularity*, applied on Equation (3.64), implies that $\pm\rho'_s$ are also the dominant singularities of $B(x)$ and that $B(x)$ admits the following expansion near ρ'_s, computed using (3.65):

$$B(x) \underset{x\to\rho'_s}{\sim} B_0 + B_2\left(1-\frac{x}{\rho'_s}\right) + B_3\left(1-\frac{x}{\rho'_s}\right)^{3/2}, \tag{3.70}$$

where $B_0 = 0.0197095812$, $B_2 = -0.2451905284$, and $B_3 = 0.7487510188$. It also implies that $B(x)$ is analytic in a dented domain at $\pm\rho'_s$. So that an application of the theorem of *sim-transfer* on (3.70) gives the following estimate on s'_n (for n even):

$$s'_n = [x^n]B(x) \underset{n\to\infty}{\sim} s' \cdot n^{-5/2} \cdot \rho_s'^{-n}, \tag{3.71}$$

where $s' = \dfrac{3}{2\sqrt{\pi}} \cdot B_3 \approx 0.6336562882$.

3.9 Conclusion

To conclude this chapter, we give here some tables presenting the main enumerative results in a compact way. We first consider the following exponential generating functions:

- $G(x) = \sum_{n\geq 0} g_n x^n/n!$, where g_n $(n \geq 0)$ is the number of cubic planar graphs on $[n]$,
- $G_1(x) = \sum_{n\geq 0} c_n x^n/n!$, where c_n $(n \geq 0)$ is the number of labelled connected cubic planar graphs on $[n]$,
- $G_2(x) = \sum_{n\geq 0} b_n x^n/n!$, where b_n $(n \geq 0)$ is the number of labelled 2-connected cubic planar graphs on $[n]$,
- $G_3(x) = \sum_{n\geq 0} t_n x^n/n!$, where t_n $(n \geq 0)$ is the number of labelled 3-connected cubic planar graphs on $[n]$.

In Table 3.1 are displayed their respective six first non-zero coefficients. Notice that the first discrepancies appear at $n = 8$. Indeed, the first cubic planar graph that is 2-connected but not 3-connected is the (unrooted) series-composition of two K_4's. And the first disconnected cubic planar graph is the disjoint union of two K_4's. Whereas in Table 3.2 are displayed the approximate values of the different orders in the asymptotic estimate of their respective coefficients.

We consider next the two exponential generating functions counting multigraphs:

- $M(x) = \sum_{n\geq 0} m_n \frac{x^n}{n!}$ counting labelled cubic planar multigraphs,
- $M_1(x) = \sum_{n\geq 0} m'_n \frac{x^n}{n!}$ counting connected labelled cubic planar multigraphs,

together with the two ordinary generating functions counting simple maps:

- $C(x) = \sum_{n\geq 0} s_n x^n$ counting rooted simple cubic planar maps,
- $B(x) = \sum_{n\geq 0} s'_n x^n$ counting 2-connected rooted simple cubic planar maps.

In Table 3.3 are displayed their respective first coefficients. The two cubic planar multigraphs with two vertices are the (unrooted) 3-bond and double-loop. Whereas in Table 3.4 are displayed the approximate values of the different orders in the asymptotic estimate of their respective coefficients.

n	g_n	c_n	b_n	t_n
4	1	1	1	1
6	60	60	60	60
8	13475	13440	13440	10920
10	5826240	5813640	5700240	4112640
12	4124741775	4116420000	3996669600	2654467200
14	4379810575140	4371563196000	4217639025600	2625727104000

Table 3.1: Numbers of arbitrary, connected, 2-connected and 3-connected labelled cubic planar graphs with n vertices.

Coefficient	Constant factor	Exponential growth
g_n	0.0610098696	3.1325905979
c_n	0.0609730610	3.1325905979
b_n	0.0592436837	3.1296662937
t_n	0.0407168760	3.0792014357

Table 3.2: The approximate values of the linear and exponential orders in the asymptotic estimate of the numbers of arbitrary, connected, 2-connected and 3-connected labelled cubic planar graphs.

n	m_n	m'_n	s_n	s'_n
2	2	2	0	0
4	47	35	1	1
6	4710	3540	7	3
8	1239875	967680	43	19
10	669496590	542694600	294	128
12	634267800705	529970364000	2129	909
14	946240741175730	808665504847200	16133	6737

Table 3.3: Numbers of arbitrary and connected labelled cubic planar multigraphs and arbitrary and 2-connected simple cubic planar maps with n vertices.

Coefficient	Constant factor	Exponential growth
m_n	0.2247427548	3.9855373662
m'_n	0.2094103951	3.9855373662
s_n	0.9367499783	3.2231120230
s'_n	0.6336562882	3.1931414661

Table 3.4: The approximate values of the linear and exponential orders in the asymptotic estimate of the numbers of arbitrary and connected labelled cubic planar mutigraphs and aribtrary and 2-connected simple cubic planar maps.

Chapter 4

Random cubic planar graphs

4.1 Introduction

A uniform random cubic planar graph on n vertices is the typical object chosen uniformly at random among all cubic planar graphs on n vertices. In this chapter, we will study some of the classical properties verified by this object. We will first focus on properties related to connectivity, then concern ourselves with the number of appearances of some given subgraphs.

If one can both enumerate a family $\mathcal{G}_n$ of graphs on n vertices, and the subfamily composed of the graphs in $\mathcal{G}_n$ satisfying a given property P, then one can compute the probability p_n that P holds in $\mathcal{G}_n$ by taking the ratio of the two. To speak deterministically of such a property, one then *takes the limit*, that is: computes when it exists the limiting probability $\lim p_n$, as $n \to \infty$. To that end, we will make an extensive use of the asymptotic results proved in the previous chapter.

4.1.1 Results on connectivity

For instance, a direct consequence of our enumerative results is an estimate of the probability of connectivity for both cubic planar graphs and multigraphs.

Theorem 21. *The limiting probability p that a uniformly at random cubic planar graph is connected is equal to*

$$p = \frac{c}{g} \approx 0.9993966774,$$

where c and g are as in Theorems 5 and 6 respectively. A similar estimate holds for a uniformly at random cubic planar multigraph. The limiting probability of connectivity in this case is equal to

$$p_m = \frac{m'}{m} \approx 0.93177818003,$$

where m' and m are as in Theorem 7.

Let us remark that the actual value of p was not computed in [7]. Indeed, it requires an application of the theorem of *dissymetry for tree-decomposable classes* to access the constant terms in the estimates for the number of both connected and arbitrary cubic planar graphs.

Theorem 22. *Let X_n and Y_n be two random variables respectively counting the number of connected components in a unifromly at random cubic planar graph and multigraph on n vertices. Then as $n \to \infty$ they are both distributed following shifted Poisson distributions:*

$$X_n \sim 1 + Po(G_1(\rho)) \qquad \text{and} \qquad Y_n \sim 1 + Po(M_1(\rho_m)),$$

where $G_1(\rho) = G_{1,0} \approx 0.0006035047$ and $M_1(\rho_m) = M_{1,0} \approx 0.0706604969$.

Observe then that theorem 21 becomes a corollary of Theorem 22, as in the case of simple graphs, we have that $\mathbb{P}[X_n = 1] \sim e^{-G_1(\rho)}$, as $n \to \infty$.

Our second result concerns the limiting distributions of some other parameters related to connectivity that have been studied for planar graphs and related classes of graphs (see for example the survey [48]).

Theorem 23. *For a random cubic planar graph, the following parameters are asymptotically Gaussian with linear expectation and variance:*

1. *the number of cut-vertices,*
2. *the number of isthmuses,*
3. *and the number of blocks.*

The values of both constants μ for the expectation and λ for the variance can be approximated to any digits. In the next table, we present such approxiamtions up to the tenth decimal:

Parameter	μ	λ
Cut-vertices	0.0018774448	0.0037934519
Isthmuses	0.0009389848	0.0009496835
Blocks	0.0018777072	0.0037958302

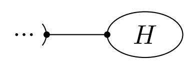


Figure 4.1: A generic cherry build from the graph H.

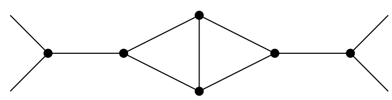


Figure 4.2: An appearance of the smallest near-brick, K_4 minus an edge, in a cubic planar graph.

4.1.2 Results on the number of subgraphs

Our next two results concern the number of copies of graphs which are close to being cubic. We define a *cherry* as a planar graph in which all vertices have degree 3 except for one distinguished vertex of degree 1. The smallest cherry has 6 vertices and is obtained by subdividing an edge of K_4 and adding one vertex of degree one. A generic cherry is pictured in Figure 4.1.

Let A be a fixed cherry with $aut(A)$ automorphisms. A copy of A in a cubic planar graph is defined as an appareance of A as a unlabelled subgraph, i.e. it is like counting appareances of A as a labelled subgraph but weigthed by $|A|!/aut(A)$.

Theorem 24. *Let $X_{A,n}$ be the number of copies of a fixed cherry A, with $aut(A)$ automorphisms, in a random cubic planar graph. Then $X_{A,n}$ is asymptotically normal with moments*

$$\mathbf{E}X_{A,n} \sim \mu n, \qquad \mathbf{Var}X_{A,n} \sim \eta n,$$

where for $\rho \approx 0.3192246062$, we have:

$$\mu \approx 1.6799126922 \cdot \frac{\rho^{|A|}}{aut(A)} \quad \textit{and} \quad \eta \approx 3.0530105678 \cdot \left(\frac{\rho^{|A|}}{aut(A)}\right)^2 + \mu.$$

We now define a *near-brick* as a graph obtained from a 3-connected cubic planar graph by removing one edge. The smallest near-brick that can appear in a cubic planar graph is K_4 minus one edge, as depicted in Figure 4.2.

Let B be a fixed near-brick with $aut(B)$ automorphisms. A copy of B in a cubic planar graph is defined as for copies of a cherry.

Theorem 25. *Let $X_{B,n}$ be the number of apparances of a fixed near-brick B, with $aut(B)$ automorphisms, in a random cubic planar graph. Then $X_{B,n}$ is asymptotically normal with moments*

$$\mathbf{E}X_{B,n} \sim \mu n, \qquad \mathbf{Var}X_{B,n} \sim \eta n,$$

where

$$\mu \approx 0.0000000289 + 1.7444626168\frac{\rho^{|B|}}{aut(B)}$$

$$\text{and } \eta \approx \mu - 3.1810806588\left(\frac{\rho^{|B|}}{aut(B)}\right)^2.$$

Our final results deal with the distribution of the number of triangles. First in a uniformly at random 3-connected cubic planar graph:

Theorem 26. *Let Z_n be the number of triangles in a uniformly at random 3-connected cubic planar graph on $2n$ vertices. Denoting by μ_n and λ_n^2 the expectation and variance of X_n, we have that*

$$\frac{Z_n - \mu_n}{\lambda_n} \to N(0,1),$$

where $N(0,1)$ is the standard normal distribution. Additionally, the mean μ_n and the variance λ_n^2 satisfy

$$\mu_n = \frac{27}{128}n(1+o(1)), \qquad \lambda_n^2 = \frac{3267}{32768}n(1+o(1)).$$

Thus recovering a result from [43]. We then extend this result to a uniformly at random arbitrary cubic planar graph:

Theorem 27. *Let X_n be the number of triangles in a random cubic planar graph. Then X_n is asymptotically normal with moments*

$$\mathbf{E}X_n \sim \mu n, \qquad \mathbf{Var}X_n \sim \lambda n,$$

where

$$\mu \approx 0.1219742813, \qquad \lambda \approx 0.0649847862.$$

It was proved in [7] that X_n is linear with high probability. Our result is a considerable sharpening of this fact. We wish to remark that this is the first time one is able to determine precisely the number of copies of a fixed graph H containing a cycle in classes of random planar graphs. The proof based on an enriched network-decomposition to apply the Quasi-Powers Theorem is technically involved and we are not able to extend it, for instance, to the number of quadrilaterals.

4.2 Statistics on connectivity

In a first part of this section, we will show how one can directly use the asymptotic estimates for the number of connected cubic planar graphs and multigraphs to compute the associated distributions for the number of connected components.

In the rest of this section, we will specify an *enriched decomposition* of connected cubic planar graphs in terms of networks for each of the following parameters: the number of cut-vertices, isthmuses and blocks. Each of those parameter will respectively be encoded by the variable v, e and w. And we will denote by $C(x,v) = \sum_{n,k} c_{n,k} v^k x^n/n!$, $C(x,e) = \sum_{n,s} c_{n,s} e^s x^n/n!$ and $C(x,w) = \sum_{n,t} c_{n,t} w^t x^n/n!$ the three associated bivariate generating functions counting connected cubic networks enriched by the new parameter.

In each case, we will then use the network-decompositon to compute the so-called *singularity curves* $\rho_1(v)$, $\rho_2(e)$ and $\rho_3(w)$, such that $\rho_1(1) = \rho_2(1) = \rho_3(1) = \rho$. Then prove, using the Quasi-Powers Theorem, that the associated sequences of random variables converges in distribution to Gaussian laws. And finally compute the approximated values of each first two moments.

4.2.1 The number of connected components

Proof of Theorem 21. Recall that $G(x) = \sum_{n\geq 0} g_n x^n/n!$ and $G_1(x) = \sum_{n\geq 0} c_n x^n/n!$ are the exponential generating functions respectively counting arbitrary and connected cubic planar graphs. The probability that a random cubic planar graph on n vertices is connected is then equal to

$$\frac{n![x^n]G_1(x)}{n![x^n]G(x)} = \frac{c_n}{g_n}.$$

Using, both estimates (3.28) and (3.37), the limiting probability that a random cubic planar graph is connected is then

$$p = \frac{c \cdot n^{-7/2} \cdot \rho^{-n} \cdot n!}{g \cdot n^{-7/2} \cdot \rho^{-n} \cdot n!} = \frac{c}{g} \approx 0.9993966774.$$

Recall similarly that $M(x) = \sum_{n\geq 0} m_n x^n/n!$ and $M_1(x) = \sum_{n\geq 0} m'_n x^n/n!$ are the exponential generating functions respectively counting arbitrary and connected cubic planar multigraphs. As above, the probability that a random cubic planar multigraph on n vertices is connected is then equal to

$$\frac{n![x^n]M_1(x)}{n![x^n]M(x)} = \frac{m'_n}{m_n}.$$

And using, both estimates (3.49) and (3.52), the limiting probability that a random cubic planar multigraph is connected is now

$$p_m = \frac{m' \cdot n^{-7/2} \cdot \rho_m^{-n} \cdot n!}{m \cdot n^{-7/2} \cdot \rho_m^{-n} \cdot n!} = \frac{m'}{m} \approx 0.93177818003.$$

This concludes the proof.

Looking at Table 3.2, we conclude from this proof that a random cubic planar graph is with high probability not 2-connected. The limiting probability that a random planar graph is connected was determined in [28] as ≈ 0.96325. That the probability for cubic planar graphs is larger can be explained intuitively as follows. The smallest cubic graph is K_4, and the probability that the fragment is K_4 tends to $\rho^2/24 \approx 0.0042460146$. Hence $p < 1 - 0.0042460146 = 0.9957539854$. This value is very close to p, since the contribution of largest fragments is extremely small. On the other hand, there are two cubic multigraphs with two vertices, and the probability that the fragment has size two tends to $\rho_m^2 \approx 0.0629544204$. Hence the probability of connectedness for multigraphs is $p_m < 1 - 0.0629544204 = 0.9370455796$, which is close to the actual value of p_m.

Proof of Theorem 22. We only present here the proof for simple graphs, as it is the exact same for multigraphs, modulo the appropriate notation. For positive integers $n, k \geq 1$, let $g_{n,k}$ be the number of labelled cubic planar graphs on n vertices and with k connected components. Notice that this number can be obtained by counting all the possible sets $\{\gamma_1, \ldots, \gamma_k\}$, where γ_i is a connected cubic planar graph (for $i \in [k]$), and such that $\sum_{i=1}^{k} |\gamma_i| = n$. So that $g_{n,k} = [x^n]G_1(x)^k/k!$. And using (3.35), the local expansion of $G_1(x)$ in a neighbourhood of its dominant singularity ρ, it holds that

$$[x^n]G_1(x)^k \underset{x \to \rho}{\sim} [x^n]kG_1(\rho)^{k-1}G_1(x).$$

If now X_n is the discrete random variable counting the number of connected components in a uniformly at random cubic planar graph on n vertices, then we have

$$\mathbb{P}[X_n = k] = \frac{g_{n,k}}{g_n} = \frac{1}{k!}\frac{[x^n]G_1(x)^k}{[x^n]G(x)} \underset{x\to\rho}{\sim} \frac{kG_1(\rho)^{k-1}}{k!}\frac{[x^n]G_1(x)}{[x^n]G(x)}$$
$$\underset{n\to\infty}{\sim} \frac{G_1(\rho)^{k-1}}{(k-1)!}\frac{G_{1,5}}{G_5},$$

where by (3.30) we have $G_5 = e^{G_1(\rho)}G_{1,5}$, so that

$$\mathbb{P}[X_n = k] \underset{n\to\infty}{\sim} \frac{G_1(\rho)^{k-1}}{(k-1)!}e^{-G_1(\rho)}.$$

This implies that $X_n - 1 \sim Po(G_1(\rho))$.

4.2.2 The number of cut-vertices

Following the decomposition of a connected cubic planar graph in terms of cubic networks, we enrich every generating functions in the system (3.2) of a new variable v marking the number of cut-vertices. Notice that, as cut-vertices only appear in loop or isthmus cubic networks, we only need to modify the equations for $L(x,v)$ and $I(x,v)$ in (3.2) as follows:

$$L(x,v) = \frac{v^2x^2}{2}\left(C(x,v) + I(x,v) - L(x,v)\right) \qquad \text{and} I(x,v) = \frac{L(x,v)^2}{v^2x^2},$$

where the term v^2 in the first equation arises from the fact that in a loop cubic network, both the root vertex and its neighbour are cut-vertices. And the term v^{-2} in the second equation encodes the suppression of both root vertices of the two loop cubic networks used in any isthmus construction.

This gives after some manipulations the following polynomial system of equations, where all the generating functions involved are bivariate:

$$\begin{array}{ll} v^2x^2(I+C-L) - 2L = 0, & L^2 - v^2x^2I = 0, \\ L+S+P+H-C = 0, & C^2 - (1+C)S = 0, \\ x^2(2C+C^2) - 2P = 0, & x^2(1+C)^3 - z = 0, \\ T - x^2(1+C)^3 - 2(1+C)H = 0. & \end{array} \tag{4.1}$$

So that using (2.7), one can eliminate every other variables from (4.1) in order to obtain a triavariate annhilating polynomial $\Phi(C,x,v)$ of $C(x,v)$. This polynomial is irreducible and has degree 12 in $C(x,v)$. It is too big to be presented here. We can however present the only factor of the discriminant of $\Phi(C, x, v)$, with respect to C, for which the polynomial induced by the projection $v = 1$ contains (3.24) as a factor. It is the singularity curve of $C(x, v)$:

$$\begin{aligned} &23328x^{12}v^6 - 34992x^{12}v^4 + 279936x^{10}v^6 + 17496x^{12}v^2 \\ &-279936x^{10}v^4 - 1300104x^8v^6 - 2916x^{12} + 69984x^{10}v^2 \\ &+3209868x^8v^4 + 303264x^6v^6 - 1880496x^8v^2 - 10425024x^6v^4 \\ &-19683x^4v^6 + 565596x^8 + 12174624x^6v^2 + 1012581x^4v^4 \\ &-12257649x^4v^2 - 17910289x^4 + 1119744x^2v^2 + 186624 = 0, \end{aligned} \tag{4.2}$$

where $x = \rho_1(v)$ and by construction $\rho_1(1) = \rho$. By the theorem of *transfer of singularity*, this is also the singularity curve for the generating function counting cubic planar graphs. So that it is enough to study the distribution for the number of cut-vertices in cubic networks.

Proof of the first part of Theorem 23. Let v_0 be a positive real number in the neighbourhood of 1. By continuity and as argued in Section 2.3.2 of Chapter 2, the coefficients of $C(x, v_0)$ admits the following asymptotic estimates:

$$n![x^n]C(x, v_0) \underset{n\to\infty}{\sim} c_1(v_0) \cdot n^{-3/2} \cdot \rho_1(v_0)^{-n} \cdot n!,$$

where the value of $c_1(v_0)$ depends on the fifth coefficient of the singular expansion of $C(x, v_0)$ around $\rho_1(v_0)$. In particular, $c_1(1) = c$, the same c is as in (3.28).

Let now X_n be the random variable counting the number of cut-vertices in a cubic planar graph with n vertices. Using (3.28), the probability generating functions associated to the sequence $\{X_n\}_{n\geq 0}$ are then given for $n \geq 0$ by:

$$\begin{aligned} p_n(v_0) = \frac{[x^n]C(x, v_0)}{[x^n]C(x)} &\underset{n\to\infty}{\sim} \frac{c_1(v_0) \cdot n^{-3/2} \cdot \rho_1(v_0)^{-n}}{c \cdot n^{-3/2} \cdot \rho^{-n}} \\ &= \frac{c_1(v_0)}{c} \cdot \left(\frac{\rho_1(v_0)}{\rho}\right)^{-n}. \end{aligned} \tag{4.3}$$

After setting $A_1(v_0) = \frac{c_1(v_0)}{c}$ and $B_1(v_0) = \frac{\rho}{\rho_1(v_0)}$ observe that we fulfill the assumptions of the *Quasi-Powers Theorem* (we will later numerically check that $B_1'(1) + B_1''(1) - B_1'(1)^2 \neq 0$). This means in particular that the sequence $\{X_n\}_n$ converges in distribution, after renormalisation, to a Gaussian law.

We will now compute the two first moments by taking the two derivatives of (4.2) with respect to v. Setting $v = 1$ in the first derivative and replacing $\rho_1(1)$ by its numerical approximation, we obtain a linear equation for $\rho_1'(1)$ that we can numerically solve. Doing the same for the second derivative, then replacing $\rho_1'(1)$ by its newly obtained numerical approximation, we similarly obtain a linear equation for $\rho''(1)$. The numerical solutions are given by:

$$\rho_1'(1) \approx -0.0005993266 \qquad \text{and} \qquad \rho_1''(1) \approx -0.0006105114,$$

and the first estimate of Theorem 23 holds. This also implies that $B_1'(1) + B_1''(1) - B_1'(1)^2 \neq 0$ as claimed.

4.2.3 The number of isthmuses

The study of the number of isthmuses is similar to that of cut-vertices, as they also only appear in loop or isthmus cubic networks. Indeed, if the loop construction arises from a loopless cubic network, then the non-loop edge adjacent to the root vertex is an isthmus. When it arises from an isthmus network, then one cut-edge is replaced by three new cut-edges. And, as an isthmus construction always arises from deleting the root vertices of two loop cubic networks then connecting their respective non-cubic vertex, we suppress two cut-edges and create one. Modifying in consequence the equations for $L(x,e)$ and $I(x,e)$ in (3.2) gives

$$L(x,e) = \frac{x^2}{2}\left(e^2 I(x,e) + e(C(x,e) - L(x,e))\right) \quad \text{and} \quad I(x,e) = \frac{L(x,e)^2}{ex^2}.$$

Similarly to the case of cut-vertices, we can rewrite (3.18) into an algebraic system of bivariate equations. And after elimiation, one can obtain the minimal polynomial of $C(x,e)$. It is of degree 10 and is also too big to be presented here. Using the same method as above, we can compute the singularity curve of $C(x,e)$:

$$\begin{aligned}
&729x^{12}e^3 + 8748x^{10}e^3 + 8748x^{10}e^2 + 26244x^8e^3 + 87480x^8e^2 \\
&+34992x^8e + 846936x^6e^3 - 660312x^6e^2 + 279936x^6e \\
&-4512544x^4e^3 + 46656x^6 - 139968x^4e^2 - 2921184x^4e \\
&+903744x^2e^3 + 279936x^4 - 2711232x^2e^2 + 5772384x^2e \\
&-46656e^3 - 3684960x^2 + 279936e^2 - 559872e + 373248 = 0,
\end{aligned} \tag{4.4}$$

where $x = \rho_2(e)$ and $\rho_2(1) = \rho$. By the same argument as before, we remark that this is also the singularity curve of the generating function counting cubic planar graphs. And it is again enough to study the distribution for the number of istmuses in a random cubic network.

Proof of the second part of Theorem 23. Let $e_0 \sim 1$ be a positive real number. The same continuity argument as for the number of cut-vertices tells us that the following estimate on the coefficients of $C(x, e_0)$ holds: the estimate

$$n![x^n]C(x, e_0) \underset{n\to\infty}{\sim} c_2(e_0) \cdot n^{-3/2} \cdot \rho_2(e_0)^{-n} \cdot n!,$$

where the value of $c_2(e_0)$ depends on the coefficients of the singular expansion of $C(x, e_0)$ around $\rho_2(e_0)$. In particular, $c_2(1) = c$.

Let Y_n be the random variable counting the number of cut-vertices in a cubic planar graph with n vertices. Using (3.28), the probability generating functions associated to the sequence $\{Y_n\}_{n\geq 0}$ are then given for $n \geq 0$ by:

$$\begin{aligned} p_n(e_0) = \frac{[x^n]C(x, e_0)}{[x^n]C(x)} &\underset{n\to\infty}{\sim} \frac{c_2(e_0) \cdot n^{-3/2} \cdot \rho_2(e_0)^{-n}}{c \cdot n^{-3/2} \cdot \rho^{-n}} \\ &= \frac{c_2(e_0)}{c} \cdot \left(\frac{\rho_2(e_0)}{\rho}\right)^{-n}. \end{aligned} \tag{4.5}$$

Again setting $A_2(e_0) = \frac{c_1(v_0)}{c}$ and $B_2(e_0) = \frac{\rho}{\rho_2(e_0)}$ we observe that we fulfill the assumptions of the *Quasi-Powers Theorem* (we will also later numerically check that $B_2'(1) + B_2''(1) - B_2'(1)^2 \neq 0$). This means in particular that the sequence $\{Y_n\}_n$ converges in distribution, after renormalisation, to a Gaussian law.

We will now compute its two first moments. Setting $e = 1$ as before in the two derivatives of the singularity curve of $C(x, e)$, and substituting $\rho_2(1) = \rho$ by its numerical approximation, we obtain the following two approximations:

$$\rho_2'(1) \approx -0.0002997471 \qquad \text{and} \qquad \rho_2''(1) \approx -0.0000031338.$$

And the second estimate of Theorem 23 holds. This also implies that $B_2'(1) + B_2''(1) - B_2'(1)^2 \neq 0$ as claimed.

4.2.4 The number of blocks

Recall that by *block* we mean a maximal 2-connected subgraph. For enumerative purposes, when encoding blocks the variable w will count the number

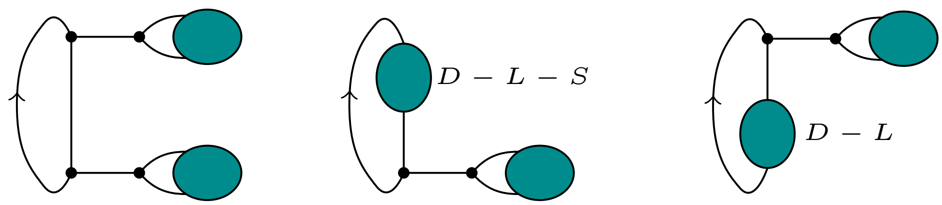


Figure 4.3: The three types of series compositions involving loop cubic networks.

of blocks in a cubic network as the root edge is removed. For example, as in the two cases counted by the generating functions $L(x, w)$ and $I(x, w)$, the root edge forms a block by itself (isthmuses and loops are blocks). And w will count the number of blocks minus one.

We also need to distinguish for each class of cubic networks, whether or not a loop cubic networks is used in its construction. The reason is that when using loop cubic networks to build larger cubic networks, we are not decreasing the number of blocks while in the rest of the cases the number of blocks in the final network decreases. For instance, the three different series constructions involving at least one loop cubic network are illustrated in Figure 4.3:

- Left is when two loop cubic networks are pasted in series, thus creating one new block.

- Middle is when the first cubic network is rooted at a loop, forcing the second to be non-series and non-loop (to exclude the previous case), thus creating no new block.

- And right is when the first cubic network is not rooted at a loop (but could be series) but the second one is rooted at a loop, thus creating again no new block.

For a parallel construction, we take the 3-bond graph rooted at an edge and substitute one or two of its two non-root edges. The special case when only one edge of the 3-bond graph is substituted by a loop cubic network is detailed in Figure 4.4: Left is when the cubic network was rooted at a loop,

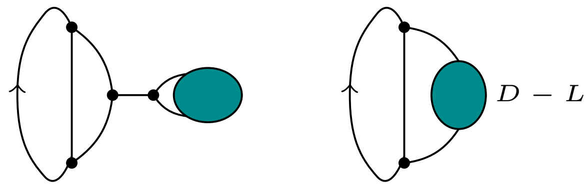


Figure 4.4: The two types of parallel constructions where only one edge of the 3-bond graph is substituted by a cubic network.

thus creating a new block. Right is when it was loopless but connected, thus not creating any new block. Adapting (3.2) to account for the new variable, we obtain the following system of equations:

$$\begin{aligned}
&C = L + S + P + H,\\
&L = x^2(w^2 I + w(C - L))/2,\\
&I = L^2/(wx^2),\\
&S = wL^2 + L(C - L + C - L - S) + (C - L)(C - L - S)/w,\\
&P = x^2(C - L + wL) + \frac{x^2}{2}(wL^2 + 2(C - L)L + (C - L)^2/w),\\
&H = \frac{w\vec{G}_3(x, 1 + L + (C - L)/w)}{1 + L + (C - L)/w},
\end{aligned}$$

where the terms $wx^2L^2/2$ in the fourth equation and w in the last one respectively encode the facts that the 3-bond itself is a block and that a 3-connected core is in particular a block. The above system can be rewritten into the polynomial system of equations, where all the generating functions invlolved are bivariate:

$$\begin{aligned}
&L + S + P + H - C = 0,\\
&x^2(w^2 I + w(C - L)) - 2L = 0,\\
&L^2 - wx^2 I = 0,\\
&w^2L^2 + w(2C - 2L - S)L + (C - L)(C - L - S) - wS = 0,\\
&x^2(C + (w - 1)L)(2w + C + (w - 1)L) - 2wP = 0,\\
&w^2(T(z) - z) - 2(w + C + (w - 1)L)H = 0,\\
&x^2(w + C + (w - 1)L)^3 - z = 0.
\end{aligned} \tag{4.6}$$

As before, eliminating from the system (4.6) gives us an annihilating polynomial of $C(x,w)$. And after factorising, and choosing the right factor by computing the first terms of the Taylor expansion of its roots, we obtain the minimal polynomial of $C(x,w)$. It has degree 12 and is also too big to be presented here. Again, following the same method as above, we compute the singularity curve of $C(x,w)$:

$$\begin{aligned}
&23328x^{12}w^6 - 34992x^{12}w^5 + 17496x^{12}w^4 + 279936x^{10}w^6\\
&-2916x^{12}w^3 - 419904x^{10}w^5 + 349920x^{10}w^4 - 1300104x^8w^6\\
&-174960x^{10}w^3 + 1810188x^8w^5 + 34992x^{10}w^2 + 638928x^8w^4\\
&+303264x^6w^6 - 1044036x^8w^3 - 979776x^6w^5 + 629856x^8w^2\\
&-8098056x^6w^4 - 19683x^4w^6 - 139968x^8w + 11305008x^6w^3\\
&+102789x^4w^5 + 455544x^6w^2 + 1246671x^4w^4 - 1119744x^6w\\
&-10004689x^4w^3 + 186624x^6 - 30995136x^4w^2 - 69984x^2w^4\\
&+9375264x^4w + 2775168x^2w^3 + 1119744x^4 - 7835616x^2w^2\\
&+20990016x^2w - 186624w^3 - 14739840x^2 + 1119744w^2\\
&-2239488w + 1492992 = 0,
\end{aligned} \tag{4.7}$$

where $x = \rho_3(w)$ and $\rho_3(1) = \rho$.

Proof of the final part of Theorem 23. The same reasoning as for the number of cut-vertices tells us that the sequence of random variables associated with the number of blocks converges in distribution to a Gaussian law. We compute the first two moments using the same method as in the case of cut-vertices and the numerical approximation of $\rho_3(1) = \rho$. In particular, we obtain:

$$\rho_3'(1) \approx -0.0005994104 \qquad \text{and} \qquad \rho_3''(1) \approx -0.0006111865.$$

This concludes the proof of Theorem 23.

4.3 Subgraphs statistics

In this section, we will prove that the number of appareances of a given cheery, of a given near-brick and of triangles in a uniformly at random cubic planar graph are all distributed following a Gaussian law. We will first deal with quasi-cubic subgraphs, that are cherries and near-bricks, then with triangles.

4.3.1 The number of quasi-cubic subgraphs

In both cases, by the theorem of *transfer of singularity*, we will be able to restrict the study of the number of occurences of a given quasi-cubic subgraph in a random cubic network. To that end, we shall explicit in each case a decomposition of a connected cubic planar graph rooted at a directed edge in terms of cubic networks, to compute the singularity curve. Then use the Quasi-Powers Theorem to obtain a limiting distribution and compute the first moments.

The number of cherries. Let A be a given cherry with $aut(A)$ automorphisms. Notice that the unique vertex of degree one in a cherry must be fixed by every automorphism. Let then $c_{n,\ell}$ be the number of connected cubic networks with n vertices and containing ℓ different appareances of the cherry A. So that $C(x,a) = \sum_{n,\ell\geq 0} c_{n,\ell}\, a^{\ell} x^n/n!$ is the associated bivariate generating function counting cubic networks. For the generating functions of the other families of cubic networks: $\mathcal{L}$, $\mathcal{I}$, $\mathcal{S}$, $\mathcal{P}$ and $\mathcal{H}$, we associate similarly a bivariate generating function with the same letter.

Observe that during the network-decomposition, a cherry A can by definition only arise from a loop construction. And we only need to modify the equation for $L(x,a)$ in the system (3.2), as follows:

$$L(x,a) = \frac{x^2}{2}(I(x,a) + C(x,a) - L(x,a)) + \frac{x^{|A|}}{\mathrm{aut}(A)}(a-1).$$

This is because occurrences of A are encoded by the monomial $\frac{|A|!}{aut(A)} \cdot \frac{x^{|A|}}{|A|!} = \frac{x^{|A|}}{aut(A)}$, since $|A|!/aut(A)$ is the number of ways of labelling A, and each occurrence is marked by a. If we then set $\kappa := x^{|A|}/aut(A)$, then we can rewrite (3.18) into an algebraic system of bivariate generating functions by replacing the equation for $L(x,a)$ by the following:

$$x^2(C + I - L) + 2(a-1)\kappa - 2L = 0. \tag{4.8}$$

Using now (2.7), the minimal polynomial of $T(z)$, one can eliminate every other generating function from the above system to obtain an annihilating polynomial of $C(x,a)$ that is irreducible and of degree 9. It is the minimal polynomial of $C(x,a)$. Taking finally its discriminant with respect to C and choosing the right factor using the same method as for the number of cut-vertices, we obtain the singularity curve of $C(x,a)$:

$$\begin{aligned}
&-729x^{12} + 17496\kappa x^8 a - 17496x^{10} - 17496\kappa x^8 - 139968\kappa^2 x^4 a^2 \\
&+279936\kappa x^6 a - 148716x^8 + 279936\kappa^2 x^4 a - 279936\kappa x^6 \\
&+373248\kappa^3 a^3 - 139968\kappa^2 x^4 - 1119744\kappa^2 x^2 a^2 + 1259712\kappa x^4 a \\
&-513216x^6 - 1119744\kappa^3 a^2 + 2239488\kappa^2 x^2 a - 1259712\kappa x^4 \\
&+1119744\kappa^3 a - 1119744\kappa^2 x^2 - 559872\kappa^2 a^2 + 1119744\kappa x^2 a \\
&+7293760x^4 - 373248\kappa^3 + 1119744\kappa^2 a - 1119744\kappa x^2 - 559872\kappa^2 \\
&+279936\kappa a - 279936x^2 - 279936\kappa - 46656 = 0,
\end{aligned} \tag{4.9}$$

where again $\kappa = x^{|A|}/aut(A)$ behaves as a constant, and $x = \rho_4(a)$ where by construction $\rho_4(1) = \rho$.

Proof of Theorem 24. If now $a_0 \sim 1$, then the same continuity argument as for the number of cut-vertices gives the following asymptotic estimate for the coefficients of $C(x, a_0)$:

$$n![x^n]C(x,a_0) \underset{n\to\infty}{\sim} c_4(a_0) \cdot n^{-3/2} \cdot \rho_4(a_0)^n \cdot n!,$$

where $c_4(a_0)$ depends on the third coefficient of the singular expansion of $C(x,a_0)$ around $\rho_4(a_0)$. In particular, $c_4(1) = c$. As before, this implies that the sequence of random variables associated to the number of appearances of the given cherry A in a cubic planar graph on n vertices converges in distribution to a Gaussian limit law, as $n \to \infty$.

To compute the first two moments of this limiting distribution, notice first that by setting $\kappa(a) := \rho_3(a)^{|A|}/aut(A)$, then we get:

$$\kappa = \kappa(1) = \rho_4(1)^{|A|}/aut(A) = \rho^{|A|}/aut(A),$$

where $\rho \approx 0.3192246062$. So that setting $a = 1$ in the first two derivatives of (4.9) gives the following numerical approximations:

$$\mu = -\frac{\rho_4'(1)}{\rho_4(1)} \cdot \frac{\rho_3(1)^{|A|}}{aut(A)} \approx 1.6799126922 \frac{\rho^{|A|}}{aut(A)},$$

and

$$\eta \approx 3.0530105678 \left(\frac{\rho^{|A|}}{aut(A)}\right)^2 + \mu,$$

as claimed.

The number of near-bricks. Let B be a given near-brick admitting $aut(B)$ automorphisms, with the convention that its two vertices of degree two are distinguishable. Let then $c_{n,m}$ be the number of connected cubic networks with n vertices and containing m different appareances of the near-brick B. So that $C(x,b) = \sum_{n,m\geq 0} c_{n,m}\, b^m x^n/n!$ is the associated bivariate generating function counting cubic networks. For the generating functions of the other families of cubic networks: $\mathcal{L}$, $\mathcal{I}$, $\mathcal{S}$, $\mathcal{P}$ and $\mathcal{H}$, we associate similarly a bivariate generating function with the same letter.

Observe that a brick $B \neq K_4^-$ can only arise from an h-network isomorphic to B in which no edge is replaced. The modified bivariate generating function of h-networks becomes then:

$$H(x,b) = \frac{\overrightarrow{G}_3(x, 1 + C(x,b))}{1 + C(x,b)} + \frac{x^{|B|}}{aut(B)}(b-1). \tag{4.10}$$

But when $B = K_4^-$, it now appears as a parallel composition of two loop networks, and we have to modify the bivariate generating function in consequence:

$$P(x,b) = x^2\left(C(x,b) + \frac{C(x,b)^2}{2}\right) + \frac{x^2}{2}(b-1)L(x,b)^2. \tag{4.11}$$

One can now rewrite Equations (4.10) and (4.11) and plugg them into (3.18) to obtain an algebraic system of equations involving bivariate generating functions, at the exception of $T \equiv T(z)$, as follows:

$$\begin{aligned}
&L + S + P + H - C = 0,\\
&x^2(I + C - L) - 2L = 0,\\
&L^2 - x^2 I = 0,\\
&C^2 - (1 + C)S = 0,\\
&2x^2C + x^2C^2 + 2(b-1)\kappa L^2 - 2P = 0,\\
&T - x^2(1 + C)^3 + 2(b-1)(1 + C)\kappa - 2(1 + C)H = 0,\\
&x^2(1 + C)^3 - z = 0.
\end{aligned} \tag{4.12}$$

where we now set $\kappa := x^{|B|}/aut(B)$. And after finally eliminating every other variables from (4.12), one can obtain an irreducible annihilating polynomial, i.e. the minimal polynomial of $C(x,b)$. It has degree 16 and is too big to be presented here. So is the singularity curve $\sigma(C,x,b)$ of $C(x,b)$, for $x = \rho_5(b)$ and $\rho_5(1) = \rho$, that one can compute using the same method as for the number of cut-vertices.

Proof of Theorem 25. Using the same arguments as those developed for the number of cut-vertices, it holds that for any $b_0 \sim 1$, the coefficients of $C(x, b_0)$ are asymptotically

$$n![x^n]C(x, b_0) \underset{n\to\infty}{\sim} c_5(b_0) \cdot n^{-3/2} \cdot \rho_5(b_0)^n \cdot n!,$$

where $c_5(1) = c$. And as before, the sequence of random variables associated to the number of near-bricks B in a uniformly at random cubic planar graph on n vertices converges in distribution to a Gaussian limit law.

We now set $\kappa(b) := \rho_5(b)^{|B|}/aut(B)$, so that $\kappa(1) = \rho^{|B|}/aut(B)$. And computing the two derivatives of the singularity curve σ with respect to b gives the two following numerical approximations:

$$\mu \approx 0.0000000289 + 1.7444626168\frac{\rho^{|B|}}{aut(B)}$$

and

$$\eta \approx \mu - 3.1810806588\left(\frac{\rho^{|B|}}{aut(B)}\right)^2,$$

as claimed.

4.3.2 The number of triangles

In order to study the distribution of the number of triangles in random cubic planar graphs, we start with 3-connected cubic graphs. By duality this amounts to studying vertices of degree 3 in triangulations. We are able to do it, by adapting the composition scheme presented in Section 2.4.2 of Chapter 2, which derives 3-connected triangulations from the irreducible ones, following an idea from [27, Section 5] where the authors studied the case $u = 0$, i.e. triangle-free 3-connected rooted cubic planar maps. The key point is that an irreducible triangulation admits no cubic vertices, as they only begin to appear in 3-connected triangulations.

We then use this study to count triangles in 3-connected cubic planar graphs. And then by adapting the decomposition of a connected cubic planar graph rooted at a directed edge, we can finally prove the limiting distribution. The key remark is that, at the exception of K_4, two triangles in a 3-connected cubic planar graph do not share a vertex. And the number of different configurations in which two triangles of a 2-connected or connected cubic planar graph intersect is finite. This means that a network decomposition is particularly suited for keeping track of the number of triangles in a cubic planar graph.

Cubic vertices in triangulations. Let $t_{n,k}$ be the number of 3-connected triangulations on $n+2$ vertices among which are with k cubic vertices, and let $T(z,u) = \sum_{n,k\geq 0} z^n u^k$ be the associated bivariate (ordinary) generating function. Our goal is to refine the composition scheme in Equation (2.11) by counting vertices of degree three.

To that end, recall that an *internal* vertex in a triangulation is a vertex not incident with the root face, otherwise it is called *external*. Observe then, that when pasting a triangulation with an external cubic vertex v on the inner face of an irreducible triangulation, the degree of v in the resulting triangulation strictly increases. This justifies the introduction of the generating function $T^*(z,u)$ counting 3-connected triangulations (where for commodity we now exclude the single triangle) but where the variable u now encodes inner cubic vertices only. Using this notation, one can deduce from (2.11) the following implicit equation:

$$T^*(z,u) = \frac{T_4(z(1+z^{-1}T^*)^2)}{1+z^{-1}T^*} + z^2(1+z^{-1}T^*)^3 + z^2(u-1). \qquad (4.13)$$

The only difference comes from the second term associated to K_4: when none of the internal faces is replaced with a triangulation, the central vertex has degree 3 and the configuration is encoded as u. Notice that this is exactly where cubic vertices appear in triangulations. We can derive the bivariate generating function $T(z,u)$ in a similar way:

$$T - z = \frac{T_4(z(1+z^{-1}T^*)^2)}{1+z^{-1}T^*} + z^2(1+z^{-1}T^*)^3 + 3z(u-1)T^* + z^2(u^4-1). \qquad (4.14)$$

We will define now two other bivariate generating functions counting triangulations, which will be of use later. Let $\mathcal{T}_0$ be the set of triangulations (except K_3 and K_4) in which the degree of the root vertex is equal to three, and $\mathcal{T}_1$ those where the degree is greater than three. In particular, $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1 \cup \{K_3, K_4\}$. Let then $T_0(z,u)$ and $T_1(z,u)$ be the associated bivariate generating functions, where u now counts the *total* number of cubic vertices, including the external ones. So that $T(z,u) = T_0(z,u) + T_1(z,u) + z + z^2u^4$. Notice then that when removing the root vertex (and the three adjacent edges) of a triangulation in $\mathcal{T}_1$, we obtain a smaller triangulation. The reverse operation is to take a triangulation, draw a vertex on its root face, join it with the three vertices on the external face, and re-root the resulting map. This gives the following equation:

$$T_1(z,u) = uz \cdot T^*(z,u). \tag{4.15}$$

And using (4.13) to eliminate $T_4(z(1+z^{-1}T^*)^2)$ from (4.14), we can write $T(z,u)$ as a function of $T^*(z,u)$ only. The generating function $T_0(z,u)$ then follows from the identity $T = T_1 + T_0 + z + z^2u^4$:

$$T_0(z,u) = (1+2zu-3z)\cdot T^*(z,u) - z^2u \tag{4.16}$$

Proof of Theorem 26. Let $u_0 \sim 1$. By continuity and using the theorem of *transfer of singularity* on (3.3), it is direct to see that $G_3(x,u_0)$ and $T^*(z,u_0)$ both share the same dominant singularity $\tau(u_0)$ and the same type of expansion for x near $\tau(u_0)$ (although G_3 will have a Puiseux expansion of branching type 5/2 instead of 3/2).

Let us then study the singularity curve of $T^*(z,u)$. To that end, we rewrite (4.13) as a polynomial system composed of the two following equations:

$$\begin{aligned} &z^2T_4 + (z+T^*)((z+T^*)^3 + z^3(u-1) - zT^*) = 0,\\ &(z+T^*)^2 - zy = 0, \end{aligned} \tag{4.17}$$

where $T_4 \equiv T_4(y)$ and $T^* \equiv T^*(z,u)$. From there we use (2.12), the minimal polynomial of $T_4(y)$ to eliminate T_4 and y and obtain an annihilating polynomial of $T^*(z,u)$:

$$\begin{aligned} &u^3z^7 + 8u^3z^6 - 3u^2z^7 + 16u^3z^5 - 21u^2z^6 + 3uz^7 - 21u^2z^5\\ &+18uz^6 - z^7 + 28u^2z^4 - 3uz^5 - 5z^6 - 26uz^4 + 8z^5 + 11z^3u\\ &-z^4 - uz^2 + (4u^3z^6 + 16u^3z^5 - 12u^2z^6 - 36u^2z^5 + 12uz^6\\ &+57u^2z^4 + 24uz^5 - 4z^6 - 20u^2z^3 - 102uz^4 - 4z^5 + 106uz^3\\ &+45z^4 - 34uz^2 - 82z^3 + 59z^2 - 14z + 1)T^* + (6u^3z^5 + 8u^3z^4\\ &-18u^2z^5 - 6u^2z^4 + 18uz^5 + 33u^2z^3 - 12uz^4 - 6z^5 - 48uz^3\\ &+10z^4 + 42uz^2 + 15z^3 + 3uz - 36z^2 + 14z + 3)(T^*)^2\\ &+(4u^3z^4 - 12u^2z^4 + 12u^2z^3 + 12uz^4 + 3u^2z^2 - 24uz^3 - 4z^4\\ &+6uz^2 + 12z^3 + 6uz - 9z^2 - 2z + 3)(T^*)^3 + (u^3z^3 - 3u^2z^3\\ &+3u^2z^2 + 3uz^3 - 6uz^2 - z^3 + 3uz + 3z^2 - 3z + 1)(T^*)^4 = 0. \end{aligned} \tag{4.18}$$

This polynomial being irreducible, it is the minimal polynomial of $T^*(z,u)$. The only factor of its discriminant with respect to T^*, for which the projection $u = 1$ factorises to $256z - 27$, is the singularity curve of $T^*(u,v)$, as follows:

$$256z(1 + z(u - 1))^2 - 27, \tag{4.19}$$

where $z = \tau(u)$ with $\tau(1) = \tau = 27/256$.

By setting $u = u_0 \sim 1$, one can now compute the Puiseux expansion of $T^*(z, u_0)$ directly from (4.18). Applying then the *transfer theorem*, we obtain an estimate on the coefficients of $T^*(z, u_0)$, as follows:

$$[z^n]T^*(z, u_0) \underset{n\to\infty}{\sim} t(u_0) \cdot n^{-3/2} \cdot \tau(u_0)^{-n},$$

where $t(u_0)$ can be computed from the third coefficient of the singular expansion of $T^*(z, u_0)$ and $t(1) = 1/8$. The same $1/8$ as the constant in (2.10). As before and using the *quasi-powers theorem*, this implies that the sequence of random variables counting the number of triangles in a unifromly at random 3-connected cubic planar graph on n vertices converges in distribution to a Gaussian limit law, as $n \to \infty$.

On can then compute the two first moments by setting $u = 1$ in the two first derivatives of (4.19), the singularity curve of $T^*(z,u)$, with respect to u. This gives the following:

$$\sigma'(1) = -\frac{729}{32768} \qquad \text{and} \qquad \sigma''(1) = \frac{137781}{8388608}.$$

An application of the *quasi-powers theorem* concludes the proof.

Triangles in cubic networks. Let $C(x,u) = \sum_{n\geq 0} r_n \frac{x^n}{n!} u^t$ be the exponential generating function counting connected cubic networks, where the variable z marks vertices and u marks triangles. We consider similarly the bivariate exponential generating functions of the other families of cubic networks: $\mathcal{L}$, $\mathcal{I}$, $\mathcal{S}$, $\mathcal{P}$ and $\mathcal{H}$.

Notice that when removing the root edge of a network, notice that any triangle adjacent to it will disappear. This justifies the following notation: to the generating function associated to a class of cubic network we add the index $i \in \{0, 1, 2\}$ to indicate the number of triangles in the networks that are incident with the root edge. For example, the generating function $C_i(x,u)$ will count connected cubic networks where the root-edge belongs to exactly

i triangles. The same convention applies to series, parallel and h-networks. The special case when the 3-connected core of an h-network is K_4 is encoded in the generating functions W_i. We let $E(x,u)$ be the generating function of networks where triangles incident to the root edge are not counted, that is,

$$E(x,u) = C_0(x,u) + u^{-1}C_1(x,u) + u^{-2}C_2(x,u). \tag{4.20}$$

The next two lemmas provide the expressions for the series $C_i \equiv C_i(x,u)$, $S_i \equiv S_i(x,u)$, $P_i \equiv P_i(x,u)$, $W_i \equiv W_i(x,u)$ $(i = 0,1,2)$, I, L and for H_0, H_1.

Lemma 28. *The following system of equations involving only bivariate generating functions holds:*

$$\begin{aligned}
&C_0 = S_0 + P_0 + W_0 + L + H_0,\\
&C_1 = S_1 + P_1 + W_1 + H_1,\\
&C_2 = P_2 + W_2,\\
&I = L^2/x^2,\\
&L = x^2\left(I + E - L\right)/2 + x^2(u-1)\left(x^2(E-L) + ux^2L + L^2\right)/2,\\
&P_0 = x^2(E-L) + x^2(E-L)^2/2,\\
&P_1 = ux^2L(E-L) + u^2x^2L,\\
&P_2 = u^2x^2L^2/2,\\
&S_0 = \left(E - (S_0 + u^{-1}S_1)\right)E - u^{-1}S_1,\\
&S_1 = uL^3 + 2ux^2(E-L)L + 2u^2x^2L^2,\\
&W_0 = x^4(2(1+u)E^2 + 8E^3 + 5E^4 + E^5)/2,\\
&W_1 = x^4(4u^2E + 6uE^2 + 2uE^3)/2,\\
&W_2 = x^4(u^4 + u^2E)/2.
\end{aligned}$$

Proof. Equations for $C_0(x,u)$, $C_1(x,u)$ and $C_2(x,u)$ are clear, since $S_2(x,u) = H_2(x,u) = 0$. The equation for $I(x,u)$ is the same as in the univariate case, as isthmus constructions do not create nor destruct triangle.

The equation for $L(x,u)$ is obtained as follows: triangles arising from loop constructions when pasting a network with a double edge on the non-root edge of the double-loop. So that from the main term $x^2(I(x,u) + E(x,u) - L(x,u))/2$ we need to consider separately three situations, which are drawn in Figure 4.5. The corresponding generating functions are $ux^4(E(x,u) - L(x,u))/2$, $u^2x^4L(x,u)/2$ and $ux^2L(x,u)^2/2$ respectively.

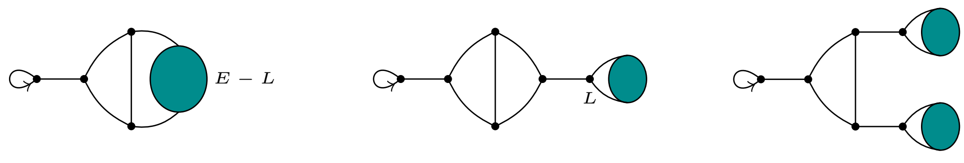


Figure 4.5: Triangles in loop networks.

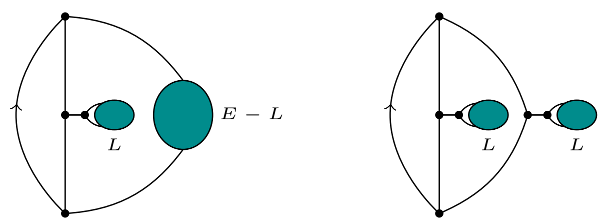


Figure 4.6: Triangles in parallel networks.

In the case of a parallel network γ, when pasting loop networks we create triangles incident with the root edge of γ. The possible such contributions to $P_1(x, u)$ and $P_2(x, u)$ are respectively counted by the generating functions $ux^2(E(x, u) - L(x, u))L(x, u)$ and $x^2u^2L(x, u)^2/2$. They are drawn in Figure 4.6 in that respective order.

The equation for $S_1(x, u)$ is derived by the fact that a triangle is created when two networks are pasted in series and one is rooted at a loop while the other one is rooted at a double edge. The three possible configurations are counted by the following generating functions: $uL(x, u)^3$, $ux^2(E(x, u) - L(x, u))L(x, u)$, $u^2x^2L(x, u)^2$, and are respectively depicted in Figure 4.7. The equation for $S_0(x, u)$ is obtained similarly to the univariate case, but now we also need to subtracte the term $u^{-1}S_1(x, u)$.

Finally, the equations for $W_0(x, u)$, $W_1(x, u)$ and $W_2(x, u)$ are obtained by considering all cases for which K_4 is the core of the h-network. □

We say that a triangle in a network is *external* if it is incident with the root edge. The non-root edges of an external triangle are called *special*. We denote by $\mathcal{M}_0$ and $\mathcal{M}_1$ the families of 3-connected cubic planar graphs (except K_4) rooted at a directed and respectively without external triangle, or with exactly one external triangle. And let $M_0(x, y, u)$ and $M_1(x, y, u)$

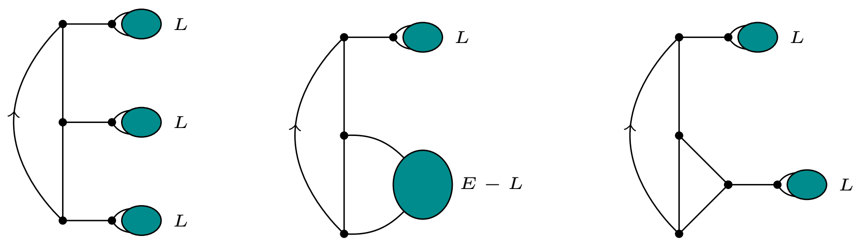


Figure 4.7: Triangles incident with the root edge of a series network.

be the associated generating functions, where variables x, y and u mark the number of vertices, edges and triangles, respectively. Similarly to Equation (3.3) it holds that:

$$2M_0(x,y,u) = T_0(x^2y^3,u) \quad \text{and} \quad 2M_1(x,y,u) = T_1(x^2y^3,u). \tag{4.21}$$

We can now prove the following lemma:

Lemma 29. *The following system of equations holds, with $E \equiv E(x,u)$:*

$$H_1(x,u) = \frac{uM_1\left(x, 1+E, 1+\frac{u-1}{(1+E)^3}\right)}{(1+E)^3+u-1}, \tag{4.22}$$

$$H_0(x,u) = \frac{M_0\left(x, 1+E, 1+\frac{u-1}{(1+E)^3}\right)}{1+E} + (2E+E^2)\frac{H_1(x,u)}{u}. \tag{4.23}$$

Proof. A network in $\mathcal{H}_1$ is obtained from a 3-connected core rooted at a triangle, i.e. a graph γ in $\mathcal{M}_1$, in which we possible replace each non-root edge with a connected network, and where the three edges of the external triangle of γ are not replaced. The term $u+3E+3E^2+E^3 = (1+E)^3+u-1$ encodes the substitution of networks over 3-sets of edges defining triangles (except the external triangle and the corresponding edges, which are not substituted). This translates into the first equation, where the denominator, which is the substitution $uy^3 = (1+E)^3+u-1$ (where y and u are the second and the third variable of $M_1(x,y,u)$, guaranties that the triangle incident to the root edge is not substituted.

Let us now consider a network in $\mathcal{H}_0$. It can be obtained in two different ways: either from a core without an external triangle, or from a core with an external triangle in which some special edges are replaced with a non-empty network. Using a similar encoding argument as before we arrive at the second equation, where the factor $2E + E^2$ in the second summand corresponds to the substitution of networks on the pair of special edges. $\square$

Triangles in rooted connected cubic planar graphs. The bivariate generating function of connected cubic planar graphs rooted at a vertex $C^{\bullet}(x,u)$ can finally be described in terms of networks, similarly as in Equation (3.21), as follows:

$$3G_1^{\bullet}(x,u) = C(x,u) + I(x,u) - L(x,u) - x^2C(x,u) - L(x,u)^2, \qquad (4.24)$$

where $C(x,u) = C_0(x,u) + C_1(x,u) + C_2(x,u)$.

One can now directly deduce from Lemma (28) that every generating function in the right-hand side of Equation (4.24) can be written as a polynomial in $E(x,u)$ (it will be made more clear in the next paragraph). And for a fixed $u_0 \sim 1$, using our now classical continuity argument and the theorem of *transfer of singularity* on Equation (4.24), it holds that each bivariate generating function $G(x,u)$, $G_1(x,u)$ and $G_1^{\bullet}(x,u)$ both share the same dominant singularity $\rho_t(u_0)$ and the same sort of expansion for x near $\rho_t(u_0)$ as $E(x,u)$. Although $E(x,u)$ and $G_1^{\bullet}(x,u)$ will admit a Puiseux expansion of branching type 3/2, while for both $G(x,u)$ and $G_1(x,u)$ the Puiseux expansion will be of branching type 5/2. It will also imply that all the above generating will be analytic in a dented domain at $\pm\rho_t(u_0)$.

In particular, to prove Theorem 27, it will suffice to study the singularity curve of $E(x,u)$ as well as the singular behaviour of $E(x,u_0)$ for x near $\rho_t(u_0)$, for any positive real number u_0 in the neighbourhood of one.

The singularity curve. Using the two equations of (4.21), one can rewrite Equation (4.20) and the equations of Lemmas 28 and (29) to obtain the following polynomial system involving bivariate generating functions:

$$\begin{aligned}
&u^2C_0 + uC_1 + C_2 - u^2E = 0,\\
&S_0 + P_0 + W_0 + L + H_0 - C_0 = 0,\\
&S_1 + P_1 + W_1 + H_1 - C_1 = 0,\\
&L^2 - x^2B = 0,\\
&x^2(B + E - L) + x^2(u-1)(L^2 + x^2(L(u-1) + E)) - 2L = 0,\\
&2x^2(E - L) + x^2(E - L)^2 - 2P_0 = 0,\\
&ux^2(E - L)L + u^2x^2L - P1 = 0,\\
&u^2x^2L^2 - 2P_2 = 0,\\
&uE^2 - (1 + E)(uS_0 + S_1) = 0,\\
&uL^3 + 2ux^2(E - L)L + 2u^2x^2L^2 - S_1 = 0,\\
&x^4(2(1+u)E^2 + 8E^3 + 5E^4 + E^5) - 2W_0 = 0,\\
&x^4(4u^2E + 6uE^2 + 2uE^3) - 2W_1 = 0,\\
&x^4(u^4 + u^2E) - 2W_2 = 0,\\
&uT_1 - 2((1 + E)^3 + u - 1)H_1 = 0,\\
&uT_0 + 2(1 + E)((2E + E^2)H_1 - uH_0) = 0,\\
&x^2(1 + E)^3 - z = 0,\\
&(u - 1)(3E + 3E^2 + E^3) = 0,
\end{aligned} \tag{4.25}$$

where $T_0 \equiv T_0(z, u)$ and $T_1 \equiv T_1(z, u)$.

Using now the two equations (4.16) and (4.15), together with its minimal polynomial (4.18) of $T^*(z, u)$, one can now eliminate every other variables from the system (4.25) to obtain an annhihilating polynomial for $E(x, u)$ irreducible and of degree 24. It is the minimal polynomial of $E(x, u)$ but is too big to be presented here. So is its singularity curve $\sigma(x, u)$, with $x = \rho_t(u)$ so that $\rho_t(1) = \rho$.

Proof of Theorem 27. Let now $u_0 \sim 1$. Setting $u = u_0$ on the minimal polynomial of $E(x, u)$ allow us to compute its Puiseux expansion for x near $\rho_t(u_0)$. It is of the form:

$$E(x, u_0) \underset{x\to\rho_t(u_0)}{\sim} E_0(u_0) + E_2(u_0)\left(1 - \frac{x}{\rho_t(u_0)}\right) + E_3(u_0)\left(1 - \frac{x}{\rho_t(u_0)}\right)^{3/2}. \tag{4.26}$$

As mentionned before, $E(x, u_0)$ is analytic in a dented domain at $\pm\rho_t(u_0)$, so that an application of the *transfer theorem* on (4.26) implies the following estimate on its coefficients:

$$[x^n]E(x, u_0) \underset{n\to\infty}{\sim} c(u_0) \cdot n^{-3/2} \cdot \rho_t(u_0)^{-n},$$

where $c(u_0) = 3E_3(u_0)/(2\sqrt{\pi})$ and $c(1) = c$.

We now satisfy every conditions of the *quasi-powers* theorem (apart, as before, for an inequality that one can verify after computing an approximation of the second moment). This implies that the sequence of random variables associated to the number of triangles in a unifromly at random cubic planar graph on n vertices converges in distribution to a Gaussian limit law. Its two first moment are computed by setting $u = 1$ in the two derivatives of $\sigma(x, u)$, the singularity curve of $E(x, u)$, with respect to u. This gives the following approximations:

$$\rho_t'(1) \approx -0.0389371919 \qquad \text{and} \qquad \rho_t''(1) \approx 0.0229417852,$$

which concludes the proof.

4.4 Conclusion

We comment briefly on the notable differences of this model with the classical model of random cubic (not necessarily planar) graphs [72]. A random cubic graph is with high probability 3-connected. Also, the number of triangles follows a Poisson law with expectation 4/3. Finally a random cubic graph is 3-colourable, whereas a random cubic graph has a (small) probability of containing K_4 as a component, hence of being 4-chromatic.

To conclude, we illustrate in Table 4.1 the approximate values of the first and the second moments of the following Gaussian limit laws: the number of cut-vertices, isthmuses, blocks and triangles in a random cubic planar graph. We do not reproduce here the first two moments of the limiting distributions

Parameter	Expectation	Variance
# cut-vertices	$0.0018774448 \cdot n$	$0.0037934519 \cdot n$
# cut-edges	$0.0009389848 \cdot n$	$0.0009496835 \cdot n$
# blocks	$0.0018777072 \cdot n$	$0.0037958302 \cdot n$
# triangles	$0.1219742813 \cdot n$	$0.0649847862 \cdot n$

Table 4.1: The first and the second moments of the random variables associated with the number of several parameters in a random cubic planar graph on n vertices.

for the number of cherries H and near-bricks B, as they directly depend on the knowledge of the respective automorphism group of H and B.

Chapter 5

Enumeration of 4-regular planar graphs

5.1 Introduction

In this chapter we provide the first scheme for counting labelled 4-regular planar graphs through a complete recursive decomposition. Let $C(x) = \sum c_n x^n/n!$ be the exponential generating function of labelled connected 4-regular planar graphs counted according to the number of vertices. We show that the derivative $C'(x)$ can be computed effectively as the solution of a (rather involved) system of algebraic equations. In particular $C'(x)$ is an algebraic function. Using a computer algebra system we can extract the coefficients of $C'(x)$, hence also of $C(x)$. If $G(x) = \sum g_n \frac{x^n}{n!}$ is now the generating function of all 4-regular planar graphs, the exponential formula $G(x) = e^{C(x)}$ allows us to find the coefficients g_n as well.

Our main result is the following:

Theorem 30. *The generating function $C'(x)$ is algebraic and is expressible as the solution of a system of algebraic equations from which one can compute effectively their coefficients.*

As a corollary, and using Lemma 2, we obtain:

Corollary 31. *The sequence* $\{g_n\}_{n\geq 0}$ *is* P*-recursive, that is, it satisfies a linear recurrence with polynomial coefficients.*

We also determine the generating function $\tau(x) = \sum t_n \frac{x^n}{n!}$ of 3-connected 4-regular graphs and show that its derivative is algebraic

Theorem 32. *The generating function* $\tau'(x)$ *is algebraic and is expressible as the solution of a system of algebraic equations from which one can compute effectively their coefficients.*

To obtain our results we follow the classical technique introduced by Tutte: take a graph rooted at a directed edge and classify the possible configurations arising from the removal of the root edge. This produces several combinatorial classes that are further decomposed, typically in a recursive way. The combinatorial decomposition translates into a system of equations for the associated generating functions, which in our case is considerably involved. Next we provide a brief overview of the combinatorial scheme in our solution.

Using a variant of the classical decomposition of 2-connected graphs into 3-connected components in the spirit of [7], we find an equation linking $C(x)$ to the generating function $T(u, v)$ of 3-connected 4-regular planar graphs, counted according to the number of simple edges and the number of double edges. Actually, T will be the generating function of rooted 3-connected *maps* (a rooted map is an embedding of a planar graph where a directed edge is distinguished) but by Whitney's theorem, 3-connected planar graphs have a unique embedding in the oriented sphere and in this situation counting graphs is equivalent to counting maps. As will be seen later, it is essential to count 3-connected maps according to simple and double edges, otherwise there is not enough information to obtain $C(x)$.

Once we have access to $T(u, v)$ we can compute the coefficients of $C(x)$ to any order. In order to compute $T(u, v)$ we apply the reverse procedure working with maps instead of graphs. The starting point is the fact that the number M_n of rooted 4-regular maps with $2n$ edges is well known, since they are in bijection with rooted (arbitrary) maps on n edges, and equal to

$$M_n = \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n}.$$

Using again the decomposition into 3-connected components, one can obtain an equation linking $T(u,v)$ and $M(z) = \sum M_n z^n$. However this is not sufficient since $T(u,v)$ is a bivariate series and cannot be recovered uniquely from the univariate series $M(z)$. In order to overcome this situation, we enrich the combinatorial scheme and count maps according to a secondary parameter: the number of isolated faces of degree 2, namely, those not incident with another face of degree 2. Notice that this parameter, when restricted to 3-connected 4-regular planar graphs, is precisely the number of double edges.

If $M(z,w)$ is the associated series, where w marks the new parameter, then we can enrich the corresponding equations and obtain an algebraic relation of the form

$$T(f(z,w,M(z,w)), g(z,w,M(z,w))) = h(z,w,M(z,w)),$$

where f, g and h are explicit functions. The transformation

$$u = f(z,w,M), \qquad v = g(z,w,M)$$

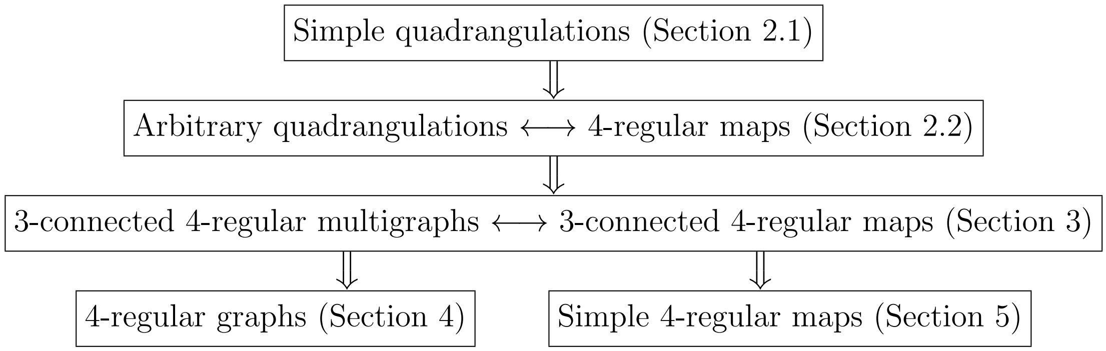
turns out to have non-zero Jacobian and can be inverted explicitly. This allows us to express $T(u,v)$ as a power series whose coefficients can be computed in terms of those of $M(z,w)$ and the inverse mapping $(z,w) \to (u,v)$. From here, we can compute the coefficients of $T(u,v)$ to any order. In particular, the coefficients of $T(u,0)$ give the numbers T_n of simple rooted 3-connected 4-regular planar maps. By double counting, we obtain the number of labelled 3-connected 4-regular planar graphs as $t_n = T_n(n-1)!/8$.

It remains to compute $M(z,w)$. To this end, we use the dual bijection between 4-regular maps and quadrangulations, and count quadrangulations according to the number of faces and the number of vertices of degree 2 not adjacent to another vertex of degree 2. This is technically demanding but can be achieved using the decomposition of quadrangulations along faces and edges, refined to take into account for the new parameter.

Once we have access to $T(u,v)$, we can also enumerate *simple* 4-regular maps, a result of independent interest (the enumeration of simple 3-regular maps can be found in [27]).

Theorem 33. *The generating function of rooted simple 4-regular maps is algebraic, and is expressible as the solution of a system of algebraic equations from which one can compute its coefficients effectively.*

The steps towards the proofs of Theorems 30 and 33 are summarized in the following diagram:



We remark that the final equations relating the power series $M(z, w)$, $T(u, v)$ and $C(x)$ are not written down explicitly. Instead, we work with several intermediate systems of equations that allow us to extract the coefficients of the corresponding series. This is computationally demanding but it is within the capabilities of a computer algebra system such as `Maple`.

Here is a summary of the chapter. In Section 5.2, we determine the series $M(z, w)$ as the solution of a system of polynomial equations. In Section 5.3, we obtain $T(u, v)$ as a computable function of $M(z, w)$, thus proving the second part of Theorem 30. In Section 5.4, we find an equation connecting $C(x)$ and $T(u, v)$, which allows us to compute the coefficients of $C(x)$, that is, the number of connected 4-regular planar graphs, and to complete the proof of Theorem 30. From the relation $G(x) = \exp(C(x))$, we obtain the coefficients of $G(x)$. Finally, in Section 5.5 we count simple 4-regular maps.

5.2 Counting quadrangulations

A *diagonal* in a quadrangulation is a path of length 2 joining opposite vertices of the external face. If uv is the root edge, there are two kind of diagonals, those incident with u and those incident with v. By planarity both cannot be present at the same time. A cycle of length 4 which is not the boundary of a face is called a *separating quadrangle*. Vertices in the outer face are called *external* vertices. A vertex of degree 2 is *isolated* if it is not adjacent to another vertex of degree 2. An isolated vertex of degree 2 is called a *2-vertex*.

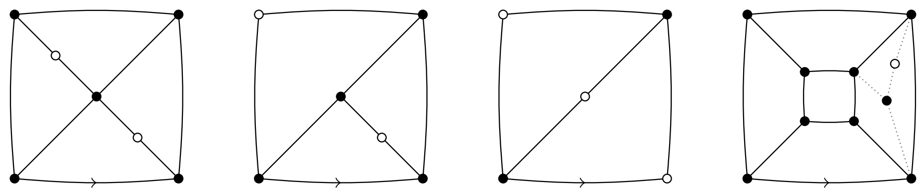


Figure 5.1: *Examples of simple quadrangulations. The first three, from left to right, belong to* $\mathcal{N}_0$, $\mathcal{N}_1$ *and* $\mathcal{N}_2$, *respectively. The last one is in* $\mathcal{R}$: *a quadrangulation from* $\mathcal{S}$ *in which one face has been replaced with a quadrangulation in* $\mathcal{N}_1$. *Isolated vertices of degree 2 are shown in white.*

5.2.1 Simple quadrangulations

All quadrangulations in this section are simple, that is, have no multiple edges. This implies in particular that all faces are quadrangles, that is, simple 4-cycles. We will now refine the system of equation from [47] or Section 2.4.2 of Chapter 2, to include a second variable marking the number of 2-vertices in simple quadrangulations. To that end, let us consider the following classes of quadrangulations, illustrated in Figure 5.1:

- $\mathcal{Q}$ are all (simple) quadrangulations.
- $\mathcal{S}$ are quadrangulations without diagonals or separating quadrangles.
- $\mathcal{N}$ are quadrangulations containing a diagonal incident with the root vertex. By symmetry they are in bijection with quadrangulations containing a diagonal not incident with the root vertex.
- $\mathcal{N}_i$ are quadrangulations in $\mathcal{N}$ with exactly i external 2-vertices, for $i = 0, 1, 2$.
- $\mathcal{R}$ are quadrangulations obtained from $\mathcal{S}$ by possibly replacing each internal face with a quadrangulation in $\mathcal{Q}$.

In the following generating functions, z marks internal faces and w marks 2-vertices. For each class of quadrangulations we have the corresponding generating function written with the same letter. For instance $Q(z, w)$ is associated with the class $\mathcal{Q}$, and so on. The exception is $S(z)$, since a quadrangulation in $\mathcal{S}$ has no vertex of degree 2, and variable w does not appear.

Lemma 34. *Let*

$$N = N_0 + N_1 + N_2, \qquad \widetilde{N} = N_0 + \frac{N_1}{w} + \frac{N_2}{w^2}.$$

Then the following system of equations holds:

$$\begin{aligned}
Q =& \quad z + 2N + R, \\
R =& \quad S(z + 2\widetilde{N} + R), \\
N_0 =& \quad (\widetilde{N} + R)\left(\widetilde{N} + R + N_0 + \frac{N_1}{2w}\right), \\
N_1 =& \quad 2zw\left(\widetilde{N} + R + N_0 + \frac{N_1}{2}\right), \\
N_2 =& \quad z^2w^3 + zw\left(N_2 + \frac{N_1}{2}\right).
\end{aligned} \tag{5.1}$$

Moreover, the system and has a unique solution with non-negative coefficients.

Proof. First we check that the system has non-negative coefficients. As $S(z)$ has non-negative Taylor coefficients, we only need to argue on the terms N_1/w and N_2/w^2. But by definition, quadrangulations in $\mathcal{N}_i$ have at least i 2-vertices, and the generating function N_i has w^i as a factor. It is immediate to check that all the right-hand terms are divisible by z, hence Lemma 1 guarantees the last claim in the statement.

The first equation follows from the fact that a quadrangulation, not reduced to a single quadrangle, either has a diagonal or is obtained from a quadrangulation in $\mathcal{S}$ by replacing internal faces with arbitrary quadrangulations in $\mathcal{Q}$.

The second equation expresses the recursive nature of the class $\mathcal{R}$. Notice that the substitution inside S contains the term $\widetilde{N}$ instead of N. The reason is that vertices of degree 2 in the outer face become vertices of degree more than two after substitution.

For the rest of the proof, notice that the left-hand terms in the last three equations correspond to quadrangulations in $\mathcal{N}$, whose diagonal is incident to the root vertex. They are decomposed following their rightmost diagonal, that is, the first diagonal to the left of the root edge. Let A be the quadrangulation consisting of a single quadrangle with a diagonal adjacent to the root vertex. It has two inner faces: f_1, incident with the root edge, and f_2. A quadrangulation in $\mathcal{N}$ is obtained by replacing the inner faces of A with simple quadrangulations, such that the quadrangulation replacing f_1 does not have a diagonal incident with the root vertex, as otherwise the decomposition would not be unique. the diagonal of A would not be the rightmost diagonal of the resulting quadrangulation, contrary to the construction. In what follows, the *top vertex* is the external vertex adjacent to the root vertex and not incident with the root edge.

Equation for N_0. Both f_1 and f_2 can be replaced either with quadrangulations in $\mathcal{R}$ or those with a diagonal not adjacent to the root vertex (which are in bijection with $\mathcal{N}$), hence the factor $\widetilde{N} + R$. In addition, f_2 can be replaced by a quadrangulation with a diagonal adjacent to its root vertex, but only if the top vertex is not of degree 2, that is, any quadrangulation in $\mathcal{N}_0$ and half of the ones in $\mathcal{N}_1$ (those in which the top vertex is not of degree 2).

Equation for N_1. In this case, either has f_1 or f_2 is empty. Notice that both cases are symmetric with respect to the diagonal of A (but in the second case, the diagonal of A does not have to be the rightmost diagonal, instead it is the leftmost). We will hence only explicit the first case, where f_1 is replaced by the single quadrangle zw while for f_2 it is a quadrangulation counted by $\widetilde{N}+R+N_0+N_1/2$. Observe that contrary to the previous equation we have the term $N_1/2$ instead of $N_1/(2w)$: this is because the middle vertex of the diagonal of A remains of degree 2.

Equation for N_2. In this case face f_1 is not replaced. If neither is f_2, we get A, hence the term z^2w^3. Else, f_2 is replaced with a quadrangulation in $\mathcal{N}$ whose top vertex is of degree 2, corresponding to the term $N_2 + N_1/2$, as before. □

From the previous system of equations we can compute the coefficients to any order of all the series involved by iteration. As we are going to see, a modified version of the series Q, N_0, N_1, N_2 is needed in Section 5.2.2.

5.2.2 Arbitrary quadrangulations

A quadrangulation of a 2-cycle is a rooted map in which each face is of degree 4 except the outer face which is of degree 2. One of the two edges in the outer face is taken as the root edge, and its tail is the root vertex. An arbitrary quadrangulation is obtained from a simple quadrangulation

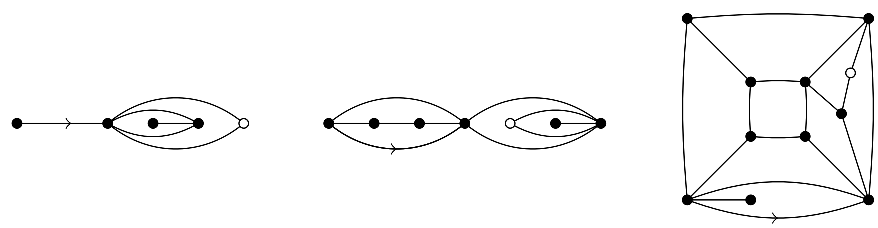


Figure 5.2: *The different types of quadrangulations in $\mathcal{B}$, depending on the nature of the root face. Isolated vertices of degree 2 are shown in white.*

by replacing edges with quadrangulations of a 2-cycle. Conversely, given an arbitrary quadrangulation, collapsing all maximal 2-cycles, one obtains a simple quadrangulation. Notice that among the simple quadrangulations, we need to include the degenerate case consisting of a path of length 2, which we denote by P_3 (see Figure 5.2 and the corresponding caption).

In an arbitrary quadrangulation there are three possibilities for the shape of the root face: it is either a quadrangle, the result of gluing two 2-cycles through a vertex, or gluing one 2-cycle and one edge (see Figure 5.2).

We now define the following classes of quadrangulations:

- $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$ are quadrangulations of a 2-cycle. $\mathcal{A}_1$ are those whose root vertex is a 2-vertex (by symmetry, they are in bijection with those in which the other external vertex is a 2-vertex), and $\mathcal{A}_0$ are those without external 2-vertices.

- $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_0^* \cup \mathcal{B}_1$ are arbitrary quadrangulations. $\mathcal{B}_1$ are those in which the root edge is incident to exactly one 2-vertex, and $\mathcal{B}_0 \cup \mathcal{B}_0^*$ are those in which the root edge is not incident to a 2-vertex. Furthermore, $\mathcal{B}_0^*$ are the quadrangulations obtained by replacing one of the two edges incident with the root edge on the single quadrangle, as illustrated in Figure 5.3. The class $\mathcal{B}_0^*$ is introduced for technical reasons that will become apparent in the next section, when we consider the dual class of 4-regular maps.

In the generating functions $A_0(z, w)$ and $A_1(z, w)$, variables z and w mark, respectively, internal faces and 2-vertices, whereas in $B_0(z, w)$, $B_1(z, w)$ and $B_0^*(z, w)$ variable z marks *all* faces: it is important to keep in mind this distinction when checking the equations satisfied by the various generating

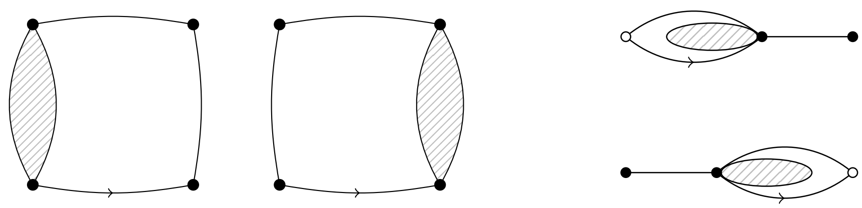


Figure 5.3: *On the left are the two types of quadrangulations in $\mathcal{A}_0^*$. On the right are two types of substitutions of P_3 counted in $\mathcal{A}_1$.*

functions. An exception, which again becomes clear when passing to the dual, is the term $2z$ encoding the path P_3 (which can be rooted in two different ways), where the middle vertex is not considered to be a 2-vertex.

The next generating function encodes the substitution of edges in simple quadrangulations:

$$\widetilde{A} = A_0 + \frac{2A_1}{w}.$$

The reason for w in the denominator is that, after substitution, the 2-vertex in $\mathcal{A}_1$ no longer has degree 2, and the factor 2 is because this vertex can be any of the two endpoints of the root edge.

To encode the substitution of edges by objects from $\mathcal{A}$, we first remark that the number of edges in a quadrangulation is twice the number of faces, hence there is a bijection between faces and pairs of edges. Moreover, since 2-vertices are isolated, pairs of edges incident with a 2-vertex are uniquely determined. Indeed, one can interpret w as to mark all pairs of edges incident with 2-vertices, while z marks pairs of remaining edges except the pair that corresponds to the external face in the aforementioned bijection. These considerations justify the following change of variables:

$$s = s(z, w) = (1 + \widetilde{A})^2, \qquad t = t(z, w) = \frac{w + 2\widetilde{A} + \widetilde{A}^2}{(1 + \widetilde{A})^2},$$

whose meaning is the following. The term $(1 + \widetilde{A})^2$ in s encodes pairs of edges that are possibly substituted. Pairs of edges incident with a 2-vertex must be treated differently: if any of them is replaced with an object from $\mathcal{A}$, the vertex no longer has degree 2, hence the term $w + 2\widetilde{A} + \widetilde{A}^2$ in t. This must be corrected with the term $(1+\widetilde{A})^2$ in the denominator of t since those edges where already counted in s.

Consider now the system (5.1) from the previous section with the change of variables

$$z = zs, \qquad w = t.$$

We remark that the factor z in zs encodes faces in the initial simple quadrangulation. We then partition the resulting quadrangulations into three families. The first one is associated to the generating function $E(z,w)$ which counts those whose initial simple quadrangulation is the single quadrangle. The second and third are counted by $Q_1(z,w)$ and $Q_0(z,w)$, which respectively count those whose root edge is incident or not with a 2-vertex, and whose initial simple quadrangulations is not the single quadrangle. In all three generating functions, the variable z marks inner faces and the variable w marks 2-vertices.

Lemma 35. *The following equalities hold:*

$$\begin{aligned} Q_1 &= \frac{1}{t}(N_1(zs,t) + 2N_2(zs,t)), \\ Q_0 &= s(2N_0(zs,t) + N_1(zs,t) + R(zs,t)) + (2\widetilde{A} + \widetilde{A}^2)Q_1, \\ E &= z(1+\widetilde{A})^4 - 4z\widetilde{A}^2 + 4zw\widetilde{A}^2, \end{aligned}$$

where z marks inner faces and w marks 2-vertices.

Proof. Let uv be the root edge. In all three cases we consider the simple quadrangulation obtained when collapsing all 2-cycles. The first equation holds because quadrangulations where there is one 2-vertex incident with the root edge are encoded by $N_1 + 2N_2$, and the factor $1/t$ because we do not replace any of its two incident edges.

In the second equation, we need to distinguish whether either u or v in the initial simple quadrangulation were of degree 2 or not. If this is the case, we must replace at least one of its two incident edges with a quadrangulation of a 2-cycle, obtaining the correcting factor $(2\tilde{A} + \tilde{A}^2)$, hence the term $(2\tilde{A} + \tilde{A}^2)Q_1$. If neither u nor v were of degree 2 we get $s(2N_0 + N_1 + R)$, where the factor s accounts for replacing the pair of edges that corresponds to the external face.

As for the last equation, vertices of degree 2 in the root face can become isolated after replacing two consecutive edges of a quadrangle by quadrangulations of a 2-cycle. There are four possibilities for this situation, encoded

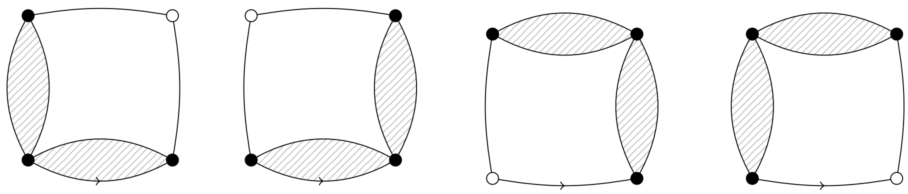


Figure 5.4: *The four types of quadrangulations obtained by replacing two consecutive edges of a quadrangle with quadrangulations of a 2-cycle, where a 2-vertex (in white) is created. The first two on the left are in $\mathcal{B}_0$, while the two on the right are in $\mathcal{B}_1$.*

by $4z\widetilde{A}^2$ in the expansion of $z(1+\widetilde{A})^4$ (see Figure 5.4). Hence the term $z(1+\widetilde{A})^4 - 4z\widetilde{A}^2 + 4zw\widetilde{A}^2$. □

We treat separately the generating function encoding the substitution of P_3, the path on 3 vertices, which corresponds to

$$\widehat{A} = A_0 + A_1 + \frac{A_1}{w}. \tag{5.2}$$

The difference with $\widetilde{A}$ is that the two edges of P_3 have one endpoint of degree 1, and when the external 2-vertex of a quadrangulation in $\mathcal{A}_1$ is identified with one of them, its degree does not increase (see Figures 5.2 and 5.3).

Lemma 36. *Let Q_0, Q_1 and E be as in Lemma 35. Then the following equations hold and have a unique solution with non-negative coefficients:*

$$\begin{aligned} A_1 &= zw(1+\widehat{A}), \\ A_0 &= 2z\widetilde{A}(1+\widehat{A}) + z(Q_0 + Q_1 + E + 2z\widetilde{A}(w-1) + 2z\widetilde{A}^2(1-w)). \end{aligned} \tag{5.3}$$

Moreover, the system has a unique non-zero solution with non-negative coefficients.

Proof. The right-hand terms are clearly divisible by z. The right-hand side of the second equation has non-negative coefficients. The term $-2z\widetilde{A}$ cancels with the first term, and $-2zw\widetilde{A}$ cancels with a corresponding term in E.

Lemma 1 then guarantees the uniqueness of the solution with non-negative coefficients.

We first observe that a quadrangulation of a 2-cycle can be thought as an ordinary quadrangulation adding an edge parallel to the root edge.

When removing the root vertex of a quadrangulation of a 2-cycle in $\mathcal{A}_1$, we obtain either an edge (term zw), a quadrangulation of a 2-cycle in $\mathcal{A}_0$ (term zwA_0), a quadrangulation of a 2-cycle in $\mathcal{A}_1$ (term zA_1), or its symmetric. In the last case, we create a 2-vertex, hence the factor w in zwA_1. The reverse operation consists of starting from a quadrangulation γ encoded in $1 + \widehat{A}$, adding a new vertex v in the outer face, connecting v to the root vertex v' of γ by two edges, and rooting the resulting map at vv' so that the outer face is a digon.

The term $2z\widetilde{A}(1 + \widehat{A})$ in the equation for A_0 encodes quadrangulations of the 2-cycle arising from a quadrangulation whose external face is not a simple 4-cycle (illustrated in the two leftmost parts of Figure 5.2 and on the right of Figure 5.3). The term $zQ_0 + zQ_1 + zE$ arises from the corresponding quadrangulations when building the 2-cycle. The term zE has to be adjusted because either we create a 2-vertex, or we remove a 2-vertex. The first case is counted by the term $2z\widetilde{A}(w-1)$, illustrated by the two graphs on the left-hand side of Figure 5.3, while the second is counted by the term $2z\widetilde{A}^2(1-w)$, illustrated by the two graphs on the right-hand side of Figure 5.4. □

From the previous lemma we can obtain the generating functions associated to $\mathcal{B}_0$, $\mathcal{B}_0^*$ and $\mathcal{B}_1$.

Lemma 37. *Let Q_0, Q_1, E, A_0, A_1 be as in the previous two lemmas and $\widehat{A}$ as defined in Equation* (5.2)*. Then B_0, B_1 and B_0^* are given by*

$$\begin{aligned} B_0 &= 2z(1+\widehat{A})(1+\widehat{A}-A_1) + z(Q_0 + E - 2zw\widetilde{A}^2 - 2z\widetilde{A}), \\ B_1 &= 2z(1+\widehat{A})A_1 + zw(Q_1 + 2z\widetilde{A}^2), \\ B_0^* &= 2z^2\widetilde{A}, \end{aligned} \tag{5.4}$$

where z marks faces and w marks 2-vertices.

Proof. Recall that an arbitrary quadrangulation is obtained by substituting edges by quadrangulations of a 2-cycle in a simple quadrangulation. The factor z in all equations is used to encode the outer face, which has not been considered in the previous generating functions.

The term $2z(1+\widehat{A})(1+\widehat{A}-A_1)$ encodes quadrangulations obtained from P_3. Notice that quadrangulations counted by the term $2z(1+\widehat{A})A_1$ are in $\mathcal{B}_1$ and must be removed from $\mathcal{B}_0$. The second term $z(Q_0+E-2zw\widetilde{A}^2-2z\tilde{A})$ encodes quadrangulations obtained from simple quadrangulations whose root face is a 4-cycle. In this situation, we have to remove from zE the terms $2z^2\widetilde{A}$ and $2z^2w\widetilde{A}^2$, which contribute to B_0^* and B_1, respectively (see the two leftmost maps in Figure 5.3 and the two rightmost maps in Figure 5.4, respectively). Finally, there is an extra contribution to B_1 with the term zwQ_1. $\square$

From the Systems (5.1) and (5.4) we can compute, by iteration, the coefficients to any order of all the series involved. In particular we can compute the coefficients of the series B_1, B_0 and B_0^*, which are needed in the next section.

5.3 Rooted 3-connected 4-regular planar maps

In this section we count 3-connected 4-regular planar maps according to the number of simple and double edges. Because they have a unique embedding on the oriented sphere, this is equivalent to counting labelled 3-connected 4-regular planar graphs. We notice that a 3-connected 4-regular map cannot have triple edges, and double edges must be vertex disjoint. In addition, all maps in this section are rooted.

For brevity, a face of degree 2 not adjacent to another face of degree 2 is called a *2-face*. We say that an edge is in a 2-face if it is one of its two boundary edges. An edge is *ordinary* if it is not in the boundary of a 2-face. Since the number of edges is even, the number of ordinary edges is also even. Maps are counted according to two parameters: the number of 2-faces, marked by variable w, and half the number of ordinary edges, marked by q. Setting $w=q$ one recovers the enumeration of 4-regular maps according to half the number of edges. Observe that the dual of a quadrangulation with ℓ 2-vertices is a 4-regular map with ℓ 2-faces.

We need to define the replacement of edges by maps. Let M be a rooted map. Consider a fixed orientation of the edges in M; since in a rooted map all vertices and edges are distinguishable we can define such an orientation unambiguously.

Replacement of simple edges. Let let $e=uv$ be an edge of M and N a map whose root edge is simple. The replacement of e with N is the map obtained by the following operation. Subdivide e twice transforming it in to the path $uu'v'v$, remove the edge $u'v'$, and identify u' and v' with the end vertices of the (previously deleted) root edge of N, respecting the orientations. An example is shown in Figure 5.5, bottom right.

Replacement of 2-faces. Let (e, e') be the endpoints of a 2-face f and N a map whose root is a 2-face f'. The replacement of f with N is the map obtained by identifying (e, e') with the endpoints of f' after having deleted the two edges of f', while preserving the orientation and the embedding. On the bottom of Figure 5.5, an example of a map rooted at a 2-face is illustrated by the second picture from the left, while the third picture from the left exhibits a 2-face that has been replaced.

Notice that if M and N in the above are 4-regular, then the maps obtained by replacement of simple edges or 2-faces are also 4-regular.

We now consider the following families of (rooted) 4-regular maps.

- $\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_0^* \cup \mathcal{M}_1$ are 4-regular maps. $\mathcal{M}_0 \cup \mathcal{M}_0^*$ are 4-regular maps in which the root edge is not incident with a 2-face, and $\mathcal{M}_1$ are those for which the root edge is incident with exactly one 2-face. $\mathcal{M}_0^*$ are maps in which the root is one of the extreme edges of a triple edge, corresponding to dual maps of quadrangulations on the left of Figure 5.3. These classes are in bijection with the classes $\mathcal{B}_0$, $\mathcal{B}_0^*$ and $\mathcal{B}_1$ from the previous seciton, and Lemma 37 gives access to the associated generating functions.

The next classes are all subclasses of $\mathcal{M}$. Given a map M, we let M^- be the map obtained by removing the root edge st. In accordance with the terminology introduced in the next section, the *poles* are the endpoints s, t of the root edge.

- $\mathcal{L}$ are *loop* maps: the root-edge is a loop.

- $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$ are *series* maps: M^- is connected and there is an edge in M^- that separates the poles. As above, the index $i = 0, 1$ refers to the number of 2-faces incident with the root edge.

- $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1$ are *parallel* maps: M^- is connected, there is no edge in M^- separating the poles, and either $M - \{s, t\}$ is disconnected or st is a double or a triple edge of M^-. The index $i = 0, 1$ has the same meaning as in the previous class.

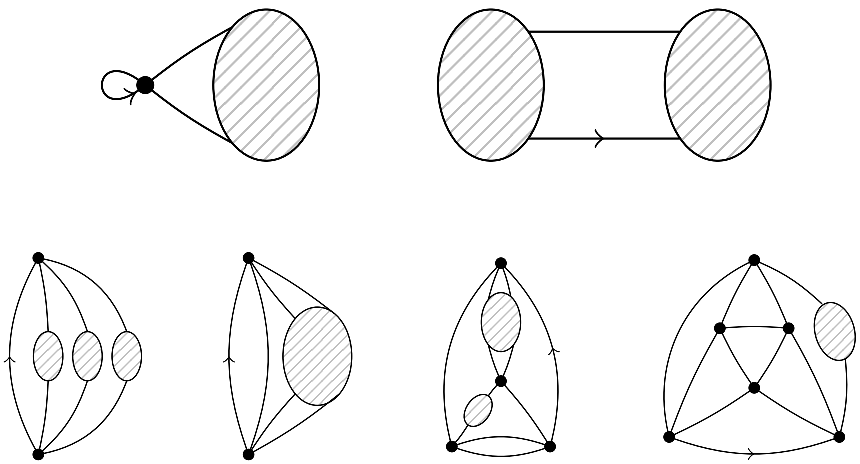


Figure 5.5: Root-decomposition of 4-regular maps. On top a loop map (left) and a series map (right). Bottom, from left to right: parallel map, map in $\mathcal{F}$, and two maps in $\mathcal{H}$.

- $\mathcal{F}$ are maps M such that the face to the right of the root-edge is a 2-face, and such that $M - \{s,t\}$ is connected; see Figure 5.5. Maps in $\mathcal{F}$ are those meant to be pasted on a 2-face. Notice that in this case, the information whether the 2-face lays on the right of the root edge or on its left will be lost. We hence chose one of those two rootings so we don't need to carry over a factor 1/2 when later doing the algebraic inversion. The maps with a 2-face laying on the left of the root edge are in $\bar{\mathcal{F}}$ and are, by symmetry, in bijection with those in $\mathcal{F}$.
- $\mathcal{H}$ are h-maps: they are obtained from a 3-connected 4-regular map C (the *core*) and possibly replacing every non-root edge of C with a map in $\mathcal{M}$. The root edge of C must be a simple edge.

Lemma 38. *The former classes partition* $\mathcal{M}$*, that is,*

$$\mathcal{M} = \mathcal{L} \cup \mathcal{S} \cup \mathcal{P} \cup \mathcal{H} \cup \mathcal{F} \cup \bar{\mathcal{F}}.$$

Proof. Let G be a 4-regular map rooted at $e = st$, and suppose e is not a loop, so we are not in the class $\mathcal{L}$. Consider the 2-connected component C

containing e. As before, C is either a series, parallel or h-composition. Series and h-compositions correspond, repectively, to classes $\mathcal{S}$ and $\mathcal{H}$. Parallel compositions are either in $\mathcal{P}$, in $\mathcal{F}$ or in $\bar{\mathcal{F}}$. □

The next step is to determine the algebraic relations among the generating functions of the previous classes. Variable q marks half the number of ordinary edges, and w the number of 2-faces. We need to introduce an auxiliary generating function D. It is combinatorially equivalent to the generating function associated to the class $\mathcal{M}$, but it will instead count maps that are *ready to be pasted on a single edge*. That is maps whose root edge has already been suppressed. The difference in terms of generating function is as follows. If the root edge was a single or a quadruple edge, i.e. the map belongs to $\mathcal{M}_0$, then it does not change anything. Now if the root edge is in a double edge forming a 2-face, i.e. the map belongs to $\mathcal{M}_1$, then removing it will make the other edge ordinary. This is accounted for by dividing the generating function by w then multiplying it by q. Finally if it belongs to a triple edge, then we two cases arise: either the two edges remaining after the suppression of the root form a 2-face or not. The latter are maps in $\mathcal{M}_0$, and it does not change anything for the associated generating function, while the former are the maps in $\mathcal{M}_0^*$, and is accounted for by dividing the generating function by q then multiplying it by w. This is a technical device that simplifies the forthcoming equations.

Let $T(u, v)$ be the generating function of 3-connected 4-regular maps in which the root edge is simple, where u marks half the number of simple edges and v marks the number of double edges (let us recall again that the number of edges in a 4-regular graph is even). The next result provides a link between the known series M_0, M_0^* and M_1, and the series T we wish to determine.

Lemma 39. *The following system of equations holds, where q marks half the*

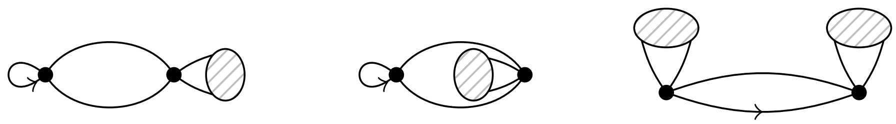


Figure 5.6: *On the left the two cases of loop maps. On the right is a map in S_1, in which two loop maps are connected in series, thus creating a 2-face.*

number of ordinary edges, and w the number of 2-faces.

$$\begin{aligned}
M_0 &= S_0 + P_0 + L + H, \\
M_1 &= \frac{w}{q}(S_1 + P_1 + 2qF), \\
M_0^* &= 2q^2 D, \\
D &= M_0 + \frac{q}{w}M_1 + \frac{w}{q}M_0^*, \\
L &= 2q(1 + D - L) + L(w + q), \\
S_0 &= D(D - S_0 - S_1) - \frac{L^2}{2}, \\
S_1 &= \frac{L^2}{2}, \\
P_0 &= q^2(1 + D + D^2 + D^3) + 2qDF, \\
P_1 &= 2q^2D^2, \\
H &= \frac{T\left(q(1+D)^2, w + q(2D + D^2) + F\right)}{1 + D}.
\end{aligned} \tag{5.5}$$

Proof. The first four equations follow directly from the definitions. Consider now loop maps. The double-loop is the map consisting of a single vertex and two loops. A loop map is obtained by possibly replacing the non-root loop of the double-loop with a map in $\mathcal{M}$. We have two possiblities illustrated in Figure 5.6, thus giving:

$$L = 2q(1 + D - L) + L(q + w).$$

A series map is obtained by taking a map in $\mathcal{M}$ with poles s_1 and t_1, a map in $\mathcal{M}\backslash\mathcal{S}$ with poles s_2 and t_2, and then replacing s_1t_1 and s_2t_2 with edges

t_1s_2 and s_1t_2, the latter being the new root. When the two maps connected in series are in $\mathcal{L}$, a 2-face is created containing the root (see Figure 5.6), and we obtain a map in $\mathcal{S}_1$. There are four ways to connect two loop maps in series, but only two of them produce a map in $\mathcal{S}_1$: when they are rooted either both on a face of degree one, or both on a face of degree more than one. In terms of generating functions we have

$$S_0 = D(D - S) - S_1 \quad \text{and} \quad S_1 = \frac{L^2}{2}.$$

A parallel map can be obtained in two different ways. First, from a map in $\mathcal{F}$ with double root edge r and replacing exactly one edge in r with a map in $\mathcal{M}$. This is encoded by $2qFD$ (see Figure 5.7). Secondly, we take the 4-bond (an edge of multiplicity 4) and replace any of its three non-root edges with maps in $\mathcal{M}$. This is encoded by $q^2(1 + D)^3$ (see Figure 5.5), but two particular cases must be considered:

1) When exactly two edges of the 4-bond are substituted, encoded by P_1 (see Figure 5.7), where a double edge is created;

2) When exactly one edge of the 4-bond is substituted.

Again there are two instances, encoded by M_0^* (see Figure 5.7), where the root edge belongs to exactly one of the two faces of degree 2. Notice that these faces are not isolated and hence are encoded as three ordinary edges. When the root edge is removed, the face of degree two containing it is removed and the remaining one becomes isolated, i.e. a 2-face. Summing this up gives the expressions for P_0 and P_1.

A map in $\mathcal{H}$ is obtained by possibly replacing the non-root simple edges of a core by a map in $\mathcal{M}$, and the double edges with either:

1) A map in $\mathcal{M}$ on one of the edges of the double edge;

2) Two maps in $\mathcal{M}$, one on each edge;

3) A map in F (see Figure 5.5).

This gives

$$H = \frac{T\left(q(1 + D)^2, w + q(2D + D^2) + F\right)}{1 + D}.$$

□

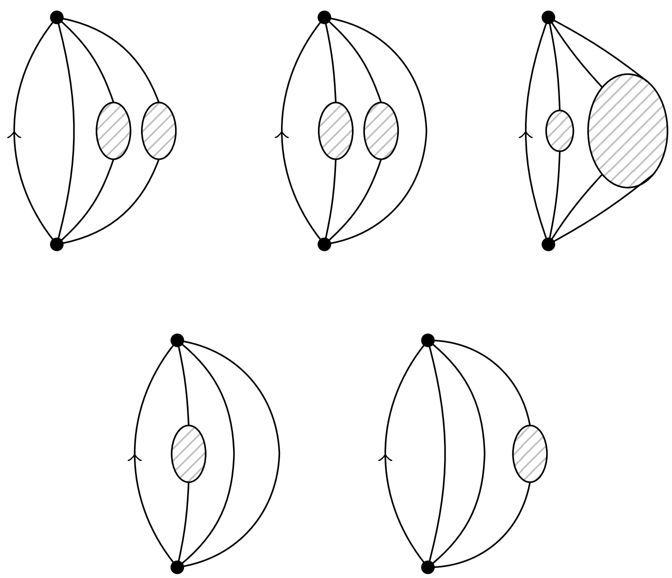


Figure 5.7: *Top from left to right: two types of maps in $\mathcal{P}_1$, and a map in $\mathcal{P}_0$ obtained by taking a map in $\mathcal{F}$ and substituting its root edge by a map in $\mathcal{M}$. Bottom: the two types of maps in $\mathcal{M}_0^*$.*

Proof of Theorem 32. From the knowledge of M and the M_i in the previous section we can determine $D = M_0^*/(2q^2)$. Hence we can also determine L, then S_0 and S_1. We also know $P_1 = 2wD^2$ and from $M_1 = S_1 + P_1 + 2wF$ we obtain F. This is enough to compute P_0, and from $M_0 = S_0 + P_0 + L + H$ we determine H. Since M and the M_i were algebraic functions, so are all the functions in the previous system.

We are ready for the final step. Consider the following change of variables

$$u = q(1 + D)^2, \qquad v = w + q(2D + D^2) + F,$$

relating T and H. The first terms in the expansion of u and v in q and w are

$$u = q + \cdots, \qquad v = w + \cdots$$

It follows that the Jacobian at $(0, 0)$ is equal to 1 and the system can be inverted, in the sense that we can determine uniquely the coefficients of the inverse series. Computationally, this can be explicitly obtained using Gröbner basis (we are grateful to Manuel Kauers for this observation).

Let the inverse of the system be

$$q = a(u, v), \qquad w = b(u, v).$$

Since D and F are algebraic functions, so are the inverse functions a and b. Now we use the last equation in Lemma 39 to get

$$T(u, v) = (1 + D(a(u, v), b(u, v))H(a(u, v), b(u, v)).$$

This equation determines T. Since all the series involved are algebraic, so is T.

Recall that t_n is the number of labelled 3-connected 4-regular planar graphs. Let $T_n = [u^k]T(u, 0)$ be the number of simple rooted 3-connected 4-regular maps. Then we have the relation

$$8nt_n = n!T_n.$$

This follows by double counting. We can label the vertices of a rooted map in $n!$ different ways, since vertices in a rooted map are distinguishable, and on the other hand, from a labelled graph we obtain $8n$ rooted maps: 4n choices for the directed root edge, and 2 choices for the root face. As a consequence,

$$8x\tau'(x) = T(u, 0).$$

Since T is algebraic, so is $\tau'(x)$. □

Figure 5.8: *The two types of networks in $\mathcal{F}$. On the left a network in $\mathcal{S}_2$. On the right a network in $\mathcal{H}_2$.*

5.4 Labelled 4-regular planar graphs

In this section we complete the proof of Theorem 30. In the sequel all graphs are labelled. First we define networks. A *network* is a connected 4-regular multigraph G with an ordered pair of adjacent vertices (s, t), such that the graph obtained by removing the edge st is simple. Vertices s and t are the *poles* of the network.

We now define several classes of networks, similar to the classes introduced in the previous section. We use the same letters, but they now represent classes of labelled *graphs* instead of classes of maps. No confusion should arise since in this section we deal with graphs.

- $\mathcal{D}$ is the class of all networks.
- $\mathcal{L}, \mathcal{S}, \mathcal{P}$ correspond as before to loop, series and parallel networks. We do not need to distinguish between $\mathcal{S}_0$ and $\mathcal{S}_1$ and between $\mathcal{P}_0$ and $\mathcal{P}_1$.
- $\mathcal{F}$ is the class of networks in which the root edge has multiplicity exactly 2 and removing the poles does not disconnect the graph.
- $\mathcal{S}_2$ are networks in $\mathcal{F}$ such that after removing the two poles there is a cut vertex, see Figure 5.8.
- $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ are h-networks: in $\mathcal{H}_1$ the root edge is simple and in $\mathcal{H}_2$ it is double, see Figure 5.8.

Lemma 40. *The two classes $\mathcal{S}_2$ and $\mathcal{H}_2$ partition $\mathcal{F}$, that is*

$$\mathcal{F} = \mathcal{S}_2 \cup \mathcal{H}_2.$$

Proof. Clearly $\mathcal{S}_2$ and $\mathcal{H}_2$ are contained in $\mathcal{F}$. Now take a network in $\mathcal{F}$ and remove its two poles. By definition, the resulting graph must be connected and has either a cut-vertex or is at least 2-connected. If it has a cut-vertex then it belongs to $\mathcal{S}_2$. Otherwise, by the same argument as in Lemma 38 it is obtained from a 3-connected core rooted at a double edge, hence if belong to $\mathcal{H}_2$. □

The generating functions of networks are of the exponential type, for instance $D(x) = \sum D_n \frac{x^n}{n!}$. We need in addition the generating functions

$$T_i(x,u,v) = \sum T^{(i)}_{n,k,\ell} u^k v^\ell \frac{x^n}{n!}$$

of 3-connected 4-regular planar graphs rooted at a directed edge, where $i = 1, 2$ indicates the multiplicity of the root, x marks vertices and u, v mark, respectively, half the number of simple and the number of double edges. The coefficients $T^{(i)}_{n,k,\ell}$ of T_i are easily obtained from those of $T = \sum t_{k,\ell} u^k v^\ell$, the generating function of 3-connected 4-regular maps computed in the previous section. By double counting we have

$$T^{(1)}_{n,k,\ell} = n! \frac{t_{k,\ell}}{2}, \qquad \ell\, T^{(2)}_{n,k,\ell} = k\, T^{(1)}_{n,k,\ell}, \qquad n = \ell + k/2.$$

Since we can compute the coefficients $t_{k,\ell}$ as in the previous section, we can compute the coefficients $T^{(i)}_{n,k,\ell}$ as well.

In terms of generating functions this amounts to

$$T^{(1)}(x,u,v) = \frac{1}{2} T(u^2 x, vx), \qquad u \frac{\partial}{\partial u} T^{(2)}(x,u,v) = v \frac{\partial}{\partial u} T^{(1)}(x,u,v). \quad (5.6)$$

Since T is an algebraic function, $T^{(1)}$ is algebraic too, but to prove that $T^{(2)}$ is algebraic needs a separate argument.

Similarly to Lemma 40 but in the maps setting, $\mathcal{F}$ (as a class of maps) is partitioned into $\mathcal{S}_2$ and $\mathcal{H}_2$. Although redundant for the purpose of extracting the coefficients of T, this decomposition can be added to the system of equations (5.5) as follows:

$$\begin{aligned} F &= S_2 + H_2, \\ S_2 &= (w + q(2D + D^2) + F)(w + q(2D + D^2) + F - S_2), \\ H_2 &= \frac{T_2\left(q(1+D)^2, w + q(2D + D^2) + F\right)}{w + q(2D + D^2) + F} \end{aligned} \tag{5.7}$$

where $T_2(u, v)$ counts 3-connected 4-regular *maps* rooted at a double edge and with the face of degree two on the right of the root edge. Since F and D are algebraic by Lemma 39, so are S_2 and H_2. And so is T_2 since it can be derived form H_2 by the same algebraic inversion as before. Finally we have

$$T^{(2)}(x, u, v) = \frac{1}{2} T_2(u^2 x, vx), \tag{5.8}$$

where the division by two encodes the choice of the root face.

Lemma 41. *The following system of equations among the previous series holds:*

$$\begin{aligned} D &= L + S + P + H_1 + F \\ L &= \frac{x}{2}(D - L) \\ S &= D(D - S) \\ P &= x^2\left(\tfrac{D^2}{2} + \tfrac{D^3}{6}\right) + FD \\ F &= S_2 + H_2 \\ S_2 &= \tfrac{1}{x}\left(F + x^2\left(D + \tfrac{D^2}{2}\right)\right)\left(F + x^2\left(D + \tfrac{D^2}{2}\right) - S_2\right) \\ H_1 &= \frac{T^{(1)}\left(x, 1 + D, D + \tfrac{D^2}{2} + \tfrac{F}{x^2}\right)}{1 + D} \\ H_2 &= \frac{T^{(2)}\left(x, 1 + D, D + \tfrac{D^2}{2} + \tfrac{F}{x^2}\right)}{D + \tfrac{D^2}{2} + \tfrac{F}{x^2}} \end{aligned} \tag{5.9}$$

Proof. The first equation follows from a direct adaptation of Lemma 38 to the context of graphs. The remaining equations follow by adapting the proof of Lemma 39 form maps to graphs. We briefly indicate the differences.

In the equation for L we must take into account that graphs are not embedded, hence the division by two, and also that the double loop is not admissible as a network. In the equation for S the only difference with Lemma 39 is that the family $\mathcal{S}_1$ is no longer needed. Similar considerations apply to the equation for P. The equation for F follows directly from Lemma 40.

The equation for $\mathcal{S}_2$ describes the decomposition of a network in $\mathcal{S}_2$. It is essentially a series composition of two networks, one in $\mathcal{F}$ and one in $\mathcal{F} \setminus \mathcal{S}_2$, in which the second pole of the first one is identified with the first pole of the second one, hence the division by x. In addition we have the cases were one of the two networks in the series composition is a *fat polygon*, that is, a cycle in which each edge is doubled, and every double edge is replaced with one or two networks in $\mathcal{D}$. Notice that contrary to double edges of networks counted in $T^{(1)}$ and $T^{(2)}$, double edges of the fat polygon are not replaced with networks in $\mathcal{F}$, as this would create a series composition with two networks in $\mathcal{S}_2$.

The last two equations are similar to that for H in Lemma 39 with the difference that double edges must be replaced with either: a network in $\mathcal{D}$; two networks in $\mathcal{D}$, encoded by $D^2/2$; or a network in $\mathcal{F}$ for which the two poles were removed, encoded by F/x^2. □

Proof of Theorem 30. Equations (5.9) can be rewritten as a system with non-negative coefficients using the identities $D - L = S + P + H_1 + F$, $D - S = L + P + H_1 + F$, and $F - S_2 = H_2$. Hence it has a unique solution with non-negative terms, which can be computed by iteration from the knowledge of $T^{(1)}$ and $T^{(2)}$. Since all the functions involved are algebraic, the solution consists of algebraic functions.

Let $C(x)$ now be the generating function of labelled 4-regular planar graphs. There is a simple relation between $C(x)$ and the series $D(x)$ of networks, namely

$$4xC'(x) = D(x) - L(x) - L(x)^2 - F(x) - \frac{x^2}{2}D(x)^2.$$

The series on the left corresponds to labelled graphs with a distinguished vertex v in which one of the 4 edges incident with v is selected. These correspond precisely to networks, except for the fact that since we are counting simple graphs we have to remove from $D(x)$ networks containing either loops or double edges, which correspond to the terms subtracted.

Finally, since D, L and F are algebraic functions, so is $C'(x)$. □

5.5 Simple 4-regular maps

The enumeration of *simple* 4-regular maps is obtained by adapting the arguments in the previous section to maps instead of graphs. We define the various classes of simple maps exactly as we did for networks, keeping the same notation. The decomposition scheme starts from the series $T_1(x,u,v)$ and $T_2(x,u,v)$ of 3-connected 4-regular maps, where the indices and variables have the same meaning as in the previous section, with the exception that now the rooted (double) edge of T_2 does not need to have a face of degree two on its right hand-side. As before, they are accessible from the series $T(u,v)$:

$$T_1(x,u,v) = T(xu^2, xv) \qquad \text{and} \qquad u\frac{\partial}{\partial u}T_2(x,u,v) = 2v\frac{\partial}{\partial v}T_1(x,u,v),$$

where multiplication by 2 on the right-hand side of the second equation is because a double edge can be rooted at any of its two edges (as discussed above).

Lemma 42. *The following equations hold:*

$$\begin{aligned}
D &= L + S + P + H_1 + 2F \\
L &= 2x(D - L) \\
S &= D(D - S) \\
P &= x^2(3D^2 + D^3) + 2FD \\
F &= S_2 + \frac{H_2}{2} \\
S_2 &= \frac{1}{x}(F + x^2(2D + D^2))(F + x^2(2D + D^2) - S_2) \\
H_1 &= \frac{T_1\left(x, 1 + D, 2D + D^2 + \frac{F}{x^2}\right)}{1 + D} \\
H_2 &= \frac{T_2\left(x, 1 + D, 2D + D^2 + \frac{F}{x^2}\right)}{2D + D^2 + \frac{F}{x^2}}
\end{aligned} \tag{5.10}$$

Proof. The proof is essentially the same as that of Lemma 41, with the following differences. Maps are embedded, hence there are $\binom{m}{k}$ ways to substitute

k maps in $\mathcal{D}$ for an edge of multiplicity m. This justifies the term $3D^2$ in the fourth equation and the term $2D$ in the last three equations. Finally, the fact that maps have a root face explains the factors 2 in the first, second and fourth equations. □

Proof of Theorem 33. From the knowledge of $T(u, v)$ of Section 5.3 and from Lemma 42, we can compute the coefficients of the series D, L, S, P, S_2, F, H_1 and H_2 up to any order. By removing maps having a loop or a multiple edge, the series $M(x)$ of rooted 4-regular simple maps is equal to

$$M(x) = D(x) - L(x) - L(x)^2 - 3x^2D(x)^2 - 2F(x). \tag{5.11}$$

Since D, L and F are algebraic, so is M. □

5.6 Conclusion

As an illustration we present in Table 5.1 the numbers of 4-regular planar graphs up to 24 vertices. We see that the first discrepancy is at $n = 12$. For this number of vertices there are 4-regular planar graphs which are either disconnected (the union of two octahedra) or are connected but not 3-connected (gluing two octahedra via two parallel edges). Table 5.2 and 5.3 give, respectively, the numbers of rooted 3-connected 4-regular maps, and simple 4-regular maps.

n	g_n	c_n	t_n
6	15	15	15
7	0	0	0
8	2520	2520	2520
9	30240	30240	30240
10	1315440	1315440	1315440
11	39916800	39916800	39916800
12	1606755150	1606651200	1546776000
13	66356690400	66356690400	63826963200
14	3068088823800	3067975310400	2879997120000
15	152398096250400	152395825982400	142057025510400
16	8196374895508800	8196176020032000	7534165871232000
17	472595587079616000	472586324386176000	430559631710208000
18	29138462100216869400	29137847418231552000	26287924131076608000
19	1912269800864459836800	1912231517504083776000	1710786280874711040000
20	133143916957026288112800	133141260589657512192000	118162522829227548672000
21	9803331490189678577136000	9803140616698955285760000	8635690901034837319680000
22	761176404797020723326816000	761161832514030029322240000	665819208405772061921280000
23	62162810722904469623293248000	62161644432203364801392640000	54014719048912416098304000000
24	5327113727746428410913561441000	5327015666189741660374318080000	4599666299608288403199344640000

Table 5.1: Numbers of arbitrary, connected and 3-connected labelled 4-regular planar graphs with n vertices.

$\ell\backslash k$	2	3	4	5	6	7	8	9	10	11	12
0					1		4	6	29	88	310
1					12	28	128	396	1460	5148	18696
2	2	6	16	40	156	546	2192	8316	32380	125510	489708
3		8	56	260	1152	4900	21344	92160	397960	1708300	7303040
4			46	510	3630	21350	115440	593622	2959160	14407250	68862960
5				312	4920	46508	347984	2282544	13791064	78760836	431601120
6					2388	48860	579736	5267640	40819100	284712736	1843137520
7						19728	498352	7123464	76274560	683057672	5415222384
8							172374	5190462	86891050	1072179834	10906813890
9								1571096	54988280	1055746780	14758457040
10									14800940	590784084	12801068400
11										143190896	6422227344
12											1415859276

Table 5.2: Coefficients of $T(u,v) = \sum t_{k,\ell} u^k v^\ell$: $t_{k,\ell}$ is the number of rooted 3-connected 4-regular maps with ℓ double edges and $2k$ simple edges, in which the root edge is simple.

n	$t_{n,0}$	M_n
6	1	1
7	0	0
8	4	4
9	6	6
10	29	29
11	88	88
12	310	334
13	1066	1196
14	3700	4386
15	13036	16066
16	46092	59164
17	164628	218824
18	591259	812503
19	2137690	3028600
20	7770968	11329468
21	28396346	42527120
22	104256321	160148795
23	384446150	604932614
24	1423383358	2291617406

Table 5.3: $t_{n,0}$ is the number of simple rooted 3-connected 4-regular maps; M_n is the number of simple 4-regular maps. As for graphs, the first discrepancy is at $n = 12$. The numbers $t_{n,0}$ match those give in Table 1 from [12] given up to $n = 15$.

Chapter 6

Further researches

Let us finally conclude this thesis by mentioning some possible further researches.

Starting with the content of Chapter 3, one could adapt again adapt the network-decomposition and the tree-decomposition, inherent to the Dissymetry Theorem for tree-decomposable classes, to fully estimate the number of 2-connected cubic planar multigraphs. Note that the Dissymetry Theorem seems here necessary as, similarly to the connected case and due to the presence of loops and multiple edges, there is no known way to describe cubic planar multigraphs rooted at an edge in function of those rooted at a vertex.

Then concerning the content of Chapter 4, the problems of understanding the distribution for the number of appearances of a given subgraph, say C_4, in a random cubic planar graph remains open. It is in fact open in a general random planar graph.

Adapting the scheme developed for triangles, one can enumerate the class of triangle-free cubic planar graphs. It is done in [54] but was not reproduced due to our effort of keeping the thesis relatively short. Considering the essence of the connectivity-decomposition, this in fact naturally extends the enumeration of 3-connected triangle-free cubic planar graphs done in [27, Section 5]. With a similar method, one can also enumerate triangle-free simple cubic planar maps. It is in fact done in [22].

Similarly to [44], one could ask for the distribution of the size of the largest 2-connected components in a random cubic planar graph. We believe that the associated random variables should converge to a Airy distribution, as in [2].

And we finish with Chapter 5, where due to the complexity of the system of equations involved in the connectivity-decomposition scheme, we were unable to obtain the minimal polynomial satisfied by the generating function of 3-connected 4-regular maps with two variables. This is necessary to obtain an asymptotic estimate on the number of 4-regular planar graphs. With this, one could also start the study of random 4-regular planar graphs. We believe this to be strictly a computational problem.

Adapting the techniques developed in Chapter 5, one could also try to enumerate the family of 5-regular planar graphs. Notice that when the cores for 4-regular graphs can have double edges, a 3-connected 5-regular map can have double and triple edges. So we would need to enrich the scheme with two new variables. The starting point of the scheme for 4-regular graphs was the family of quadrangulations without separating quadrangle. The point was that, because such quadrangulation do not have 2-vertices, one could control their apparition in arbitrary quadrangulations. One should then start by designing an analogue configuration for pentagulations. A good starting point could be the work of Bernardi and Fusy in [5], where they gave a bijection to compute the generating function of pentagulations with girth five.

Zusammenfassung

Das zentrale Thema dieser Dissertation sind Familien von regulären planaren Graphen und Karten. Insbesondere sind wir an daran interessiert, diese zu zählen und die Zusammenhänge zu deren zufälligen Gegenstücken zu erforschen.

Im ersten Teil geben wir sowohl eine rekursive als auch eine asymptotische Abzählung von kubischen, planaren Graphen, Multigraphen und einfachen Karten, durch eine Dekomposition entlang deren Komponenten. Im zweiten Teil wenden wir diese Resultate auf zufällige kubische planare Graphen an. Insbesondere berechnen wir die Wahrscheinlichkeit von Zusammenhängigkeit, und beweisen das einige bedeutende Parameter normalverteilt sind: die Anzahl der cut-vertices, isthmuses, Blöcke, cherries, near-bricks und Dreiecke. Im dritten und letzten Teil entwickeln wir das erste kombinatorisches Schema, basierend auf einem Dekompositionsschema das ähnlich zu dem im Kontext von kubischen planaren Graphen ist, das zur rekursiven Abzählung von 4-regulären planaren Graphen und einfachen Karten führt.

Selbständigkeitserklärung

Gemäß §7(4) der Promotionsordnung des Fachbereichs Mathematik und Informatik der Freien Universität Berlin versichere ich hiermit, dass ich alle Hilfsmittel und Hilfen angegeben und auf dieser Grundlage die Arbeit selbständig verfasst habe. Des Weiteren versichere ich, dass ich diese Arbeit nicht schon einmal zu einem früheren Promotionsverfahren eingereicht habe.

Berlin, den

Clément Requilé

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