



Composite fermions and their pair states in a strongly-coupled Fermi liquid

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Abstract

Our goal is to understand the phenomena arising in optical lattice fermions at low temperature in an external magnetic field. Varying the field, the attraction between any two fermions can be made arbitrarily strong, where composite bosons form via so-called Feshbach resonances. By setting up strong-coupling equations for fermions, we find that in spatial dimension $d > 2$ they couple to bosons which dress up fermions and lead to new massive composite fermions. At low enough temperature, we obtain the critical temperature at which composite bosons undergo the Bose–Einstein condensate (BEC), leading to BEC-dressing massive fermions. These form tightly bound pair states which are new bosonic quasi-particles producing a BEC-type condensate. A quantum critical point is found and the formation of condensates of complex quasi-particles is speculated over.

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1. Introduction

The attraction between any two fermions can be tuned, as a function of an external magnetic field, and be made so strong that the coupling constant reaches the unitarity limit of infinite s -wave scattering length “ a ” via a Feshbach resonance. At that point, a smooth BCS–BEC

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crossover takes place, the Cooper pairs which form in the weak-coupling limit at low temperature and make the system a BCS superconductor, become so strongly bound that they behave like bosonic quasi-particles with a pseudogap at high temperature T^* , and form a new type of BEC at the critical temperature T_c . The recent article [1] reviews the successful progresses of BCS–BEC theories and experiments of dilute Fermi gases, whose thermodynamics can be expressed as scaling functions of a and T , independently of all microscopic details. In this letter, as opposed to dilute Fermi gases, we study strongly interacting fermions in an optical lattice for ongoing experiments [2] and yet completely understood theoretical issues, such as quasi-particle spectra, phase structure and critical phenomena, as well as thermodynamical and transport properties, which can be very different from that of better-studied dilute Fermi gases. We use the approach of strong-coupling expansion to find the massive spectra of not only composite bosons but also composite fermions, and obtain the critical line and phase diagram in the strong-coupling region. Some preliminary discussions are presented on the relevance of our results to experiments.

2. Lattice fermions

We consider fermions in an underlying lattice with a spacing ℓ . In order to address strong-coupling fermions at finite temperature T , we incorporate the relevant s -wave scattering physics via a “ ℓ_0 -range” contact potential in the Hamiltonian for spinor wave function $\psi_{\uparrow,\downarrow}(i)$, which represents a fermionic neutral atom of fermion number “ e ” that we call “charge”, and $\psi_{\uparrow,\downarrow}^\dagger(i)$ represents its “hole” state “ $-e$ ”,

$$\beta\mathcal{H} = \beta \sum_{i,\sigma=\uparrow,\downarrow} (\ell^d)\psi_\sigma^\dagger(i) \left[-\nabla^2/(2m\ell^2) - \mu \right] \psi_\sigma(i) - g\beta \sum_i (\ell^d)\psi_\uparrow^\dagger(i)\psi_\downarrow^\dagger(i)\psi_\downarrow(i)\psi_\uparrow(i), \tag{1}$$

$\beta = 1/T$, each fermion field $\psi_\sigma(i)$ of length dimension $[\ell^{-d/2}]$, mass m and chemical potential μ is defined at a lattice site “ i ”. The index “ i ” runs over all lattice sites. The Laplace operator ∇^2 is defined as ($\hbar = 1$)

$$\nabla^2\psi_\sigma(i) \equiv \sum_{\hat{\ell}} \left[\psi_\sigma(i + \hat{\ell}) + \psi_\sigma(i - \hat{\ell}) - 2\psi_\sigma(i) \right], \tag{2}$$

where $\hat{\ell} = 1, \dots, d$ indicate the orientations of lattice spacing to the nearest neighbors. Tuned by optical lattice and magnetic field, the s -wave attraction between the up- and down-spins is characterized by a bare coupling constant $g(\ell_0) > 0$ of length dimension $[\ell^{d-1}]$ and the range $\ell_0 < \ell$.

Inspired by strong-coupling quantum field theories [3,4], we calculate the two-point Green functions of composite boson and fermion fields to effectively diagonalize the Hamiltonian into the bilinear form of these composite fields, and find the composite-particle spectra in the strong-coupling phase.

3. Composite bosons

We first consider a composite bosonic field $\mathcal{C}(i) = \psi_\downarrow(i)\psi_\uparrow(i)$ and study its two-point function on a lattice [4],

$$G(i) = \langle \psi_{\downarrow}(0)\psi_{\uparrow}(\ell), \psi_{\uparrow}^{\dagger}(i)\psi_{\downarrow}^{\dagger}(i) \rangle = \langle C(0), C^{\dagger}(i) \rangle. \tag{3}$$

We find (methods) that in the strong-coupling effective Hamiltonian, $C = \psi_{\downarrow}\psi_{\uparrow}$ represents a massive composite boson with propagator

$$gG(q) = \frac{g [2m/(\beta\ell)]^2}{4\ell^{-2} \sum_{\hat{\ell}} \sin^2(q\hat{\ell}/2) + M_B^2} \tag{4}$$

with pole of mass M_B and residue of form factor gR_B^2 :

$$M_B^2 = \left[g(2m)^2(\ell/\beta) - 2d \right] \ell^{-2} > 0, \quad R_B^2 = (2m/\beta\ell)^2. \tag{5}$$

From Eq. (4), the effective Hamiltonian of the composite boson field C on a lattice can be written as

$$\mathcal{H}_{\text{eff}}^B = \sum_i (\ell^3) Z_B^{-1} C^{\dagger}(i) \left[-\nabla^2/(2M_B\ell^2) - \mu_B \right] C(i). \tag{6}$$

The chemical potential is $\mu_B = -M_B/2$ and the wave function renormalization is $Z_B = gR_B^2/2M_B$. Provided Z_B is finite, we renormalize the fermion field and the composite boson field as

$$\psi \rightarrow (gR_B^2)^{-1/4} \psi, \quad \text{and} \quad C \rightarrow (2M_B)^{1/2} C, \tag{7}$$

so that the composite boson $C(i)$ behaves like a quasi-particle. This is a pair in a tightly bound state on a lattice, analogous to the Feshbach resonance at the unitarity limit of continuum theory, and contrary to the loosely-bound Cooper pair in the weak-coupling region.

Analogously to $C(i)$, we consider the composite field of fermion and hole, i.e., the plasmon field $\mathcal{P}(i) = \psi_{\downarrow}^{\dagger}(i)\psi_{\uparrow}(i)$. We perform a similar calculation to the two-point Green function $G_{\mathcal{P}}(i) = \langle \mathcal{P}(0), \mathcal{P}^{\dagger}(i) \rangle$, and obtain the same result as (4) and (5), indicating a tightly bound state of plasmon field, whose Hamiltonian is (6) with $C(i) \rightarrow \mathcal{P}(i)$. This is not surprised since the pair field $C(i)$ and the plasmon field $\mathcal{P}(i)$ fields are symmetric in the strong-interacting Hamiltonian (1). However, the charged pair field $C(i)$ and neutral plasmon $\mathcal{P}(i)$ field can be different up to a relative phase of field $\theta(i)$. We select the relative phase field as such that $\langle |\mathcal{P}(i)| \rangle = \langle |C(i)| \rangle$. We also obtain the identically vanishing two-point Green function $\langle \mathcal{P}(0), C^{\dagger}(i) \rangle$, as $C(i)$ is charged ($2e$) and $\mathcal{P}(i)$ is neutral ($e - e = 0$).

The bound states C are composed of two constituent fermions $\psi_{\downarrow}(k_1)$ and $\psi_{\uparrow}(k_2)$ around the Fermi surface. Assuming $k_1 \approx k_2 \approx k_F$ and $k_2 - k_1 = q \ll k_F \sim \ell^{-1}$, Eq. (4) becomes

$$gG(q) = \frac{gR_B^2/(2M_B)}{(q^2/2M_B) + M_B/2} \Rightarrow \frac{gR_B^2}{q^2 + M_B^2}, \quad (q\ell \ll 1), \tag{8}$$

where “ q ” indicates filled levels around the Fermi surface that are involved in paring.

In Eq. (6) for $M_B^2 > 0$, the wave-function renormalization $Z_B = gR_B^2/2M_B \propto gT^2$ relates to the bound-state size ξ_{boson} . As effective coupling gT^2 decreases, Z_B decreases and ξ_{boson} increases. The number of paring-involved states around Fermi surface $q \sim \xi_{\text{boson}}^{-1}$ decreases. The lattice pair field $C(i)$ turns to describe Feshbach resonance at the unitarity limit of continuum theory, then to describe loose Cooper pair in the weak-coupling region. The vanishing form factor indicates that the bosonic bound state pole dissolves into two fermionic constituent cut, i.e., two unpaired fermions, as discussed in Refs. [5,6]. This qualitatively shows the feature of smooth cross-over transition from tightly bound pair field $C(i)$ first to Feshbach resonance, then

to Cooper pair, and then to unpaired fermions. The transition to a normal Fermi liquid of unpaired fermions takes place at the dissociation scale of pseudogap temperature $T^*(g)$ that we shall quantitatively study in future.

4. Phase transition

On the other hand, as the effective coupling gT^2 varies, the mass term $M_B^2 \mathcal{C}\mathcal{C}^\dagger$ in Eqs. (4) and (5) possibly changes its sign from $M_B^2 > 0$ to $M_B^2 < 0$ and the pole M_B becomes imaginary, implying the second-order phase transition from the symmetric phase to the condensate phase [4], where the nonzero condensation of pairing field $\langle \mathcal{C}(i) \rangle \neq 0$ is developed and will be duly discussed below. $M_B^2 = 0$ gives the critical line:

$$m^2 g_c T_c = d/(2\ell), \tag{9}$$

where g_c is the critical value of bare coupling $g(\ell_0)$ defined at the short-distance scale $\ell_0 < \ell$, and T_c is the critical temperature. Note that this result is qualitatively consistent with the strong-coupling behavior $T_c \sim t^2/U$ in the attractive Hubbard model [7].

In order to discuss the critical behaviors of the second-order phase transition, we focus our attention on the neighborhood (scaling domain) of the critical line (9), where the characteristic correlation length $\xi \sim M_B^{-1}$ is much larger than the lattice spacing ℓ , thus microscopic details of lattice are physically irrelevant. Therefore, in the scaling domain of critical line (9) we approximately treat the lattice field theory (1) and (2) as a continuum field theory describing Grand Canonical Ensembles at finite temperature [8,9]. In this framework, we obtain the relation between the critical “bare” coupling g_c in Eq. (9) and the “renormalized” coupling described by the s -wave scattering length a via the two-particle Schrödinger equation at critical temperature T_c ,

$$\frac{m}{4\pi a} = -\frac{1}{g_c(\Lambda)} + \frac{T_c}{V} \sum_{\omega_n, |\mathbf{k}| < \Lambda} \frac{1}{\omega_n^2 + \epsilon_{\mathbf{k}}^2}, \quad \Lambda = \pi \ell_0^{-1}. \tag{10}$$

Here continuum spectrum $\epsilon_{\mathbf{k}} = |\mathbf{k}|^2/2m$ denotes the energy of free fermions and $\sum_{\omega_n, |\mathbf{k}| < \Lambda}$ contains the phase-space integral and the sum over the Matsubara frequencies $\omega_n = 2\pi T n$, obtained in Eq. (7A.3) of textbook [10]. This can be written in d dimensions as

$$\frac{4\pi a}{m g_c(\Lambda)} = \frac{a k_F}{4\pi b} \mathcal{S}_d(T_c) - 1. \tag{11}$$

In $d = 3$ dimensions, we approximately adopt the half-filling fermion density $n \approx 1/\ell^3 \approx k_F^3/3\pi^2$, Fermi momentum $k_F \approx (3\pi^2)^{1/3}/\ell$, and Fermi energy $\epsilon_F = k_F^2/2m$. Moreover we introduce the dimensionless and optical lattice tunable parameter $b = 2^{-1}(3\pi^2)^{1/3}\ell_0/\ell < 1$, that measures the fraction of filled levels around ϵ_F , contributing to the pairing, and find $\mathcal{S}_3(T_c) \equiv \int_0^1 dt \tanh\left[\frac{\epsilon_F}{T_c} \frac{\pi^2 t^2}{8b^2}\right]$, with $\mathcal{S}_3(0) = 1$. Equations (10) and (11) reduce properly to the well-known $a - g(\Lambda)$ relation in the continuum theory at zero temperature. Nevertheless, it is worthwhile in future study to use the lattice version of Eq. (10), where the continuum spectrum $\epsilon_{\mathbf{k}}$ is replaced by the lattice spectrum (2) of free fermions and the phase-space summation $\sum_{\omega_n, |\mathbf{k}|}$ is over entire Brillouin zone. We expect a numerical modification of the phase-space function $\mathcal{S}_d(T_c)$ in Eq. (11), and it does not qualitatively change the critical behavior discussed below, since the critical line (9) is obtained from lattice calculations.

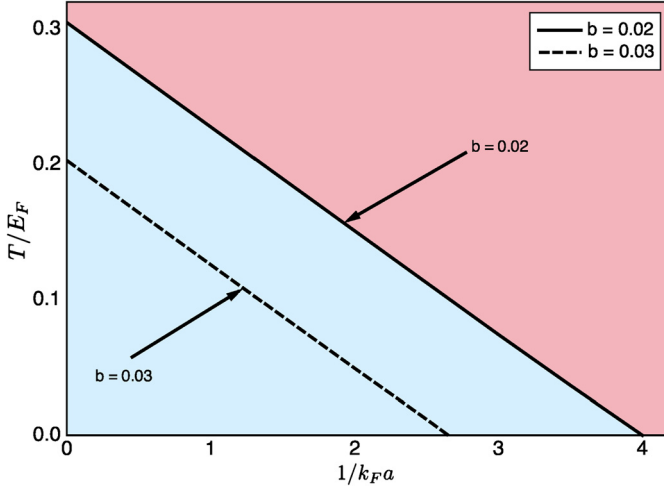


Fig. 1. The phase diagram in strong coupling region. Transition temperature T_c/ϵ_F is plotted as a function of $1/k_F a \geq 0$ for the selected parameters $b = 0.02, 0.03$. The quantum critical point is shown to lie at one of the corresponding zeros $(1/k_F a)_{T_c=0} = 4.0, 2.7$. Above the critical line is a normal liquid consisting of massive composite bosons and fermions. Below the critical line lies a superfluid phase with a new type of BEC involving composite massive fermions.

From the critical line (9) and approximate $a - g_c(\Lambda)$ relation (11) at critical temperature T_c , we find for large $1/ak_F$ or $g_c(\Lambda)$,

$$T_c = T_c^u(T_c) \left[1 - \frac{4\pi b}{S_d(T_c)} \frac{1}{ak_F} \right], \quad (12)$$

where $T_c^u(T_c)/\epsilon_F \equiv (3\pi^2)^{-1/d} dS_d(T_c)/(4\pi)^2 b$ and $T_c^u = T_c^u(T_c^u)$ is the critical temperature at $1/ak_F = 0$. In the superfluid phase ($T < T_c$), the boson field $\mathcal{C}(i)$ develops a nonzero expectation value $\langle \mathcal{C}(i) \rangle$ and undergoes BEC. To illustrate the critical line (12) separating two phases, in Fig. 1, we plot numerical results of (12) in $d = 3$ for the parameters $b = 0.02, 0.03$ corresponding to the ratios $\ell_0/\ell = 0.013, 0.02$. The “linear” critical line in Fig. 1 is due to the weak T_c -dependence in the highly nonlinear relation (12).

Our strong-coupling result shows a decreasing critical temperature T_c for large g_c or $1/ak_F$, where the length a and size ξ_{boson} can approach the lattice spacing ℓ and become even smaller. Taking the limit $g_c \rightarrow \infty$ at constant $T_c g_c$, we find $T_c \rightarrow 0$, implying a *quantum critical point*. This happens at $1/ak_F \rightarrow (1/ak_F)_{\text{qc}} \equiv S_d(0)/4\pi b = 1/4\pi b$, where composite particles are most tightly bound states, locating at the lowest energy level of the “ ℓ_0 -range” contact potential with $a = 2\pi\ell_0$, their thermal fluctuations are negligible. The figure shows that for smaller b , more fermions in the Fermi sphere are involved in the pairing, resulting in a larger composite-boson density and a higher T_c .

Our results (Fig. 1) are valid only in the very strong-coupling region $g_c \gg 1$, i.e., $0 < 1/ak_F < (1/ak_F)_{\text{qc}}$, since our approach of strong-coupling expansion to interacting lattice fermions is particularly appropriate in this region. Nevertheless, we extrapolate the critical line in Fig. 1 to the critical temperature $T_c^u/\epsilon_F \approx 0.31, 0.2$ at the unitarity limit $1/ak_F = 0$, so as to compare with and contrast to the results obtained by continuum theory for dilute Fermi gas. The experimental value $T_c^u/\epsilon_F \approx 0.167$ of dilute Fermi gas [11] can be achieved for $b \approx 0.037$. However, such a simple extrapolation is not expected to be quantitatively correct, and high-order strong-coupling

calculations ($\mathcal{O}(1/g^n)$, $n > 1$) and proper summation over “ n ” are needed, since the critical coupling g_c is not very large in the neighborhood of the unitarity limit $1/ak_F = 0$. In fact, for $1/ak_F \gtrsim 0$ our result of monotonically decreasing T_c/ϵ_F , as shown in Fig. 1, is in contrast not only with the constancy BEC limit $T_c/\epsilon_F = 0.218$ obtained by considering mean-field value and Gaussian fluctuation in the normal states of Fermi gas [12], but also with non-monotonic T_c/ϵ_F obtained by Quantum Monte Carlo simulations [13].

However, we have to stress the validity of our results of monotonically decreasing T_c/ϵ_F that leads to a quantum critical point in very strong-coupling region ($0 < 1/ak_F < (1/ak_F)_{qc}$), i.e., beyond the unitarity limit $1/ak_F = 0$. The reasons are that these results are obtained by considering the strong-coupling limit ($g \rightarrow \infty$) of interacting lattice fermions, calculating the first order correction $\mathcal{O}(1/g)$ and nontrivial recursion relation (see methods). Our calculations take into account contributions from strong-correlating collective modes of lattice fermions at the strong-coupling limit. Therefore, our approach is very different from the approach of considering Gaussian fluctuations of normal states of continuum fermion gas upon the mean-field limit, which is not valid in the strong-coupling limit. It is deserved to use Quantum Monte Carlo simulations to study the lattice field theory (1) and (2) in such a very strong-coupling region.

To end this section, we make the following speculations on the phases beyond the quantum critical point $(1/ak_F)_{qc}$. The quantum phase transition undergoes from the phase $(1/ak_F) < (1/ak_F)_{qc}$ to the phase $(1/ak_F) > (1/ak_F)_{qc}$, and the latter possibly involves the formation and condensation of more complex composite quasi-particles of higher spin-angular momentum pairing, e.g., spin triplet $\mathcal{C}^{tri} \equiv (\psi_\uparrow\psi_\uparrow, \psi_\uparrow\psi_\downarrow, \psi_\downarrow\psi_\downarrow)$ with mass gap $M_B^{tri}(T)$ and phase factor $\mathcal{S}_d^{tri}(T)$, and the bosonic triplet \mathcal{C}^{tri} dresses up a fermion to form a three-fermion state discussed below. The critical line $T_c^{tri}(1/ak_F)$ obtained from $M_B^{tri}(T_c^{tri}) = 0$ separates the quasi-particle \mathcal{C}^{tri} formation from their condensate phases. This critical line $T_c^{tri}(1/ak_F)$ starts from the quantum critical point $T_c^{tri} = 0$ and $(1/ak_F) = (1/ak_F)_{qc}$, and increases as $(1/ak_F)$ increases (g decreases) for $(1/ak_F) > (1/ak_F)_{qc}$. The first-order phase transition should occur at the quantum critical point, which should be an IR fixed point. All these aspects will be studied in future.

5. Composite fermions

In this section, we show that the strong-coupling attraction between fermions forms not only composite bosons but also composite fermions. To exhibit the presence of composite fermions, using the paring field $\mathcal{C}(i)$ we calculate the two-point Green functions [4]:

$$S_{LL}(i) \equiv \langle \psi_\uparrow(0), \psi_\uparrow^\dagger(i) \rangle, \tag{13}$$

$$S_{ML}(i) \equiv \langle \psi_\uparrow(0), \mathcal{C}^\dagger(i)\psi_\downarrow(i) \rangle, \tag{14}$$

$$S_{ML}^\dagger(i) \equiv \langle \psi_\downarrow^\dagger(0)\mathcal{C}(0), \psi_\uparrow^\dagger(i) \rangle \tag{15}$$

$$S_{MM}(i) \equiv \langle \psi_\downarrow^\dagger(0)\mathcal{C}(0), \mathcal{C}^\dagger(i)\psi_\downarrow(i) \rangle. \tag{16}$$

We find (methods) that in the strong-coupling effective Hamiltonian, the propagator

$$S_{\text{Fermion}}(p) = \frac{2}{4\ell^{-2} \sum_{\hat{\ell}} \sin^2(p\hat{\ell}/2) + M_F^2}, \tag{17}$$

represents a composite fermion that is the superposition of the fermion ψ_\uparrow and the three-fermion state $\mathcal{C}(i)\psi_\downarrow^\dagger(i)$ [4],

$$\begin{aligned} \Psi_{\uparrow}(i) &= R_B^{-1/2} \psi_{\uparrow}(i) + R_B^{-3/2} \mathcal{C}(i) \psi_{\downarrow}^{\dagger}(i) \\ &\Rightarrow g^{1/4} \psi_{\uparrow}(i) + g^{3/4} \mathcal{C}(i) \psi_{\downarrow}^{\dagger}(i), \end{aligned} \tag{18}$$

where the three-fermion state $\mathcal{C}(i) \psi_{\downarrow}^{\dagger}(i)$ is made of a hole $\psi_{\downarrow}^{\dagger}(i)$ “dressed” by a cloud of composite bosons $\mathcal{C}(i)$. The associated two-point Green function reads

$$\begin{aligned} \langle \Psi_{\uparrow}(0), \Psi_{\uparrow}^{\dagger}(i) \rangle &= \langle \psi_{\uparrow}(0), \psi_{\uparrow}^{\dagger}(i) \rangle + \langle \psi_{\uparrow}(0), \mathcal{C}^{\dagger}(i) \psi_{\downarrow}(i) \rangle \\ &\quad + \langle \mathcal{C}(0) \psi_{\downarrow}^{\dagger}(0), \psi_{\uparrow}^{\dagger}(i) \rangle + \langle \mathcal{C}(0) \psi_{\downarrow}^{\dagger}(0), \mathcal{C}^{\dagger}(i) \psi_{\downarrow}(i) \rangle, \end{aligned} \tag{19}$$

whose momentum transformation satisfies (36). A similar result holds for the spin-down composite fermion $\Psi_{\downarrow}(i) = R_B^{-1/2} \psi_{\downarrow}(i) + R_B^{-3/2} \mathcal{C}(i) \psi_{\uparrow}^{\dagger}(i)$. They can be represented in the effective Hamiltonian

$$\mathcal{H}_{\text{eff}}^F = \sum_{i,\sigma=\uparrow\downarrow} (\ell^3) Z_F^{-1} \Psi_{\sigma}^{\dagger}(i) \left[-\nabla^2 / (2M_F \ell^2) - \mu_F \right] \Psi_{\sigma}(i). \tag{20}$$

Here $\mu_F = -M_F/2$ is the chemical potential and $Z_F = g/M_F$ the wave-function renormalization. Following the renormalization (7) of fermion fields, we renormalize composite fermion field $\Psi_{\uparrow,\downarrow} \Rightarrow (Z_F)^{-1/2} \Psi_{\uparrow,\downarrow}$, which behaves as a quasi-particle in Eq. (20), analogously to the composite boson (6). The negatively charged (e) three-fermion state is a negatively charged ($2e$) paring field $\mathcal{C}(i) = \psi_{\downarrow}(i) \psi_{\uparrow}(i)$ of two fermions combining with a hole $\psi_{\downarrow}^{\dagger}(i)$. These negatively charged (e) composite fermions $\Psi_{\uparrow\downarrow}(i)$ are composed of three-fermion states $\mathcal{C} \psi_{\uparrow}^{\dagger}$ or $\mathcal{C} \psi_{\downarrow}^{\dagger}$ and a fermion ψ_{\uparrow} or ψ_{\downarrow} . Similarly, positively charged ($-e$) composite fermions $\Psi_{\uparrow}^{\dagger}(i)$ or $\Psi_{\downarrow}^{\dagger}(i)$ are composed by three-fermion states $\mathcal{C}^{\dagger} \psi_{\uparrow}$ or $\mathcal{C}^{\dagger} \psi_{\downarrow}$ combined with a hole $\psi_{\uparrow}^{\dagger}$ or $\psi_{\downarrow}^{\dagger}$. Suppose that two constituent fermions $\psi_{\downarrow}(k_1)$ and $\psi_{\uparrow}(k_2)$ of the paring field $\mathcal{C}(q)$, one constituent hole $\psi_{\uparrow}^{\dagger}(k_3)$ are around the Fermi surface, $k_1 \approx k_2 \approx k_3 \approx k_F$, then the paring field $\mathcal{C}(q)$ for $q = k_2 - k_1 \ll k_F$, and three-fermion state $p = k_1 - k_2 + k_3 \approx k_3 \approx k_F$ is around the Fermi surface. As a result, the composite fermions $\Psi_{\uparrow\downarrow}$ live around the Fermi surface as well. Suppose that the three-fermion state $p = k_1 - k_2 + k_3 \approx k_3 \ll k_F$, the composite fermion propagator (17) reduces to its continuum version $S_{\text{Fermion}}(p) \sim (p^2 + M_F^2)^{-1}$.

As discussed below Eq. (8), in the regime of small coupling g , the form factor Z_B of composite boson vanishes and the bosonic bound state pole dissolves into two fermionic constituent cut. Analogously, the form factor Z_F of composite fermion vanishes and the three-fermion bound state pole dissolves into three fermionic constituent cut [6], i.e., three unpaired fermions. The smooth cross-over transition takes place.

The same results (13)–(20) are obtained for the plasmon field $\mathcal{P}(i) = \psi_{\downarrow}^{\dagger}(i) \psi_{\uparrow}(i)$ combined with another fermion or hole, and the associated composite fermion

$$\begin{aligned} \Psi_{\uparrow}^{\mathcal{P}}(i) &= R_B^{-1/2} \psi_{\uparrow}(i) + R_B^{-3/2} \mathcal{P}(i) \psi_{\downarrow}(i) \\ &\Rightarrow g^{1/4} \psi_{\uparrow}(i) + g^{3/4} \mathcal{P}(i) \psi_{\downarrow}(i), \end{aligned} \tag{21}$$

whose two-point Green function,

$$\begin{aligned} \langle \Psi_{\uparrow}^{\mathcal{P}}(0), \Psi_{\uparrow}^{\mathcal{P}\dagger}(i) \rangle &= \langle \psi_{\uparrow}(0), \psi_{\uparrow}^{\dagger}(i) \rangle + \langle \psi_{\uparrow}(0), \mathcal{P}^{\dagger} \psi_{\downarrow}^{\dagger}(i) \rangle \\ &\quad + \langle \mathcal{P} \psi_{\downarrow}(0), \psi_{\uparrow}^{\dagger}(i) \rangle + \langle \mathcal{P} \psi_{\downarrow}(0), \mathcal{P}^{\dagger} \psi_{\downarrow}^{\dagger}(i) \rangle. \end{aligned} \tag{22}$$

The same is for $\Psi_{\downarrow}^{\mathcal{P}}(i) = R_B^{-1/2}\psi_{\downarrow}(i) + R_B^{-3/2}\mathcal{P}(i)\psi_{\uparrow}(i)$ the spin-down field. They can be represented in the effective Hamiltonian (20) with $\Psi_{\sigma}(i) \rightarrow \Psi_{\sigma}^{\mathcal{P}}(i)$, following the renormalization (7) of fermion fields, and renormalization $\Psi_{\uparrow,\downarrow}^{\mathcal{P}} \Rightarrow (Z_F)^{-1/2}\Psi_{\uparrow,\downarrow}^{\mathcal{P}}$. The charged three-fermion states $\mathcal{P}\psi_{\uparrow\downarrow}$ or $\mathcal{P}^{\dagger}\psi_{\uparrow\downarrow}^{\dagger}$ are composed of a fermion or a hole combined with a neutral plasmon field $\mathcal{P}(i) = \psi_{\downarrow}^{\dagger}(i)\psi_{\uparrow}(i)$ or $\mathcal{P}^{\dagger}(i) = \psi_{\uparrow}^{\dagger}(i)\psi_{\downarrow}(i)$ of a fermion and a hole. The composite fermions $\Psi_{\uparrow,\downarrow}^{\mathcal{P}}(i)$ are composed of a three-fermion state $\mathcal{P}\psi_{\uparrow,\downarrow}$ in combination with a fermion ψ_{\uparrow} or ψ_{\downarrow} . The same thing is true for its charge-conjugate state. Suppose that constituent fermion $\psi_{\downarrow}(k_1)$ and hole $\psi_{\downarrow}^{\dagger}(k_2)$, another constituent fermion $\psi_{\uparrow}(k_3)$ are all around the Fermi surface, $k_1 \approx k_2 \approx k_3 \approx k_F$, and the plasmon field $q = k_2 - k_1 \ll k_F$ and composite fermion $p = k_1 - k_2 + k_3 \approx k_3 \approx k_F$ is around the Fermi surface as well. The three-fermion states in Eqs. (18) and (21) are related, $\mathcal{C}(i)\psi_{\downarrow}^{\dagger}(i) = -\mathcal{P}(i)\psi_{\downarrow}(i)$. This implies that the three-fermion states $\mathcal{C}(i)\psi_{\downarrow}^{\dagger}(i)$ and $\mathcal{P}(i)\psi_{\downarrow}(i)$ are the same up to a definite phase factor $e^{i\pi}$. Thus the composite fermions $\Psi_{\sigma}(i)$ (18) and $\Psi_{\sigma}^{\mathcal{P}}(i)$ (21) are indistinguishable up to a definite phase factor.

6. Conclusion and remarks

We present some discussions of the critical temperature (12), effective Hamiltonians (6) and (20). (i) In the regime $T^* > T > T_c$, there is a mixed liquid of composite bosons and fermions with the pseudogap $M_{F,B}(T)$, which is expected to dissolve to normal unpaired Fermi gas at the crossover temperature T^* . The non-interacting composite bosons and fermions are either charged or neutral, obeying Bose–Einstein or Fermi–Dirac distribution. They behave as superfluids up to a relatively high crossover temperature T^* . (ii) In the regime $T_c > T$, composite bosons develop a nonzero $\langle \mathcal{C} \rangle$ and undergo BEC, the ground state contains the BEC-dressing fermions $\Psi_{\uparrow}^{\mathcal{B}}$ and $\Psi_{\downarrow}^{\mathcal{B}}$, which are composite fermions of Eqs. (18)–(20) by substituting composite boson \mathcal{C} for its expectation value $\langle \mathcal{C} \rangle$. These massive fermionic quasi-particles pair tightly to new bosonic quasi-particles $\Phi_{\mathcal{B}} = \Psi_{\uparrow}^{\mathcal{B}}\Psi_{\downarrow}^{\mathcal{B}}$ or $\Psi_{\uparrow}^{\mathcal{B}\dagger}\Psi_{\downarrow}^{\mathcal{B}}$, which undergo a BEC condensate $\langle \Phi_{\mathcal{B}} \rangle \neq 0$ to minimize the ground-state energy. This may be the origin of high- T_c superconductivity and a similar form of composite superfluidity. If there is a smooth crossover transition from BSC to BEC, these features, although discussed for $1/ak_F > 0$, are expected to be also true in $1/ak_F \ll 0$ with much smaller scale $M_{F,B}(T)$. Due to the presence of composite fermions in addition to composite bosons, we expect a further suppression of the low-energy spectral weight for single-particle excitations and the material follows harder equation of state. Its observable consequences include a further T -dependent suppression of heat capacity and gap-like dispersion in the density-of-states and spin susceptibility. The measurements of entropy per site [2], vortex-number [14], and shift in dipole oscillation frequency [15] can be probes into composite-boson and -fermion spectra, and pairing mechanism.

It is known that the limit $1/ak_F \ll 0$ produces an IR-stable fixed point, and its scaling domain is described by an effective Hamiltonian of BCS physics with the gap scale $\Delta_0 = \Delta(T_c)$ in $T \sim T_c \lesssim T^*$. This is analogous to the IR-stable fixed point and scaling domain of an effective Lagrangian of Standard Model (SM) with the electroweak scale in particle physics [16,17].

The unitarity limit $1/ak_F \rightarrow 0^{\pm}$ representing a scale invariant point [18] was formulated in a renormalization group framework [19], implying a UV-stable fixed point of large coupling, where the running coupling g approaches g_{UV} , as the energy scale becomes large. In the scaling domain of this UV-fixed point $1/ak_F \rightarrow 0^{\pm}$ and $T \rightarrow T_c^u$, an effective Hamiltonian of composite bosons and fermions is realized with characteristic scale

$$M_{B,F}(T) = \left(\frac{T - T_c^u}{T_c^u} \right)^{\nu/2} \frac{(2d)^{1/2}}{(3\pi^2)^{1/d}} k_F, \quad T \gtrsim T_c^u \quad (23)$$

with a critical exponent $\nu = 1$ [20]. This shows the behavior of the correlation length $\xi \propto M_{B,F}^{-1}$, and characterizes the size of composite particles via the wave function renormalization factor $Z_{B,F} \propto M_{B,F}^{-1}$. This domain should be better explored experimentally. The analogy was discussed in particle physics with anticipations of the UV scaling domain at TeV scales and effective Lagrangian of composite particles made by SM elementary fermions [21].

7. Methods

Strong-coupling limit and expansion. To calculate the expansion in strong-coupling limit, we relabel $\beta\ell^d \rightarrow \beta$ and $2m\ell^2 \rightarrow 2m$, so the lattice spacing ℓ is set equal to unity, and rescales $\psi_\sigma(i) \rightarrow (\beta g)^{1/4} \psi_\sigma(i)$ and $\psi_\sigma^\dagger(i) \rightarrow (\beta g)^{1/4} \psi_\sigma^\dagger(i)$, so that the Hamiltonian (1) can be written as $\beta\mathcal{H} = \sum_i [h\mathcal{H}_0(i) + \mathcal{H}_{\text{int}}(i)]$, where the hopping parameter $h \equiv \beta/(\beta g)^{1/2}$ and

$$\mathcal{H}_0(i) = \sum_{\sigma=\uparrow,\downarrow} \psi_\sigma^\dagger(i) (-\nabla^2/2m - \mu) \psi_\sigma(i), \quad (24)$$

$$\mathcal{H}_{\text{int}}(i) \equiv -\psi_\uparrow^\dagger(i) \psi_\downarrow^\dagger(i) \psi_\downarrow(i) \psi_\uparrow(i), \quad (25)$$

and the partition function with expectation values

$$\mathcal{Z} = \prod_{i,\sigma} \int d\psi_\sigma(i) d\psi_\sigma^\dagger(i) \exp(-\beta\mathcal{H}), \quad (26)$$

$$\langle \dots \rangle = \mathcal{Z}^{-1} \prod_{i,\sigma} \int d\psi_\sigma(i) d\psi_\sigma^\dagger(i) (\dots) \exp(-\beta\mathcal{H}). \quad (27)$$

Fermion fields ψ_\uparrow and ψ_\downarrow are one-component Grassmann variables with $\psi_\sigma(i)\psi_{\sigma'}(j) = -\psi_{\sigma'}(j)\psi_\sigma(i)$ and integrals $\int d\psi_\sigma(i)\psi_{\sigma'}(j) = \delta_{\sigma,\sigma'}\delta_{ij}$, $\int d\psi_\sigma^\dagger(i)\psi_{\sigma'}^\dagger(j) = \delta_{\sigma,\sigma'}\delta_{ij}$. All others vanish.

In the limit $h \rightarrow 0$ for $g \rightarrow \infty$ and finite T , the kinetic terms (24) are neglected, and the partition function (26) has a nonzero strong-coupling limit

$$\prod_i \int_{i\downarrow} \int_{i\uparrow} \exp(-\mathcal{H}_{\text{int}}) = -\prod_i \int_{i\downarrow} \psi_\downarrow(i)^\dagger \psi_\downarrow(i) = (1)^{\mathcal{N}}, \quad (28)$$

where \mathcal{N} is the total number of lattice sites, $\int_{i\uparrow} \equiv \int [d\psi_\uparrow^\dagger(i) d\psi_\uparrow(i)]$ and $\int_{i\downarrow} \equiv \int [d\psi_\downarrow^\dagger(i) d\psi_\downarrow(i)]$. The strong-coupling expansion can now be performed in powers of the hopping parameter h .

Green functions of composite particles. The leading strong-coupling approximation to Green function (3) is $G(i) = \delta^{(d)}(i)/\beta g$. The first correction is obtained by using the one-site partition function $Z(i)$ and the integral

$$\begin{aligned} \langle \psi_\uparrow \psi_\downarrow \rangle &\equiv \frac{1}{Z(i)} \int_{i\downarrow} \int_{i\uparrow} \psi_\uparrow(i) \psi_\downarrow(i) e^{-h\mathcal{H}_0(i) - \beta\mathcal{H}_{\text{int}}(i)} \\ &= h^2 \sum_{\hat{\ell}}^{\text{ave}} \psi_\uparrow(i; \hat{\ell}) \sum_{\hat{\ell}'} \psi_\downarrow(i; \hat{\ell}') \approx h^2 \sum_{\hat{\ell}}^{\text{ave}} \psi_\uparrow(i; \hat{\ell}) \psi_\downarrow(i; \hat{\ell}), \end{aligned} \quad (29)$$

where the non-trivial result needs $\psi_{\uparrow,\downarrow}^{\dagger}(i)$ fields in the hopping expansion of $e^{-h\mathcal{H}_0(i)}$, and $\sum_{\hat{\ell}}^{\text{ave}} \psi_{\sigma}(i; \hat{\ell}) \equiv \sum_{\hat{\ell}} \left[\psi_{\sigma}(i + \hat{\ell}) + \psi_{\sigma}(i - \hat{\ell}) \right]$. In Eq. (3), integrating over fields $\psi_{\uparrow,\downarrow}(i)$ at the site “ i ”, the first corrected version reads:

$$G(i) = \frac{\delta^{(d)}(i)}{\beta g} + \frac{1}{\beta g} \left(\frac{\beta}{2m} \right)^2 \sum_{\hat{\ell}}^{\text{ave}} G^{\text{nb}}(i; \hat{\ell}), \tag{30}$$

where $\delta^{(d)}(i)$ is a spatial δ -function and $G^{\text{nb}}(i \pm \hat{\ell})$ is the Green function (3) without integration over fields ψ_{σ} at the neighbor site i . Note that the nontrivial contributions come only from kinetic hopping terms $\propto (h/2m)^2 = (1/\beta g)(\beta/2m)^2$, excluding the chemical potential term $\mu \psi_{\sigma}^{\dagger}(i) \psi_{\sigma}(i)$ in the Hamiltonian (24).

Replacing $G^{\text{nb}}(i \pm \hat{\ell})$ by $G(i \pm \hat{\ell})$ converts Eq. (30) into a recursion relation for $G(i)$, which actually takes into account of high-hopping corrections in a strong-coupling expansion. Going to momentum space we approximately obtain $G(q) = \frac{1}{\beta g} + \frac{2}{\beta g} \left(\frac{\beta}{2m} \right)^2 G(q) \sum_{\hat{\ell}} \cos(q\hat{\ell})$, solved by

$$G(q) = \frac{[2m/(\beta\ell)]^2}{4\ell^{-2} \sum_{\hat{\ell}} \sin^2(q\hat{\ell}/2) + M_B^2}. \tag{31}$$

Here we have resumed the original lattice spacing ℓ by setting back $\beta \rightarrow \beta\ell^3$ and $2m \rightarrow 2m\ell^2$.

By the analogy to (29)–(31), we calculate Green functions (13)–(16) for composite fermions and obtain three recursion relations

$$S_{LL}(p) = \frac{1}{\beta g} \left(\frac{\beta}{2m} \right)^3 \left[2 \sum_{\hat{\ell}} \cos(p\hat{\ell}) \right] S_{ML}(p), \tag{32}$$

$$S_{ML}(p) = \frac{1}{\beta g} + \frac{1}{\beta g} \left(\frac{\beta}{2m} \right) \left[2 \sum_{\hat{\ell}} \cos(p\hat{\ell}) \right] S_{LL}(p), \tag{33}$$

$$S_{MM}(p) = \frac{1}{\beta g} \left(\frac{\beta}{2m} \right) \left[2 \sum_{\hat{\ell}} \cos(p\hat{\ell}) \right] S_{ML}^{\dagger}(p). \tag{34}$$

We solve these recursion relations and obtain

$$S_{ML}(p) = \frac{(1/\beta g)}{1 - (1/\beta g)^2 (\beta/2m)^4 \left[2 \sum_{\hat{\ell}} \cos(p\hat{\ell}) \right]^2}, \tag{35}$$

$S_{LL}(p)$ and $S_{MM}(p)$. Define for the propagator of the composite fermion $S_{\text{Fermion}}(p)$ the quantity $gS(p)$,

$$\begin{aligned} S(p) &= R_B^{-1} S_{LL}(p) + 2R_B^{-2} S_{ML}(p) + R_B^{-3} S_{MM}(p) \\ &= \frac{2}{4\ell^{-2} \sum_{\hat{\ell}} \sin^2(p\hat{\ell}/2) + M_F^2}, \end{aligned} \tag{36}$$

where R_B and $M_F^2 = M_B^2$ follow Eq. (5). For some more details, please see Ref. [22].

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