Einstein's constraints: a dynamical approach

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EINSTEIN CONSTRAINTS:
A DYNAMICAL APPROACH

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If you want to build a ship, don't drum up people to collect wood and don't assign them tasks and work, but rather teach them to long for the endless immensity of the sea.

Antoine de Saint-Exupéry

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1

INTRODUCTION

This goal of this thesis is to study certain properties of a quasilinear parabolic differential equation with spherical domain, namely

$$u_t = a(\theta, \phi, u, \nabla u)\Delta_{\mathbb{S}^2} u + f(\theta, \phi, u, \nabla u) \tag{1}$$

with initial data $u(0, \theta, \phi) = u_0(\theta, \phi)$ such that $f \in C^2$, $a \in C^1$, the parabolicity condition $a > 0$ holds and $\Delta_{\mathbb{S}^2}$ is the Laplace-Beltrami operator on the sphere $\mathbb{S}^2$. In coordinates, the angle variables are $(\theta, \phi) \in [0, \pi] \times [0, 2\pi]$ with Neumann boundary condition in θ and periodic boundary in ϕ.

This thesis has three main results. The first two describes the asymptotics as $t \to \infty$ when the spatial domain is unidimensional. The third proves a symmetry property about equilibria solutions of (1), and how the symmetry of the spherical domain influences the symmetry of solutions.

Suppose that solutions are *axisymmetric*, that is, $u(t, \theta, \phi)$ that are independent of rotations with respect to the angle ϕ and depend only in θ, as in $u(t, \theta)$. Hence, u solves the following equation

$$u_t = a(\theta, u, u_\theta)\left[u_{\theta\theta} + \frac{u_\theta}{\tan(\theta)}\right] + f(\theta, u, u_\theta) \tag{2}$$

with initial data $u(\theta, 0) = u_0(\theta)$, where $\theta \in [0, \pi]$ has Neumann boundary. Even though the equation has a degenerate coefficient at the boundary, solutions are still regular.

The equation (2) defines a semiflow denoted by $(t, u_0) \mapsto u(t)$ in the Banach space $X := C_w^{2\beta+\alpha}([0, \pi])$ where $\alpha, \beta \in (0, 1)$ are respectively a fractional power and the Hölder exponent, and $w = \sin(\theta)$ is a weight induced by the spherical metric. The appropriate functional setting for the semilinear case is described in Chapter 2, and in Chapter 3 for the quasilinear case. See also [49].

In order to study the long time behaviour of (2), we suppose that f satisfies the following conditions, where $p := u_\theta$,

$$f(x, u, 0) \cdot u < 0 \tag{3}$$

$$|f(x, u, p)| < f_1(u) + f_2(u)|p|^\gamma \tag{4}$$

$$\frac{|a_\theta|}{1 + |p|} + |a_u| + |a_p| \cdot [1 + |p|] \leq f_3(|u|) \tag{5}$$

$$0 < \epsilon \leq a(\theta, u, p) \leq \delta \tag{6}$$

where the first condition holds for $|u|$ large enough uniformly in θ, the second for all (θ, u, p) for continuous f_1, f_2 and $\gamma < 2$, and the third for continuous f_3.

Those conditions imply that $|u|$ and $|u_x|$ are bounded. Hence bounded solutions are global in time and the flow is dissipative: trajectories $u(t)$ eventually enter a large ball in the phase-space X. See Chapter 6, Section 5 in [84]. Also [56] and [12].

Moreover, these hypothesis guarantee the existence of a nonempty global attractor $\mathcal{A}$ of (2), which is the maximal compact invariant set. Equivalently, it is the set of bounded trajectories $u(t)$ in the phase-space X that exist for all $t \in \mathbb{R}$. See [12].

For the statement of the main theorem that describes the global attractor $\mathcal{A}$, denote by the *zero number* $z(u_*)$ the number of strict sign changes of a continuous function $u_*(\theta)$.

Recall that the *Morse index* $i(u_*)$ of an equilibrium u_* is given by the number of positive eigenvalues of the linearized operator at such equilibrium, that is, the dimension of the unstable manifold of said equilibrium.

We say that two different equilibria u_-, u_+ of (2) are *adjacent* if there does not exist an equilibrium u_* of (2) such that $u_*(0)$ lies between $u_-(0)$ and $u_+(0)$, and

$$z(u_- - u_*) = z(u_- - u_+) = z(u_+ - u_*).$$

This notion was described by Fiedler and Rocha [29] and refined in Wolfrum [86]. The zero number of difference of solutions roughly describes the number of intersections between those solutions.

Both the zero number and Morse index can be computed from a permutation of the equilibria, as it was done in [33] and [29]. Such permutation is called the *Sturm Permutation*. We construct an analogous permutation for the case of boundary singularity in Section (2.3), as in (8). For such, it is required that the flow of the equilibria equation of (8) exists for all $\theta \in [0, \pi]$. Sufficient conditions for boundedness are given in [63], which in turn implies global existence.

Theorem 1.0.1. Sturm Attractors [Lappicy ('17)]
Consider $a \in C^1$ and $f \in C^2$ satisfying the growth conditions (3). *Suppose that all equilibria for the equation* (2) *are hyperbolic. Then,*

1. *the global attractor $\mathcal{A}$ consists of equilibria $\mathcal{E}$ and heteroclinics $\mathcal{H}$.*
2. *there exists a heteroclinic $u(t) \in \mathcal{H}$ between $u_-, u_+ \in \mathcal{E}$ such that*

$$u(t) \to_{t \to \pm\infty} u_\pm$$

if, and only if, u_- and u_+ are adjacent and $i(u_-) > i(u_+)$.

The first claim follows due to the existence of a Lyapunov functional constructed by Matano [53] and Zelenyak [87]. A modification of such functional for the case of degenerate coefficients is done in Chapter 2, whereas the quasilinear case was already done by Matano.

The second claim answers the question of which equilibria connects to which other. This geometric description was carried out by Hale and do Nascimento [37] for the Chafee Infante problem, by Brunovský and Fiedler [27] for some $f(x,u)$ and by Fiedler and Rocha [29] for certain $f(x,u,u_x)$. Such attractors are known as *Sturm attractors*.

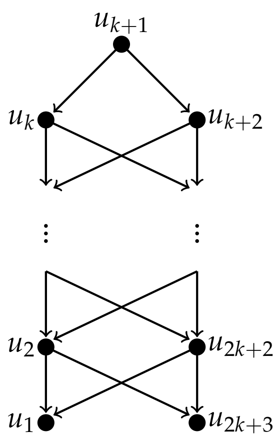
There are two difficulties in proving such theorem: it is a quasilinear equation, and the diffusion has degenerate coefficients at the north and south pole $\theta = 0, \pi$. The singularities are dealt by considering weighted Banach spaces in Chapter 2 so that the degenerate elliptic operator becomes regular. The quasilinearity is dealed by choosing appropriate Banach spaces X, in Chapter 3. It is the aim of this thesis to modify the existing theory for such cases and still obtain a Sturm attractor.

In particular, we compute the attractor explicitely for the example of Chafee-Infante type nonlinearity with singular boundary coefficients and quasilinear diffusion.

Corollary 1.0.2. Chafee-Infante Attractor [Lappicy ('17)]

Consider $a(u) = u^n$ and $f(\lambda, u) = \lambda u^{n+1}(1-u^2)$ for some $n \in \mathbb{N}_0$ in the equation (2)*. Let $\lambda \in (\lambda_k, \lambda_{k+1})$, where λ_k is the k-th eigenvalue of the axially symmetric Laplacian with $k \in \mathbb{N}_0$. Consider the phase space $X := C^{2\beta+\alpha}_w([0,\pi]) \cap \{u > \epsilon\}$ with weigth $w := \sin(\theta)$ and fixed $\epsilon > 0$.*

Then, there are $2k+3$ hyperbolic equilibria $u_1, ..., u_{2k+3}$ and its attractor $\mathcal{A}$ in X is as the below figure, where arrows denote heteroclinics.



We note that even though the Chapters 2 and 3, for degenerate coefficients and quasilinear diffusion, are written separately, their methods can be combined in order to prove the above Theorem. We prefer to split the discussion in two chapters so that the reader knows which problem arises in each case, and how this problem is dealt differently.

The last result of this thesis studies how the symmetry of the spherical domain influences solutions of elliptic equations on such domain.

Consider the quasilinear elliptic equation

$$0 = a(\theta, \phi, u, \nabla u)\Delta_{\mathbb{S}^2} u + f(u) \tag{7}$$

where $(\theta, \phi) \in \mathbb{S}^2$, a and f are analytic.

We say that a function $u \in C^2(\mathbb{S}^2)$ has *axial extrema* if its extrema occur as axis from the north to south pole. Mathematically, if $u_\phi(\theta_0, \phi_0) = 0$ for a fixed $(\theta_0, \phi_0) \in \mathbb{S}^2$, then $u_\phi(\theta, \phi_0) = 0$ for any $\theta \in [0, \pi]$. In that case, the extrema depend only at the position in ϕ.

Moreover, we say that the axial extrema are *leveled* if all axial maxima ϕ_i have the same value $u(\theta, \phi_i) = M(\theta)$, and all axial minima ϕ_i also have the same value $u(\theta, \phi_i) = m(\theta)$.

Since a, f are analytic, so is u, as in [21]. Therefore there are finitely many leveled axial extrema and we denote them by $\{\phi_i\}_{i=0}^N$.

Theorem 1.0.3. Symmetry of Certain Equilibria [Lappicy ('17)]
Suppose that u is a non constant equilibrium of (7) *such that all its extrema are leveled and axial. Then* $\phi_i = \frac{\phi_{i-1}+\phi_{i+1}}{2}$ *and*

$$u(\theta, \phi) = u(\theta, R_{\phi_i}(\phi))$$

for all $i = 0, , ..., N$, *where* $\phi_{-1} := \phi_N$ *and* $\phi_{N+1} := \phi_0$, $R_{\phi_i}(\phi) := 2\phi_i - \phi$ *is the reflection of* ϕ *with respect to* ϕ_i *and* $(\theta, \phi) \in [0, \pi] \times [\phi_{i-1}, \phi_i]$.

For positive solutions of elliptic equations on a ball with Dirichlet boundary conditions such symmetrization result was obtained by Gidas, Ni and Nirenberg [34], using the moving plane method developed by Alexandrov [2] and further by Serrin [75]. We give a brief sketch of this method in Section 4.1.

Proving the symmetrization property in the sphere carry some difficulties. In particular, it has no boundary and it is not clear where to start the moving plane method. This problem was solved partially by Padilla [58] for particular convex subsets of the sphere, and later considered by Kumaresan and Prajapat [45] for subsets of the sphere contained in a hemisphere. Later by Brock and Prajapat [17] for subsets containing hemisphere, but still not the full sphere. All these methods rely on a stereographic projection, so that domains within the sphere are transformed to domains in the Euclidean space and one can apply the moving plane method. Moreover, such results in the sphere deal with positive solutions.

From now on, the thesis is organized as follows. Chapter 2 constructs the axisymmetric global attractor in case the equation (2) is semilinear and has degenerate coefficients. Chapter 3 constructs the axisymmetric global attractor in case the equation (2) is quasilinear and has regular coefficients. Chapter 4 rigorously proves the symmetry property for certain equilibria. Chapter 5 discuss an application of such equations and attractors: the Einstein Hamiltonian constraint equation. Each chapter is independent of the other, having its own introduction and discussion sessions.

2

AXISYMMETRIC DYNAMICS

The goal of this chapter is to study the Sturm attractors of semilinear parabolic equations with degenerate coefficients on the boundary, as it was done without the degenerate term by Brunovský and Fiedler [18], and later by Fiedler and Rocha [29].

Consider the scalar semilinear parabolic differential equation

$$u_t = \Delta_{\mathbb{S}^2} u + f(\theta, \phi, u, \nabla u)$$

with initial data $u(0,\theta,\phi) = u_0(\theta,\phi)$ such that $f \in C^2$ and $\Delta_{\mathbb{S}^2}$ is the Laplace-Beltrami operator on the sphere $\mathbb{S}^2$. In coordinates, the angle variables are $(\theta,\phi) \in [0,\pi] \times [0,2\pi]$ with Neumann boundary condition in θ and periodic boundary in ϕ.

Suppose that solutions $u(t,\theta,\phi)$ are *axisymmetric*, that is, they are independent of rotations with respect to the angle ϕ and depend only in θ. Hence, $u(t,\theta)$ solves the following equation

$$u_t = u_{\theta\theta} + \frac{u_\theta}{\tan(\theta)} + f(\theta, u, u_\theta) \tag{8}$$

with initial data $u(0,\theta) = u_0(\theta)$, where $\theta \in [0,\pi]$ has Neumann boundary. Even though the equation has a degenerate coefficient at the boundaries $\theta = 0$ or π, solutions are still regular.

The equation (8) defines a semiflow denoted by $(t, u_0) \mapsto u(t)$ in a Banach space X. The appropriate functional setting is described in Section 2.2.

In order to study the long time behaviour of (8), we suppose that f satisfies the following conditions

$$f(\theta, u, 0) \cdot u < 0 \tag{9}$$

$$|f(\theta, u, u_\theta)| < f_1(u) + f_2(u)|p|^\gamma \tag{10}$$

where the first condition holds for $|u|$ large enough uniformly in θ, and the second for all (θ, u, p) for continuous f_1, f_2 and $\gamma < 2$.

The first dissipative condition implies that $|u|$ is bounded, and hence bounded solutions are global in time. The second dissipative condition implies that $|u_x|$ is bounded. Hence the flow is dissipative: trajectories $u(t)$ eventually enter a large ball in the phase-space X. See [3], [46] and [12].

Moreover, these hypothesis guarantee that there exists a nonempty global attractor $\mathcal{A}$ of (8), which is the maximal compact invariant set.

Equivalently, it is the set of bounded trajectories $u(t,.)$ in the phase-space X that exist for all $t \in \mathbb{R}$. See [12].

For the statement of the main theorem of this chapter that describes the global attractor $\mathcal{A}$, denote by the *zero number* $z(u_*)$ the number of strict sign changes of a continuous function $u_*(\theta)$. In Section 2.4 it is given a rigorous definition.

We say that two different equilibria u_-, u_+ of (8) are *adjacent* if there does not exist an equilibrium u_* of (8) such that $u_*(0)$ lies between $u_-(0)$ and $u_+(0)$, and

$$z(u_- - u_*) = z(u_- - u_+) = z(u_+ - u_*).$$

This notion was described by Fiedler and Rocha [29] and refined in Wolfrum [86].

Recall that the *Morse index* $i(u_*)$ of an equilibrium u_* is given by the number of positive eigenvalues of the linearized operator at such equilibrium, that is, the dimension of the unstable manifold of u_*.

Both the zero number and Morse index can be computed from a permutation of the equilibria, as it was done in [33] and [29]. Such permutation is called the *Sturm Permutation*. We construct an analogous permutation for the case of boundary singularity in Section (2.3), as in [29]. For such, it is required that the flow of the equilibria equation of (8) exists for all $\theta \in [0, \pi]$. Sufficient conditions for boundedness are given in [63], which in turn implies global existence.

Theorem 2.0.1. Sturm Attractor [Lappicy ('17)]
Consider $f \in C^2$ satisfying the growth conditions (9). *Suppose that all equilibria for the equation* (8) *are hyperbolic. Then,*

1. *the global attractor $\mathcal{A}$ of* (8) *consists of equilibria $\mathcal{E}$ and heteroclinic orbits $\mathcal{H}$.*
2. *there exists a heteroclinic $u(t) \in \mathcal{H}$ between $u_-, u_+ \in \mathcal{E}$ such that*

$$u(t) \to_{t \to \pm\infty} u_\pm$$

if, and only if, u_- and u_+ are adjacent and $i(u_-) > i(u_+)$.

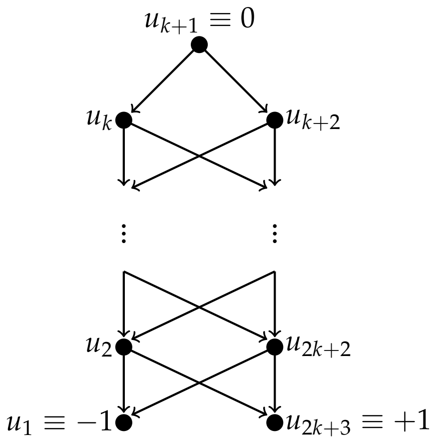
The first claim follows due to the existence of a Lyapunov functional constructed by Matano [53] and Zelenyak [87]. A modification of such functional for the case of degenerate coefficients is done in Section 2.2.

The second claim answers the question of which equilibria connects to which other. This geometric description was carried out by Hale and do Nascimento [37] for the Chafee Infante problem, by Brunovský and Fiedler [27] for $f(x,u)$ and by Fiedler and Rocha [29] for $f(x,u,u_x)$. Such attractors are known as *Sturm attractors*.

Constructing the Sturm attractor for the equation (8) is problematic due to its degenerate coefficient. It is the aim of this chapter to modify the existing theory for such boundary singularity and still obtain a Sturm attractor.

In particular, we compute the attractor explicitely for the example of Chafee-Infante type nonlinearity with degenerate boundary coefficients.

Corollary 2.0.2. Chafee-Infante Attractor [Lappicy ('17)]
Consider $f(\lambda, u) = \lambda u(1-u^2)$ *in the equation* (8). *Let* $\lambda \in (\lambda_k, \lambda_{k+1})$, *where* λ_k *is the k-th eigenvalue of the axisymmetric Laplacian with* $k \in \mathbb{N}_0$.
Then, there are $2k+3$ *hyperbolic equilibria* $u_1, ...u_{2k+3}$ *and its attractor* $\mathcal{A}$ *is below, where arrows denote heteroclinics.*



This corollary is proved by constructing the Sturm permutation of the axisymmetric Chafee-Infante, yielding the the same as the usual Chafee-Infante problem. Hence, their attractors are geometrically (connection-wise) the same and their only difference lies in the equilibria and the domain of the parameter λ.

The remaining sections are organized as follows.

We firstly introduce the functional setting in Section 2.1. Further in Section 2.2, we construct a Lyapunov functional for the singular case by modifying Matano's arguments from [51]. In particular this implies that the attractor consists of equilibria and heterolinics.

Then, we focus on the connection problem. All the necessary information about the adjacency, namely the zero numbers and Morse indices, are encoded in a permutation of the equilibria, which is described in Section 2.3. This was done firstly by [33], and here is modified for the singular case.

In Section 2.5, it is proven the dropping lemma for the singular case, as well as some consequences. This is a fundamental result for the attractor construction that dates back to Sturm and is done by modifying arguments of Chen and Polácik [62], where they proved such result for a degenerate coefficient at only one boundary value. Then all the previous tools are put together to construct the attractor in Section 2.5, as it was done [29].

Lastly, Section 2.6 gives an example of the developed theory and constructs the attractor for the axisymmetric Chafee-Infante problem.

2.1 FUNCTIONAL SETTING

The Banach space used in the upcoming theory consists of the subspaces of $L^2(\mathbb{S}^2)$ depending only on the angle $\theta \in [0, \pi]$. A more precise description is given below.

The equation (8) can be rewritten as an abstract differential equation on a Banach space. Consider the operator

$$A : D(A) \to X$$
$$u \mapsto Au := \frac{1}{\sin(\theta)} \partial_\theta \left(\sin(\theta)\partial_\theta u\right)$$

where we consider the weighted space $X := L^2_w([0, \pi])$ with weight $w := \sin(\theta)$, and $D(A) \subset X$ is the domain of the operator A. The weight is chosen to tame the singular term, and it is exactly the metric on the axially symmetric arc within the sphere, parametrized by the angle $\theta \in [0, \pi]$. The norm inherited in this space is

$$||u(\theta)||_{L^2_w([0,\pi])} = \left(\int_0^\pi u^2(\theta) \sin(\theta) d\theta \right)^{\frac{1}{2}}$$

The operator A is a self-adjoint singular Sturm-Liouville operator on the space X. Its spectrum consists of real and simple eigenvalues $\lambda_k = k(k+1)$ for $k \in \mathbb{N}_0$ with corresponding eigenfunctions being the Legendre polynomials $\phi_k = P_k(\cos\theta)$, which form an orthonormal basis of X. Hence, A is a sectorial operator and generates a compact, dissipative and analytic semigroup, as in [41].

In particular, it settles the theory of existence, uniqueness and qualitative properties of solutions. For instance, the equation (8) with the dissipative conditions (9) defines a dynamical system on some subspaces of X containing $D(A)$. Namely, on fractional power spaces X^α for particular choices of $\alpha \in (0, 1)$. For instance, it is convenient to chose $\alpha > 3/2$ so that one obtains the Sobolev embedding $X^\alpha \subset C^1$.

Moreover, one can prove the existence and certain properties of invariant manifolds tangent to the linear eigenspaces spanned by ϕ_k.

Theorem 2.1.1. Filtration of Invariant Manifolds [41]

Let u_* be a hyperbolic equilibrium of (8) *with Morse index $n := i(u_*)$. Then there exists a filtration of the* unstable manifold

$$W^u_0(u_*) \subset ... \subset W^u_{n-1}(u_*) = W^u(u_*)$$

where each W^u_k has dimension $k + 1$ and tangent space spanned by $\phi_0, ..., \phi_k$.

Analogously, there is a filtration of the stable manifold

$$... \subset W^s_{n+1}(u_*) \subset W^s_n(u_*) = W^s(u_*)$$

where each W^s_k has codimension k and tangent space spanned by $\phi_k, \phi_{k+1}, ...$

Note that the above index labels are not in agreement with the dimension of each submanifold within the filtration, but it is with the number of zeros an eigenfunction has. For example, an eigenfunction ϕ_k corresponding to the eigenvalue $\lambda_k > 0$ has k simple zeroes, whereas the $\dim(W^u_k) = k+1$.

An important property is the behaviour of solutions within each submanifold of the above filtration of the unstable or stable manifolds.

Theorem 2.1.2. Linear Asymptotic Behaviour [41], [5], [18]

Consider a hyperbolic equilibrium u_* with Morse index $n := i(u_*)$ and a trajectory $u(t)$ of (8)*. The following holds,*

1. *If $u(t) \in W^u_k(u_*) \backslash W^u_{k-1}(u_*)$ with $k = 0, ..., i(u_*) - 1$, then*

$$\frac{u(t) - u_*}{||u(t) - u_*||} \xrightarrow{t \to -\infty} \pm \phi_k.$$

2. *If $u(t)$ in $W^s_k(u_*) \backslash W^s_{k+1}(u_*)$ with $k \geq i(u_*)$, then*

$$\frac{u(t) - u_*}{||u(t) - u_*||} \xrightarrow{t \to \infty} \pm \phi_k..$$

where the convergence takes place in C^1_w, and $W^u_{-1}(u_*) = \emptyset$.

The conclusions of 1. and 2. also hold true by replacing the difference $u(t) - u_*$ with the tangent vector u_t.

The reason this theorem works for both the tangent vector $v := u_t$ or the difference $v := u_1 - u_2$ of any two solutions u_1 and u_2 of the nonlinear equation (8) is because they satisfy a linear equation of the type

$$v_t = v_{\theta\theta} + \frac{v_\theta}{\tan(\theta)} + b(t,\theta) v_\theta + c(t,\theta) v \tag{11}$$

where $\theta \in (0, \pi)$ has Neumann boundary conditions, $b(t,\theta)$ and $c(t,\theta)$ are bounded, and the self-adjoint operator $A := \frac{1}{\sin(\theta)} [\sin(\theta) u_\theta]$ is bounded from below and has eigenvalues $\lambda_k = k(k+1)$ in the Hilbert space $L^2_w([0,\pi])$ with weight $w := \sin(\theta)$.

Proof We give a sketch of the appendix in [5]. This proof works for the case that $b \equiv 0$, in particular when $f = f(\theta, u)$. For the most general proof of this fact, see [18].

Let the trajectory $v(t)$ with initial data $v_0 \not\equiv 0$ be either the tangent vector $u_t(t)$ or the different $u_1(t) - u_2(t)$ of two solutions of (8), hence v is a solution of (11).

Consider γ_k a spectral gap, that is, a number in $\mathbb{R}$ between the eigenvalues λ_k and λ_{k+1}. Denote by P_k the projections of A on the space spanned by the eigenfunctions with associated eigenvalues less than γ_k, and its complement by $Q_k := Id_X - P_k$.

Define $\hat{v}(t) := \frac{v(t)}{||v(t)||_{C^1_w}}$. Note by linearity of the projection P_k and the decomposition of X into the images of P_k and Q_k,

$$||P_k \hat{v}(t)||^2 = \frac{||P_k v(t)||^2}{||v(t)||^2} = \frac{||P_k v(t)||^2}{||P_k v(t)||^2 + ||Q_k v(t)||^2}.$$

Due to the spectral gap, it can be proven that the projected trajectories $P_k v(t)$ and its complement $Q_k v(t)$ have strict different asymptotic rates, and one is faster than the other. Moreover, there is an unique k_0 such that for $k < k_0$ one of the projected trajectories is faster, whereas for $k \geq k_0$ the same projected trajectories is slower.

Mathematically, $\lim_{t\to\infty} \frac{||P_k v(t)||}{||Q_k v(t)||}$ is either 0 or ∞, for each $k > 1$. This can be applied to the above, yielding

$$\lim_{t\to\infty} ||P_k(\hat{v})|| = 0 \text{ or } 1.$$

Moreover, there exists an unique k_0 such that the above limit is 0 for all $k < k_0$, and 1 for $k \geq k_0$. See the Lemma 7 in [5]. That is,

$$\lim_{t\to\infty} e^{\lambda_k t}\hat{v}(t) = 0, \text{ for all } k < k_0$$
$$\lim_{t\to\infty} e^{\lambda_k t}\hat{v}(t) \neq 0, \text{ for all } k \geq k_0$$

Hence the asymptotic direction of $v(t)$ is on the direction of the eigenfunction ϕ_{k_0-1}.

■

2.2 VARIATIONAL STRUCTURE

In this section, we show that there exists a Lyapunov functional, as it was done by Zelenyak [87] and Matano [52]. We modify Matano's construction bearing in mind that the metric on the sphere induces a space with weighted norms, and this weight should be incorporated into the construction of the Lyapunov functional. As a consequence of the Lyapunov functional, bounded trajectories tend to equilibria.

Lemma 2.2.1. Lyapunov Functional

There exists a Lagrange functional L such that

$$E := \int_0^\pi L(\theta, u, u_\theta)\sin(\theta)d\theta \tag{12}$$

is a Lyapunov functional for the equation (8).

Note that in the case that the nonlinearity f does not depend on u_θ, then the Lagrange functional $L(\theta, u, u_\theta) := \frac{1}{2}u_\theta^2 - F(\theta, u)$ yields a Lyapunov functional E, where F is the primitive function of f. Indeed,

$$\frac{dE}{dt} = -\int_0^\pi (u_t)^2 \sin(\theta)d\theta.$$

For nonlinearities of the type $f(\theta, u, u_\theta)$, Matano's idea yields a Lyapunov functional such that

$$\frac{dE}{dt} := -\int_0^\pi (u_t)^2 L_{pp} \sin(\theta) d\theta \tag{13}$$

where $p := u_\theta$ and L satisfy the convexity condition $L_{pp} > 0$. Hence, the case that f does not depend on u_θ is seen as a particular case when $L_{pp} = 1$.

Proof Let $p := u_\theta$ and differentiate (12) with respect to t,

$$\frac{dE}{dt} = \int_0^\pi \left[L_u u_t + L_p u_{\theta t}\right] \sin(\theta) d\theta.$$

Integrating the second term by parts and noticing that the $\sin(\theta)$ is 0 at the boundaries,

$$\begin{aligned}\frac{dE}{dt} &= \int_0^\pi \left[L_u \sin(\theta) - \frac{d}{d\theta}(L_p \sin(\theta))\right] u_t d\theta \\ &= \int_0^\pi \left[(L_u - L_{p\theta} - L_{pu}u_\theta - L_{pp}u_{\theta\theta}) \sin(\theta) - L_p \cos\theta\right] u_t d\theta.\end{aligned}$$

Substitute (8) casted as $u_{\theta\theta} \sin(\theta) = u_t \sin(\theta) - f \sin(\theta) - u_\theta \cos(\theta)$,

$$\begin{aligned}\frac{dE}{dt} = &\int_0^\pi (L_u - L_{p\theta} - L_{pu}u_\theta + L_{pp}f) \sin(\theta) u_t d\theta \\ &+ \int_0^\pi (L_{pp}u_\theta - L_p) \cos(\theta) u_t d\theta \\ &- \int_0^\pi L_{pp} u_t^2 \sin(\theta) d\theta.\end{aligned}$$

To obtain the desired equality (13), one now has to guarantee that there exists a function L satisfying

$$(L_u - L_{p\theta} - L_{pu}p + L_{pp}f) \sin(\theta) + (L_{pp}p - L_p) \cos(\theta) = 0 \tag{14}$$

for all $u, p \in \mathbb{R}$ and $\theta \in [0, \pi]$.

Differentiating this equation with respect to p, some of the terms cancel, yielding

$$(-L_{pp\theta} - L_{ppu}p + L_{ppp}f + L_{pp}f_p) \sin(\theta) + (L_{ppp}p) \cos(\theta) = 0. \tag{15}$$

To make sure that $L_{pp} > 0$, Matano makes an Ansatz by introducing a function $g = g(\theta, u, p)$ through $L_{pp} = \exp(g) > 0$. Hence, g satisfies the following linear first order differential equation,

$$[(g_\theta + g_u p - g_p f - f_p) \sin(\theta) - (g_p p) \cos(\theta)] \exp(g) = 0. \tag{16}$$

Or equivalenty,

$$\left[g_\theta + g_u p + g_p \left(-f - p\frac{\cos(\theta)}{\sin(\theta)}\right)\right] \sin(\theta) = f_p \sin(\theta).$$

This can be solved through the method of characteristics: along the solutions of the ordinary differential equation

$$\begin{cases} u_\theta &= p \\ p_\theta &= -f - p\frac{\cos(\theta)}{\sin(\theta)} \end{cases}$$

the function g must satisfy

$$\frac{dg}{d\theta} = f_p.$$

Note that the characteristic equation is the equation for equilibria. Under the assumption that solutions of such equations exist for all initial conditions $(u, p) \in \mathbb{R}^2$ at $\theta = 0$, and all $\theta \in [0, \pi]$, one obtain a global solution g of (16) with some initial data, for example, $g(0, u, p) \equiv 0$.

It is still needed to ascend from a function g satisfying (16) to a function L satisfying (14). A choice for L such that $L_{pp} = \exp(g)$ can be obtained by integrating this relation twice with respect to p, yielding

$$L(\theta, u, p) := \int_0^p \int_0^{p_1} \exp(g(\theta, u, p_2))dp_2 dp_1 + G(\theta, u)$$

and this is a solution of (15).

To show that such L is also a a solution of (14), we have to restrict which G are allowed.

Recall that (15) was obtained through differentiating (14) with respect to p. That means that the left-hand side of (14) is independent of p, since it is equal to 0. Hence it is satisfied for all p if holds for $p = 0$.

At $p = 0$, the construction of L yields that $L_p = L_{p\theta} = 0$ and $L_u = G_u$. Plugging it in the equation (14) at $p = 0$, it yields

$$(G_u + L_{pp} f) \sin(\theta) = 0.$$

Hence, $G_u + L_{pp} f = 0$, that is, $G_u = -\exp(g) f$. Integrating in u,

$$G(\theta, u) := -\int_0^u f(\theta, u_1, 0) exp(g(\theta, u, 0))du_1$$

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Note that one can do a similar construction of a Lyapunov function without assuming that the $\sin(\theta)$ appears in the integrand, as in (12). But such coefficient will appear once the differential equation is plugged in the Ansatz for the Lyapunov functional.

Moreover, Matano's construction can be adapted to more general singular Sturm Liouville operators of the form $\frac{\partial_\theta(r(\theta)\partial_\theta)}{w(\theta)}$, if the weight $\sin(\theta)$ within the integrand is replaced by $w(\theta)$.

Therefore, the LaSalle invariance principle holds and implies that bounded solutions converge to equilibria, and any ω-limit set consists of a single equilibrium. See [53]. Moreover, the global attractor can be characterized as follows, yielding the first part of the main result.

Proposition 2.2.2. Attractor Decomposition [41], [12]
If the equation (8) *has a Lyapunov functional and a discrete set of equilibria $\mathcal{E}$, then the global attractor $\mathcal{A}$ is decomposed as*

$$\mathcal{A} = \bigcup_{v \in \mathcal{E}} W^u(v)$$

and consists only of the set of equilibria and heteroclinics orbits.

Note that hyperbolic equilibria must be isolated. Moreover, there must be finitely many due to dissipativity.

2.3 STURM PERMUTATION

The next step on our quest to find the Sturm attractor is to construct a permutation associated to the equilibria, which is done using shooting methods. This enables the computation of the Morse indices and zero number of equilibria. That was firstly done by Fusco and Rocha [33] using methods also described by Fusco, Hale and Rocha in [66], [39], [67], [69] and [32].

The equilibria equation associated to (8) can be rewritten as

$$0 = \frac{1}{\sin(\theta)} \frac{d}{d\theta}[u_\theta \sin(\theta)] + f(\theta, u, u_\theta) \tag{17}$$

for $\theta \in [0, \pi]$ with Neumann boundary conditions.

In order to get rid of the singularities at $\theta = 0$ and π, rescale the system by $\tau(\theta) := \ln(\tan(\theta/2)) \in (-\infty, \infty)$, which maps the singularities at $\theta = 0, \pi$ to $\tau = \pm\infty$. Moreover, reduce the system to first order through $p := u_\tau$. Lastly, add the extra equation $\theta_\tau = \sin(\theta)$ to obtain an autonomous system. Hence,

$$\begin{cases} u_\tau &= p \\ p_\tau &= -f\left(\theta, u, \frac{p}{\sin(\theta)}\right) \sin^2(\theta) \\ \theta_\tau &= \sin(\theta) \end{cases} \tag{18}$$

where the Neumann boundary condition becomes $\lim_{\tau \to \pm\infty} p(\tau) = 0$, since the Neumann boundary in changed of coordinates yields

$$0 = \lim_{\theta \to 0, \pi} u_\theta = \lim_{\tau \to \pm\infty} u_\tau \frac{d\tau}{d\theta} = \lim_{\tau \to \pm\infty} p(\tau) \cosh(\tau) \tag{19}$$

and $\lim_{\tau \to \pm\infty} \cosh(\tau) \to \infty$. This forces exponential decay of p.

Note that the term $\sin^2(\theta)$ cuts off the reaction f, being 1 at the equator and decaying to 0 near the poles. This means that the diffusion near the poles are stronger. Also, f is a function of $\frac{p}{\sin(\theta)}$, which seems to be singular, but $p := u_\tau = u_\theta \sin(\theta)$.

Recall we assumed that solutions of (18) are defined for all $\theta \in [0, \pi]$ and any initial data (u, p).

In the nonsingular case, the idea to find equilibria (8) is as follows. They must lie in the line

$$L_0 := \{(\theta, u, p) \in \mathbb{R}^3 \mid (\theta, u, p) = (0, a, 0) \text{ and } a \in \mathbb{R}\}$$

due to Neumann boundary at $\theta = 0$. Then, evolve this line under the flow of the equilibria differential equation and intersect it with an analogous line L_π at $\theta = \pi$, so that it also satisfies Neumann at $\theta = \pi$. This reasoning does not work for the singular case, since L_0 is a line of equilibria and is invariant under the shooting flow (18). A new approach is needed.

In the singular case, the linearization of (18) at each point in L_0 has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = 0$. with respective generalized eigenvectors $v_1 = (0, 0, 1), v_2 = (1, 0, 0), v_3 = (0, 1, 0)$. Hence, there is an one dimensional unstable direction given by the θ-axis, and two center directions given by the invariant plane $\{(u, p, 0) \in \mathbb{R}^3\}$.

Furthermore, each point $(0, a, 0) \in L_0$ has an associated one dimensional strong unstable manifold $W^u(0, a, 0)$, which is locally a graph $\{(\theta, u^u(\theta, a), p^u(\theta, a)) \in \mathbb{R}^3\}$. See [35]. The collection of all these strong unstable manifolds defines the *unstable shooting manifold* M^u,

$$M^u := \bigcup_{a \in \mathbb{R}} W^u(0, a, 0).$$

Similarly, each point $(0, b, 0) \in L_\pi$ has a one-dimensional strong stable manifold given locally by the graph $\{(\theta, u^s(\theta, b), p^s(\theta, b)) \in \mathbb{R}^3\}$, and its collection defines the *stable shooting manifold* M^s,

$$M^s := \bigcup_{b \in \mathbb{R}} W^s(0, b, 0).$$

Note that the unstable (or stable) manifolds will exist globally for any initial data $(0, a, 0)$ (or $(0, b, 0)$), since solutions are global and it is possible to extend such invariant manifolds through the flow of (18).

Denote by M^u_θ the cross-section of M^u for some fixed $\theta \in [0, \pi]$. This is a curve parametrized by $a \in \mathbb{R}$. Similarly, M^s_θ is a curve parametrized by b.

We obtain the following characterization of equilibria through the shooting manifolds and its relation with the Morse indices and zero numbers, similar to [66] and [36].

Lemma 2.3.1. ***Equilibria Through Shooting***

1. *The set of equilibria $\mathcal{E}$ of* (8) *is in one-to-one correspondence with $M^u_\theta \cap M^s_\theta$ for any $\theta \in [0, \pi]$.*

2. *An equilibrium point corresponding to fixed $a \in \mathbb{R}$ and $b \in \mathbb{R}$ is hyperbolic if, and only if, $W^u(0, a, 0)$ intersects $W^s(0, b, 0)$ transversely.*

3. *If u_* correspond to a hyperbolic equilibrium of (8), then its Morse index is given by $i(u_*) = 1 + \lfloor \frac{\zeta(\theta_0)}{\pi} \rfloor$ where $\zeta(\theta_0)$ is the angle between M^u and M^s measured clockwise at their intersection point θ_0, and $\lfloor . \rfloor$ denotes the floor function.*

Proof To prove 1), note that a point in $M^u_\theta \cap M^s_\theta$ satisfies the equilibria equation by definition of the shooting manifolds. Moreover, the Neumann boundary conditions are also satisfied since solutions are in the appropriate stable/unstable manifolds.

Conversely, consider an equilibrium of (8). It must satisfiy the Neumann boundary conditions (19), which requires exponential convergence rate to 0. This implies that the equilibrium must be both in the strong unstable M^u and strong stable M^s manifolds. Moreover, such manifolds intersect for some $\theta \in [0, \pi]$, because the equilibrium is continuous. By uniqueness and invariance of the shooting manifolds, they must also intersect for all $\theta \in [0, \pi]$.

Due to the uniqueness of the shooting differential equation (18), such correspondence above is one-to-one.

To prove 2), consider an equilibrium u_* corresponding to $a, b \in \mathbb{R}$. We compare the eigenvalue problem for u_* and the differential equation satisfied by the angle of the tangent vectors of the shooting manifold.

Introducing the τ variable, the eigenvalue problem for u_* is

$$\begin{cases} \lambda u \sin^2(\theta) & = u_{\tau\tau} + [D_u f(\theta, u_*, p_*).u + D_p f(\theta, u_*, p_*).p] \sin^2(\theta) \\ \theta_\tau & = \sin(\theta) \end{cases}$$

with boundary conditions $\lim_{\tau \to \pm\infty} u_\tau(\tau) = 0$. From now on, the coordinates of Df are suppressed.

Rewriting the above system as a system of first order by $p := u_\tau$,

$$\begin{cases} u_\tau & = p \\ p_\tau & = -[D_u f.u + D_p f.p - \lambda u] \sin^2(\theta) \\ \theta_\tau & = \sin(\theta) \end{cases}$$

with boundary conditions $\lim_{\tau \to \pm\infty} p(\tau) = 0$.

In polar coordinates $(u, p) =: (r \cos(\mu), -r \sin(\mu))$, the angle given by $\mu := \arctan(\frac{p}{u})$ satisfies

$$\begin{cases} \mu_\tau & = \sin^2(\mu) + [D_u f.u + D_p f.p - \lambda] \sin^2(\theta) \cos^2(\mu) \\ \theta_\tau & = \sin(\theta) \end{cases} \tag{20}$$

with $\lim_{\tau = -\infty} \mu(\tau) = 0$ and $\lim_{\tau \to \infty} \mu(\tau) = k\pi$ for some $k \geq 0$.

On the other hand, M^u_θ is parametrized by $a \in \mathbb{R}$ and its tangent vector $(\frac{\partial u(\theta,a)}{\partial a}, \frac{\partial p(\theta,a)}{\partial a})$ satisfies the following linearized equation,

$$\begin{cases} (u_a)_\tau &= p_a \\ (p_a)_\tau &= -[D_u f(\theta, u^u, p^u).u_a + D_p f(\theta, u^u, p^u).p_a] \sin^2(\theta) \\ \theta_\tau &= \sin(\theta) \end{cases} \tag{21}$$

with initial data $\lim_{\tau\to-\infty}(u_a, p_a) = (1, 0)$. Note that the linearization is considered along the unstable manifold given by the graph $\{(\theta, u^u(\theta), p^u(\theta)) \in \mathbb{R}^3\}$, but from now on we supress this dependence of Df.

In polar coordinates $(u_a, p_a) =: (\rho\cos(\nu), -\rho\sin(\nu))$, where ν is the clockwise angle of the tangent vector of M^u_θ with the u-axis,

$$\begin{cases} \nu_\tau &= \sin^2(\nu) + [D_u f.u + D_p f.p] \sin^2(\theta)\cos^2(\nu) \\ \theta_\tau &= \sin(\theta) \end{cases} \tag{22}$$

with initial data $\lim_{\tau\to-\infty} \nu(\tau, a) = 0$.

Similarly, the angle $\tilde{\nu}$ of the tangent vector of M^s_θ with the u-axis satisfies the equation (22), but with initial data $\lim_{\tau\to\infty} \nu(\tau, b) = 0$.

Note that the equation (22) that both angles ν and $\tilde{\nu}$ of the tangent vector satisfy is the same equation as the eigenvalue problem in polar coordinates (20) with $\lambda = 0$, where each ν or $\tilde{\nu}$ encodes the boundary condition at $\tau = -\infty$ of ∞.

By hypothesis, the equilibrium u_* corresponds to the pair of initial data $a, b \in \mathbb{R}$. That means that $M^u_{\theta_0}$ intersects $M^s_{\theta_0}$ for some fixed $\theta_0 \in [0, \pi]$.

Suppose that u_* is not hyperbolic, that is, $\lim_{\tau\to\infty} \mu(\tau) = k\pi$ for $\lambda = 0$ and some $k \in \mathbb{N}$. We compare this value with the angle between the shooting curves at θ_0. More precisely, it is proven that

$$\lim_{\tau\to\infty} \mu(\tau) = \nu(\theta_0) - \tilde{\nu}(\theta_0). \tag{23}$$

Indeed, for $\theta \in [0, \theta_0]$ the equations (20) and (22) are the same, since both of them are linearized at the same orbit u_*, which corresponds to the unstable manifold of $(0, a, 0) \in \mathbb{R}^3$. Since both of them have the same initial data, uniqueness implies

$$\mu(\theta_0) = \nu(\theta_0).$$

To obtain a relation between μ and $\tilde{\nu}$, consider the change of coordinates in the eigenvalue problem (20) as $\tilde{\mu} := \mu - k\pi$. The equation (20) is invariant under this transformation, since $\sin^2(\tilde{\mu} + k\pi) = \sin^2(\tilde{\mu})$. But the boundary condition changes at $\theta = \pi$, namely, $\lim_{\tau\to\infty} \tilde{\mu}(\tau) = 0$. Therefore, $\tilde{\mu}$ satisfies the same equation as the angle $\tilde{\nu}$, for $\theta \in [\theta_0, \pi]$. Hence, by uniqueness,

$$\mu(\theta_0) - k\pi = \tilde{\mu}(\theta_0) = \tilde{\nu}(\theta_0).$$

Subtracting these last two equations yields $k\pi = \nu(\theta_0) - \tilde{\nu}(\theta_0)$, that is, the intersection of the shooting manifolds is not transverse at their intersection point θ_0.

Conversely, if the shooting manifolds are not transverse at some intersection point for θ_0, then $k\pi = \nu(\theta_0) - \tilde{\nu}(\theta_0)$.

Concatenate the solution ν within M^u from $\theta \in [0, \theta_0]$ with the initial data $\lim_{\tau=-\infty} \nu(\tau) = 0$, and $\tilde{\nu}$ within M^s for $\theta \in [\theta_0, \pi]$ with initial data $\tilde{\nu}(\theta_0) = \nu(\theta_0) - k\pi$. Hence, the previous boundary conditions $\lim_{\tau=\infty} \tilde{\nu}(\tau) = 0$ implies that $\lim_{\tau=\infty} \tilde{\nu}(\tau) = k\pi$, by considering the new initial data at $\theta = \theta_0$. Note such concatenated solution satisfy the equation (20) for the angle μ of the eigenvalue problem with $\lambda = 0$. This implies there exists a solution μ of (20) and hence $\lambda = 0$ is an eigenvalue. Thus, the equilibrium u_* is not hyperbolic.

To prove 3), consider the solution $\mu(\tau, \lambda)$ of the eigenvalue problem in polar coordinates (20). The Sturm oscillation theorem implies that

$$\psi(\lambda) := \lim_{\tau \to \infty} \mu(\tau, \lambda) \tag{24}$$

is decreasing so that $\lim_{\lambda \to -\infty} \psi(\lambda) = \infty$ and $\lim_{\lambda \to \infty} \psi(\lambda) = -\pi/2$. Hence, there exists a decreasing sequence $\{\lambda_k\}_{k \in N}$ to $-\infty$ such that $\psi(\lambda_k) = k\pi$ for $k \in \mathbb{N}$. This implies that there exists a solution of (20) for each λ_k such that $\psi(\lambda_k) = k\pi$, and hence $\{\lambda_k\}_{k \in N}$ are the eigenvalues.

Recall that the Morse index $i(u_*)$ is the number of positive eigenvalues of the linearization at u_*, that is

$$... < \lambda_{i(u_*)} < 0 < \lambda_{i(u_*)-1} < ... < \lambda_0.$$

Since $\psi(\lambda)$ is decreasing and $\lambda_{i(u_*)}$ are eigenvalues, then

$$i(u_*)\pi = \psi(\lambda_{i(u_*)}) > \psi(0) > \psi(\lambda_{i(u_*)-1}) = (i(u_*) - 1)\pi.$$

Divide the above by π and consider the integer value, yielding that $i(u_*) = \lfloor \frac{\psi(0)}{\pi} \rfloor + 1$. It was noted in (23) that $\psi(0) = \nu(\theta_0) - \tilde{\nu}(\theta_0)$, which is exactly the angle between M^u and M^s.

■

Hence, one can obtain a *Sturm permutation* σ by labeling the intersection points $u_i \in M^u_{\frac{\pi}{2}} \cap M^s_{\frac{\pi}{2}}$ firstly along $M^u_{\frac{\pi}{2}}$ following its parametrization given by $(\frac{\pi}{2}, u^u(\frac{\pi}{2}, a), p^u(\frac{\pi}{2}, a))$ as a goes from $-\infty$ to ∞. Namely,

$$u_1 < ... < u_N$$

where N denotes the number of equilibria. Secondly, label the intersection points along $M^s_{\frac{\pi}{2}}$ following its parametrization by $b \in \mathbb{R}$,

$$u_{\sigma(1)} < ... < u_{\sigma(N)}$$

The Morse indices of equilibria and the zero number of difference of equilibria can be calculated through the Sturm permutation σ, as in [68] and [29]. This yields all necessary information for adjacency. The main tool for such proofs is the third part of the above Lemma: the rotation along the shooting curve increases the Morse index.

Lemma 2.3.2. Adjacency Through Permutation [29]

The Sturm permutation σ yields the Morse indices of equilibria and zero numbers of difference of equilibria. Mathematically,

1. *For any $1 \leq n \leq N$,*

$$i(v_n) = \sum_{j=1}^{n-1} (-1)^j sgn(\sigma^{-1}(j+1) - \sigma^{-1}(j))$$

2. *For any $1 \leq n < m \leq N$,*

$$z(v_n - v_m) = i(v_m) + \frac{1}{2}[(-1)^n (sgn(\sigma^{-1}(n) - \sigma^{-1}(m))) - 1] + \sum_{j=m+1}^{n-1} sgn(\sigma^{-1}(j+1) - \sigma^{-1}(m))$$

where empty sums denote 0*.*

2.4 DROPPING LEMMA

Let the *zero number* $z^t(u)$ count the number of strict sign changes in θ of a C^1 function $u(t,\theta) \not\equiv 0$, for each fixed t. More precisely,

$$z^t(u) := \sup_k \left\{ \begin{array}{l} \exists \text{ partition } \{\theta_j\}_{j=1}^k \text{ of } [0,\pi] \text{ such that} \\ u(t,\theta_j)u(t,\theta_{j+1}) < 0 \text{ for all } j = 1,...,k \end{array} \right\}.$$

and $z^t(u) = -1$ if $u \equiv 0$. In case u does not depend on t, we simply write $z^t(u) = z(u)$.

A point $(t_0, \theta_0) \in \mathbb{R} \times [0,\pi]$ such that $u(t_0,\theta_0) = 0$ is said to be a *simple zero* if $u_\theta(t_0,\theta_0) \neq 0$ and a *multiple zero* if $u_\theta(t_0,\theta_0) = 0$.

The following result shows that the zero number of certain solutions of (8) is nonincreasing in time t, and decreases whenever a multiple zero occur. Different versions of this well known fact are due to Sturm [81], Matano [52], Angenent [6] and others. See [47] for a more recent account.

Lemma 2.4.1. Dropping Lemma

Consider $v \not\equiv 0$ a solution of the linear equation (11) *for $t \in [0,T)$. Then, its zero number $z^t(v)$ satisfies*

1. *$z^t(v) < \infty$ for any $t \in (0,T)$.*

2. *$z^t(v)$ is nonincreasing in time t.*

3. *$z^t(v)$ decreases at multiple zeros (t_0, θ_0) of v, that is,*

$$z^{t_0-\epsilon}(v) > z^{t_0+\epsilon}(v)$$

for any sufficiently small $\epsilon > 0$.

Recall that both the tangent vector u_t and the difference $u_1 - u_2$ of two solutions u_1, u_2 of the nonlinear equation (8) satisfy a linear equation as (11). Hence, the dropping lemma deals with the zero number of such solutions.

Below we give the idea of two different proofs. The first is an adaptation of Chen and Poláčik [62], where the dropping lemma was proved for the case of a singular coefficient at one boundary point. The second by Angenent [6], where this lemma was proved for the case of regular coefficients. We also note that it is also possible to adapt the Newton polygon method done in Angenent [8] and Angenent with Fiedler [11], but this is not pursued here.

2.4.1 *Proof 1*

This proof adapts Chen and Poláčik [62]. We cut off solutions nearby each boundary point so that it satisfies a differential equation with only one boundary singularity, and then apply the dropping lemma for such equations as it was proved in [62].

We say two functions $u(t, \theta)$ and $v(t, \theta)$ have the *same type of zeros* if for each fixed t, their zeros in θ coincide, together with their property of being simple or multiple. Mathematically, $u(t, \theta_0) = 0$ if, and only if $v(t, \theta_0) = 0$, for fixed t. Moreover, consider a zero θ_0 of u and v for fixed t, then $u_\theta(t, \theta_0) = 0$ if, and only if $v_\theta(t, \theta_0) = 0$, .

Lemma 2.4.2. *Suppose $u \not\equiv 0$ is a solution of (11). Then, there exists bounded functions v and d on $[t_1, t_2] \times [0, \pi]$ satisfying*

$$v_t = v_{\theta\theta} + \frac{v_\theta}{\theta} + d(t, \theta)v \tag{25}$$

where $\theta \in (0, \pi)$ has Neumann boundary conditions. Moreover, for a fixed $\theta_1 \in (0, \pi)$, the functions u and v have the same type of zeros for $\theta \in [0, \theta_1]$, whereas $v \neq 0$ for all $\theta \in [\theta_1, \pi]$.

Proof The idea is to localize the solution $u(t, \theta)$ for each t and θ near the boundary $\theta = 0$, and cut off whatever is far from it. Vaguely, this defines $v(t, \theta)$, and $d(t, \theta)$ is chosen accordingly so that one obtains the desired equation (25).

Since the solution $u \not\equiv 0$, choose a point θ_1 such that the solution is not zero at θ_1 for a nonempty small interval of time $[t_1, t_2]$, by continuity in t. Moreover, due to continuity in θ, choose $\theta_2 \in (\theta_1, \pi)$ such that $u(t, \theta) \neq 0$ for $[t_1, t_2] \times [\theta_1, \theta_2]$. Without loss of generality, suppose that u is positive for $[t_1, t_2] \times [\theta_1, \theta_2]$. Otherwise, consider $u(t, \theta) \mapsto -u(t, \theta)$.

Expand the singular term in power series as $\frac{1}{\tan(\theta)} = \frac{1}{\theta} + b(\theta)$, where $b(\theta) = \sum_{n=0}^{\infty} b_n \theta^{2n+1}$ is analytic in $\theta \in [0, \pi)$ and its coefficients b_n are related to the Bernoulli numbers. Plugging this in (11), yields

$$u_t = u_{\theta\theta} + \frac{u_\theta}{\theta} + b(\theta)u_\theta + c(t,\theta)u.$$

Since $b(\theta)$ converges for $\theta \in [0, \pi)$ but not for $\theta = \pi$, this is how the singularity at $\theta = \pi$ is encoded in the new equation.

In order to get rid of $b(\theta)$, rescale the solution for $\theta \in [0, \theta_2]$ by $\tilde{u}(t,\theta) := \exp\left(\frac{1}{2}\int_0^\theta b(y)dy\right)u(t,\theta)$. Note $u(t,\theta)$ and $\tilde{u}(t,\theta)$ have the same type of zeros. The chain rule implies

$$\tilde{u}_t = \tilde{u}_{\theta\theta} + \frac{\tilde{u}_\theta}{\theta} + \tilde{c}(t,\theta)\tilde{u}$$

for $\theta \in [0, \theta_2]$, where $\tilde{c}(t,\theta) := c(t,\theta) - \frac{b(\theta)}{2\theta} + \frac{b^2(\theta)}{4} - \frac{b_\theta(\theta)}{2}$. Note the term $\frac{b(\theta)}{\theta}$ is not singular at $\theta = 0$ due to the nature of $b(\theta)$, that is, its first order term is $b_0\theta$.

Next, the rescaled solution will be cut off. Define the cut off function $\eta : [0, \pi] \to [0, 1]$ given by

$$\eta(\theta) := \begin{cases} 1 & \text{for } \theta \in [0, \theta_1] \\ 0 < \eta(\theta) < 1 & \text{for } \theta \in (\theta_1, \theta_2) \\ 0 & \text{for } \theta \in [\theta_2, \pi] \end{cases}$$

which transitions smoothly from 1 to 0.

Let $v : [t_1, t_2] \times [0, \pi] \to \mathbb{R}$ be defined by

$$v(t,\theta) := \begin{cases} \eta(\theta)[\tilde{u}(t,\theta) - 1] + 1 & \text{for } \theta \in [0, \theta_2] \\ 1 & \text{for } \theta \in (\theta_2, \pi]. \end{cases}$$

That is, $v(t,\theta) = \tilde{u}(t,\theta)$ for $\theta \in [0, \theta_1]$. For $\theta \in [\theta_1, \theta_2]$ there is a transition phase from $\tilde{u}$ to the constant function 1. For $\theta \in [\theta_2, \pi]$, the singularity at $\theta = \pi$ does not play a role anymore, since $v(t,\theta) \equiv 1$ satisfies a trivial equation.

The chain rule says that $v(t,\theta)$ satisfies

$$v_t = v_{\theta\theta} + \frac{v_\theta}{\theta} - \frac{\eta_\theta[\tilde{u} - 1]}{\theta} + \eta\tilde{c}\tilde{u} - \eta_{\theta\theta}[\tilde{u} - 1] - 2\eta_\theta\tilde{u}_\theta.$$

Now $d(t,\theta)$ is defined so that $v(t,\theta)$ satisfies the desired equation (25). For $\theta \in [0, \theta_1]$, the only term that does not vanish is $\tilde{c}\tilde{u}$, since $\eta \equiv 1$ and $\eta_\theta \equiv 0 \equiv \eta_{\theta\theta}$. This defines d in this interval. For $\theta \in (\theta_1, \theta_2)$, define most terms on the right hand side by $d(t,\theta)v$, as below. For $\theta \in [\theta_2, \pi]$, the function $v \equiv 1$ and $v_t = v_{\theta\theta} = \frac{v_\theta}{\theta} = 0$. Hence, it satisfies a trivial equation and define $d := 0$. More precisely,

$$d(t,\theta) := \begin{cases} \tilde{c} & \text{for } \theta \in [0, \theta_1] \\ \frac{1}{v}\left[-\frac{\eta_\theta[\tilde{u}-1]}{\theta} + \eta\tilde{c}\tilde{u} - \eta_{\theta\theta}[\tilde{u} - 1] - 2\eta_\theta\tilde{u}_\theta\right] & \text{for } \theta \in (\theta_1, \theta_2) \\ 0 & \text{for } \theta \in [\theta_2, \pi] \end{cases}$$

is bounded, since all terms $\tilde{u}, \tilde{u}_\theta, \eta, \eta_\theta, \eta_{\theta\theta}, \tilde{c}$ are bounded for $\theta \in [0, \theta_2]$. Also, note $v > 0$ for $\theta \in [\theta_1, \theta_2]$ and hence $\frac{1}{v}$ is well defined and bounded. Indeed, the solution u is positive in this interval, and so is $\tilde{u}$, since they have the same type of zeros. If $\tilde{u} \geq 1$ it is clear that $v > 0$ by its definition, and if $1 > \tilde{u} > 0$, one also obtains that $v > 0$ by noticing that $\eta \in [0, 1]$ for $\theta \in [0, \theta_2]$.

Hence, we have defined v and d satisfying (25) such that v and u have the same type of zeros and $v \equiv 1$ for $\theta \in [\theta_2, \pi]$.

■

In order to apply the dropping lemma to functions $v(t, \theta)$ satisfying the equation (25), as in [62], one still needs two adaptations. Firstly, the dropping lemma is proved for $\theta \in [0, 1]$ and this can be circumvented by stretching the interval through $\theta \mapsto \pi\theta$. Secondly, in [62] it is considered Dirichlet boundary condition at the regular boundary $\theta = 1$, but their proof works similarly for the Neumann case by changing the odd reflection done at the regular boundary $\theta = 1$ to an even reflection. Such choice of reflections is done explicitly in [6], for different boundary conditions.

Proof of Lemma 2.4.1 (dropping lemma) Firstly, we prove that u has finitely many zeros. The Lemma 2.4.2 implies that one can construct a v satisfying (25) with same type of zeros of u. Due to the dropping Lemma in [62], v has finitely many zeros and consequently u has finitely many zeros for $\theta \in [0, \theta_1]$.

To conclude that u also has finitely many zeros for $\theta \in [\theta_1, \pi]$, consider the change of coordinates $\tilde{\theta} := \pi - \theta$. The solution $u(t, \tilde{\theta})$ satisfies the equation (25) with $\tilde{\theta} \in [0, \pi - \theta_1]$, and by the dropping lemma in [62], it also has finitely many zeros for $\tilde{\theta} \in [0, \pi - \theta_1]$. Equivalently, u has finitely many zeros for $\theta \in [\theta_1, \pi]$.

Secondly, we prove that multiple zeros must drop. Suppose (t_0, θ_0) is a multiple zero of a solution $u \not\equiv 0$ of (11). By the Lemma 2.4.2, there is a function $v(t, \theta)$ having zeros of the same type as $u(t, \theta)$ for $\theta \in [0, \theta_1]$ and some fixed $\theta_1 \in (0, \pi)$.

If $\theta_0 \leq \theta_1$, then the dropping lemma in [62] implies that the number of zeros of $v(t, \theta)$ should drop. Since $v(t, \theta)$ is not zero for $\theta \in [\theta_1, \pi]$, then the zero that dropped should have occured for $\theta \in [0, \theta_1]$. This implies that some zero of $u(t, \theta)$ must have dropped, since they have the same type of zeros.

If $\theta_0 > \theta_1$, then consider the change of coordinates $\tilde{\theta} := \pi - \theta$ and the same arguments as above show that the multiple zero of $u(t, \tilde{\theta})$ must have dropped for $\tilde{\theta} \in [0, \pi - \theta_1]$.

Thirdly, we prove that the zero number is not increasing in time. We already know that it must drop at multiple zeros. Suppose (t_0, θ_0) is a simple zero, that is $u(t_0, \theta_0) = 0$ and $u_\theta(t_0, \theta_0) \neq 0$. Hence, the implicit function theorem says that $u(t, \theta(t)) = 0$ for an unique curve

$\theta(t)$ in small neighborhood of t_0 such that $\theta(t_0) = \theta_0$. Hence, the simple zero persists and no new zeros are created.

■

2.4.2 *Proof 2*

This proof is an adaptation of Angenent [6], by rescaling the solution nearby a multiple zero of multiplicity n and showing that there are n zero curves backwards in time, and less curves forwards in time. We give a sketch, for a more detailed account see [47].

For $t_0 > 0$, the *localization* of the solution $v(t,\theta)$ of (11) nearby the multiple zero (t_0, θ_0),

$$w(\tau, \xi) := e^{-\frac{\xi^2}{2}} v(t_0 - e^{-2\tau}, \theta_0 + 2e^{-\tau}\xi)$$

for $\tau \geq -\frac{1}{2}\log(t_0) =: \tau_0$. Due to the properly chosen parabolic rescaling, $w(\tau, \xi)$ satisfies

$$w_\tau = \frac{1}{2} w_{\xi\xi} + \frac{1}{2\tan(\theta_0 + 2e^{-\tau}\xi)} w_\xi - \frac{1}{2}(\xi^2 - 1)w + q(\tau, \xi)w$$

where $(\tau, \xi) \in (\tau_0, \infty) \times \mathbb{R}$ and $q(\tau, \xi)$ is bounded and decay with τ.

There are two cases: either the multiple zero is in the interior $\theta_0 \in (0, \pi)$ or in one of the boundaries $\theta_0 = 0, \pi$.

In the first case, the tangent term is regular and one can rescale this w_ξ term out by an appropriate multiplying w by an appropriate exponential. Then the arguments of Angenent [6] hold.

In the second case, there is a singular term only at one of the boundaries it is being zoomed in. One can reflect solutions along the other boundary, which is regular, and rescale the bounded terms to obtain

$$w_\tau = \frac{1}{2} w_{\xi\xi} + \frac{1}{2\xi} w_\xi - \frac{1}{2}(\xi^2 - 1)w + q(\tau, \xi)w$$

for $x \in \mathbb{R}_+$.

The operator $\frac{1}{2}w_{\xi\xi} + \frac{1}{2\xi}w_\xi$ is self-adjoint in $L^2_r([0,\infty))$ with weigth r. Due to Sturm-Liouville, the spectrum of such operator consists of simple eigenvalues and respective eigenfunctions $\phi_n(\xi) = e^{-\xi^2/2}L_n(\xi)$, where L_n is a multiple of the n-th Laguerre polynomial. This eigenvalue problem is also known in the literature as the quantum harmonic oscillator in spherical coordinates. One can then follow the proof of Angenent by simply changing the functional spaces and its basis. For a more detailed account see [47].

2.4.3 *Consequences of the dropping lemma*

Two results follow by combining the dropping lemma 2.4.1 and the asymptotic description in Theorem 2.1.2. The first is a result relating the zero number within invariant manifold and the Morse indices of equilibria. The second is the Morse-Smale property.

Theorem 2.4.3. Zero number within Invariant Manifolds [82], [18]
Consider a equilibria $u_\pm \in \mathcal{E}$ and a trajectory $u(t) \not\equiv u_\pm$ of (8)*. Then,*

1. *If $u(t) \in W^u(u_-)$, then $i(u_-) > z^t(u - u_-)$.*
2. *If $u(t) \in W^s_{loc}(u_+)$, then $z^t(u - u_+) \geq i(u_+)$.*
3. *If $u(t) \in W^u(u_-) \cap W^s_{loc}(u_+)$, then*

$$i(u_+) \leq z^t(u - u_\pm) < i(u_-).$$

These results also hold by replacing $u(t) - u_*$ with the tangent vector u_t.

The above theorem implies that (8) has no homoclinic orbits. Indeed, if there were any, then $i(u_*) < i(u_*)$, which is a contradiction.

Proof The proof from [18] is sketched.

Firstly we prove the claim regarding the unstable manifold, since it is finite dimensional. Consider time $t > -t_j$, where $t_j > 0$. Noticing that a normalization factor doesn't change the number of zeros and using the dropping lemma 2.4.1, we obtain

$$z^{-t_j}\left(\frac{u - u_-}{||u - u_-||}\right) = z^{-t_j}(u - u_-) \geq z^t(u - u_-)$$

Theorem 2.1.2 guarantees that if $u(t) \in W^u_k(u_-)\backslash W^u_{k-1}(u_-)$, then $\frac{u-u_-}{||u-u_-||} \to \pm\phi_k$ as $-t_j \to -\infty$, in $C^1_w([0,\pi])$, where the eigenfunction $\pm\phi_k$ has k simple zeros, due to Sturm-Liouville theory. Moreover, $z^{-t_j}(.)$ is constant in a C^1_w neighborhood of any C^1_w function with only simple zeros. This implies that the above left hand side equals $z^{-t_j}(\pm\phi_k) = k$, for t_j sufficiently large, yielding

$$k \geq z^t(u - u_-). \tag{26}$$

One can repeat this argument for each layer within the filtration of the unstable manifold $W^u(u_-)$ as in Theorem 2.1.1, yielding the inequality (26) for each $k \leq i(u_-) - 1$. Therefore, the zero number in the unstable manifold $W^u(u_-)$ is less than the biggest of all these values, namely, $i(u_-)$. This yields the first part of the theorem.

Now we prove the second claim of the theorem regarding the stable manifold. Similar arguments as before hold, yielding

$$z^t(u - u_+) \geq k$$

for each finite dimensional $W^s_k(u_+)\backslash W^s_{k+1}(u_+)$.

This does not guarantee that such inequality holds on the whole stable manifold $W^s(u_+)$, since the set $\cap_{k \geq i(u_+)} W^s_k(u_+) \subseteq W^s(u_+)$ is not necessarily empty. Hence, one can not deduce the inequality in the full stable manifold from the same inequality for sets within its filtration.

Note that the attractor $\mathcal{A}$ is finite dimensional. Hence $W^s(u_+) \cap \mathcal{A}$ is finite dimensional, and therefore the above arguments from the unstable manifold holds for subsets of the stable manifold within the attractor.

But we continue to give a sketch of the proof for the most general case, that is, for solutions in the infinite dimensional $W^s(u_+)$. To circumvent the problem of infinite dimensionality, consider $u(t) \in W^s_k(u_+)$ with $k \geq i(u_+)$. By the dropping lemma, there exists a $t_* > 0$ such that $u(t) - u_+$ has finitely many zeros for all $t > t_*$, and

$$z^t(u - u_+) \geq z^{t_*}(u - u_+)$$

for $t < t_*$.

Moreover, it is shown in [19] that $u(t) - u_+$ has only simple zeros for an open dense set in $[t_*, \infty)$. Due to the dropping lemma, $u(t) - u_+$ has finitely many zeros and is non-increasing for $t > t_*$, decreasing strictly at multiple zeros. Therefore, it can only drop finitely many times. Hence, no dropping occurs for large enough t_*, and the number of zeros is constant for $t > t_*$. From now on we suppose that t_* is large so that these remarks hold.

Let t_* be large enough, then $||u(t_*) - u_+||_{C^1_w}$ is small enough, and hence there exists $\tilde{u} \in W^s_m(u_+) \backslash W^s_{m+1}(u_+)$ such that

$$z(\tilde{u} - u_+) = z^{t_*}(u - u_+)$$

for some $m \in \mathbb{N}_0$, since $\dim(W^s_m(u_+) \backslash W^s_{m+1}(u_+)) = 1$ and all elements in this space have m zeros for $t > t_*$, since no more dropping occurs.

For $\tilde{u} \in W^s_m(u_+) \backslash W^s_{m+1}(u_+)$, the bound

$$z(\tilde{u} - u_+) \geq m$$

holds. Moreover, this holds for any $m \geq i(u_+)$, which proves the desired theorem.

For the last part, we apply the first two claims together as follows. For $u(t) \in W^u(u_-)$, then $i(u_-) > z^t(u - u_-)$ for all $t \in \mathbb{R}$. By the dropping lemma, for $t < \tau$, we have that $z^t(u - u_-) \geq z^\tau(u - u_-)$. In particular, for large $\tau > 0$, we obtain $z^\tau(u - u_-) = z^\tau(u_- - u_+)$, yielding

$$i(u_-) > z^t(u - u_-) \geq z^\tau(u_- - u_+)$$

for all $t < \tau$. Repeating the arguments above, one obtain that

$$z^t(u_- - u_+) \geq i(u_+).$$

This proves that $i(u_-) > z^t(u - u_-) \geq i(u_+)$ for all $t \in [-\tau, \tau]$ with sufficiently large τ.

■

Theorem 2.4.4. Morse-Smale Property [42], [5], [33]
Consider two hyperbolic equilibria u_- *and* u_+ *with respective Morse indices* $i(u_-), i(u_+)$. *If* $W^u(u_-) \cap W^s(u_+) \neq \emptyset$, *then such intersection is transverse. Moreover,* $W^u(u_-) \cap W^s(u_+)$ *is an embedded submanifold of dimension* $i(u_-) - i(u_+)$.

This last theorem implies that if the semigroup has a finite number of equilibria, in which all are hyperbolic, then it is a Morse-Smale system in the sense of [38]. Note that this property can hold even in case the equilibria are not hyperbolic, as in [42].

Proof The sketch of the proof follows [5].

Consider $u_0 \in W^u(u_-) \cap W^s(u_+)$, that is, there exists a trajectory $u(t)$ with initial data u_0 such that $\lim_{t\to\pm\infty} u(t) = u_\pm$. To prove trsversality, we need to show that

$$X = T_{u_0}W^u(u_-) + T_{u_0}W^s(u_+).$$

The idea is to construct a linear subspace $L \subseteq T_{u_0}W^u(u_-)$ such that $\dim(L) = \operatorname{codim}(T_{u_0}W^s(u_+))$ and L does not intersect $T_{u_0}W^s(u_+)$.

The hyperbolicity of u_+ yields that

$$X = T_{u_0}W^u(u_+) + T_{u_0}W^s(u_+)$$

and hence $\dim(L) = i(u_+)$. Since L does not intersect $T_{u_0}W^s(u_+)$, we have that

$$X = L + T_{u_0}W^s(u_+)$$

which finishes the proof of transversality, since $L \subseteq T_{u_0}W^s(u_-)$.

We now construct a space L with the above properties. Theorem 2.4.3 implies that $i(u_-) \geq z(u_t(0)) > i(u_+)$, and hence we consider the linear subspace

$$L := span\{\phi_k \mid k = 0, ..., i(u_+) - 1\} \subset T_{u_0}W^u(u_-)$$

where ϕ_k are the eigenfunctions of the linearization at u_0. Hence $\dim(L) = i(u_+)$ and since L is a linear subspace,

$$z(\psi) = z\left(\sum_{k=0}^{i(u_+)-1} c_k\phi_k\right) < i(u_+)$$

for any $\psi \in L$ with $c_k \in \mathbb{R}$, as in [18].

On the other hand, suppose towards a contradiction that ψ also lies in $T_{u_0}W^s(u_+)$. Consider a solution $\psi(t)$ of the linear equation (11) with initial data ψ, hence the dropping lemma implies that

$$z^0(\psi) \geq z^t(\psi) \geq i(u_+)$$

for any $t \geq 0$, a contradiction.

Therefore,

$$L \cap T_{u_0} W^s(u_+) = \emptyset.$$

Lastly, $W^u(u_-)$ and $W^s(u_+)$ are embedded submanifolds. Hence their intersection is also an embedded submanifold and has dimension as follows

$$\begin{aligned}\dim(W^u(u_-) \cap W^s(u_+)) &= \dim(W^u(u_-)) - \dim(W^s(u_+)^{\perp}) \\ &= i(u_-) - i(u_+).\end{aligned}$$

which is positive, since $i(u_-) \geq z(u_t(0)) > i(u_+)$ as in Theorem 2.4.3.

■

2.5 STURM GLOBAL STRUCTURE

This section gathers all the tools developed in the previous sections in order to construct the attractor for the parabolic equation with degenerate coefficients (8) and prove the second part of the main Theorem 2.0.1.

Its proof is a consequence of two propositions. Firstly, due to the *cascading principle*, it is enough to construct all heteroclinics between equilibria such that their Morse indices differ by 1. Secondly, on one direction, the *blocking principle*: some conditions imply that there does not exist a heteroclinic connection; on the other direction, the *liberalism principle*: if those conditions are violated, then there exists a heteroclinic.

The cascading and blocking principles follow from the dropping lemma and Morse-Smale property from Section 2.4, and we give a sketch as in [29]. There is only a mild modification in the proof of the liberalism principle in Proposition 2.5.2.

Proposition 2.5.1. Cascading Principle [29]
There exists a heteroclinic between two equilibria u_- and u_+ such that $n := i(u_-) - i(u_+) > 0$ if, and only if, there exists a sequence (cascade) of equilibria $\{v_k\}_{k=0}^n$ with $v_0 := u_-$ and $v_n := u_+$, such that the following holds for all $k = 0, ..., n-1$

1. $i(v_{k+1}) = i(v_k) + 1$
2. *There exists a heteroclinic from v_{k+1} to v_k*

Proposition 2.5.2. Blocking and Liberalism Principles [29]
There exists a heteroclinic connection between equilibria u_- and u_+ with $i(u_-) = i(u_+) + 1$ if, and only if,

1. Morse blocking: $z(u_- - u_+) = i(u_+)$,
2. Zero number blocking: $z(u_- - u_*) \neq z(u_+ - u_*)$ *for all equilibria u_* between u_- and u_+ along M^u_θ for some $\theta \in [0, \pi]$.*

The blocking and liberalism principles assert that the Morse indices $i(.)$ and zero numbers $z(.)$ construct the global structure of the attractor explicitly. Those numbers can be obtained through the Sturm permutation, as in Section 2.3.

In particular, one can check the zero number blocking for $\theta = 0$ as it is done in [29]. We prefer to state the condition for some $\theta \in [0, \pi]$ because the Sturm permutation in Section 2.3 labels the equilibria along M^u_θ and M^s_θ for some $\theta \in [0, \pi]$. Moreover, those curves are computed for $\theta = \pi/2$ for the Chafee-Infante example in Section 2.6

We now show that u_* lies in between u_- and u_+ at $\theta = 0$ if, and only if it is also between $u_\pm$ along M^u_θ for any $\theta \in [0, \pi]$. Indeed, due to continuity with respect to the initial data $(0, a, 0) \in \mathbb{R}^3$ of the shooting flow (18), the curve M^u_θ for fixed $\theta \in [0, \pi)$ is continuous and the order of $a \in \mathbb{R}$ induces an order along M^u_θ, hence the parametrization respects its labeling. At $\theta = \pi$, continuity also yields an ordering of the equilibria within M^u_θ.

Note one can replace M^u_θ in the zero number blocking by M^s_θ, since imilar arguments as above hold and show that u_* lies in between u_- and u_+ at $\theta = \pi$ if, and only if it is also between $u_\pm$ along M^s_θ for some $\theta \in [0, \pi]$.

Proof of Proposition 2.5.1

($\Longleftarrow$) Theorem 2.4.4 guarantees that the dynamical system generated by (8) is Morse-Smale, which has the following transitivity relation: if there exists a heteroclinics between u_- and v, and another between v and u_+, then there also exists a heteroclinic connecting u_- and u_+. See [59].

($\Longrightarrow$) Consider two hyperbolic equilibria u_- and u_+ with Morse indices $i(u_-)$ and $i(u_+)$. Denote set of heteroclinic connections by

$$\Sigma := W^u(u_-) \cap W^s(u_+)) \neq \emptyset$$

where $\dim(\Sigma) = i(u_-) - i(u_+) =: n$. The proof follows by induction on n. The statement is trivial for $n = 1$. Suppose that the theorem holds for all $1, ..., n-1$ with $n \geq 2$, and it is shown that it also holds for n.

Denote the boundary $\partial\Sigma := \overline{\Sigma} \backslash \Sigma$, where the closure is understood in the $H^2_w([0, \pi])$-topology of $X = L^2_w([0, \pi])$ with weight $w = \sin(\theta)$. Note that $\partial\Sigma$ is closed, $\overline{\Sigma}$ is invariant and $u_-, u_+ \in \partial\Sigma$.

Firstly, it is shown that there exists an equilibrium $v \in \partial\Sigma \backslash \{u_-, u_+\}$. Suppose towards a contradiction that it does not exist such v. Then, one can construct $u_0 \in \Sigma$ near u_- such that $z(u_0 - u_-) = i(u_+)$ and $u_0 - u_-$ have a different sign than $u_+ - u_-$ at $\theta = 0$, see [29]. Consider a trajectory $u(t) \in \Sigma$ through u_0 which satisfies

$$z^t(u - u_-) \geq i(u_+) = z(u_0 - u_-)$$

since $u(t) \in W^s(u_-) \subset \Sigma$ and it is known how the zero number behaves within invariant manifolds by Theorem 2.4.3.

On the other hand, the different sign condition at $\theta = 0$ implies $u(t) - u_-$ has a zero at the boundary $\theta = 0$ for some time t_0. Moreover, Neumann boundary conditions enforce this zero to be a multiple zero. The dropping lemma 2.4.1 implies

$$z^t(u - u_-) < z(u_0 - u_-)$$

for $t > t_0$. Both these inequalities yield the desired contradiction.

Lastly, for equilibria $v \in \partial\Sigma \backslash \{u_-, u_+\}$, there are heteroclinic conections from u_- to v, and from v to u_+. This holds for Morse-Smale systems, see [14]. Moreover, connections occur from an equilibrium with higher to another with lower Morse indices, hence $i(u_-) > i(v)$. Subtracting $i(u_+)$ from both sides, yields $n > i(v) - i(u_+)$, by definition of n. The induction hypothesis implies that there is a cascade between v and u_+. Similarly, one obtain a cascade from u_- to v. Joining both these cascades yield the full cascade.

■

For the proof of the liberalism theorem, it is used the Conley index to detect orbits between u_- and u_+. We give a brief introduction of Conley's theory, and how it can be applied in this context. See Chapters 22 to 24 in [80] for a brief account of the Conley index, and its extension to infinite dimensional systems in [71].

Consider the space $\mathcal{X}$ of all topological spaces and the equivalence relation given by $Y \sim Z$ for $Y, Z \in \mathcal{X}$ if, and only if Y is homotopy equivalent to Z, that is, there are continuous maps $f : Y \to Z$ and $g : Z \to Y$ such that $f \circ g$ and $g \circ f$ are homotopic to id_Z and id_Y, respectively. Then, the quotient space $\mathcal{Y}/\sim$ describes the homotopy equivalent classes $[Y]$ of all topological spaces which have the same homotopy type. Intuitively, $[Y]$ describes all other topological spaces which can be continuously deformed into Y.

Suppose Σ is an invariant isolated set, that is, it is invariant with respect to positive and negative time of the semiflow, and it has a closed neighborhood N such that Σ is contained in the interior of N with Σ being the maximal invariant subset of N.

Denote $\partial_e N \subset \partial N$ the *exit set of* N, that is, the points which are not strict ingressing in N,

$$\partial_e N := \{u_0 \in N \mid u(t) \notin N \text{ for all sufficiently small } t > 0\}.$$

The *Conley index* is defined as

$$C(\Sigma) := [N/\partial_e N]$$

namely the homotopy equivalent class of the quotient space of the isolating neighborhood N relative to its exit set $\partial_e N$. Such index is homotopy invariant and does not depend on the particular choice of isolating neighborhood N.

We compute the Conley index for two examples.

Firstly, the Conley index of a hyperbolic equilibria u_+ with Morse index n. Consider a closed ball $N \subset X$ centered at u_+ without any other equilibria in N, as isolating neighborhood. The flow provides a homotopy that contracts along the stable directions to the equilibria u_+. Then, N is homotoped to a n-dimensional ball B^n in the finite dimensional space spanned by the first n eigenfunctions, related to the unstable directions. Note the exit set $\partial_e B^n = \partial B^n = \mathbb{S}^{n-1}$, since after the homotopy there is no more stable direction and the equilibria is hyperbolic. Therefore, the quotient of a n-ball and its boundary is an n-sphere,

$$C(u_+) = [N/\partial_e N] = [B^n/\partial_e B^n] = [B^n/\mathbb{S}^{n-1}] = [\mathbb{S}^n].$$

Secondly, the Conley index of the union of two disjoint invariant sets, for example u_- and u_+ with respective disjoint isolating neighborhoods N_- and N_+. Then, $N_- \cup N_+$ is an isolating neighborhood of $\{u_-, u_+\}$. By definition of the wedge sum

$$\begin{aligned} C(\{u_-, u_+\}) &= \left[\frac{N_- \cup N_+}{\partial_e(N_- \cup N_+)}\right] \\ &= \left[\frac{N_-}{\partial_e N_-} \vee \frac{N_+}{\partial_e N_+}\right] = C(u_-) \vee C(u_+). \end{aligned}$$

The Conley index can be applied to detect heteroclinics as follows. Construct a closed neighborhood N such that its maximal invariant subspace is the closure of the set of heteroclinics between $u_\pm$,

$$\Sigma = \overline{W^u(u_-) \cap W^s(u_+)}.$$

Suppose, towards a contradiction, that there are no heteroclinics connecting u_- and u_+, that is, $\Sigma = \{u_-, u_+\}$. Then, the index is given by the wedge sum $C(\Sigma) = [\mathbb{S}^n] \vee [\mathbb{S}^m]$, where n, m are the respective Morse index of u_- and u_+.

If, on the other hand, one can prove that $C(\Sigma) = [0]$, where $[0]$ means that the index is given by the homotopy equivalent class of a point. This would yield a contradiction and there should be a connection between u_- and u_+. Moreover, the Morse-Smale structure excludes connection from u_+ to u_-, and hence there is a connection from u_- to u_+.

Hence, there are three ingredients missing in the proof: the Conley index can be applied at all, the construction of a isolating neighborhood N of Σ and the proof that $C(\Sigma) = [0]$.

Proof of Proposition 2.5.2

($\Longrightarrow$) This part is called blocking in [29]. Consider a heteroclinic $u(t)$ connecting u_- to u_+ such that $i(u_-) = i(u_+) + 1$.

In order to prove the Morse blocking, note that for large $t > 0$,

$$z^t(u_- - u) = z(u_- - u_+) = z^{-t}(u - u_+).$$

Combining with Theorem 2.4.3,

$$i(u_+) \leq z(u_- - u_+) < i(u_-) = i(u_+) + 1.$$

To prove the zero number blocking, suppose towards a contradiction that there exists an equilibrium u_* between u_- and u_+ at $\theta = 0$ such that $z(u_- - u_*) = z(u_+ - u_*)$. For large $t > 0$, we have that

$$z^{-t}(u - u_*) = z(u_- - u_*) = z(u_+ - u_*) = z^t(u - u_*).$$

On the other hand, for large $t > 0$, $u(-t)$ is close to u_-, and $u(t)$ is close to u_+ at $\theta = 0$. This means that if u_* is between u_- and u_+ at $\theta = 0$, then $u(t) - u_*$ changes sign at $\theta = 0$, as t increases. Hence the profile $u(t) - u_*$ has a zero at $\theta = 0$ for some time, and it is a multiple zero due to Neumann boundary conditions. The dropping lemma 2.4.1 implies that the number of zeros drops,

$$z^{-t}(u - u_*) > z^t(u - u_*)$$

for large $t > 0$. The last two equations yield a contradiction.

($\Longleftarrow$) This is also called liberalism in [29]. Consider hyperbolic equilibria u_-, u_+ such that $i(u_-) = i(u_+) + 1$ and satisfies both the Morse and the zero number blocking. Without loss of generality, assume $u_-(0) > u_+(0)$.

It is used the Conley index to detect orbits between u_- and u_+. Note that the semiflow generated by the equation (8) on the Banach space X is admissible for the Conley index theory in the sense of [71], due to a compactness property that is satisfied by the parabolic equation (8), namely that trajectories are precompact in phase space. See Theorem 3.3.6 in [41].

As mentioned above, in order to apply the Conley index concepts we need to construct appropriate neighborhoods and show that the Conley index is $[0]$.

Consider the closed set

$$K(u_\pm) := \left\{ u \in X \mid \begin{array}{c} z(u - u_-) = i(u_+) = z(u - u_+) \\ u_+(0) \leq u(0) \leq u_-(0) \end{array} \right\}$$

Consider also closed ϵ-balls $B_\epsilon(u_\pm)$ centered at $u_\pm$ such that they don not have any other equilibria besides $u_\pm$, respectively, for some $\epsilon > 0$.

Define

$$N_\epsilon(u_\pm) := B_\epsilon(u_-) \cup B_\epsilon(u_+) \cup K(u_\pm).$$

The zero number blocking condition implies there are no equilibria in $K(u_\pm)$ besides possibly u_- and u_+. Hence, $N_\epsilon(u_\pm)$ also has no equilibria besides u_- and u_+.

Denote Σ the maximal invariant subset of N_ϵ. We claim that Σ is the set of the heteroclinics from u_- to u_+ given by $\overline{W^u(u_-) \cap W^s(u_-)}$.

On one hand, since Σ is globally invariant, then it is contained in the attractor $\mathcal{A}$, which consists of equilibria and heteroclinics. Since there are no other equilibria in $N_\epsilon(u_\pm)$ besides $u_\pm$, then the only heteroclinics that can occur are between them.

On the other hand, Theorem 2.4.3 implies that along a heteroclinic $u(t) \in \mathcal{H}$ the zero number satisfies $z^t(u - u_\pm) = i(u_+)$ for all time, since $i(u_-) = i(u_+) + 1$. Therefore $u(t) \in K(u_\pm)$ and the closure of the orbit is contained in $N_\epsilon(u_\pm)$. Since the closure of the heteroclinic is invariant, it must be contained in Σ.

Lastly, it is proven that $C(\Sigma) = [0]$ in three steps, yielding the desired contradiction and the proof of the theorem. We modify the first and second step from [29], whereas the third remain the same.

In the first step, a model is constructed displaying a saddle-node bifurcation with respect to a parameter μ, for $n := z(u_+ - u_-) \in \mathbb{N}$ fixed,

$$0 = [v_{\xi\xi} + \frac{1}{\tan(\xi)} v_\xi] P_n^2 + \lambda_n v + g_n(\mu, \xi, v, v_\xi) \tag{27}$$

where $\xi \in [0, \pi]$ has Neumann boundary conditions, $\lambda_n = n(n+1)$ are the eigenvalues of the axisymmetric laplacian with the Legendre polynomials $P_n(cos(\xi))$ as eigenfunctions, and

$$g_n(\mu, \xi, v) := \left[v^2 - \mu P_n^2\right] P_n.$$

For $\mu > 0$, the equilibria of (27) are $v_\pm = \pm\sqrt{\mu} P_n(\cos(\xi))$, since P_n are the eigenfunctions of the axially symmetric Laplacian.

$$z(v_+ - v_-) = z(u_+ - u_-) =: n \tag{28}$$

since the n intersections of v_- and v_+ will be at its n zeroes.

Moreover, $v_\pm$ are hyperbolic equilibria for small $\mu > 0$, such that $i(v_+) = n + 1$ and $i(v_-) = n$. Indeed, parametrize the bifurcating branches by $\mu = s^2$ so that $v(s, \xi) = s P_n(\cos(\xi))$, where $s > 0$ correspond to v_+ and $s < 0$ to v_-. Linearizing at the equilibrium $v_\pm$ yields the following linear operator

$$L_n(s) v := \left[v_{\xi\xi} + \frac{1}{\tan(\xi)} v_\xi + (\lambda_n + 2s) v \right] P_n^2.$$

This operator can be seen as a Sturm-Liouville eigenvalue problem in the space L^2_w with weight $w(\xi) = \sin(\xi) P_n^2$, namely

$$(\eta - 2s) \sin(\xi) P_n^2 v = \left[v_\xi \sin(\xi) \right]_\xi + \sin(\xi) \lambda_n v \right] P_n^2.$$

Hence, for each n fixed, $\eta_n(s) = 2s$ is an eigenvalue with $P_n(\cos(\xi))$ its corresponding eigenfunction, since the terms on the right hand side yield the eigenvalue problem for the axisymmetric Laplacian and vanish.

We now use a perturbation argument in Sturm-Liouville theory. For $\mu = s^2 = 0$, the eigenvalues of $L_n(0)$ in L^2_w coincide with the eigenvalues of the usual axisymmetric laplacian such that there is one eigenvalue $\eta_n(0) = 0$ and n positive eigenvalues. For small $\mu < 0$, the number of positive eigenvalues persist, and there is no eigenvalue 0, since $\eta(\mu) < 0$; whereas for small $\mu > 0$, the number of positive eigenvalues increases by 1, and there is no eigenvalue 0, since $\eta(\mu) > 0$. This yields the desired claim about hyperbolicity and the Morse index.

Now consider the semilinear parabolic equation such that (27) is its equilibria equation. The equilibria $v_\pm$ together with their connecting orbits of the corresponding evolution equation form an isolated invariant set

$$\Sigma_\mu := \overline{W^u(v_-) \cap W^s(v_+)}$$

with isolating neighborhood $N_\epsilon(v_\pm)$, and the bifurcation parameter can also be seen as a homotopy parameter. Hence the Conley index is of a point by homotopy invariance as desired, that is,

$$C(\Sigma_\mu) = C(\Sigma_0) = [0]. \tag{29}$$

In the second step, the equilibria v_- and v_+ are transformed respectively into u_- and u_+ via a diffeomorphism which is not a homotopy.

Recall $n = z(v_- - v_+) = z(u_+ - u_-)$. Hence, choose $\theta(\xi)$ a smooth diffeomorphism of $[0, \pi]$ that maps the zeros of $v_- - v_+$ to the zeros of $u_- - u_+$. Therefore, from now on we suppose that the zeros of $v_- - v_+$ and $u_- - u_+$ occur in the same points in $\theta \in [0, \pi]$.

Consider the transformation

$$\begin{aligned} \Theta : X &\to X \\ v &\mapsto \alpha[v - v_-] + u_- \end{aligned}$$

where α is defined pointwise through

$$\alpha := \begin{cases} \frac{u_+ - u_-}{v_+ - v_-} & \text{, if } v_+ \neq v_- \\ \frac{\partial_\theta(u_+ - u_-)}{\partial_\theta(v_+ - v_-)} & \text{, if } v_+ = v_- \end{cases}$$

such that the coefficient α is smooth and nonzero due to the l'Hôpital rule. Hence, $\Theta(v_-) = u_-$ and $\Theta(v_+) = u_+$ as desired.

Moreover, the number of intersections of functions is invariant under the maps Θ and θ, that is,

$$z(\Theta(v(\theta) - \tilde{v}(\theta))) = z(v(\xi) - \tilde{v}(\xi)) \tag{30}$$

and hence $K(v_\pm)$ is mapped to $K(u_\pm)$ under Θ.

Through the map θ, the model equation (27) is mapped to the equation (8) with different diffusion coefficients and nonlinearity, namely

$$u_t = (\theta_\xi)^2 u_{\theta\theta} + \frac{\theta_\xi}{\tan(\theta)} u_\theta + f(\theta, u, u_\theta \theta_\xi) + u_\theta \theta_{\xi\xi}$$

where the Neumann boundary conditions are preserved.

Through the map Θ, the model equation (27) is mapped to

$$\begin{aligned} u_t = \alpha u_{\theta\theta} + \frac{\alpha}{\tan(\theta)} u_\theta + f(\theta, \Theta(u), \partial_\theta \Theta(u)) \\ + 2\alpha_\theta \partial_\theta [u - v_-] + \alpha_{\theta\theta}[u - v_-] + \partial_{\theta\theta} u_- \\ + \frac{\alpha_\theta [u - v_-] + \partial_\theta u_- - \partial_{\theta\theta} v_-}{\tan(\theta)} \end{aligned}$$

where the Neumann boundary conditions are preserved.

In other words, the maps Θ and θ yield and equation of the type

$$u_t = a(\theta) u_{\theta\theta} + b(\theta) \frac{u_\theta}{\tan(\theta)} + \tilde{f}(\theta, u, u_\theta)$$

where the coefficients $a(\theta)$ and $b(\theta)$ are obtained by combining the diffusion within the above two equations, whereas the remaining terms are collected in order to define the reaction term $\tilde{f}$.

Moreover, the set Σ_μ is transformed into an isolated invariant set $\tilde{\Sigma}$ with invariant neighborhood $N_\epsilon(u_\pm)$ such that

$$C(\Sigma_\mu) = C(\Theta(\Sigma_\mu)) = C(\tilde{\Sigma}). \tag{31}$$

In the third step, we homotope the diffusion coefficients $a(\theta), b(\theta)$ and nonlinearity $\tilde{f}$ to be the standard axisymmetric diffusion and the desired reaction f from the equation (8). Indeed, consider the semilinear parabolic equation

$$u_t = a^\tau(\theta) u_{\theta\theta} + b^\tau(\theta) \frac{u_\theta}{\tan(\theta)} + \tilde{f}^\tau(\theta, u, u_\theta)$$

where

$$\begin{aligned} a^\tau(\theta) &:= \tau a(\theta) + 1 - \tau \\ b^\tau(\theta) &:= \tau b(\theta) + 1 - \tau \\ f^\tau(\theta, u, u_\theta) &:= \tau \tilde{f} + (1 - \tau) f + \sum_{i=-\,,\,+} \chi_{u_i} \mu_{u_i}(\tau)[u - u_i(\theta)] \end{aligned}$$

and χ_{u_i} are cut-offs begin 1 nearby u_i and zero far away, the coefficients $\mu_i(\tau)$ are zero near $\tau = 0$ and 1 and shift the spectra of the linearization at $u_\pm$ such that uniform hyperbolicity of these equilibria is guaranteed during the homotopy.

Consider u_-, u_+ and their connecting orbits during this homotopy,

$$\Sigma^\tau := \overline{W^u(u_-^\tau) \cap W^u(u_+^\tau)}.$$

Note that $\Sigma^\tau \subseteq K(u_\pm)$, since the dropping lemma holds throughout the homotopy. Variying τ, the equilibria, $u_\pm$ do not bifurcate due to normal hyperbolicity. Choosing $\epsilon > 0$ small enough, the neighborhoods $N_\epsilon(u_\pm)$ is an isolating neighborhood of Σ^τ throughout the homotopy. Indeed, Σ^τ can never touch the boundary of $K(u_\pm)$, except at the points $u_\pm$ by the dropping lemma. Once again the Conley index is preserved by homotopy invariance, yielding

$$C(\Sigma) = C(\Sigma^0) = C(\Sigma^\tau) = C(\Sigma^1) = C(\tilde{\Sigma}). \tag{32}$$

Finally, the equations (29), (31) and (32) yield that the Conley index of Σ is the homotopy type of a point, and hence the desired result.

■

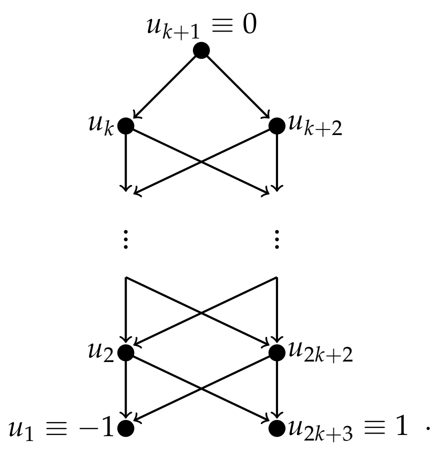
2.6 EXAMPLE: AXISYMMETRIC CHAFEE-INFANTE

In this section it is given an example of the theory above, namely, it is constructed the attractor of the axially symmetric Chafee-Infante problem,

$$u_t = u_{\theta\theta} + \frac{1}{\tan(\theta)} u_\theta + \lambda u[1 - u^2] \tag{33}$$

where $\theta \in [0, \pi]$ has Neumann boundary conditions and initial data $u_0 \in H^\alpha_w([0, \pi])$ with $\alpha > 3/2$ and weight $w(\theta) := \sin(\theta)$, so that the equation generates a dynamical system in such space, as in [41].

Theorem 2.6.1. *For $\lambda \in (\lambda_k, \lambda_{k+1})$, where λ_k is the k-th eigenvalue of the axially symmetric Laplacian with $k \in \mathbb{N}_0$, the axially symmetric Chafee Infante problem* (33) *has $2k + 3$ hyperbolic equilibria $u_1, ..., u_{2k+3}$ and the attractor $\mathcal{A}$ is given by*



We will prove that the Sturm permutation for the axially symmetric case is the same as the Sturm permutation for the regular case in [29]. Hence, both attractors are geometrically (connection-wise) the same. The only difference lies in the equilibria, and the parameter λ must lie between two eigenvalues of the appropriate diffusion operator.

The proof is divided in the upcoming subsections. Following the shooting arguments in Section 2.3, we explicitly construct the shooting manifolds. Then we count how many times they intersect, yielding all equilibria, and if such intersections are transverse, yielding hyperbolicity. Lastly those intersection points are labeled accordingly, yielding the permutation σ and hence the attractor $\mathcal{A}$.

The equilibria equation describing the shooting curves is

$$\begin{cases} u_\tau &= p \\ p_\tau &= -\lambda u[1-u^2]\sin^2(\theta) \\ \theta_\tau &= \sin(\theta). \end{cases} \tag{34}$$

Solutions of the shooting (34) exist for all $\theta \in [0,\pi]$ and all initial data $a \in L_0 := \{(0,a,0) \mid a \in \mathbb{R}\}$ or $b \in L_\pi := \{(\pi,b,0) \mid b \in \mathbb{R}\}$. Indeed, solutions of (34) are bounded, since one can compare solutions of the axially symmetric Chafee-Infante equation (34) with the usual Chafee-Infante ODE, which is known to have global bounded solutions. Indeed, $p_\tau = f(u)\sin^2(\theta) < f(u) = q_\tau < \infty$ where p_τ is related to the axially Chafee-Infante and q_τ to the regular Chafee-Infante. Hence, bounded solutions exist globally in time.

This system possesses two symmetries, namely invariance under

$$\textit{time reversal: } \tau \mapsto -\tau, \tag{35}$$

$$\textit{reflection: } u \mapsto -u, \tag{36}$$

where both symmetries also changes the sign of $p := u_\tau$.

2.6.1 *Construction of the shooting curves*

Note that the stable shooting manifold M^s is obtained through the time reversal (35), which is simply a reflection in the p-axis of the unstable shooting manifold M^u.

In order to construct the unstable shooting manifold M^u, we analyze four regions for the initial data $(0,a,0) \in \mathbb{R}^3$ constrained to the trivial equilibria $a \equiv -1,0,1$, for all $\lambda > 0$.

Note also that part of the unstable shooting manifold M^u, namely when $a < 0$, is obtained through a rotation by π, fixing the origin, of the the piece of the shooting manifold M^u when $a > 0$, due to the reflectional symmetry (36).

If $a > 1$, then the corresponding solution remains bigger than 1 for small time by continuity. Hence, the shooting flow (34) implies that $p_\tau > 0$ and the shooting manifold $M^u|_{a>1}$ increases in the p direction as θ increases.

For $a \in (0,1)$, we will show that the unstable shooting manifold $M^u|_{a\in(0,1)}$ winds around the trivial equilibria $a \equiv 0$. More precisely, the angle and radius of the shooting manifold in polar coordinates are monotone with respect to its parametrization given by the initial data $a \in \mathbb{R}$.

This was proved in [39] using the Hamiltonian structure of the Chafee-Infante system, which can not be applied for the system (34), since it is nonautonomous. Instead, we adapt ideas of [48].

Indeed, the shooting flow (34) in polar coordinates with the clockwise angle, $(u, p) =: (\rho\cos(\mu), -\rho\sin(\mu))$, is given by

$$\begin{cases} \rho_\tau &= \rho\sin(\mu)\cos(\mu)\left[\lambda(1-\rho^2\cos^2(\mu))\sin^2(\theta)-1\right] \\ \mu_\tau &= \sin^2(\mu)+\lambda[1-\rho\cos^2(\mu)]\sin^2(\theta)\cos^2(\mu) \\ \theta_\tau &= \sin(\theta) \end{cases} \tag{37}$$

with $\lim\limits_{\tau=-\infty}\mu(\tau) = 0$ describing L_0. Note that

$$\mu_\tau > 0 \tag{38}$$

for $|\rho| < 1$, that is, the angle μ is increasing in τ and each solution within the shooting manifolds are winding around the trivial equilibria 0 as τ increases.

Consider the map $F(\lambda, \theta, \rho, \mu) : R^4 \to R^3$, where each coordinate F_i correspond to the i-th line of the right-hand side in (37). Note that F is a Lipschitz function, since it is a composition of Lipschitz maps.

Now, we show the monotonicity of the angle μ with respect to the initial data $a \in (0,1)$, that is, the angle μ decreases as a increases. This means that the bigger the initial data $a \in (0,1)$, smaller the angle, hence outer orbits rotate slower than inner orbits.

Lemma 2.6.2. *Let $(\rho(\tau), \mu(\tau))$ and $(\tilde\rho(\tau), \tilde\mu(\tau))$ be solutions of (37) corresponding to different initial data, that is, $\lim_{\tau\to-\infty}(\rho(\tau), \mu(\tau)) = (a, 0)$ and $\lim_{\tau\to-\infty}(\tilde\rho(\tau), \tilde\mu(\tau)) = (\tilde a, 0)$ such that $a < \tilde a$, for $a, \tilde a \in (0,1)$.*

Then

$$\mu(\tau) > \tilde\mu(\tau) \tag{39}$$

and

$$\rho(\tau) < \tilde\rho(\tau) \tag{40}$$

for all $\tau \in \mathbb{R}$. Moreover, if $\lambda > \tilde\lambda$ in (37), then

$$\mu(\lambda) > \mu(\tilde\lambda) \tag{41}$$

for all $\tau \in \mathbb{R}$ and fixed initial data a.

Proof Firstly we show a weaker version of (39) with a non strict inequality, namely

$$\mu(\tau) \geq \tilde\mu(\tau) \tag{42}$$

for all $\tau \in (-\infty, \infty)$.

Suppose, towards a contradiction, that

$$\mu(\tau_1) < \tilde\mu(\tau_1) \tag{43}$$

for some $\tau_1 \in (-\infty, \infty)$. We will extend such inequality for $\tau \in (\tau_2, \tau_1)$ for some $\tau_2 < \tau_1$.

In order to extend (43), note that for τ large and negative, the flow of the angle in (37) is given by its linearization,

$$\mu_\tau = \sin^2(\mu) + \lambda[Df(a,0).\rho\cos(\mu)]\sin^2(\theta)\cos^2(\mu) \tag{44}$$

where the linearization is given by $Df(a,0) = 1 - 3a^2$. The angle $\tilde{\mu}$ satisfies a similar equation with linearization given by $Df(\tilde{a},0) = 1 - 3\tilde{a}^2$.

Indeed, nearby a non-hyperbolic fixed point, the flow (37) is topologically equivalent to a decoupled system as in [77], where the first equation describes the flow on the center manifold, and the second describes the linear hyperbolic dynamics. If the equilibria is hyperbolic, there is no center manifold and this breaks down to the Hartman-Grobman theorem. Since the shooting manifolds are the strong unstable and stable manifolds, there is no center direction within them, and the flow is topological equivalent to its corresponding hyperbolic part of the linearization,

Note that $Df(a,0) > Df(\tilde{a},0)$, since $a < \tilde{a}$. By the comparison theorem in [20], one obtains that for such linearizations,

$$\mu(\tau) > \tilde{\mu}(\tau) \tag{45}$$

for all $\tau \in (-\infty, \tau^*)$ with τ^* negative and large such that $\tau^* < \tau_1$, that is, so that the nonlinear system is topological equivalent to the linear one.

By the intermediate value theorem, there exists $\tau_2 \in (-\tau^*, \tau_1)$ such that $\mu(\tau_2) = \tilde{\mu}(\tau_2)$. We can choose the biggest of those values, due to continuity of those functions up to τ_1, yielding

$$\mu(\tau) < \tilde{\mu}(\tau) \tag{46}$$

for $\tau \in (\tau_2, \tau_1]$, which extends the inequality (43) as claimed.

On the other hand, the integral formulation of (37) yields that

$$\mu(\tau_1) - \mu(\tau_2) = \int_{\tau_2}^{\tau_1} F_2(\lambda, \theta, \rho, \mu) d\tau \tag{47}$$

and similarly for $\tilde{\mu}$.

Consider the difference $\tilde{\mu} - \mu$ of the above representation. Notice that $\mu(\tau_2) = \tilde{\mu}(\tau_2)$ and the right-hand side is a Lispchitz map in μ and ρ, while λ and θ are fixed,

$$|\tilde{\mu}(\tau_1) - \mu(\tau_1)| \leq c(\lambda, \theta) \int_{\tau_2}^{\tau_1} \sqrt{|\tilde{\rho} - \rho|^2 + |\tilde{\mu} - \mu|^2} d\tau.$$

Note that the square root of a sum is less than the sum of the square roots. Moreover, the solutions $\rho, \tilde{\rho}$ of (34) are bounded, hence $|\tilde{\rho} - \rho|$

is bounded. Lastly, one can get rid of the norms in $|\tilde{\mu} - \mu|$, due to (46). These considerations yield

$$\tilde{\mu}(\tau_1) - \mu(\tau_1) \leq c_1 \int_{\tau_2}^{\tau_1} d\tau + c_2 \int_{\tau_2}^{\tau_1} (\tilde{\mu} - \mu) d\tau. \tag{48}$$

By the mean value theorem, there exists some $\tilde{\tau} \in (\tau_1, \tau_2)$ such that $\tau_1 - \tau_2 = \frac{1}{\tilde{\mu}(\tilde{\tau}) - \mu(\tilde{\tau})} \int_{\tau_2}^{\tau_1} (\tilde{\mu} - \mu) d\tau$. Since $\mu, \tilde{\mu}$ are continuous functions on a compact interval, the fractional term in this formula is well defined and a fixed bounded value. We substitute such formula for $\tau_1 - \tau_2$ above, yielding

$$\tilde{\mu}(\tau_1) - \mu(\tau_1) \leq c_3 \int_{\tau_2}^{\tau_1} (\tilde{\mu} - \mu) d\tau \tag{49}$$

where $c_3 := \frac{c_1}{\tilde{\mu}(\tilde{\tau}) - \mu(\tilde{\tau})} + c_2$.

The integral Grönwall inequality implies that $\tilde{\mu}(\tau_1) - \mu(\tau_1) \leq 0$, which contradicts the definition of τ_1 in (43) and proves the non strict inequality (42).

Now we show the strict inequality (39). Suppose on the contrary that there exists a $\tau_3 \in \mathbb{R}$ such that $\mu(\tau_3) = \tilde{\mu}(\tau_3)$.

Let τ^* be obtained by studying the linear flow as before, such that the strict inequality (45) holds for all $\tau \in (-\infty, \tau^*]$. Hence $\tau^* < \tau_3$. Due to the non strict inequality (42), we have in particular that $\mu(\tau) \geq \tilde{\mu}(\tau)$ for $\tau \in (\tau^*, \tau_3)$.

Integrate backwards from τ_3 to τ^*. Indeed, reverse the orientation of $\tau \in [\tau^*, \tau_3]$ through $\tilde{\tau} := -\tau$, so that $\tilde{\tau} \in [\tau_3, \tau^*]$. The integral formulation of the ODE yields

$$\mu(\tau^*) - \mu(\tau_3) = \int_{\tau_3}^{\tau^*} F_2(\lambda, \theta, \rho, \mu) d\tilde{\tau}$$

with similar equation for $\tilde{\mu}$.

Hence, the same methods from equations (47) and (49) can be applied for the difference $\mu(\tau^*) - \tilde{\mu}(\tau^*)$, yielding the inequality $\mu(\tau^*) - \tilde{\mu}(\tau^*) \leq 0$, which contradicts the definition of τ^*. This proves the inequality (39).

Analogously, the above arguments can be used to prove a monotonicity in the radial coordinate. There are two mild adaptations in such proof. Firstly, one does not need to study the linearized flow for the radius, since the initial data is already ordered by $a < \tilde{a}$. Secondly, to obtain (48), one needed to bound $|\tilde{\rho} - \rho|$. Here, we need to bound $|\tilde{\mu} - \mu|$, which is a continuous function on the compact interval $[\tau_2, \tau_1]$ and hence attains a maximum. Then, the mean value theorem is used for $|\tilde{\rho} - \rho|$ and the proof is analogous.

The monotonicity in the parameter λ is seen by comparing the shooting flow (34) as λ increases.

■

2.6.2 *Intersection of shooting curves: finding equilibria*

The shooting curves $M^u_{\pi/2}$ and $M^s_{\pi/2}$ always intersect at the constant equilibria $a \in \{-1, 0, 1\}$.

If $a > 1$, the shooting curves $M^u_{\pi/2}$ and $M^s_{\pi/2}$ are monotone in the initial data. The former increase in the p direction as θ increases, whereas the latter decreases in the p direction, for any $\lambda \in \mathbb{R}_+$. Hence, they do not intersect. Analogously for $a < 1$.

Consider the case that $|a| < 1$. We show that intersections of the shooting curves only occur either at the u or p-axis. Then we show how many intersections there are with those axis.

Lemma 2.6.3. *$M^u \cap M^s \subseteq \{(\theta, u, p) \in \mathbb{R}^3 \mid p = 0$ or $u = 0\}$.*

Proof Towards a contradiction, suppose that there is an intersection point $(u, p) \in M^u \cap M^s$ which is not in these axis.

If $(u, p) \in M^u$, then $(-u, -p) \in M^u$, due to reflection symmetry (36). Similarly, if $(u, p) \in M^s$, then $(-u, -p) \in M^s$. Therefore,

$$(-u, -p) \in M^u \cap M^s.$$

Also, if $(u, p) \in M^u$, then $(u, -p) \in M^s$, due to the construction of M^s, which is done by the time reversal (35) of M^u. Similarly, if $(u, p) \in M^s$, then $(u, -p) \in M^u$, and hence

$$(u, -p) \in M^u \cap M^s.$$

The same arguments in the above two paragraph using both the time reversal (35) and reflection symmetry (36) yield

$$(-u, p) \in M^u \cap M^s.$$

Therefore there are four points with the same radius in the intersection $M^u \cap M^s$ and none of those lie in the u or p-axis. The pigeon hole principle guarantees that at least two of those four points were constructed with the initial data a either in $(0, 1)$ or $(-1, 0)$, contradicting the monotonicity of the radius (40), and proving the lemma.

■

The next step is to find exactly how many intersections there are between the stable and unstable shooting curves, for each $\lambda \in \mathbb{R}_+$. As λ increases, the shooting curves change due to the continuous dependence on the parameter, yielding a different attractor. See [41] for the dependence of the attractor on parameters.

There are always three trivial equilibria $0, \pm 1$ in the intersection of the shooting curves. A new pair of equilibria appears when λ crosses an eigenvalue of the spherical laplacian λ_k. This characterizes the pitchfork bifurcations that occur at each λ_k and gives a different proof of such results, as in [22].

Lemma 2.6.4. *Consider* $\lambda \in (\lambda_k, \lambda_{k+1})$*, where* λ_k *is the k-th eigenvalue of the axially symmetric Laplacian with* $k \in \mathbb{N}_0$*.*

Then there are $2k+3$ *intersections of* $M^u \cap M^s$*, and the angle of the tangent vector of the unstable shooting curve at* $(0,0) \in M^u_{\pi/2}$ *is given by* $\mu(\lambda_{k+1}) = \frac{\pi}{2}(k+1)$*.*

Proof The proof follows by induction on $k \in \mathbb{N}_0$. For the basis of induction, $k = 0$, it is proved that there are three equilibria for $\lambda \in (0, \lambda_1)$ and that $\mu(\lambda_1) = \frac{\pi}{2}$.

For $\lambda_0 = 0$, the shooting flow (34) implies that $p \equiv 0$ and hence the unstable shooting manifold is given by the u-axis. Therefore $\mu(0) = 0$. By continuous dependence on λ, this curve changes a little for λ small. Moreover, due to the monotonicities (38), (39) and (40) for $a \in (0,1)$, the unstable shooting manifold spirals clockwise towards the trivial equilibria 0. Considering the appropriate reflections through Symmetries 1 and 2, one obtains the full unstable and stable curves as below.

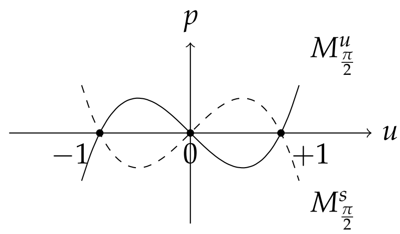


Figure 1: Shooting curves of (34) for $\lambda \in (0, \lambda_1)$

Consider function $\mu(\lambda, a)$ describing the angle of the tangent vector of the unstable shooting curve $M^u_{\frac{\pi}{2}}$ corresponding to the initial data $u_0 \in [0,1]$ and the parameter $\lambda \in \mathbb{R}_+$.

Recall that the angle μ is also monotone in λ for fixed $a \in \mathbb{R}$ as in (41). Moreover, $\lim_{\lambda\to\infty} \mu(\lambda, a) = \infty$, for any $\tau \in \mathbb{R}$ and $a \in (-1,1)$ fixed. Indeed, it follows by combining that (24) is increasing in λ, and (23) with the Symmetry 1, which implies that the stable angle is minus the unstable angle.

Therefore, there is a $\lambda_* > 0$ such that $\mu(\lambda_*, a) = \frac{\pi}{2}$. We have to prove that $\lambda_* = \lambda_1$ and that there are no new equilibria for $\lambda \in (0, \lambda_*)$.

The angle monotonicity (39) implies that the biggest value that μ attains is at $a = 0$. Together with the monotonicity in λ, we have that $\mu(\lambda, a) < \mu(\lambda_*, a) < \mu(\lambda_*, 0) = \frac{\pi}{2}$ for $\lambda < \lambda_*$ and $a \in (0,1)$. Hence, there is no intersection of the unstable shooting curve with the negative p-axis. Since μ is continuous and monotone, in order to reach the positive p-axis, described by the angle $3\pi/2$ in polar coordinates, the shooting curve would have to cross firstly the negative part of the p-axis, which we already showed is not possible.

By the construction of the remaining part of the unstable manifold $M^u_{\pi/2}$ for $a \in (-1, 0)$, through Symmetry 2, there is also no intersection of this piece of the unstable shooting curve with the p-axis. Hence, the only intersection points of the unstable shooting curve lies in the trivial equilibria $a = -1, 0, 1$.

Moreover, due to Symmetry 1 and the construction of the shooting stable manifold $M^s_{\pi/2}$, there are no intersection points of the shooting stable manifold with the p-axis, except the ones regarding the trivial equilibria related to the initial data $b = -1, 0, 1$.

This proves that there are no other equilibria for $\lambda \in (0, \lambda_*)$. In order to show that $\lambda_* = \lambda_1$, recall that the angle of the tangent of $M^u_{\pi/2}$ is $\mu(\lambda_*, 0) = \pi/2$.

Due to the Symmetry 1, the angle of the tangent of the stable manifold $M^s_{\pi/2}$ at $b = 0$ will be $-\mu(\lambda_*, 0) = -\frac{\pi}{2}$. Hence, the angle between those tangent vectors is π, as in (23). This is the definition of an eigenvalue λ_1 through the eigenvalue problem in polar coordinates (20). This shows $\lambda_* = \lambda_1$.

This proves the basis of induction. For the induction step, suppose that for $\lambda \in (\lambda_{k-1}, \lambda_k)$, there are $2(k-1)+3$ equilibria and $\mu(\lambda_k) = \frac{\pi}{2}k$. Note the last condition informs how many times the unstable shooting curve has crossed the u and p axis. We shall prove that for $\lambda \in (\lambda_k, \lambda_{k+1})$, there are $2k+3$ equilibria, $\mu(\lambda_{k+1}) = \frac{\pi}{2}(k+1)$ and λ_{k+1} is the $(k+1)$-th eigenvalue.

The monotonicity in λ as in (41) implies that there exists a $\lambda^* > \lambda_k$ such that $\mu(\lambda^*, 0) = \frac{\pi}{2}(k+1)$. The arguments to show that $\lambda^* = \lambda_{k+1}$ and that two new equilibria appear for $\lambda \in (\lambda_k, \lambda^*$ are analogous as the basis of induction.

There are two cases, depending on the parity of k. This influences which axis the shooting curve intersects and where the new equilibria appear, as λ croses λ_k.

If k is odd, then the new equilibria appear in the p-axis. We illustrate such case in the figure below, when λ crosses λ_1.

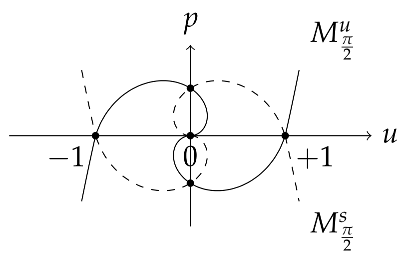


Figure 2: Shooting curves of (34) for $\lambda \in (\lambda_1, \lambda_2)$

This can be seen as follows. By the induction hypothesis, we have that $\mu(\lambda_k) = \frac{\pi}{2}k$, which means that the tangent of the shooting $M^u_{\pi/2}$ at $a = 0$ is parallel to the p-axis for odd k. Since μ is increasing in λ and the shooting curve is continuous, then the shooting curve nearby

$a = 0$ moves from the quadrants $\{p > 0, u < 0\}$ and $\{p < 0, u > 0\}$ to its compliment, as λ crosses λ_k. This creates two new intersections of the shooting curve with the p-axis. Due to the construction of the stable shooting curve $M^s_{\pi/2}$, it also intersects the p-axis in the same points.

Then, one repeat the arguments in the induction step in order to show there is no intersection of the shooting curves with the u-axis.

The only remaining claim to be proven is that equilibria can't disappear, after they appear. The only possibility for this to happen is if two equilibria within the u or p axis collide. Note that neighboring equilibria come from different parts of the initial data: either a is in $(0,1)$ or $(-1,0)$. Hence, if they collide, it contradicts uniqueness of the shooting flow (34), since their initial data is different.

The other case when k is even yields new equilibria in the u-axis. We illustrate this in the example below, as λ crosses λ_2 and two new equilibria appear in the u-axis.

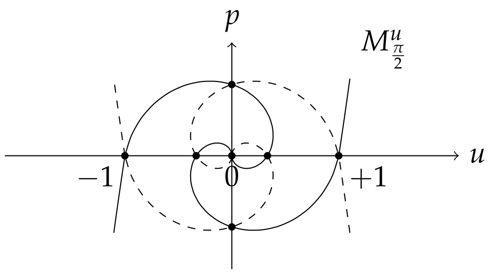


Figure 3: Shooting curves of (34) for $\lambda \in (\lambda_2, \lambda_3)$

■

2.6.3 *Hyperbolicity: all intersections are transverse.*

It is enough to check that the unstable and stable manifolds M^u and M^s are not tangent to the u or p axis. Indeed, M^u is tangent to u-axis if, and only if M^s is also, since one is obtained from the other through to the reflection $p \mapsto -p$. Similarly, M^u is tangent to p-axis if, and only if there is another tangency of M^u with the p-axis, due to the rotation $(u, p) \mapsto (-u, -p)$. Moreover, M^s is obtained from M^u through the reflection $p \mapsto -p$, hence M^s is also tangent to the p-axis.

Recall that the tangent vector of the unstable shooting manifold is given by (u_a, p_a) and satisfy the equation (21). Hence, the tangent vector is tangent to the u-axis if it is horizontal, that is, if the first coordinate $u_a = 0$. On the other hand, the coordinate in polar coordinates is $u = \rho \cos(\mu)$, and the chain rule within the tangency condition implies

$$0 = u_a = (\rho \cos(\mu))_a = \rho_a \cos(\mu) - \rho \sin(\mu) \mu_a$$

Algebraic manipulation yields $\mu = \arctan(\frac{\rho_a}{\mu_a}\rho)$. Note that the monotonicity properties (39) and (40) implies ρ_a and μ_a are nonzero with different signs, for both cases that a is either in $(0,1)$ or $(-1,0)$. Moreover, the radius $\rho > 0$. Therefore, the argument $\frac{\rho_a}{\mu_a}\rho$ is strictly negative, and hence $\mu \in (-\pi/2, 0)$. That is, the point where the tangency occurs is neither at the u, nor the p-axis, because those in polar coordinates are given by $\mu = \frac{\pi}{2}k$. This contradicts that intersections must occur at the u-axis.

Similarly a tangency occurs at the p-axis, if it the vector is vertical, namely $p_a = 0$. In polar coordinates $p = -\rho\sin(\mu)$, the tangency condition and the chain rule implies that $\mu = \arctan(\frac{\rho_a}{-\mu_a}\rho)$. By a similar analysis as above, the argument is strictly positive and hence $\mu \in (0, \pi/2)$. That is, the intersection does not occur in the p-axis, yielding a contradiction.

2.6.4 *Obtaining the permutation.*

We construct the Sturm permutation for $\lambda \in (\lambda_k, \lambda_{k+1})$ by induction on k. As mentioned before, the idea is to label the intersections of the unstable and stable manifolds, firstly along M^u following its parametrization given by the initial data a from $-\infty$ to ∞. Then label them along M^s, also following its parametrization given by b from $-\infty$ to ∞.

For $k = 0$, that is, $\lambda \in (\lambda_0, \lambda_1)$, there are no other intersections of the shooting curves, except the trivial equilibria $a = -1, 0, 1$. Noticing how the shooting curve was constructed before, this is exactly their order along both M^u and M^s, since their parametrization goes from $-\infty$ to ∞. Hence, the permutation is the identity $\sigma = id$, since their oder is the same along M^u and M^s.

For the induction step, we find the permutation for $\lambda \in (\lambda_k, \lambda_{k+1})$, with $k \geq 1$, supposing that the permutation for $\lambda \in (\lambda_{k-1}, \lambda_k)$ is given by

$$\sigma = (2, 2k)(4, 2k-2)... \tag{50}$$

where (j, l) denotes a transposition in the group of permutations of S_N with appropriate N.

Notice that for $\lambda < \lambda_k$ with small $|\lambda - \lambda_k|$, there are $N = 2k+1$ equilibria, and $\lceil k/2 \rceil$ transpositions in the above permutation, for $k \geq 1$, where $\lceil . \rceil$ denotes the ceiling function. Moreover, note that such permutation has all even numbers less or equal $N = 2k+1$.

There are two cases: either k is even or odd. The previous construction of the shooting curve implies that it rotates clock-wise around the trivial equilibria 0. The parity of k tells how the shooting curve behaves for $\lambda < \lambda_k$, in particular, if the equilibria nearby the trivial equilbiria 0 is obtained by an intersection with the u or p axis.

Suppose k is odd. Labeling the equilibria along M^u and M^s for $\lambda < \lambda_k$, the trivial equilibria 0 is labeled $k+1$, since there are k equilibria before it along the unstable manifold. Hence, the nearby equilibria are labeled by k for the equilibria before, and $k+2$ for the equilibria after it. Moreover, the last transposition in the permutation (50) is $(k+1, k+1)$, since $k+1$ is even.

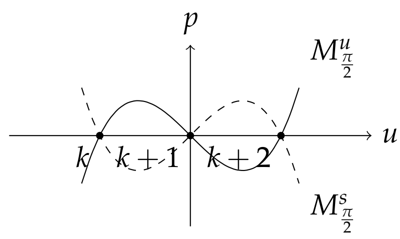


Figure 4: $\lambda < \lambda_k$ with k odd

The labeling within the unstable manifold for the equilibria labeled less than $k+1$ will not change. Moreover, as λ cross λ_k, two new equilibria appear, one on each side along the unstable manifold. The trivial equilibria 0, which was labeled $k+1$ for $\lambda < \lambda_k$, will be shifted by 1, yielding $k+2$ for $\lambda > \lambda_k$. All other having label bigger than $k+1$ will be shifted by two. See the figure below.

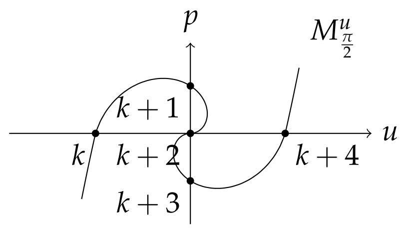


Figure 5: $\lambda > \lambda_k$ with k odd

Similarly, the change of labeling is similar along the stable manifold, since the the stable manifold is obtained by a reflection in the u-axis of the unstable manifold.

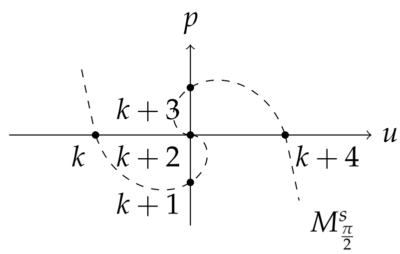


Figure 6: $\lambda > \lambda_k$ with k odd

One only has to check what happens to the permutation itself. There is a new transposition in the permutation given by $(k+1, k+3)$. Note k is odd, and hence both $k+1$ and $k+3$ are even. Moreover, the transposition, which was $(k+1, k+1)$ must be shifted to $(k+2, k+2)$, yielding the identity transposition and not changing (50).

Therefore, there number of transpositions does not change, and number $\lceil k/2 \rceil$ is the same. The only difference is the relabeling of equilibria within the permutation, described above, yielding the desired permutation. The case when the orientation of the unstable manifold is reversed have similar arguments as above.

For k even, the above argument can be adapted. Notice that there are $2k+1$ equilibria, and again the trivial equilibria 0 is labeled by $k+1$. As λ crosses λ_k, there are two new equilibria along the unstable manifold. Hence, the ones before k should not be relabeled, the origin $k+1$ for $\lambda < \lambda_k$ should be relabeled by $k+2$ for $\lambda > \lambda_k$, and all equilibria with label bigger than $k+1$ should be shifted by 2.

Again, since the stable manifold is obtained by the reflection of the unstable manifold with respect to the u-axis, then one can see that the new permutation that should be added is $(k, k+4)$. Notice those are even numbers. Again, the transposition of the origin does not change (50), since it yields the identity transposition given by $(k+2, k+2)$. The case when the orientation of the unstable manifold is reversed have similar arguments as above.

2.6.5 *Obtaining the attractor.*

The permutation obtained above is the same as the regular Chafee-Infante problem. Hence, the attractors are connection-wise the same, since the conditions for the existence of heteroclinics are the same.

2.7 DISCUSSION

The shooting method used to construct the attractor generalizes the bifurcation result in [22] for radially symmetric solutions in the disk. Indeed, not only we are able to prove the existence of bifurcating equilibria, but can also compute secondary bifurcations that might occur, hyperbolicity of all equilibria, their Morse indices and how they fit together in the attractor, by computing heteroclinic trajectories.

After the construction of the Sturm attractor for the parabolic equation with degenerate coefficients, we see that if the Sturm permutation for the degenerate case coincide with the permutation for the case of regular coefficients, then the attractors for both cases coincide. This happens since they are both constructed in the same way, yielding the same conditions about the zero numbers and Morse indices.

The construction of the Sturm attractor in the case of general degenerate diffusion is not proved here, but we believe the above arguments can be replicated. Moreover, we believe that different diffusion yields the same attractors connection-wise, whenever the parameter is in the proper range, differing only in the shape of equilibria.

3

QUASILINEAR DYNAMICS

The goal of this chapter is to study the Sturm attractors of quasilinear parabolic equations, as it was done for the semilinear case by Brunovský and Fiedler [18], and later by Fiedler and Rocha [29].

Consider the scalar quasilinear parabolic differential equation

$$u_t = a(x, u, u_x)u_{xx} + f(x, u, u_x) \tag{51}$$

with initial data $u(0, x) = u_0(x)$ such that $f \in C^2$, $a \in C^1$, the strict parabolicity condition $a > \epsilon$ holds for $\epsilon > 0$ and $x \in [0, \pi]$ has Neumann boundary.

Instead of calling the spatial variable θ to denote the spherical angle as before, we call it x and ommit the singularity in the diffusion operator.

The equation (51) defines a semiflow denoted by $(t, u_0) \mapsto u(t)$ in a Banach space X. The appropriate functional setting is described in Section 3.1.

In order to study the long time behaviour of (51), we suppose that a and f satisfy the following conditions, where $p := u_\theta$,

$$f(x, u, 0) \cdot u < 0 \tag{52}$$

$$|f(x, u, p)| < f_1(u) + f_2(u)|p|^\gamma \tag{53}$$

$$\frac{|a_\theta|}{1 + |p|} + |a_u| + |a_p| \cdot [1 + |p|] \leq f_3(|u|) \tag{54}$$

$$0 < \epsilon \leq a(\theta, u, p) \leq \delta \tag{55}$$

where the first condition holds for $|u|$ large enough, uniformly in x, the second for all (x, u, p) for continuous f_1, f_2 and $\gamma < 2$, and the third for continuou f_3 and $\epsilon, \delta > 0$.

Those conditions imply that $|u|$ and $|u_x|$ are bounded. Hence bounded solutions are global in time and the flow is dissipative: trajectories $u(t)$ eventually enter a large ball in the phase-space X. See Chapter 6, Section 5 in [84]. Also [56] and [12].

Moreover, these hypothesis guarantee the existence of a nonempty global attractor $\mathcal{A}$ of (51), which is the maximal compact invariant set. Equivalently, it is the set of bounded trajectories $u(t)$ in the phase-space X that exist for all $t \in \mathbb{R}$. See [12].

Due to a Lyapunov functional, constructed by Matano [53] and Zelenyak [87], the global attractor consists of equilibria and their heteroclinic connections within their unstable manifolds. It still persits the question of which equilibria connects to which other. This geometric description was carried out by Hale and do Nascimento [37] for the Chafee Infante problem, by Brunovsk and Fiedler [27] for $f(x, u)$ and by Fiedler and Rocha [29] for $f(x, u, u_x)$. Such attractors are known as *Sturm attractors*.

Constructing the Sturm attractor for the equation (51) is problematic due to its quasilinear nature. It is the aim of this chapter to modify the existing theory for such equations and still obtain a Sturm attractor.

For the statement of the main theorem of this chapter, denote by the *zero number* $z(u_*)$ the number of sign changes of a continuous function $u_*(x)$. Recall that the *Morse index* $i(u_*)$ of an equilibrium u_* is given by the number of positive eigenvalues of the linearized operator at such equilibrium, that is, the dimension of the unstable manifold of said equilibrium.

We say that two different equilibria u_-, u_+ of (51) are *adjacent* if there does not exist an equilibrium u_* of (51) such that $u_*(0)$ lies between $u_-(0)$ and $u_+(0)$, and

$$z(u_- - u_*) = z(u_- - u_+) = z(u_+ - u_*).$$

This notion was firstly described by Wolfrum [86].

Both the zero number and Morse index can be computed from a permutation of the equilibria, as it was done in [33] and [29]. Such permutation is called the *Sturm Permutation* and is computed in Section (3.3), as it was done [29]. For such, it is required that the flow of the equilibria equation of (51) exists for all $x \in [0, \pi]$.

Theorem 3.0.1. Sturm Attractor [Lappicy ('17)]
Consider $f \in C^2$ satisfying the growth conditions (52)*. Suppose that all equilibria for the equation* (51) *are hyperbolic. Then,*

1. *the global attractor $\mathcal{A}$ of* (51) *consists of equilibria $\mathcal{E}$ and heteroclinic connections $\mathcal{H}$.*

2. *there exists a heteroclinic orbit $u(t) \in \mathcal{H}$ between two equilibria $u_-, u_+ \in \mathcal{E}$ such that*

$$u(t) \to_{t \to \pm\infty} u_\pm$$

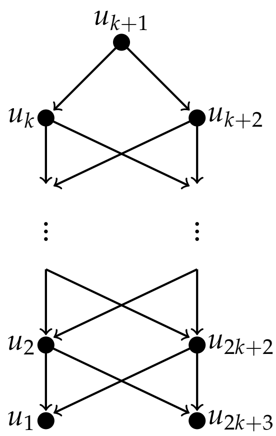
if, and only if, u_- and u_+ are adjacent and $i(u_-) > i(u_+)$.

This meets some expectations of Fiedler [26], which mentions that fully nonlinear equations yield the same type of attractors as the semilinear ones. We prove it for the quasilinear, and leave the fully nonlinear for another occasion.

In particular, we compute and comment on some explicit attractors. Firstly, when the diffusion coefficient is $a(u) = u^2$, and a reaction of the Chafee Infante type with adjusted powers, in order to match the diffusion rate. This attractor is used in Chapter 5 as an application of the Einstein Hamiltonian equation. This same diffusion with another reaction is used to model the curve shortening flow in $\mathbb{R}^2$, as in [9].

Corollary 3.0.2. Chafee-Infante Attractor [Lappicy ('17)] *Consider the equation* (51) *with* $a(u) = u^n$ *and* $f(\lambda, u) = \lambda u^{n+1}(1 - u^2)$ *for some* $n \in \mathbb{N}_0$. *Let* $\lambda \in (\lambda_k, \lambda_{k+1})$, *where* λ_k *is the k-th eigenvalue of the Laplacian with* $k \in \mathbb{N}_0$.

Then, there are $2k + 3$ *hyperbolic equilibria* $u_1, ..., u_{2k+3}$ *and its attractor* $\mathcal{A}$ *is*



The remaining is organized as follows.

We firstly introduce the functional setting in Section 3.1, including invariant manifolds for the quasilinear case. Further in Section 3.2, we construct a Lyapunov functional for the quasilinear case $a(x, u, u_x)$ using Matano's arguments from [51] for the quasilinear case $a(u)$. In particular this implies that the attractor consists of equilibria and heterolinics.

Then, we focus on the connection problem. All the necessary information about the k-adjacency, namely the zero numbers and Morse indices, are encoded in a permutation of the equilibria, which is described in Section 3.3. This was done firstly by [33]. The shooting is similar to the semilinear case, but one needs to divide the reaction term by the diffusion noticing $a > 0$.

In Section 3.4, we recall the dropping lemma and some consequences, which hold since the difference of solutions satisfy a linear equation. This is a fundamental result for the attractor construction that dates back to Sturm. Then all the previous tools are put together to construct the attractor in Section 3.5, as it was done [29].

Lastly, Section 3.6 gives an example of the developed theory and constructs the attractor of the Chafee-Infante type.

3.1 FUNCTIONAL SETTING

The Banach space used on the upcoming theory consists on subspaces of Hölder continuous functions $C^\beta([0,\pi])$ with $\beta \in (0,1)$. A more precise description is given below, following [49], [4], [12].

The equation (51) can be rewritten as an abstract differential equation on a Banach space,

$$u_t = Au + g(x, u, u_x) \tag{56}$$

where A is the linearization of the right-hand side of (51) at u_*, and g the remaining nonlinear part.

Indeed, consider the operator

$$\begin{aligned} A : D(A) &\to X \\ v \mapsto Av &= [a_u(u_*, p_*) + a_p(u_*, p_*)]v_{xx} + Df(x, u_*, p_*)v \end{aligned} \tag{57}$$

where $p_* := (u_*)_x$, $X := C^\beta([0,\pi])$ and $D(A) = C^{2+\beta}([0,\pi]) \subset X$ is the domain of the operator A. Note Df is a short for $D_u f + D_p f$. Note all coefficients depend only on x.

Note that if u_* is a homogeneous equilibria, that is, independent of x, then $g(x,u) = f(u) - Df(u_*).u$. For any other $u_*(x)$, then $g(x,u) = f(u) - Df(u_*).u + [a(u_*, p_*)\Delta u_*]u$.

As in Lunardi [49], we consider the interpolation spaces $X^\alpha = C^{2\alpha+\beta}([0,\pi])$ between $D(A)$ and X with $\alpha \in (0,1)$ such that A generates a strongly continuous semigroup in X, and hence the equation (51) with the dissipative conditions (52) defines a dynamical system in X^α.

In particular, it settles the theory of existence and uniqueness. For certain qualitative properties of solutions, such as the existence of invariant manifolds tangent to the linear eigenspaces, one needs to know more about the spectrum of A.

Indeed, note that A is self-adjoint with respect to the weighted $C^{2\alpha+\beta}$ metric, with weight given by $[a_u(u_*, p_*) + a_p(u_*, p_*)]^{-1}$. Hence, the eigenvalue problem for A is a regular Sturm-Liouville problem, since the coefficients depend only on x and are all bounded. Therefore, the spectrum $\sigma(A)$ consists of real simple eigenvalues λ_k accumulating at $-\infty$, and corresponding eigenfunctions $\phi_k(x)$ which form a an orthonormal basis of X. Moreover, there is a spectral gap between eigenvalues that allows us to get the following filtration of invariant manifolds. Note that since it is supposed that equilibria are hyperbolic, then there is no 0 as an eigenvalue and no center direction.

Theorem 3.1.1. Filtration of Invariant Manifolds [54], [49]

Let u_* be a hyperbolic equilibrium of (51) *with Morse index $n := i(u_*)$. Then there exists a filtration of the* unstable manifold

$$W^u_0(u_*) \subset ... \subset W^u_{n-1}(u_*) = W^u(u_*)$$

where each W_k^u *has dimension* $k+1$ *and tangent space spanned by* $\phi_0, ..., \phi_k$.
Analogously, there is a filtration of the stable manifold

$$... \subset W^s_{n+1}(u_*) \subset W^s_n(u_*) = W^s(u_*)$$

where each W_k^s *has codimension* k *and tangent space spanned by* $\phi_k, \phi_{k+1}, ...$.

Note that the above index labels are not in agreement with the dimension of each submanifold within the filtration, but it is with the number of zeros an eigenfunction has. For example, each eigenfunction ϕ_k has k simple zeroes, whereas the $\dim(W_k^u) = k+1$.

An important property is the behaviour of solutions within each submanifold of the above filtration of the unstable or stable manifolds.

Theorem 3.1.2. Linear Asymptotic Behaviour [41], [5], [18]
Consider a hyperbolic equilibrium u_* *with Morse index* $n := i(u_*)$ *and a trajectory* $u(t)$ *of* (51)*. The following holds,*

1. *If* $u(t) \in W_k^u(u_*) \backslash W^u_{k-1}(u_*)$ *with* $k = 0, ..., i(u_*) - 1$*. Then,*

$$\frac{u(t) - u_*}{||u(t) - u_*||} \xrightarrow{t \to -\infty} \pm\phi_k$$

2. *If* $u(t)$ *in* $W_k^s(u_*) \backslash W^s_{k+1}(u_*)$ *with* $k \geq i(u_*)$*. Then,*

$$\frac{u(t) - u_*}{||u(t) - u_*||} \xrightarrow{t \to \infty} \pm\phi_k$$

where the convergence takes place in C^1_w.
The conclusions of 1. and 2. also hold true by replacing the difference $u(t) - u_*$ *with the tangent vector* u_t.

The reason this theorem works for both the tangent vector $v := u_t$ or the difference $v := u_1 - u_2$ of any two solutions u_1, u_2 of the nonlinear equation (51) is that they satisfy a linear equation of the type

$$u_t = a(t,x)u_{xx} + b(t,x)u_x + c(t,x)u \tag{58}$$

where $x \in (0, \pi)$ has Neumann boundary conditions and the functions $a(t,x), b(t,x)$ and $c(t,x)$ are bounded.

The proof in [5] works for the case $a = a(x, u, u_x)$ and $f = (x, u, u_x)$ considering Dirichlet boundary condition; or in case $a = a(x, u)$ and $f = f(x, u)$ considering other boundary conditions. For the general case, see [18].

3.2 VARIATIONAL STRUCTURE

In this section, we show that there exists a Lyapunov functional for the quasilinear case $a(x, u, u_x)$, as it was done by Matano [52] for $a(u)$. Hence, bounded trajectories tend to equilibria.

Lemma 3.2.1. Lyapunov Functional
There exists a Lagrange functional L such that

$$E := \int_0^\pi L(x, u, u_x) dx \tag{59}$$

is a Lyapunov functional for the equation (51).

Note that in the case that the nonlinearity f does not depend on u_x, then the Lagrange functional $L(x, u, u_x) := \frac{1}{2}u_x^2 - F(x, u)$ yields a Lyapunov functional E, where F is the primitive function of $\frac{f}{a}$. Indeed,

$$\frac{dE}{dt} = -\int_0^\pi \frac{1}{a(x, u, p)}(u_t)^2 dx.$$

For nonlinearities of the type $f(x, u, u_x)$, Matano's idea yields a Lyapunov functional of the type

$$\frac{dE}{dt} := -\int_0^\pi \frac{L_{pp}}{a(x, u, p)}(u_t)^2 dx \tag{60}$$

where $p := u_x$ and L satisfy the convexity condition $L_{pp} > 0$. Hence, one needs that $a(x, u, p) > 0$. Note that Matano's construction was done for $a(u)$, and here we consider more general quasilinear equations, namely $a(x, u, p)$.

Proof Let $p := u_x$ and differentiate (59) with respect to t,

$$\frac{dE}{dt} = \int_0^\pi \left[L_u u_t + L_p u_{xt}\right] dx.$$

Integrating the second term by parts,

$$\begin{aligned} \frac{dE}{dt} &= \int_0^\pi \left[L_u - \frac{d}{dx}L_p\right] u_t dx \\ &= \int_0^\pi \left[L_u - L_{px} - L_{pu}u_x - L_{pp}u_{xx}\right] u_t dx. \end{aligned}$$

Substitute (51) casted as $u_{xx} = \frac{1}{a(x,u,u_x)}[u_t - f \sin(x)]$,

$$\frac{dE}{dt} = \int_0^\pi \left[L_u - L_{px} - L_{pu}u_x + L_{pp}\frac{f}{a}\right] u_t dx - \int_0^\pi L_{pp}u_t^2 dx.$$

To obtain the desired equality (60), one has to guarantee that there exists a function L satisfying

$$L_u - L_{px} - L_{pu}u_x + L_{pp}\frac{f}{a} = 0 \tag{61}$$

for all $u, p \in \mathbb{R}$ and $x \in [0, \pi]$.

Differentiating this equation with respect to p, some of the terms cancel, yielding

$$-L_{ppx} - L_{ppu}p + L_{ppp}f + L_{ppp}\frac{f}{a} + L_{pp}\frac{f_p a - a_p f}{a^2} = 0. \tag{62}$$

To make sure that $L_{pp} > 0$, Matano makes an Ansatz by introducing a function $g = g(x, u, p)$ through $L_{pp} = \exp(g) > 0$. Hence, g satisfies the following linear first order differential equation,

$$[g_x + g_u p - g_p \frac{f}{a} + \frac{f_p a - a_p f}{a^2}] \exp(g) = 0. \tag{63}$$

Or equivalenty,

$$\left[g_x + g_u p + g_p \left(-\frac{f}{a}\right)\right] = \frac{f_p a - a_p f}{a^2}.$$

This can be solved through the method of characteristics: along the solutions of the ordinary differential equation

$$\begin{cases} u_x &= p \\ p_x &= -\frac{f}{a} \end{cases} \tag{64}$$

the function g must satisfy

$$\frac{d}{dx} g = \frac{f_p a - a_p f}{a^2}. \tag{65}$$

We suppose that solutions of (64) exist for all initial conditions $(u, p) \in \mathbb{R}^2$ at $x = 0$, and all $x \in [0, \pi]$. Hence, we also suppose one can solve (65) to obtain a global solution g of (63) with some initial data, for example, $g(0, u, p) \equiv 0$. For example, in case a and f do not depend on p, (65) is indeed solvable.

It is still needed to ascend from a function g satisfying (63) to a function L satisfying (61). A choice for L such that $L_{pp} = \exp(g)$ can be obtained by integrating this relation twice with respect to p, yielding

$$L(x, u, p) := \int_0^p \int_0^{p_1} \exp(g(x, u, p_2)) dp_2 dp_1 + G(x, u)$$

and this is a solution of (62).

To show that such L is also a a solution of (61), we have to restrict which G are allowed.

Recall that (62) was obtained through differentiating (61) with respect to p. That means that the left-hand side of (61) is independent of p, since it is equal to 0. Hence it is satisfied for all p if holds for $p = 0$.

At $p = 0$, the construction of L yields that $L_p = L_{px} = 0$ and $L_u = G_u$. Plugging it in the equation (61) at $p = 0$, it yields

$$G_u + L_{pp} \frac{f}{a} = 0.$$

Hence, $G_u = -\exp(g)\frac{f}{a}$. Integrating in u,

$$G(x, u) := -\int_0^u \frac{f(x, \tilde{u}, 0)}{a(x, \tilde{u}, 0)} exp(g(x, u, 0)) d\tilde{u}.$$

■

Therefore, the LaSalle invariance principle holds and implies that bounded solutions converge to equilibria, and any ω-limit set consists of a single equilibrium. See [53]. Moreover, the global attractor can be characterized as follows, yielding the first part of the main result.

Proposition 3.2.2. Attractor Decomposition [12]
If the equation (51) *has a Lyapunov functional and a discrete set of equilibria $\mathcal{E}$, then the global attractor $\mathcal{A}$ is decomposed as*

$$\mathcal{A} = \bigcup_{v \in \mathcal{E}} W^u(v)$$

and consists only of the set of equilibria and connection orbits.

Note that hyperbolic equilibria must be isolated. Moreover, there must be finitely many due to dissipativity.

3.3 STURM PERMUTATION

The next step on our quest to find the Sturm attractor is to construct a permutation associated to the equilibria, which is done using shooting methods. This enables the computation of the Morse indices and zero number of equilibria. That was firstly done by Fusco and Rocha [33] using methods also described by Fusco, Hale and Rocha in [66], [39], [67], [69] and [32].

The equilibria equation associated to (51) can be rewritten as

$$0 = a(x, u, u_x)u_{xx} + f(x, u, u_x)$$

for $x \in [0, \pi]$ with Neumann boundary conditions and the parabolicity condition $a > 0$.

Reduce the system to first order through $p := u_\tau$. Lastly, add the extra equation $x_\tau = 1$ to obtain an autonomous system. Hence,

$$\begin{cases} u_\tau &= p \\ p_\tau &= -\frac{f(x,u,p)}{a(x,u,u_x)} \\ x_\tau &= 1 \end{cases} \tag{66}$$

where the Neumann boundary condition becomes $p = 0$. We suppose that solutions are defined for all $x \in [0, \pi]$ and any initial data.

The idea to find equilibria (51) is as follows. They must lie in the line

$$L_0 := \{(x, u, p) \in \mathbb{R}^3 \mid (x, u, p) = (0, b, 0) \text{ and } b \in \mathbb{R}\}$$

due to Neumann boundary at $x = 0$. Then, evolve this line under the flow of the equilibria differential equation and intersect it with an analogous line L_π at $x = \pi$, so that it also satisfies Neumann at $x = \pi$. More precisely, one can write the *shooting manifold* as

$$M := \{(x, u, p) \in [0, \pi] \times \mathbb{R}^3 \mid (x, u(x, b, 0), p(x, b, 0)), b \in \mathbb{R}\}.$$

where $(x, u(x, b, 0), p(x, b, 0))$ is the solution of (66) which evolves the initial data $(0, b, 0)$.

Denote by M_x the cross-section of M for some fixed $x \in [0, \pi]$. This is a curve parametrized by $a \in \mathbb{R}$.

We obtain the following characterization of equilibria through the shooting manifolds and its relation with the Morse indices and zero numbers, similar to [66] and [36].

Lemma 3.3.1. ***Equilibria Through Shooting***

1. *The set of equilibria $\mathcal{E}$ of* (51) *is in one-to-one correspondence with $M_\pi \cap L_\pi$.*

2. *An equilibrium point corresponding to fixed $b \in \mathbb{R}$ is hyperbolic if, and only if, M_π intersects L_π transversely at $(\pi, u(\pi, b, 0), 0)$.*

3. *If u_* correspond to a hyperbolic equilibrium of* (51)*, then its Morse index is given by $i(u_*) = 1 + \lfloor \frac{\zeta(x_0)}{\pi} \rfloor$ where $\zeta(x_0)$ is the angle between M_π and L_π measured clockwise at their intersection point x_0, and $\lfloor . \rfloor$ denotes the floor function.*

Proof To prove 1), note that a point in $M_\pi \cap L_\pi$ satisfies the equilibria equation with Neumann boundary conditions by definition of the shooting manifolds. Conversely, consider an equilibrium of (51) satisfying Neumann boundary must be in $M_\pi \cap L_\pi$. Due to the uniqueness of the shooting differential equation (66), such correspondence above is one-to-one.

To prove 2), consider an equilibrium u_* corresponding to the initial data $b \in \mathbb{R}$. We compare the eigenvalue problem for u_* and the differential equation satisfied by the angle of the tangent vectors of the shooting manifold.

The eigenvalue problem for u_* is

$$\lambda u = a(x, u_*, p_*) u_{xx} + [D_u f(x, u_*, p_*).u + D_p f(x, u_*, p_*).p]$$

with Neumann boundary conditions for $x \in [0, \pi]$. From now on, the coordinates of Df and a are suppressed.

Rewriting the above system as a system of first order by $p := u_x$,

$$\begin{cases} u_x &= p \\ p_x &= -\frac{D_u f.u + D_p f.p - \lambda u}{a} \end{cases}$$

with Neumann boundary conditions.

In polar coordinates $(u, p) =: (r\cos(\mu), -r\sin(\mu))$, the angle given by $\mu := \arctan(\frac{p}{u})$ satisfies

$$\mu_\tau = \sin^2(\mu) + \frac{D_u f.u + D_p f.p - \lambda u}{a}\cos^2(\mu) \tag{67}$$

with $\mu(0) = 0$ and $\mu(\pi) = k\pi$ for some $k \geq 0$.

On the other hand, M_x is parametrized by the initial data $b \in \mathbb{R}$ and its tangent vector $(\frac{\partial u(x,b)}{\partial b}, \frac{\partial p(x,b)}{\partial b})$ corresponding to the trajectory u_* satisfies the following linearized equation,

$$\begin{cases} (u_b)_x &= p_b \\ (p_b)_x &= -\frac{D_u f(x,u_*,p_*).u_b + D_p f(x,u_*,p_*).p_b}{a(x,u_*,p_*)} \end{cases} \tag{68}$$

with initial data $(u_b(0), p_b(0)) = (1, 0)$. From now on we suppress this dependence of Df and a.

In polar coordinates $(u_b, p_b) =: (\rho\cos(\nu), -\rho\sin(\nu))$, where ν is the clockwise angle of the tangent vector of M_x at the trajectory u_* with the u-axis,

$$\nu_x = \sin^2(\nu) + \frac{D_u f.u_b + D_p f.p_b}{a}\cos^2(\nu) \tag{69}$$

with initial data $\nu(0, b, 0) = 0$.

Note that the angle ν of the tangent vector in (69) satisfy is the same equation as the eigenvalue problem in polar coordinates (67) with $\lambda = 0$ and same boundary conditions at $x = 0$.

Suppose that u_* is not hyperbolic, that is, there exists a solution of (67) with $\mu(\pi) = k\pi$ for $\lambda = 0$ and some $k \in \mathbb{N}$. Since this is the same equation as (69), uniqueness implies that $\nu(\pi) = k\pi$. This implies that M_π and L_π are not transverse.

Conversely, if M_π and L_π are not transverse, then $\nu(\pi) = k\pi$ for some $k \in \mathbb{N}$. Again, notice this is the same equation for (67) and hence there exists a solution of (67) for $\lambda = 0$ such that $\nu(\pi) = k\pi$. Hence, $\lambda = 0$ is an eigenvalue and u_* is not hyperbolic.

To prove 3), consider the solution $\mu(x, \lambda)$ of the eigenvalue problem in polar coordinates (67). The Sturm oscillation theorem implies that

$$\psi(\lambda) := \mu(\pi, \lambda)$$

is decreasing so that $\lim_{\lambda\to-\infty}\psi(\lambda) = \infty$ and $\lim_{\lambda\to\infty}\psi(\lambda) = -\pi/2$. Hence, there exists a decreasing sequence $\{\lambda_k\}_{k\in N}$ to $-\infty$ such that $\psi(\lambda_k) = k\pi$ for $k \in \mathbb{N}$. This implies that there exists a solution of (67) for each λ_k such that $\psi(\lambda_k) = k\pi$, and hence $\{\lambda_k\}_{k\in N}$ are the eigenvalues.

Recall that the Morse index $i(u_*)$ is the number of positive eigenvalues of the linearization at u_*, that is

$$... < \lambda_{i(u_*)} < 0 < \lambda_{i(u_*)-1} < ... < \lambda_0.$$

Since $\psi(\lambda)$ is decreasing and $\lambda_{i(u_*)}$ are eigenvalues, then

$$i(u_*)\pi = \psi(\lambda_{i(u_*)}) > \psi(0) > \psi(\lambda_{i(u_*)-1}) = (i(u_*) - 1)\pi.$$

Divide the above by π and consider the integer value, yielding that $i(u_*) = \lfloor \frac{\psi(0)}{\pi} \rfloor + 1$. By definition, $\psi(0) = \nu(\pi, 0)$, which is exactly the angle between M_π and L_π.

■

Therefore, a *Sturm permutation* σ is obtained by labeling the intersection points $u_i \in M_\pi \cap L_\pi$ firstly along M_π following its parametrization given by $(\pi, u(\pi, b, 0), p(\pi, b, 0))$ as b goes from $-\infty$ to ∞,

$$u_1 < ... < u_N$$

where N denotes the number of equilibria. Secondly, label the intersection points along L_π by increasing values,

$$u_{\sigma(1)} < ... < u_{\sigma(N)}$$

The Morse indices of equilibria and the zero number of difference of equilibria can be calculated through the Sturm permutation σ, as in [68] and [29]. The main tool for such proofs is the third part of the above Lemma: the rotation along the shooting curve increases the Morse index.

3.4 DROPPING LEMMA

Let the *zero number* $0 \leq z^t(u) \leq \infty$ count the number of strict sign changes in x of a C^1 function $u(t, x) \not\equiv 0$, for each fixed t. More precisely,

$$z^t(u) := \sup_k \left\{ \begin{array}{c} \text{There is a partition } \{x_j\}_{j=1}^k \text{ of } [0,\pi] \\ \text{such that } u(t,x_j)u(t,x_{j+1}) < 0 \text{ for all } j \end{array} \right\}$$

and $z^t(u) = -1$ if $u \equiv 0$. In case u does not depend on t, we omit the index and simply write $z^t(u) = z(u)$.

A point $(t_0, x_0) \in \mathbb{R} \times [0, \pi]$ such that $u(t_0, x_0) = 0$ is said to be a *simple zero* if $u_x(t_0, x_0) \neq 0$ and a *multiple zero* if $u_x(t_0, x_0) = 0$.

The following result proves that the zero number of certain solutions of (51) is nonincreasing in time t, and decreases whenever a multiple zero occur. Different versions of this well known fact are due to Sturm [81], Matano [52], Angenent [6] and others. See [47] for a more recent account.

Lemma 3.4.1. Dropping Lemma

Consider $u \not\equiv 0$ *a solution of the linear equation* (58) *for* $t \in [0, T)$. *Then, its zero number* $z^t(u)$ *satisfies*

1. $z^t(u) < \infty$ *for any* $t \in (0, T)$.
2. $z^t(u)$ *is nonincreasing in time* t.
3. $z^t(u)$ *decreases at multiple zeros* (t_0, x_0) *of* u*, that is,*

$$z^{t_0-\epsilon}(u) > z^{t_0+\epsilon}(u)$$

for any sufficiently small $\epsilon > 0$.

Recall that both the tangent vector u_t and the difference $u_1 - u_2$ of two solutions u_1, u_2 of the nonlinear equation (51) satisfy a linear equation as (58), and the proof is exactly the same as the one for the semilinear equations. The proof even holds for fully nonlinear equations, as in [47].

We mention two consequences of the dropping lemma 3.4.1 and the asymptotic description in Theorem 3.1.2. The first is a result relating the zero number within invariant manifolds and the Morse indices of equilibria. The second is the Morse-Smale property.

Theorem 3.4.2. Zero number and Invariant Manifolds [82], [18]
Consider an equilibrium $u_* \in \mathcal{E}$ *and a trajectory* $u(t)$ *of* (51)*. Then,*

1. *If* $u(t) \in W^u(u_*)$*, then* $i(u_*) > z^t(u - u_*)$.
2. *If* $u(t) \in W^s_{loc}(u_*) \backslash \{u_*\}$*, then* $z^t(u - u_*) \geq i(u_*)$.

These results also hold by replacing $u(t) - u_*$ *with the tangent vector* u_t.

The above theorem implies that (51) has no homoclinic orbits. Indeed, if there were any, then $i(u_*) < i(u_*)$, which is a contradiction.

Theorem 3.4.3. Morse-Smale Property [42], [5], [33]
Consider two hyperbolic equilibria u_- *and* u_+ *with respective Morse indices* $i(u_-), i(u_+)$*. If* $W^u(u_-) \cap W^s(u_+) \neq \emptyset$*, then such intersection is transverse. Moreover,* $W^u(u_-) \cap W^s(u_+)$ *is an embedded submanifold of dimension* $i(u_-) - i(u_+)$.

This last theorem implies that if the semigroup has a finite number of equilibria, in which all are hyperbolic, then it is a Morse-Smale system in the sense of [38]. Note that hyperbolicity of equilibria is a generic property. See brunovsky,polacik. Also, hyperbolicity is not even necessary for transversality, as in henry.

3.5 STURM GLOBAL STRUCTURE

This section gathers all the tools developed in the previous sections, in order to construct the attractor for the quasilinear parabolic equation (51) and prove part of the main Theorem 3.0.1.

Its proof is a consequence of the following two propositions. Firstly, due to the *cascading principle*, it is enough to construct all heteroclinics between equilibria such that the Morse indices of such equilibria differ by 1. Secondly, on one direction, the *blocking principle*: some conditions imply that there does not exist a heteroclinic connection; on the other direction, the *liberalism principle*: if those conditions are violated, then there exists a heteroclinic.

The cascading and blocking principles follow from the dropping lemma and Morse-Smale property from Section 3.4, and we give a sketch as in [29]. There is only a mild modification in the proof of the liberalism principle in Proposition 2.5.2.

Proposition 3.5.1. Cascading Principle [29]

There exists a heteroclinic between two equilibria u_- and u_+ such that $n := i(u_-) - i(u_+) > 0$ if, and only if, there exists a sequence (cascade) of equilibria $\{v_k\}_{k=0}^{n}$ with $v_0 := u_-$ and $v_n := u_+$, such that the following holds for all $k = 0, ..., n-1$

1. $i(v_{k+1}) = i(v_k) + 1$
2. *There exists a heteroclinic from v_{k+1} to v_k*

Proposition 3.5.2. Blocking and Liberalism Principles [29]

There exists a heteroclinic connection between equilibria u_- and u_+ with $i(u_-) = i(u_+) + 1$ if, and only if,

1. Morse blocking: $z(u_- - u_+) = i(u_+)$,
2. Zero number blocking: $z(u_- - u_*) \neq z(u_+ - u_*)$ *for all equilibria u_* between u_- and u_+ at $x = 0$.*

The blocking and liberalism principles assert that the Morse indices $i(.)$ and zero numbers $z(.)$ construct the global structure of the attractor explicitly. Those numbers can be obtained through the Sturm permutation, as in Section 3.3.

For the proof of the liberalism theorem, it is used the Conley index to detect orbits between u_- and u_+. We give a brief introduction of Conley's theory, and how it can be applied in this context. See Chapters 22 to 24 in [80] for a brief account of the Conley index, and its extension to infinite dimensional systems in [71].

Consider the space $\mathcal{X}$ of all topological spaces and the equivalence relation given by $Y \sim Z$ for $Y, Z \in \mathcal{X}$ if, and only if Y is homotopy equivalent to Z, that is, there are continuous maps $f : Y \to Z$ and $g : Z \to Y$ such that $f \circ g$ and $g \circ f$ are homotopic to id_Z and id_Y, respectively. Then, the quotient space $\mathcal{Y}/\sim$ describes the homotopy equivalent classes $[Y]$ of all topological spaces which have the same homotopy type. Intuitively, $[Y]$ describes all other topological spaces which can be continuously deformed into Y.

Suppose Σ is an invariant isolated set, that is, it is invariant with respect to positive and negative time of the semiflow, and it has a closed neighborhood N such that Σ is contained in the interior of N with Σ being the maximal invariant subset of N.

Denote $\partial_e N \subset \partial N$ the *exit set of* N, that is, the points which are not strict ingressing in N,

$$\partial_e N := \{u_0 \in N \mid u(t) \notin N \text{ for all sufficiently small } t > 0\}.$$

The *Conley index* is defined as

$$C(\Sigma) := [N/\partial_e N]$$

namely the homotopy equivalent class of the quotient space of the isolating neighborhood N relative to its exit set $\partial_e N$. Such index is homotopy invariant and does not depend on the particular choice of isolating neighborhood N.

We compute the Conley index for two examples.

Firstly, the Conley index of a hyperbolic equilibria u_+ with Morse index n. Consider a closed ball $N \subset X$ centered at u_+ without any other equilibria in N, as isolating neighborhood. The flow provides a homotopy that contracts along the stable directions to the equilibria u_+. Then, N is homotoped to a n-dimensional ball B^n in the finite dimensional space spanned by the first n eigenfunctions, related to the unstable directions. Note the exit set $\partial_e B^n = \partial B^n = \mathbb{S}^{n-1}$, since after the homotopy there is no more stable direction and the equilibria is hyperbolic. Therefore, the quotient of a n-ball and its boundary is an n-sphere,

$$C(u_+) = [N/\partial_e N] = [B^n/\partial_e B^n] = [B^n/\mathbb{S}^{n-1}] = [\mathbb{S}^n].$$

Secondly, the Conley index of the union of two disjoint invariant sets, for example u_- and u_+ with respective disjoint isolating neighborhoods N_- and N_+. Then, $N_- \cup N_+$ is an isolating neighborhood of $\{u_-, u_+\}$. By definition of the wedge sum

$$\begin{aligned} C(\{u_-, u_+\}) &= \left[\frac{N_- \cup N_+}{\partial_e(N_- \cup N_+)}\right] \\ &= \left[\frac{N_-}{\partial_e N_-} \vee \frac{N_+}{\partial_e N_+}\right] = C(u_-) \vee C(u_+). \end{aligned}$$

The Conley index can be applied to detect heteroclinics as follows. Construct a closed neighborhood N such that its maximal invariant subspace is the closure of the set of heteroclinics between $u_\pm$,

$$\Sigma = \overline{W^u(u_-) \cap W^s(u_+)}.$$

Suppose, towards a contradiction, that there are no heteroclinics connecting u_- and u_+, that is, $\Sigma = \{u_-, u_+\}$. Then, the index is given by the wedge sum $C(\Sigma) = [\mathbb{S}^n] \vee [\mathbb{S}^m]$, where n, m are the respective Morse index of u_- and u_+.

If, on the other hand, one can prove that $C(\Sigma) = [0]$, where $[0]$ means that the index is given by the homotopy equivalent class of a point. This would yield a contradiction and there should be a connection between u_- and u_+. Moreover, the Morse-Smale structure excludes connection from u_+ to u_-, and hence there is a connection from u_- to u_+.

Hence, there are three ingredients missing in the proof: the Conley index can be applied at all, the construction of a isolating neighborhood N of Σ and the proof that $C(\Sigma) = [0]$.

Proof of Theorem 3.5.2

($\Longleftarrow$) This is also called liberalism in [29]. Consider hyperbolic equilibria u_-, u_+ such that $i(u_-) = i(u_+) + 1$ and satisfies both the Morse and the zero number blocking. Without loss of generality, assume $u_-(0) > u_+(0)$.

It is used the Conley index to detect orbits between u_- and u_+. Note that the semiflow generated by the equation (8) on the Banach space X is admissible for the Conley index theory in the sense of [71], due to a compactness property that is satisfied by the parabolic equation (8), namely that trajectories are precompact in phase space. See Theorem 3.3.6 in [41].

As mentioned above, in order to apply the Conley index concepts we need to construct appropriate neighborhoods and show that the Conley index is [0].

Consider the closed set

$$K(u_\pm) := \left\{ u \in X \mid \begin{array}{c} z(u - u_-) = i(u_+) = z(u - u_+) \\ u_+(0) \leq u(0) \leq u_-(0) \end{array} \right\}$$

Consider also closed ϵ-balls $B_\epsilon(u_\pm)$ centered at $u_\pm$ such that they don not have any other equilibria besides $u_\pm$, respectively, for some $\epsilon > 0$.

Define

$$N_\epsilon(u_\pm) := B_\epsilon(u_-) \cup B_\epsilon(u_+) \cup K(u_\pm).$$

The zero number blocking condition implies there are no equilibria in $K(u_\pm)$ besides possibly u_- and u_+. Hence, $N_\epsilon(u_\pm)$ also has no equilibria besides u_- and u_+.

Denote Σ the maximal invariant subset of N_ϵ. We claim that Σ is the set of the heteroclinics from u_- to u_+ given by $\overline{W^u(u_-) \cap W^s(u_-)}$.

On one hand, since Σ is globally invariant, then it is contained in the attractor $\mathcal{A}$, which consists of equilibria and heteroclinics. Since there are no other equilibria in $N_\epsilon(u_\pm)$ besides $u_\pm$, then the only heteroclinics that can occur are between them.

On the other hand, Theorem 2.4.3 implies that along a heteroclinic $u(t) \in \mathcal{H}$ the zero number satisfies $z^t(u - u_\pm) = i(u_+)$ for all time, since $i(u_-) = i(u_+) + 1$. Therefore $u(t) \in K(u_\pm)$ and the closure of the orbit is contained in $N_\epsilon(u_\pm)$. Since the closure of the heteroclinic is invariant, it must be contained in Σ.

Lastly, it is proven that $C(\Sigma) = [0]$ in three steps, yielding the desired contradiction and the proof of the theorem. We modify the first and second step from [29], whereas the third remain the same.

In the first step, a model is constructed displaying a saddle-node bifurcation with respect to a parameter μ, for $n := z(u_+ - u_-) \in \mathbb{N}$ fixed,

$$0 = a(v)[v_{\xi\xi} + \lambda_n v] + g_n(\mu, \xi, v, v_\xi) \tag{70}$$

where $\xi \in [0, \pi]$ has Neumann boundary conditions, $\lambda_n = -n^2$ are the eigenvalues of the laplacian with $\cos(n\xi)$ as respective eigenfunctions, and

$$g_n(\mu, \xi, v) := \left(v^2 + \frac{1}{n^2}v_\xi^2 - \mu\right)\cos(n\xi).$$

For $\mu > 0$, a simple calculation shows that $v_\pm = \pm\sqrt{\mu}\cos(n\xi)$ are equilibria of (70). Hence,

$$z(v_+ - v_-) = z(u_+ - u_-) =: n \tag{71}$$

since the n intersections of v_- and v_+ will be at its n zeroes.

Moreover, those equilibria are hyperbolic for small $\mu > 0$, such that $i(v_+) = n + 1$ and $i(v_-) = n$. Indeed, parametrize the bifurcating branches by $\mu = s^2$ so that $v(s, \xi) = s\cos(n\xi)$, where $s > 0$ correspond to v_+ and $s < 0$ to v_-. Linearizing at the equilibrium $v(s, \xi)$, the eigenvalue problem becomes

$$\eta v = a(v_\pm)[v_{\xi\xi} + \lambda_n v] + [2sv_\pm v + \frac{2s\partial_\xi v_\pm v_\xi}{-n}]v_\pm$$

where $v_\pm$ are coefficients depending on ξ, and the unknown eigenfunction is v, corresponding to the eigenvalue η.

Hence $\eta_n(s) = 2s$ is an eigenvalue with $v_\pm(\xi)$ its corresponding eigenfunction. Hence, by a perturbation argument in Sturm-Liouville theory, that is $\mu = 0$ we have the usual laplacian with n positive eigenvalues and one eigenvalue $\eta_n(0) = 0$. Hence for small $\mu < 0$, the number of positive eigenvalues persist, whereas for small $\mu > 0$, the number of positive eigenvalues increases by 1. This yields the desired claim about hyperbolicity and the Morse index.

Now consider the quasilinear parabolic equation such that (70) is the equilibria equation. The equilibria $v_\pm$ together with their connecting orbits form an isolated set

$$\Sigma_\mu := \overline{W^u(v_-) \cap W^s(v_+)}$$

with isolating neighborhood $N_\epsilon(v_\pm)$, and the bifurcation parameter can also be seen as a homotopy parameter. Hence the Conley index is of a point by homotopy invariance as desired, that is,

$$C(\Sigma_\mu) = C(\Sigma_0) = [0]. \tag{72}$$

In the second step, the equilibria v_- and v_+ are transformed respectively into u_- and u_+ via a diffeomorphism which is not a homotopy.

Recall $n = z(v_- - v_+) = z(u_+ - u_-)$. Hence, choose $\theta(\xi)$ a smooth diffeomorphism of $[0, \pi]$ that maps the zeros of $v_- - v_+$ to the zeros of $u_- - u_+$. Therefore, from now on we suppose that the zeros of $v_- - v_+$ and $u_- - u_+$ occur in the same points in $\theta \in [0, \pi]$.

Consider the transformation

$$\begin{aligned} \Theta : X &\to X \\ v &\mapsto \alpha[v - v_-] + u_- \end{aligned}$$

where α is defined pointwise through

$$\alpha := \begin{cases} \frac{u_+ - u_-}{v_+ - v_-} & , \text{ if } v_+ \neq v_- \\ \frac{\partial_\theta(u_+ - u_-)}{\partial_\theta(v_+ - v_-)} & , \text{ if } v_+ = v_- \end{cases}$$

such that the coefficient α is smooth and nonzero due to the l'Hôpital rule. Hence, $\Theta(v_-) = u_-$ and $\Theta(v_+) = u_+$ as desired.

Moreover, the number of intersections of functions is invariant under the maps Θ and θ, that is,

$$z(\Theta(v(\theta) - \tilde{v}(\theta))) = z(v(\xi) - \tilde{v}(\xi)) \tag{73}$$

and hence $K(v_\pm)$ is mapped to $K(u_\pm)$ under Θ.

Through the map ξ, the model equation (70) is mapped to the equation (51) with different diffusion coefficients and nonlinearity, namely

$$u_t = a(u)(x_\xi)^2 u_{xx} + f(x, u, u_x x_\xi) + u_x \xi_{xx}$$

where the Neumann boundary conditions are preserved.

Through the map Θ, the model equation (70) is mapped to

$$\begin{aligned} u_t = a(u)\alpha u_{xx} &+ f(x, \Theta(u), \partial_x \Theta(u)) \\ &+ 2\alpha_x \partial_x[u - v_-] + \alpha_{xx}[u - v_-] + \partial_{xx} u_- \\ &+ \alpha_x[u - v_-] + \partial_x u_- - \partial_{xx} v_- \end{aligned}$$

where the Neumann boundary conditions are preserved. We call the reaction terms $\tilde{f}$.

In other words, the maps Θ and θ yield and equation of the type

$$u_t = \tilde{a}(x) a(u) u_{xx} + \tilde{f}(x, u, u_x)$$

where the coefficients $\tilde{a}(x)$ and $b(\theta)$ are obtained by combining the diffusion within the above two equations, whereas the remaining terms are collected in order to define the reaction term $\tilde{f}$.

Moreover, the set Σ_μ is transformed into an isolated invariant set $\tilde{\Sigma}$ with invariant neighborhood $N_\epsilon(u_\pm)$ such that

$$C(\Sigma_\mu) = C(\Theta(\Sigma_\mu)) = C(\tilde{\Sigma}). \tag{74}$$

Hence, one identify the equilibria in the model constructed in the first step with the equilibria from the equation (51), by preserving neighborhoods and the Conley index.

In the third step, we homotope the diffusion coefficients $a(\theta), b(\theta)$ and nonlinearity $\tilde{f}$ to be the standard axisymmetric diffusion and the desired reaction f from the equation (8). Indeed, consider the semilinear parabolic equation

$$u_t = a^\tau(\theta)u_{\theta\theta} + b^\tau(\theta)\frac{u_\theta}{\tan(\theta)} + \tilde{f}^\tau(\theta, u, u_\theta)$$

where

$$\begin{aligned} a^\tau(\theta) &:= \tau a(\theta) + 1 - \tau \\ b^\tau(\theta) &:= \tau b(\theta) + 1 - \tau \\ f^\tau(\theta, u, u_\theta) &:= \tau\tilde{f} + (1-\tau)f + \sum_{i=-,+} \chi_{u_i}\mu_{u_i}(\tau)[u - u_i(\theta)] \end{aligned}$$

and χ_{u_i} are cut-offs begin 1 nearby u_i and zero far away, the coefficients $\mu_i(\tau)$ are zero near $\tau = 0$ and 1 and shift the spectra of the linearization at $u_\pm$ such that uniform hyperbolicity of these equilibria is guaranteed during the homotopy.

Consider u_-, u_+ and their connecting orbits during this homotopy,

$$\Sigma^\tau := \overline{W^u(u_-^\tau) \cap W^u(u_+^\tau)}.$$

Note that $\Sigma^\tau \subseteq K(u_\pm)$, since the dropping lemma holds throughout the homotopy. Variying τ, the equilibria, $u_\pm$ do not bifurcate due to normal hyperbolicity. Choosing $\epsilon > 0$ small enough, the neighborhoods $N_\epsilon(u_\pm)$ is an isolating neighborhood of Σ^τ throughout the homotopy. Indeed, Σ^τ can never touch the boundary of $K(u_\pm)$, except at the points $u_\pm$ by the dropping lemma. Once again the Conley index is preserved by homotopy invariance, yielding

$$C(\Sigma) = C(\Sigma^0) = C(\Sigma^\tau) = C(\Sigma^1) = C(\tilde{\Sigma}). \tag{75}$$

Finally, the equations (72), (74) and (75) yield that the Conley index of Σ is the homotopy type of a point, and hence the desired result.

■

3.6 EXAMPLE: QUASILINEAR CHAFEE-INFANTE

In this section it is given an example of the theory above, namely, it is constructed the attractor of a quasilinear Chafee-Infante type problem,

$$u_t = u^n u_{xx} + \lambda u^{n+1}[1 - u^2] \tag{76}$$

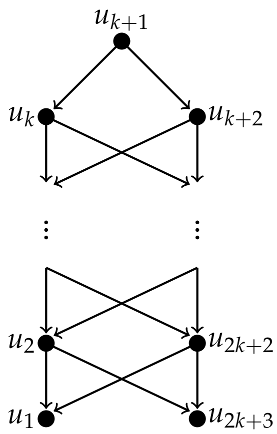
where $n \in \mathbb{N}_0$, $x \in [0, \pi]$ has Neumann boundary conditions and initial data $u_0 \in C^{2\alpha+\beta}([0,\pi]) \cap \{u > \epsilon\}$ with $\alpha.\beta \in (0,1)$ and $\epsilon > 0$ small, so that the equation generates a dynamical system in such space, as in [49].

The equilibria equation describing the shooting curves is

$$\begin{cases} u_\tau &= p \\ p_\tau &= -\frac{f(\lambda,u)}{a(u)} = -\lambda u[1-u^2] \\ x_\tau &= 1. \end{cases} \tag{77}$$

Hence the shooting is exactly the same as the semilinear one. Thus, the permutation and the attractor are the same as the semilinear Chafee-Infante problem in [29]. Therefore, both attractors are geometrically (connection-wise) the same. The only difference lies in the equilibria, and the parameter λ must lie between two eigenvalues of the appropriate diffusion operator.

Theorem 3.6.1. *For $\lambda \in (\lambda_k, \lambda_{k+1})$, where λ_k is the k-th eigenvalue of the linearized operator of (76) with $k \in \mathbb{N}_0$, the equation (76) has $2k+3$ hyperbolic equilibria $u_1, ..., u_{2k+3}$. Moreover, its Sturm attractor $\mathcal{A}_k$ is*



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3.7 DISCUSSION

We now mention two applications in the realm of general relativity and curve shortening flow, and two generalization proposals for the results in this chapter, namely a similar result for fully nonlinear parabolic equations, and fourth order parabolic equations.

It would be interesting to compute the attractor for other chosen nonlinearities f. In particular, one that has its application the construction of metrics at the event horizon of black holes, with a prescribed scalar curvature. For the case of Schwarzschild metric, the scalar curvature is chosen so that resulting parabolic equation is

$$u_t = u^2 u_{xx} - u(1 - \lambda u^2)$$

where $\lambda \in \mathbb{R}_+$. Such problem was considered in [28], where in such axially symmetric class it was shown that the equilibria $u \equiv \lambda^{-1/2}$ bifurcates in an alternating sequence of pitchfork and transcritical bifurcations.

In this setting, the bifurcating equilibria are not necessarily hyperbolic, but only normally hyperbolic, as it was seen in [28]. Their computation of the Morse indices indicate that the equilibria are hyperbolic in the subspace of axially symmetric solutions.

Moreover, the nonlinearity f above does not satisfy the growth conditions that guarantees dissipativity. Numerical simulation of the shooting curve suggest that such nonlinearity is slowly nondissipative, as it is in [70]. We postpone this discussion, until all these tools have been sharpened and adapted for such case.

Another application is to obtain results in curve shortening flow. Considering a planar jordan curve flowing with respect to its curvature flow, then its curvature changes according to a quasilinear parabolic equation

$$k_t = k^2[k_{xx} + k]$$

as in [9] and [10]. Note there are several ways of modelling this same phenomena.

Certain solutions are known to blow up in finite time. In particular, Grayson's theorem, in [7], guarantee that a strict convex curve shrinks to a point in finite time. The condition that a curve is strictly convex, mathematically is $k > \epsilon$ for $\epsilon > 0$ fixed and guarantees strict parabolicity of the equation above.

We believe that self-similar solutions self-similar solutions of the type of the ODE blow up rate yields a dissipative nonlinearity, and the attractor of such equation is a single equilibria related to the circle. This would yield another proof of Grayson's theorem.

The attractor construction in this chapter raises the question if one can construct the Sturm attractor for fully nonlinear second order parabolic and dissipative equation of the type

$$u_t = f(x, u, u_x, u_{xx})$$

satisfying the parabolicity condition $\partial_q f > 0$ for $q = u_{xx}$. Some results, such as shooting and obtaining permutations can be easily adapted. Others, like a Lyapunov functional and the full attractor construction are not so obvious. Those questions shall be treated in the near future.

A last question is if it is possible to obtain any dynamical information of the attractor for higher order parabolic equations, such as the Swift-Hohenberg,

$$u_t = -(1 + \partial_x^2)^2 u + \lambda u + f(u)$$

where $\lambda > 1$. Such equation can be used to generate spatial-temporal patterns.

This system has a Lyapunov functional. Hence, solutions either blow-up in finite time, or are global and then converge to equilibria. But the main tool from the second order equation in such case is not known, and probably does not hold: the dropping lemma. See [60].

4

THE SYMMETRIZATION PROPERTY

The goal of this chapter is to study how the symmetry of the spherical domain influences solutions of elliptic equations on such domain. The method pursued is a variant of the moving plane method, discovered by Alexandrov [2] and used for differential equations by Gidas, Ni and Nirenberg [34].

Consider the quasilinear elliptic equation

$$0 = a(u)\Delta_{\mathbb{S}^2} u + f(u) \tag{78}$$

where $(\theta, \phi) \in \mathbb{S}^2$, a and f are analytic.

We say that a function $u \in C^2(\mathbb{S}^2)$ has *axial extrema* if its extrema in ϕ occur as axis from the north to south pole. Mathematically, if $u_\phi(\theta_0, \phi_0) = 0$ for a fixed $(\theta_0, \phi_0) \in \mathbb{S}^2$, then $u_\phi(\theta, \phi_0) = 0$ for any $\theta \in [0, \pi]$. In that case, the extrema depend only at the position in ϕ. Note that there are finitely many axial extrema, since a, f are analytic, and so is u, as in [21]. Denote them by $\{\phi_i\}_{i=0}^N$.

A simpler case of the above is when axial extrema are *leveled*, that is, if all axial maxima ϕ^* have the same value $u(\theta, \phi^*) = M(\theta)$, and all axial minima ϕ_* also have the same value $u(\theta, \phi_*) = m(\theta)$.

Theorem 4.0.1. Symmetry of Certain Equilibria [Lappicy ('17)]
Suppose that u is a non constant equilibrium of (78) *such that all its extrema are leveled and axial. Then $\phi_i = \frac{\phi_{i-1}+\phi_{i+1}}{2}$ and*

$$u(\theta, \phi) = u(\theta, R_{\phi_i}(\phi)) \tag{79}$$

for all $i = 0, , ..., N$, where $\phi_{-1} := \phi_N$ and $\phi_{N+1} := \phi_0$, $R_{\phi_i}(\phi) := 2\phi_i - \phi$ is the reflection of ϕ with respect to ϕ_i and $(\theta, \phi) \in [0, \pi] \times [\phi_{i-1}, \phi_i]$.

For positive solutions of elliptic equations on a ball with Dirichlet boundary conditions, a similar symmetrization result was obtained by Gidas, Ni and Nirenberg [34], using the moving plane method developed by Alexandrov [2] and further by Serrin [75]. We give a brief sketch of this method in Section 4.1.

Proving the symmetrization property in the sphere has some difficulties. In particular, the domain has no boundary and it is not clear where to start the moving plane method. This problem was partially solved by Padilla [58] for particular convex subsets of the sphere, and later considered by Kumaresan and Prajapat [45] for subsets of the sphere contained in a hemisphere. Later by Brock and Prajapat [17] for subsets containing hemisphere, but still not the full sphere. All these methods rely on a stereographic projection, so that domains within the sphere are transformed into domains in the Euclidean space and one can apply the moving plane method. Moreover, such results in the sphere deal with positive solutions.

The following sections are organized as follows. We recall the important tools and mechanisms in the moving plane method in Section 4.1. Then, we adapt it to a moving arc method for elliptic problems in Section 4.2 and prove the main theorem in this chapter. Lastly, in Section 4.3, we discuss alternative proofs and generalizations of such method.

4.1 MECHANISMS OF SYMMETRIZATION

In this section we mention three important mechanisms in order to prove a symmetrization property using a variant of the moving plane method. Then, we give a sketch of the Gidas, Ni and Nirenberg's proof in order to compare with the moving arc method introduced in Section 4.2.

Consider the following elliptic case,

$$0 = L_\Omega u + f(u) \tag{80}$$

for some elliptic operator L_Ω in a domain Ω and f analytic.

Firstly, the domain Ω must have some symmetry. For example, the sphere is symmetric with respect to rotations and reflections of both angle variables (θ, ϕ). Those are the expected symmetries for solutions of (80).

Secondly, the linear diffusion operator L_Ω should be invariant under such symmetry. For example, the spherical Laplacian is

$$\Delta_{\mathbb{S}^2} u = u_{\theta\theta} + \frac{u_\theta}{\tan(\theta)} + \frac{u_{\phi\phi}}{\sin^2(\theta)}$$

and a change of coordinates $\tilde{\theta}(\theta)$ would also change the coefficients $\frac{1}{\tan(\theta)}$ and $\frac{1}{\sin^2(\theta)}$.

On the other hand, any reflection or rotation in the ϕ variable do not change the coefficients above, yielding the same spherical laplacian as linear operator. Hence, ϕ is the candidate variable that solutions might possess some symmetry. Therefore, the difference of solutions $w(\theta, \phi) := u(\theta, \phi) - u(\theta, R_\epsilon(\phi))$ satisfy a linear equation

$$0 = \Delta_{\mathbb{S}^2} w + c(\theta, \phi) w \tag{81}$$

with $c(\theta, \phi) := \int_0^1 \partial_u f((1-\zeta)u(\theta, R_\epsilon(\phi)) + \zeta u(\theta, \phi)) d\zeta$ and $R_\epsilon(\phi) := 2\epsilon - \phi$ is the reflection of ϕ with respect to the arc ϵ. Similarly for the difference $w(\theta, \phi) := u(\theta, \phi) - u(\theta, Rot_\epsilon(\phi))$.

Thirdly, one desires that $w \equiv 0$. This follows by choosing an appropriate subdomain of Ω so that (81) has a maximum principle and w has a sign at the boundary of such subdomain, yielding $w \geq 0$ and $w \leq 0$.

4.1.1 *Moving Plane Method*

We describe the main idea of Gidas, Ni and Nirenberg [34] for positive solutions u of the elliptic equation (80) with $x = (x_1, x_2) \in \Omega$ being the unit two dimensional ball centered at the origin and Dirichlet boundary conditions.

Consider the left-most point of the ball $x_1 = -1$ and a vertical line being its tangent. Then, move this vertical line by a little amount to the right, that is, in the x_1 direction. Call it the *ϵ-line*.

Define by Ω_ϵ the open subdomain considering the points within the ball between the left most point $x_1 = -1$ and the ϵ-line,

$$\Omega_\epsilon := \{x \in \Omega \mid x_1 \in (-1, \epsilon) \text{ and } \epsilon \in (-1, 0)\}.$$

Its boundary $\partial\Omega_\epsilon$ consists of two segments: one along the ϵ-line, another along the boundary of Ω. Moreover, the reflection with respect to the ϵ-line is given by $R_\epsilon(x) := (2\epsilon - x_1, x_2)$.

The idea is to move the ϵ-line from the left to the right, and consider an appropriate an appropriate differential equation in Ω_ϵ.

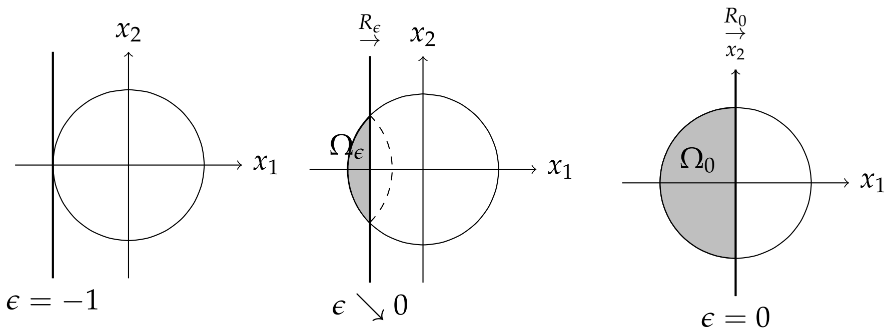


Figure 7: Moving Plane Method: start with the ϵ-line at the left-most point of the domain and move it to the right, until it reaches the symmetric line at the origin. Repeat this procedure in all radial directions.

The difference of solutions

$$w^\epsilon(x) := u(x) - u(R_\epsilon x)$$

satisfies a linear equation, where $R_\epsilon(x) := (2\epsilon - x_1, x_2)$ is the reflection of the x_1 variable along the ϵ-line,

$$\begin{cases} 0 = \Delta w^\epsilon + c(x) w^\epsilon & \text{on } \Omega_\epsilon \\ w^\epsilon \leq 0 & \text{on } \partial\Omega_\epsilon \end{cases}$$

for $x \in \Omega_\epsilon$ and $c(x) := \int_0^1 \partial_u f((1-\tau)u(\theta, R_\epsilon(\phi)) + \tau u(\theta,\phi))d\tau$.

The boundary conditions are $w^\epsilon = 0$ along the ϵ-line due to the reflection, whereas $w^\epsilon < 0$ outside the ϵ-line, due to Dirichlet boundary conditions and the assumption that $u > 0$ in the interior of Ω, which is where the reflection $R_\epsilon(x)$ lies.

Hence one can apply the weak maximum principle for thin domains, yielding that $w^\epsilon \leq 0$ on Ω_ϵ. Such maximum principle will be made precise when it is used for the spherical case.

Using once again the maximum principle, one can move the ϵ-line until the origin. In order to show this extension argument, it is crucial that the solution u is positive and not only non-negative.

Indeed, consider

$$\epsilon^* := \sup\{\epsilon \in [-1,0] \mid w^\epsilon \leq 0 \text{ in } \Omega_\epsilon\}.$$

Suppose that $\epsilon^* < 0$. In order to contradict its maximality, it is shown that

$$w^{\epsilon^*+\delta} \leq 0$$

in $\Omega_{\epsilon^*+\delta}$ for some $\delta > 0$.

In particular, the definition of ϵ^* says that $w^{\epsilon^*} \leq 0$ for the compact subset $\overline{\Omega_{\epsilon^*-\delta}} \subset \Omega_{\epsilon^*}$. Moreover, the inequality is strict,

$$w^{\epsilon^*} < 0$$

in $\overline{\Omega_{\epsilon^*-\delta}}$.

Indeed, in order to prove it, notice that the strong maximum principle guarantees the only other possibility would be that $w^{\epsilon^*} \equiv 0$ in $\overline{\Omega_{\epsilon^*-\delta}}$. This case is not possible for $\epsilon^* < 0$, because $w^{\epsilon^*} < 0$ at the boundary $\Omega \cap \partial\Omega_{\epsilon^*-\delta}$, due to Dirichlet boundary conditions and that $u > 0$ in Ω, by assumption.

By continuity in the parameter ϵ and compactness of $\overline{\Omega_{\epsilon^*-\delta}}$, such inequality still holds with a parameter increment by small $\delta > 0$, yielding

$$w^{\epsilon^*+\delta} < 0 \tag{82}$$

in $\overline{\Omega_{\epsilon^*-\delta}}$.

It is enough to prove $w^{\epsilon^*+\delta} \leq 0$ for the remaining set $\Omega_{\epsilon^*+\delta} \backslash \overline{\Omega_{\epsilon^*-\delta}}$. In this set, $w^{\epsilon^*+\delta}$ satisfies

$$\begin{cases} 0 = \Delta w^{\epsilon^*+\delta} + c(x) w^{\epsilon^*+\delta} & \text{on } \Omega_{\epsilon^*+\delta} \backslash \overline{\Omega_{\epsilon^*-\delta}} \\ w^{\epsilon^*+\delta} \leq 0 & \text{on } \partial(\Omega_{\epsilon^*+\delta} \backslash \overline{\Omega_{\epsilon^*-\delta}}) \end{cases}$$

The boundary values are $w^{\epsilon^*+\delta} = 0$ at the $(\epsilon^* + \delta)$-line due to the reflection, $w^{\epsilon^*+\delta} < 0$ at the $(\epsilon^* - \delta)$-line due to the compactness argument in (82), and $w^{\epsilon^*+\delta} < 0$ on the remaining part of the boundary that intersects the ball, due to Dirichlet and $u > 0$.

Since $\Omega_{\epsilon^*+\delta} \backslash \overline{\Omega_{\epsilon^*-\delta}}$ is still a thin domain for $\delta > 0$ small, the maximum principle implies

$$w^{\epsilon^*+\delta} \leq 0$$

in $\Omega_{\epsilon^*+\delta} \backslash \overline{\Omega_{\epsilon^*-\delta}}$.

This contradicts the maximality of $\epsilon^* < 0$, and hence $\epsilon^* = 0$. This proves $w^0 \leq 0$ in Ω_0.

The change of variables $x_1 \mapsto -x_1$ yields the same linear equation in the ball as above, due to the symmetry of the domain, but w^ϵ has the opposite sign within its boundary conditions. Using the same arguments as above, one obtains the opposite inequality $w^\epsilon \geq 0$. Alternatively, one can repeat the moving plane method by starting at $x_1 = 1$ and decrease the ϵ-line until the origin. Lastly, due to the symmetry of the ball, one can use this argument on all radial directions. This yields a sketch of the symmetrization result for elliptic equations.

4.2 SYMMETRY OF CERTAIN EQUILIBRIA

The starting point of the moving arc method is at some axial extrema ϕ_{i-1}. Without loss of generality, we can assume $\phi_{i-1} = 0$. Otherwise, consider $\phi \mapsto \phi - \phi_{i-1}$.

Define a small sector within the sphere nearby $\phi_{i-1} = 0$,

$$\Omega_\epsilon := \{(\theta, \phi) \in \mathbb{S}^2 \mid \theta \in (0, \pi), \phi \in (0, \epsilon)\}$$

with small $\epsilon > 0$. The boundary $\partial\Omega_\epsilon$ is given by two axis, namely at $\phi = 0$ and $\phi = \epsilon$. We call the latter by *ϵ-arc*.

Then, we will move the ϵ-arc by increasing ϵ, and consider reflections along it, defined by $R_\epsilon(\phi) := 2\epsilon - \phi$. Due to the periodic boundary conditions in ϕ, one has $\Omega_\epsilon, R_\epsilon(\Omega_\epsilon) \subset \mathbb{S}^2$ for arbitrarily large ϵ.

We will then show that equilibria solutions of (78) and its reflection are related by an inequality, for small $\epsilon > 0$. Then one can extend such inequality for larger ϵ. Lastly one can prove the reversed inequality. The method is better illustrated in the below picture and the following lemmata.

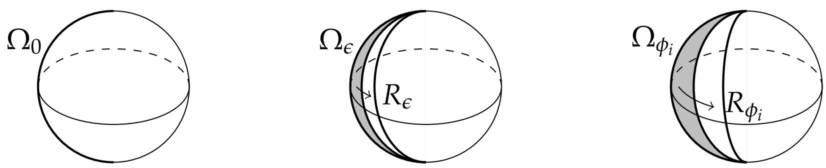


Figure 8: Moving Arc Method: start at ϕ_{i-1} and consider sectors Ω_ϵ nearby it. Reflect along the and ϵ-arc to obtain an inequality. Then extend such inequality for bigger ϵ. Repeat the process starting at ϕ_{i+1} and move the arc in the opposite direction.

Recall that a function $u \in C^2(\mathbb{S}^2)$ has *leveled axial extrema* if its extrema in ϕ occur as axis from the north to south pole, that is, if $u_\phi(\theta_0, \phi_0) = 0$ for a fixed $(\theta_0, \phi_0) \in \mathbb{S}^2$, then $u_\phi(\theta, \phi_0) = 0$ for any $\theta \in [0, \pi]$. In that case, the extrema depend only at the position in

ϕ. Since a, f are analytic, so is u and there are finitely many extrema, denoted by $\{\phi_i\}_{i=0}^N$.

Moreover, all axial maxima ϕ_i have the same value $u(\theta, \phi_i) = M(\theta)$, and all axial minima ϕ_i also have the same value $u(\theta, \phi_i) = m(\theta)$.

Lemma 4.2.1. ***Inequality for Reflections***
Consider u a nonconstant equilibrium of (78) *such that all extrema are leveled and axial. If ϕ_i is an leveled axial maximum, then*

$$u(\theta, \phi) \leq u(\theta, R_{\phi_i}(\phi))$$

for all $i = 1, , ..., N$, $(\theta, \phi) \in [0, \pi] \times [\phi_{i-1}, \phi_i]$.

Similarly, the above lemma holds with reversed inequality, if ϕ_i is an axial leveled minimum. This yields the claimed reflection property in the ϕ direction.

Lemma 4.2.2. ***Equality for Reflections***
Consider u an equilibrium of (78) *such that all extrema are leveled and axial, given by $\{\phi_i\}_{i=0}^N$. Then $\phi_i = \frac{\phi_{i-1}+\phi_{i+1}}{2}$ and*

$$u(\theta, \phi) = u(\theta, R_{\phi_i}(\phi))$$

for all $i = 1, , ..., N$, where $\phi_{-1} := \phi_N$ and $\phi_{N+1} := \phi_0$, and $(\theta, \phi) \in [0, \pi] \times [\phi_{i-1}, \phi_i]$.

The main tools to prove the above are different versions of the maximum principle. Below we state a particular version of the one for thin domains on manifolds, as in Padilla [58], and the strong unique continuation theorem for manifolds, due to Kazdan [44].

Theorem 4.2.3. Maximum Principles
Consider w a solution of the linear differential equation,

$$0 = \Delta_{\mathbb{S}^2} w + c(\theta, \phi) w \tag{83}$$

where $\Delta_{\mathbb{S}^2}$ is the Laplace-Beltrami operator on the sector $\Omega_\epsilon \subseteq \mathbb{S}^2$ with $\epsilon > 0$.

1. Thin domains: *Consider $U \subseteq \Omega_\epsilon$ such that the Lebesgue measure $|U| < \mu$ for sufficiently small μ. If $w \leq 0$ on ∂U, then $w \leq 0$ in U.*
2. Strong: *If $w \leq 0$ on Ω_ϵ, then either $w < 0$ in Ω_ϵ or $w \equiv 0$ in $\overline{\Omega_\epsilon}$.*
3. Unique continuation: *If $w \equiv 0$ in an open set $U \subseteq \mathbb{S}^2$, then $w \equiv 0$ in $\mathbb{S}^2$.*

In particular, Ω_ϵ with $\epsilon > 0$ sufficiently small i a thin domain. Also, the sector Ω_ϵ is an open set.

In order to prove the strong version, suppose that $w(\theta, \phi) = 0$ for some $(\theta, \phi) \in \Omega_\epsilon$. One can calculate the value of a function at a point by its mean value around a geodesic ball B_r in the sphere with geodesic distance $r > 0$ small. Hence,

$$0 = w(\theta, \phi) = \frac{1}{vol(B_r)} \int_{B_r} w \leq \sup_{B_r}(w) = 0$$

and the equality holds if and only if $w \equiv 0$ on B_r. One can show that $\{(\theta, \phi) \in \Omega_\epsilon \mid w(\theta, \phi) = 0\}$ is nonempty, closed and open in the connected set Ω_ϵ, hence it must be the full set Ω_ϵ. See [25] for a detailed exposition of the euclidean case.

Proof of Lemma 4.2.1. We firstly prove an inequality for a small reflection, then we extend the inequality for larger reflections.

Recall that the axial extrema are finite, since f is analytic, and so is u. See [21]. Consider three consecutive axial extrema, $\phi_{i-1} < \phi_i < \phi_{i+1}$. As mentioned, we start the moving arc at ϕ_{i-1}. Suppose that it is an axial minimum, then ϕ_i is an axial maximum and ϕ_{i+1} is an axial minimum.

Indeed, if ϕ_i was a also an axial minimum, there would be an axial maximum $\phi_* \in (\phi_{i-1}, \phi_i)$, by the extreme value theorem and noticing all extrema are axial. This contradicts that ϕ_{i-1} and ϕ_i are consecutive. Similarly for ϕ_{i+1}.

Therefore,

$$u(\theta, \phi_{i-1}) \leq u(\theta, \phi) \tag{84}$$

for all $\theta \in [0, \pi]$ and $\phi \in (\phi_{i-1}, \phi_i)$. In particular, the inequality must be strict for some value (θ, ϕ), otherwise the function would be locally constant, and due to the unique continuation theorem, globally constant.

Define the difference of the equilibrium solution u of (78) and itself with reflected angle through

$$w^\epsilon(\theta, \phi) := u(\theta, \phi) - u(\theta, R_\epsilon(\phi))$$

for $(\theta, \phi) \in \Omega_\epsilon$ with $\epsilon > 0$ sufficiently small. Then, w^ϵ satisfies

$$\begin{cases} 0 = Lw^\epsilon + c(\theta, \phi) w^\epsilon & \text{on } \Omega_\epsilon \\ w^\epsilon \leq 0 & \text{on } \partial\Omega_\epsilon \end{cases} \tag{85}$$

where $L := b(\theta, \phi)\Delta_{\mathbb{S}^2}$ is an elliptic operator defined below.

Indeed, the boundary values are $w^\epsilon(\theta, \epsilon) = 0$ due to the reflection, and $w^\epsilon(\theta, 0) \leq 0$ since $\phi_{i-1} = 0$ is an axial minimum, as in (84).

Moreover, the fundamental theorem of calculus implies that the difference of two solutions of (78) satisfy

$$0 = \int_0^1 \frac{d}{d\tau} \left[a(u^\tau) \Delta_{\mathbb{S}^2} u^\tau + f(u^\tau) \right] d\tau$$

with $u^\tau := \tau u(\theta, \phi) + (1 - \tau) u(\theta, R_\epsilon(\phi))$.

Calculating the above derivatives with respect to τ,

$$0 = b(\theta,\phi)\Delta_{\mathbb{S}^2} w^{\epsilon} + c(\theta,\phi) w^{\epsilon}$$

where the bounded coefficients are given by the Hadamard formulas,

$$b(\theta,\phi) := \int_0^1 a(u^{\tau}) d\tau$$
$$c(\theta,\phi) := \int_0^1 \partial_u f(u^{\tau}) d\tau + \int_0^1 \partial_u a(u^{\tau}) \Delta u^{\tau} d\tau.$$

The maximum principle for thin domains in Theorem 4.2.3 implies that

$$w^{\epsilon} \leq 0 \tag{86}$$

in Ω_{ϵ} for $\epsilon > 0$ sufficiently small.

Now, we extend such inequality for larger $\epsilon > 0$. Let

$$\epsilon^* := \sup\{\epsilon \in [0, 2\pi] \mid w^{\epsilon} \leq 0 \text{ in } \Omega_{\epsilon} \text{ for all } \epsilon \leq \epsilon^*\}$$

Note $\epsilon^* > 0$, using the maximum principle for thin domains as in (86). Also, $\epsilon^* \leq \phi_i$, since this is an axial maximum and hence $w^{\phi_i+\delta}(\theta,\phi) > 0$ for some $(\theta,\phi) \in \Omega_{\phi_i+\delta} \backslash \Omega_{\phi_i}$ with $\delta > 0$ sufficiently small, by applying the same arguments as in (84) for a maximum.

We claim that $\epsilon^* = \phi_i$. Suppose that $\epsilon^* < \phi_i$. In order to contradict its maximality, it is shown that

$$w^{\epsilon^*+\delta} \leq 0 \tag{87}$$

in $\Omega_{\epsilon^*+\delta}$ for some $\delta > 0$.

The extension argument is a bit different than the one presented in Section 4.1 for the Gidas, Ni and Nirenberg. The reason is that the previous extension argument works well for strict positive solutions, but not for solutions satisfying the non strict inequality (84). We adapt the method using the unique continuation theorem, as Poláčik [61].

By hypothesis, we have that $w^{\epsilon^*} \leq 0$ in Ω_{ϵ^*}. Moreover, the strong maximum principle in Theorem 4.2.3 guarantees

$$w^{\epsilon^*} < 0$$

in Ω_{ϵ^*}. The other possibility, that $w^{\epsilon^*} \equiv 0$ in $\overline{\Omega_{\epsilon^*}}$ would imply that $w^{\epsilon^*} \equiv 0$ in $\mathbb{S}^2$ due to the unique continuation theorem. Hence u would be a constant, which is a contradiction.

In particular,

$$w^{\epsilon^*} < 0$$

for a compact subset K of Ω_{ϵ^*} such that $|\Omega_{\epsilon^*} \backslash K| < \frac{\mu}{2}$.

Due to continuity in the parameter and that K is compact,

$$w^{\epsilon^*+\delta} < 0 \tag{88}$$

in K.

It is enough to prove a similar inequality for the remaining set $\Omega_{\epsilon^*+\delta}\backslash K$. In this set, $w^{\epsilon^*+\delta}$ satisfy

$$\begin{cases} 0 = Lw^{\epsilon^*+\delta} + c(\theta,\phi)w^{\epsilon^*+\delta} & \text{on } \Omega_{\epsilon^*+\delta}\backslash K \\ w^{\epsilon^*+\delta} \leq 0 & \text{on } \partial(\Omega_{\epsilon^*+\delta}\backslash K) = \partial\Omega_{\epsilon^*+\delta} \cup \partial K \end{cases}$$

with L and $C(\theta,\phi)$ as in (85). Note that one can choose $\delta > 0$ small such that $|\Omega_{\epsilon^*+\delta}\backslash\Omega_{\epsilon^*}| \leq \frac{\mu}{2}$, and hence $|\Omega_{\epsilon^*+\delta}\backslash K| \leq \mu$.

Note that since K is a compact subset of the open set $\Omega_{\epsilon^*+\delta}$, then $\partial(\Omega_{\epsilon^*+\delta}\backslash K) = \partial\Omega_{\epsilon^*+\delta} \cup \partial K$. Moreover, the boundary values are obtained in ∂K because of the inequality (88), $w^{\epsilon^*+\delta}(\theta,\epsilon^*+\delta) = 0$ due to the reflection at the $(\epsilon^*+\delta)$-line, and $w^{\epsilon^*+\delta}(\theta,0) \leq 0$ since the solution only has leveled axial extrema, and $(\theta,0)$ is one of them.

Since $\Omega_{\epsilon^*+\delta}\backslash K$ is still a thin domain for $\delta > 0$ small, the maximum principle implies that

$$w^{\epsilon^*+\delta} \leq 0$$

in $\Omega_{\epsilon^*+\delta}\backslash K$.

Combining this inequality in $\Omega_{\epsilon^*+\delta}\backslash K$ with the same inequality (88) in K, yields the desired inequality (87) in the whole set $\Omega_{\epsilon^*+\delta}$. This contradicts the maximality of ϵ^*, yielding

$$w^{\phi_i} \leq 0$$

in Ω_{ϕ_i}.

■

Proof of Lemma 4.2.2.

Consider three consecutive leveled axial extrema $\phi_{i-1} < \phi_i < \phi_{i+1}$ such that ϕ_i is a maximum and the other two are minima. From the Lemma 4.2.1,

$$u(\theta,\phi) \leq u(\theta, R_{\phi_i}(\phi)) \tag{89}$$

with $(\theta,\phi) \in [0,\pi] \times [\phi_{i-1},\phi_i]$.

It is proven the reversed inequality. This is done by moving the arc on the reversed orientation of ϕ, starting at the minimum ϕ_{i+1}. Indeed, consider the change of variable $\tilde{\phi}(\phi) := \phi_{i+1} - \phi$ with starting axis $\tilde{\phi}(\phi_{i+1}) = 0$.

The analogous of the condition (84) is that $\tilde{\phi} = 0$ is an axial minimum, namely

$$u(\theta,0) \leq u(\theta,\tilde{\phi})$$

for $\tilde{\phi} \in (0,\tilde{\phi}_i)$ and $\tilde{\phi}_i := \tilde{\phi}(\phi_i) = \phi_{i+1} - \phi_i$ is where the maximum ϕ_i lies in the $\tilde{\phi}$ coordinates.

Hence, one can apply the Lemma 4.2.1 in the new variable $\tilde{\phi}$,

$$u(\theta,\tilde{\phi}) \leq u(\theta, R_{\tilde{\phi}_i}(\tilde{\phi})) \tag{90}$$

with $(\theta,\tilde{\phi}) \in [0,2\pi] \times [0,\tilde{\phi}_i]$.

We want to compare (89) and (90), but they are valid in different domains. Consider $\phi \in [\phi_{i-1}, \phi_i]$, yielding (89). Then, either $R_{\phi_i}(\phi)$ is in $[\phi_i, \phi_{i+1}]$ or $[\phi_{i-1}, \phi_i]$, depending which interval is bigger. In the former case, one can use (90) with $\tilde{\phi} = R_{\phi_i}(\phi)$, whereas in the latter, one can use (89) again with $R_{\phi_i}(\phi)$ instead of ϕ. Both cases yield

$$u(\theta, \phi) \leq u(\theta, R_{\phi_i}(\phi)) \leq u(\theta, R_{\tilde{\phi}_i}(R_{\phi_i}(\phi)))$$

with $\phi \in [\phi_{i-1}, \phi_i]$.

Since ϕ_i and $\tilde{\phi}_i$ denote the same point in different coordinates, the composition of reflections $R_{\tilde{\phi}_i}(R_{\phi_i}(\phi)) = \phi$. Hence,

$$u(\theta, \phi) = u(\theta, R_{\phi_i}(\phi))$$

for $\phi \in [\phi_{i-1}, \phi_i]$.

In particular,

$$u(\theta, \phi_{i-1}) = u(\theta, R_{\phi_i}(\phi_{i-1})).$$

Then $R_{\phi_i}(\phi_{i-1})$ is another leveled and axial minimum, since all extrema are leveled and axial. Moreover, we chose three consecutive extrema $\phi_{i-1} < \phi_i < \phi_{i+1}$, hence $R_{\phi_i}(\phi_{i-1}) = \phi_{i+1}$ and ϕ_i is the midpoint between ϕ_{i-1} and ϕ_{i+1}.

■

4.3 DISCUSSION

Clearly, the main result also holds for the sphere S^n with $n \geq 1$. In particular, for $n = 1$, the axial extrema hypothesis is superfluous and the only remaining condition is that extrema are leveled. This implies the symmetry of the equilibria with leveled extrema in the attractor constructed by Fiedler, Rocha and Wolfrum [30], and stabilized by Schneider [73].

Moreover, it is believed that the hypothesis can be weakened yielding similar results. In particular, if the extrema are not *axial*, but are curves from the north pole to the south pole, then a similar result should hold, by using the methods in [61]. Also, if the extrema are not *leveled*, one can expect a similar result, but the domain in which the symmetrization holds is different. Lastly, if $a, f \in C^1$ it should be possible to obtain the same result. Analyticity was only used to obtain finitely many axial extrema.

Note the moving arc method is not limited to the sphere. Depending on the Laplace-Beltrami, the results here can be adapted to other domains. For general manifolds, one might need an assumption on convexity in the candidate direction for symmetrization, such as geodesic convexity. Further investigation is being carried for the torus and the hyperbolic disk.

A natural question that arises is the usual symmetry question: are positive solutions of elliptic equations on the sphere axisymmetric?

The lack of boundary implies the moving plane method is not adequate. One can try other means, for example by stereographic projection. Indeed, Frankel in [31] collects several symmetrization results for the whole plane $\mathbb{R}^2$. In this case, it is required admissible decay conditions at ∞ in order to start the moving plane method. These conditions could possibly be satisfied due to the singularity at the north pole, which imposes Neumann boundary conditions, and hence growth conditions at ∞. Another possibility to obtain axisymmetry of solutions is to adapt the work of Pacella and Weth [57] or Saldaña [72] for the sphere.

We also mention that the method presented here can be replicated for the problem in the ball with any boundary condition, yielding a symmetry result for sign changing solutions. For example the Chafee-Infante nonlinearity $f(\lambda, u) = \lambda u[1 - u^2]$ with $\lambda \in \mathbb{R}$, the Gidas, Ni and Nirenberg result concerns the trivial solutions $u \equiv 0, \pm 1$ which are clearly radially symmetric. On the other hand, there are several bifurcating solutions from 0 that change sign and their properties should be studied. Some of those bifurcating solutions can be computed numerically, as in [83] and references therein. In those cases, there are few examples of solutions which seem to have axial extrema and display the symmetrization proved here.

Another extension of the main theorem in this chapter is a symmetrization property for parabolic equations. Indeed, similar results for positive solutions of parabolic equations on the ball with Dirichlet boundary were obtained by Poláčik and Hess [43], and Babin [13]. In particular, it is proven that the global attractor $\mathcal{A}$ restricted to the subspace of positive solutions consists of radially symmetric solutions. Hence, the result for elliptic equations can be seen as a particular case of the parabolic version, since equilibria are in the attractor.

With that in mind, note that the radially symmetric attractor can be explicitly constructed from the Chapter 2, using nodal properties. But the construction of the attractor for the ball is far from being understood, since nodal properties are not available in higher dimensions. A tool that could be used to tackle that problem could be symmetry.

5

EINSTEIN HAMILTONIAN CONSTRAINT

The Einstein equations model gravity through spacetime as ten coupled partial differential equations. Six of those evolve space in time, whereas the other four constrain the initial data, or intuitively, dictate how space curves itself and is embedded in the bigger framework of spacetime.

We focus on *static time-symmetric* spacetime, namely time independent solutions such that the embedding of space in spacetime is trivial and its extrinsic curvature vanishes. Hence, those ten equations are reduced to only one that indicates how space can bend intrinsically, known as the Einstein's Hamiltonian constraint. See [64].

Exact solutions of such equation with a prescribed function T_{00} describing its energy density are called *pressureless perfect fluids*, commonly used to model idealized distributions of matter in stellar models, such as galaxies and their interactions, or the interior of a star or a black hole. See Chapter 4 in [74].

A simple case among the perfect fluids are the *spherically symmetric* ones. Mathematically, space is described by a three dimensional Riemannian manifold $\mathcal{S}$ with metric g. Assume that the space $\mathcal{S}$ can be written in spherical coordinates, that is $\mathcal{S} := \mathbb{R}_+ \times \mathbb{S}^2$ with $r \in \mathbb{R}_+$ being the radial foliation of two dimensional spheres $(\theta, \phi) \in \mathbb{S}^2$. In the simplest case, the metric splits as

$$g = u^2 dr^2 + r^2 w$$

where w is the standard round metric in $\mathbb{S}^2$ and the metric component $u = u(r, \theta, \phi)$ is an unknown function. For a list of known exact spherically symmetric solutions, see Table I and II in [16].

Computing the scalar curvature $R(g)$ of $\mathcal{S}$, Bartnik [15] claimed that u satisfies the following parabolic equation,

$$2ru_r = u^2 \Delta_{\mathbb{S}^2} u + u + \frac{r^2 R(g) - 2}{2} u^3 \tag{91}$$

where this equation fails to be parabolic at $r = 0$.

This is purely a geometric fact of the chosen space $\mathcal{S}$ and metric g, and still has no relation to the Einstein equations. This connection is made by prescribing a matter model given by a C^∞ function T_{00} and relating it with the the scalar curvature $R(g)$ through the Einstein's Hamiltonian constraint equation,

$$R(g) = 16\pi T_{00} \tag{92}$$

as in [64] and [15].

The simplest solution of (91) is given by the Schwarzschild metric when $R(g) = \frac{4}{r^2}$ and the solution is independent of the angle variables, namely $u(r) = (1/r - 1)^{-1/2}$ which blows up at $r_1 := 1$. This solution models black holes and the radius r_1 is also known as the *event horizon*, whereas there is another *singularity* at $r_0 := 0$.

We are interested in Schwarzschild self-similar solutions of (91), as

$$u(r, \theta, \phi) = \left(\frac{1}{r} - 1\right)^{-\frac{1}{2}} v(r, \theta, \phi) \tag{93}$$

and hence constructing the space of initial data for such solutions in order to rigorously study the dynamics of Einstein evolution equations, such as the stability of black holes.

The stability of black holes has been widely studied over the past years, as in [55], but still lacks some rigorous treatment. Usually only linear stability is treated, and in such case perturbations of the Schwarzschild solution still satisfy the Einstein constraint equations. For the nonlinear stability, the problem is open and it is not known if perturbations of Schwarzschild still satisfy the constraints.

Through the self-similar glasses from (93), v satisfies the following equation for some prescribed scalar curvature $R(g)$,

$$2(1 - r)v_r = v^2 \Delta_{\mathbb{S}^2} v - v + \frac{r^2 R(g)}{2} v^3.$$

Note that the parabolicity of the equation breaks down at the even horizon $r_1 := 1$, since there is no radial derivative. Moreover, it is the backwards heat equation for $r > r_1$, which is not well-posed.

For $r > r_1$, Schwarzschild self-similar solutions of (91) with certain curvature $R(g)$ were constructed by Smith in [79]. For $r \in (r_0, r_1)$ with $r_0 > 0$, and curvature $R(g) = \frac{\lambda+2}{r^2}$ with $\lambda \in \mathbb{R}_+$, it was shown that there are several non-spherical symmetric solutions bifurcating from the Schwarzschild solution, by Fiedler, Hell and Smith in [28].

It is the aim of this thesis to study the structure at the event horizon $r_1 := 1$ from a dynamical point of view, depending on the interior of the black hole, namely $r < r_1$.

For that, rescale the equation through $r = 1 - e^{-2t}$, so that the breakdown of parabolicity at $r_1 := 1$ is now represented as $t \to \infty$ in

$$v_t = v^2 \Delta_{\mathbb{S}^2} v - v + \frac{r^2 R(g)}{2} v^3. \tag{94}$$

Note this equation for v is parabolic, and hence one can study its initial value problem with initial data at $t = 0$, corresponding to $r = 0$.

We recall that we are dealing with time independent solutions of the Einstein equations. Even though t is usually called time in parabolic equations, its interpretation here is different: it is a rescaled radial distance from the black hole singularity at $t = 0$ such that the event horizon $r_1 := 1$ occurs at $t = \infty$.

There are two main results in this thesis, describing the structure of the metric at r_1. One describes the possible static axisymmetric self-similar Schwarzschild metrics at r_1, described rigorously in Chapters 2 and 3. The other, done in Chapter 4, describes some symmetries of the metric at r_1.

For the first result, suppose a priori that the prescribed curvature satisfies the following growth conditions

$$R(g) < \frac{2}{r^2 v^2} \tag{95}$$

for $|v|$ large enough uniformly in θ.

Under such assumption, the equation (94) generates a dissipative dynamical system denoted by $(t, v_0) \mapsto v(t)$ in the phase space $X := C^{2\alpha+\beta}(\mathbb{S}^2) \cap \{v > \epsilon\}$ where $\alpha, \beta \in (0,1)$ are respectively the Hölder exponent and a fractional power exponent, and $\epsilon > 0$ is a fixed value that guarantees strict parabolicity of (94). These conditions are sufficient for the existence of a global attractor $\mathcal{A}$ of (94) which attracts all bounded sets as $t \to \infty$. See [12] and [38].

Hence, for any bounded initial data $v_0 \in X$ at $t = 0$ and a scalar curvature $R(g)$ satisfying (95), there exists a metric $v(t, \theta, \phi)$ in phase space for all $t \in (0, \infty)$, by the semiflow of (94). Moreover, v will approach an equilibrium in $\mathcal{A}$ as $t \to \infty$, due to the existence of a Lyapunov function.

In other words, for any bounded initial data $v_0 \in X$ at the singularity $r_0 := 0$ of self-similar Schwarzschild solutions, there exists a metric $v(r, \theta, \phi)$ for $r \in (0, r_1)$ such that v converges to an equilibrium of $\mathcal{A}$ as $r \to r_1$. Since r_1 is the event horizon, this means that self-similar metrics at the horizon $v(r_1, \theta, \phi)$ are given by equilibria $v_1(\theta, \phi) \in \mathcal{A}$ through $v(r_1, \theta, \phi) = v_1(\theta, \phi)$ and the attractor $\mathcal{A}$ describes the possible metrics at r_1.

Then, one can use Smith's construction in [79] with such equilibria $v(r_1, \theta, \phi) \in \mathcal{A}$ as initial data at the horizon r_1, yielding a metric for $r > r_1$, if one supposes that the scalar curvature $R(g)$ is compactly supported and satisfies $R(g) < \frac{1}{r^2}$ in $(r_1, \infty) \times \mathbb{S}^2$, and $R(g) = 0$ for $[r_1, r_1 + \delta)$ for $\delta > 0$ small.

The above construction yields a metric

$$g = \frac{v(r, \theta, \phi)}{\frac{1}{r} - 1} dr^2 + r^2 \omega \tag{96}$$

such that $v(0, \theta, \phi) = v_0(\theta, \phi) \in X$, the bounded trajectory $v(r, \theta)$ lies in X for all $r \in [0, \infty)$ and $v(r_1, \theta, \phi) \in \mathcal{A}$.

We are interested into a more detailed study of the structure of the attractor $\mathcal{A}$ that describes the possible metrics at the event horizon r_1. For such, we consider axially symmetric solutions and suppose that the metric $v(r,\theta)$ is independent of the angle $\phi \in \mathbb{S}^1$.

Axially symmetric solutions in general relativity have been extensively studied and are also known in the literature as *Ernst solutions* due to [24]. For a collection of case studies, see [50]. Numerical simulation for the dynamics of interaction, pulsation or collapse of axisymmetric stars was done in [78].

Therefore restricting the semiflow to the invariant subspace of axisymmetric solutions $X_{axi} \subseteq \mathbb{X}$, one obtains a subattractor $\mathcal{A}_{axi} \subseteq \mathcal{A}$ of the flow of

$$v_t = v^2 \left[v_{\theta\theta} + \frac{v_\theta}{\tan(\theta)} \right] - v + \frac{r^2 R(g)}{2} v^3. \tag{97}$$

In order to formulate the main result describing and constructing $\mathcal{A}_{axi}$ explicitely, we still need three necessary notions.

Recall that the *Morse index* $i(v_*)$ of a hyperbolic equilibrium v_* is given by the number of positive eigenvalues of the linearized operator at such equilibrium. Denote by the *zero number* $z(v_*)$ the number of strict sign changes of a continuous function $v_*(\theta)$.

We say that two different equilibria v_-, v_+ of (97) are *adjacent* if there does not exist an equilibrium v_* of (97) such that $v_*(0)$ lies between $v_-(0)$ and $v_+(0)$, satisfying

$$z(v_- - v_*) = z(v_- - v_+) = z(v_+ - v_*).$$

Both the zero number and Morse index can be computed explicitely, as it is done in Chapters 2 and 3. Combining both these chapters yields the following construction of the subattractor $\mathcal{A}_{axi}$ within the axisymmetric subspace X_{axi}.

Theorem 5.0.1. Event Horizons and Sturm Attractors [Lappicy ('17)]
Suppose that space is given by a Riemmanian manifold $(\mathcal{S}, g)$ such that $\mathcal{S} := \mathbb{R}_+ \times \mathbb{S}^2$ and let its scalar curvature $R(g)$ be a prescribed smooth function satisfying (95). *Suppose also that all equilibria of* (97) *are hyperbolic.*

If the metric is given by the form

$$g = \frac{v(r,\theta,\phi)}{\frac{1}{r} - 1} dr^2 + r^2 \omega$$

where ω is the standard metric on $\mathbb{S}^2$, then the event horizon at $r_1 := 1$ is described by a function $v(r_1, \theta, \phi)$, which is an element of the the global attractor $\mathcal{A}$ of (94).

Moreover, if v is axisymmetric, then the global attractor $\mathcal{A}_{axi}$ of (97) *consists of equilibria $\mathcal{E}$ and heteroclinics $\mathcal{H}$ and can be computed explicitely. Namely, there is a heteroclinic $v(t) \in \mathcal{H}$ between equilibria $v_-, v_+ \in \mathcal{E}$ such that $v(t) \xrightarrow{t \to \pm\infty} v_\pm$ if, and only if, v_- and v_+ are adjacent and $i(v_-) > i(v_+)$.*

Recall that the interior region $r \in [0, r_1)$ of the horizon does not influence the Cauchy development of the exterior $r \in [r_1, \infty)$ of the horizon, see [85]. This is a claim about *time*. The above Theorem is a claim about *space*: the event horizon can not be arbitrary for each fixed time, but it depends on the metric v inside the black hole $r \in [0, r_1)$, in particular at the singularity $r_0 = 0$.

It could be of interest to consider the inverse problem: consider a metric $v(r_1, \theta, \phi)$, which is in $\mathcal{A}$ at the event horizon r_1 with prescribed $R(g)$, and find its basin of attraction. Hence, for one given metric at the event horizon, one can find a zoo of possibilities of metrics inside the horizon, including the singularity.

Also, in the above Theorem it is supposed that the only horizon that can occur lies at r_1. Indeed, horizons occur at spheres in the spatial foliation for some fixed radius, which is a minimal surface such that no other leaf has positive mean curvature, see [79]. Since each leaf $\mathbb{S}^2$ has mean curvature $H_r = \frac{2}{ru}$, horizons occur either at $r_0 := 0$ or whenever u blows up. Due to the dissipativity conditions, v is bounded for $r \in (0, r_1)$ and the only blow up point is at r_1. Moreover, there is no other horizon for $r > r_1$ as it is mentioned in Smith [79], due to the choice of the standard spherical metric for the foliation.

If one desires to study the existence of other apparent horizons and their interplay, as in [28], one should drop the growth condition on the prescribed scalar curvature (95), which yields a dissipative dynamical system. In this case, the metric v could possibly blow up for $r < r_1$ and other horizons might occur inside the event horizon; or the metric u could grow up and v is also unbounded at the event horizon r_1.

We now chose a prescribed scalar curvature so that the attractor at the event horizon r_1 can be constructed.

Corollary 5.0.2. A Prescribed Scalar Curvature [Lappicy ('17)]

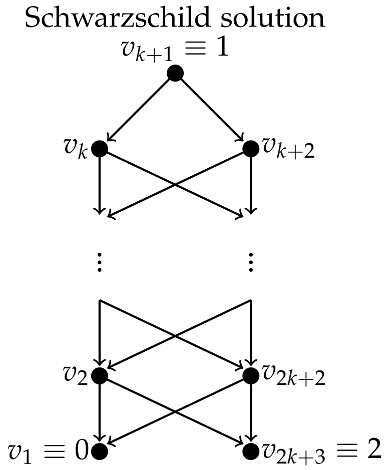
If $R = \frac{2}{r^2}[\frac{1}{v^2} + \lambda v^3 (v-1)(2-v)]$, *where* $\lambda \in (\lambda_k, \lambda_{k+1})$ *and* λ_k *is the k-th eigenvalue of the spherical Laplacian with* $k \in \mathbb{N}_0$.

Then, the axisymmetric self-similar Schwarzschild metrics at the event horizon $r_1 := 1$ *is given by one of the* $2k+3$ *equilibria* $v_1, ..., v_{2k+3}$ *within the Chafee-Infante type attractor below, where arrows denote heteroclinic connections.*

Indeed, the above choice of $R(g)$ yields the equation

$$v_t = v^2 \left[v_{\theta\theta} + \frac{v_\theta}{\tan(\theta)} + \lambda v (v-1)(2-v) \right] \tag{98}$$

for $\lambda \in \mathbb{R}$. The unkown $w := v - 1$ satisfies the Chafee-Infante equation with quasilinear diffusion coefficient $(w+1)^2$. Hence, the equilibria $v \equiv 1$, corresponding to Schwazrschild, has the role of the bifurcating equilibria $w \equiv 0$ in the usual Chafee-Infante equation and each time λ crosses an eigenvalue of the spherical Laplacian, the Schwarzschild solution bifurcates to an axisymmetric solution.



It would be interesting to compute the attractor for the prescribed scalar curvature from [28], namely $R = \frac{\lambda+2}{r^2}$. The problem is that this yields a slowly non-dissipative nonlinearity with non hyperbolic equilibria. That is, solutions v might now stay bounded as $r \to r_1$ and grow-up occurs. One could still study a noncompact attractor at r_1, as it is done in [70], but this is not pursued here.

Note that in order to construct the attractor $\mathcal{A}_{axi}$, one needs to know the *zero number* of the difference of solutions $v - \tilde{v}$, where the trivial solution $\tilde{v} \equiv 1$ represents the Schwarzschild solution in self similar variables. Roughly speaking, one needs to know how many intersections other equilibria have with $\tilde{v}$. This encodes the information of how much such equilibria deviate from $\tilde{v}$, and whenever a solution intersects with the trivial solution, it means that $v(r, \theta) = 1$ and the metric looks like the Schwarzschild solution at that fixed radius r.

The second main result of this thesis answers partially the question of how the symmetry of the sphere dictates a symmetry of some solutions in the attractor $\mathcal{A}$. This is done precisely in Chapter 4.

We say a function $u \in C^2(\mathbb{S}^2)$ has *axial extrema* if its maxima and minima in ϕ occur as axis from the north to south pole. In other words, if $u_\phi(\theta_0, \phi_0) = 0$ for a fixed $(\theta_0, \phi_0) \in \mathbb{S}^2$, then $u_\phi(\theta, \phi_0) = 0$ for any $\theta \in [0, \pi]$. In that case, the extrema depend only at the position in ϕ. Note that if $R(g)$ is analytic, then u is also, as in [21]. Then the set of axial extrema is finite, and we denote them by $\{\phi_i\}_{i=0}^N$ where $\phi_0 := \phi_N$.

A simpler case of the above is written below, in the case that the axial extrema are *leveled*, that is, if all axial maxima ϕ_i have the same value $u(\theta, \phi_i) = M(\theta)$, and all axial minima ϕ_i also have the same value $u(\theta, \phi_i) = m(\theta)$.

Therefore, we state a Theorem about a symmetry property of equilibria of (97) that satisfies the above conditions about their extrema.

Theorem 5.0.3. Symmetry within a Event Horizon [Lappicy ('17)]

Suppose that R is analytic and u is a non constant equilibria of (78) *having only leveled axial extrema. Then $\phi_i = \frac{\phi_{i-1}+\phi_{i+1}}{2}$ and*

$$u(\theta, \phi) = u(\theta, R_{\phi_i}(\phi))$$

for all $i = 1, ..., N$, where $(\theta, \phi) \in [0, 2\pi] \times [\phi_{i-1}, \phi_i]$.

The symmetrization property above informs us that at the horizon r_1, certain metrics will tend to be symmetric.

This theorem raises the mathematical question whether such result holds for other domains, such as the torus or the hyperbolic disk. Subsequently, it raises the physical question of space foliated by other two dimensional surfaces than the sphere, and if the resulting equation for the scalar curvature is still parabolic and of the same form as (91). Even though Hawking's theorem in [40] stating that event horizons for certain black holes are topologically $\mathbb{S}^2$, other stellar objects of interest could carry different topology. For example, it was found numerically that dust collapse might yield a toroidal horizon before reaching its spherical shape in [1]. The above theorem might aid such dynamical question, of a toroidal horizon becoming a spherical one.

Another idea that comes to mind is to try other types of self similarity, for example, Reissner−Nordström self-similar solutions to model charged black holes; or choosing the Schwarzschild metric in the exterior, and a regular interior in order to model dense stars. Once space of initial data has been constructed for the latter, one can rigorously study the dynamics of stars and its collapse into black holes, as in [65].

Yet another proposal for a future project is as follows. Recall that the Hamiltonian constraint above is one out of four constraints. In the non-time symmetric case, one can rewrite them as a system of a parabolic equation as above coupled with three ODEs, as in [76]. In [23] it was proved that the global attractor has finite dimension, but other dynamical properties are still out of reach. One could still pursue simpler results, such as bifurcations, or pattern formation arising from symmetry breaking and Turing instability phenomena.

Due to the no hair theorem, black holes are fully described by their mass, charge and angular momentum. The Schwarzschild self-similarity studied here describes the possible metrics at the event horizon knowing its mass inside, if the black hole is chargeless and has no momentum. The proposals above, namely studying Reissner-Nordström self-similar solutions, and studying the four constraint equations could describe the full space of initial data for black holes.

BIBLIOGRAPHY

[1] A. M. Abrahams, G. B. Cook, S. L. Shapiro, and S. A. Teukolsky. Solving Einstein's equations for rotating spacetimes: Evolution of relativistic star clusters. *Phys. Rev. D*, pages 5153–5164, (1994).

[2] A. D. Alexandrov. A characteristic property of spheres. *Annali di Matematica Pura ed Applicata*, pages 303–315, (1962).

[3] H. Amann. Existence and multiplicity theorems for semi-linear elliptic boundary value problems. *Mathematische Zeitschrift*, pages 281 – 295, (1976).

[4] H. Amann. Linear and Quasilinear Parabolic Problems, Volume I. *Birkhäuser Basel*, (1985).

[5] S. Angenent. The Morse−Smale property for a semi-linear parabolic equation. *J. Differential Equations*, pages 427–442, (1986).

[6] S. Angenent. The zero set of a solution of a parabolic equation. *J. für die reine und angewandte Math.*, pages 76–96, (1988).

[7] S. Angenent. Shortening embedded curves. *Annals of Mathematics*, pages 71–111, (1989).

[8] S. Angenent. Solutions of the one−dimensional porous medium equation are determined by their free boundary. *J. London Math. Soc*, pages 339–353, (1990).

[9] S. Angenent. Parabolic Equations for Curves on Surfaces: Part II. Intersections, Blow-up and Generalized Solutions. *Annals of Mathematics*, pages 171–215, (1991).

[10] S. Angenent. Inflection points, extatic points and curve shortening. *Hamiltonian systems with three or more degrees of freedom*, pages 3–10, (1999).

[11] S. Angenent and B. Fiedler. The dynamics of rotating waves in scalar reaction diffusion equations. *Trans. Amer. Math. Soc.*, pages 545–568, (1988).

[12] A. Babin and M. Vishik. Attractors of Evolution Equations. *Elsevier Science*, (1992).

[13] A. V. Babin. Symmetrization properties of parabolic equations in symmetric domains. *Journal of Dynamics and Differential Equations*, pages 639–658, (1994).

[14] A. Banyaga and D. Hurtubise. Lectures on Morse Homology. *Springer*, (2004).

[15] R. Bartnik. Quasi-spherical metrics and prescribed scalar curvature. *J. of Differential Geometry*, pages 31–71, (1993).

[16] P. Boonserm, M. Visser, and S. Weinfurtner. Generating perfect fluid spheres in general relativity. *Phys. Rev. D*, pages 124–137, (2005).

[17] F. Brock and J. Prajapat. Some new symmetry results for elliptic problems on the sphere and in euclidean space. *Rendiconti del Circolo Matematico di Palermo*, pages 445–462, (2000).

[18] P. Brunovský and B. Fiedler. Numbers of Zeros on Invariant Manifolds in Reaction-diffusion Equations. *Non-Linear Anal.*, pages 179–193, (1986).

[19] P. Brunovský and B. Fiedler. Simplicity of zeros in scalar parabolic equations. *Journal of Differential Equations*, pages 237–241, (1986).

[20] M. Bundinčević. A comparision theorem for differential equations. *Novi Sad J. Math.*, pages 55–56, (2010).

[21] C. Cao, M. A. Rammaha, and E. S. Titi. Gevrey regularity for nonlinear analytic parabolic equations on the sphere. *Journal of Dynamics and Differential Equations*, pages 411–433, (2000).

[22] A. S. do Nascimento. Bifurcation and stability of radially symmetric equilibria of a parabolic equation with variable diffusion. *Journal of Differential Equations*, pages 84–103, (1989).

[23] M. Efendiev and S. Zelik. Global attractor and stabilization for a coupled PDE-ODE system. *Preprint: arXiv:1110.1837*, (2011).

[24] F. J. Ernst. New formulation of the axially symmetric gravitational field problem. ii. *Phys. Rev.*, pages 1415–1417, 1968.

[25] L. C. Evans. Partial differential equations: Second edition. *American Mathematical Society*, (2010).

[26] B. Fiedler. Do global attractors depend on boundary conditions? *Doc. Math. J. DMV*, (1996).

[27] B. Fiedler and P. Brunovský. Connecting orbits in scalar reaction diffusion equations II: The complete solution. *Journal Differential Equations*, pages 106–135, (1989).

[28] B. Fiedler, J. Hell, and B. Smith. Anisotropic Einstein data with isotropic nonnegative scalar curvature. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, pages 401–428, (2015).

[29] B. Fiedler and C. Rocha. Heteroclinic orbits of semilinear parabolic equations. *Journal of Differential Equations*, pages 239–281, (1996).

[30] B. Fiedler, C. Rocha, and M. Wolfrum. Heteroclinic orbits between rotating waves of semilinear parabolic equations on the circle. *Journal of Differential Equations*, pages 99–138, (2004).

[31] L. Fraenkel. *An Introduction to Maximum Principles and Symmetry in Elliptic Problems*. Cambridge University Press, (2000).

[32] G. Fusco and J. Hale. Stable Equilibria in a Scalar Parabolic Equation with Variable Diffusion. *SIAM Journal on Mathematical Analysis*, pages 1152–1164, (1985).

[33] G. Fusco and C. Rocha. A permutation related to the dynamics of a scalar parabolic PDE. *Journal of Differential Equations*, pages 111 – 137, (1991).

[34] B. Gidas, W. M. Ni, and L. Nirenberg. Symmetry and related properties via the maximum principle. *Comm. Math. Phys.*, pages 209–243, (1979).

[35] J. Guckenheimer and P. Holmes. Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. *Springer-Verlag New York*, (1983).

[36] J. Hale. Dynamics of a scalar parabolic equation. *Canadian Applied Mathematics Quarterly*, pages 239–314, (1999).

[37] J. Hale and A. S. do Nascimento. Orbital connections in a parabolic equation. *SIAM J. Appl. Math.*, (1983).

[38] J. Hale, L. Magalhães, and W. Oliva. An Introduction to Infinite Dimensional Dynamical Systems Geometric Theory. *Springer New York*, (1984).

[39] J. Hale and C. Rocha. Bifurcations in a Parabolic Equation with Variable Diffusion. *Nonlinear Analysis*, pages 479 – 494, (1985).

[40] S. W. Hawking. Black holes in general relativity. *Communications in Mathematical Physics*, pages 152–166, (1972).

[41] D. Henry. Geometric theory of semilinear parabolic equations. *Springer, New York*, (1981).

[42] D. Henry. Some infinite dimensional Morse−Smale systems defined by parabolic differential equations. *J. Differential Equations*, pages 165–205, (1985).

[43] P. Hess and P. Poláčik. Symmetry and convergence properties for non−negative solutions of nonautonomous reaction−diffusion problems. *Proceedings of the Royal Society of Edinburgh*, pages 573–587, (1994).

[44] J. L. Kazdan. Unique continuation in geometry. *Communications on Pure and Applied Mathematics*, pages 667–681, (1988).

[45] S. Kumaresan and J. Prajapat. Serrin's result for hyperbolic space and sphere. *Duke Mathematical Journal*, pages 17–28, (1998).

[46] O. Ladyzhenskaya. *Attractors for Semi-groups and Evolution Equations*. Cambridge University Press, (1991).

[47] P. Lappicy. A note on the dropping lemma. *preprint*, (2016).

[48] N. Lebovitz. Oscillation theory and the spectra of eigenvalues. *Chapter 11 in ODE Textbook*, (2013).

[49] A. Lunardi. Analytic semigroups and optimal regularity in parabolic problems. *Springer*, (1995).

[50] M. E. MacCallum. Galaxies, axisymmetric systems and relativity. *Cambridge University Press*, (2011).

[51] H. Matano. Convergence of solutions of one−dimensional semilinear parabolic equations. *Journal of Mathematics of Kyoto University*, pages 221–227, (1978).

[52] H. Matano. Non increase of the lapnumber for a one dimensional semilinear parabolic equation. *J. Fac. Sci. Univ. Tokyo IA Math*, pages 401–441, (1982).

[53] H. Matano. Asymptotic behavior of solutions of semilinear heat equations on S^1. *Nonlinear Diffusion Equations and Their Equilibrium States II: Proceedings of a Microprogram held August 25-September 12, 1986*, pages 139–162, (1988).

[54] A. Mielke. Locally Invariant Manifolds for Quasilinear Parabolic Equations. *Rocky Mountain Journal of Mathematics*, pages 707–714, (1991).

[55] I. R. Mihalis Dafermos, Gustav Holzegel. The linear stability of the Schwarzschild solution to gravitational perturbations. *Preprint: arXiv:1601.06467*, 2016.

[56] O. A. Oleinik and S. N. Kruzhkov. Quasi-linear second-order parabolic equations with many independent variables. *Russian Math. Surveys*, pages 105–146, (1961).

[57] F. Pacella and T. Weth. Symmetry of solutions to semilinear elliptic equations via morse index. *Proc. Amer. Math. Soc.*, pages 1753–1762, (2007).

[58] P. Padilla. Symmetry properties of positive solutions of elliptic equations on symmetric domains. *Applicable Analysis*, pages 153–169, (1997).

[59] J. Palis and S. Smale. Structural Stability Theorems. *Proc. Sympos. Pure Math.*, pages 223–231, (1970).

[60] L. A. Peletier and W. C. Troy. Pattern formation described by the swift-hohenberg equation. *Surikaisekikenkyusho Kokyuroku*, (2000).

[61] P. Poláčik. On symmetry of nonnegative solutions of elliptic equations. *Annales de l'Institut Henri Poincaré (C) Non Linear Analysis*, (2012).

[62] P. Poláčik and X.-Y. Chen. Asymptotic periodicity of positive solutions of reaction diffusion equations on a ball. *Journal für die reine und angewandte Mathematik*, pages 17–52, (1996).

[63] Y. N. Raffoul. Boundedness in nonlinear functional differential equations with applications to volterra integro−differential equations. *J. Integral Equations Applications*, pages 375–388, (2004).

[64] A. D. Rendall. Partial differential equations in General Relativity. *Oxford University Press*, (2007).

[65] L. Rezzolla. An introduction to gravitational collapse to black holes. *Lecture notes for International School of Gravitation and Cosmolog, 7th−10th of Sept. of 2004*, (2004).

[66] C. Rocha. Generic Properties of Equilibria of Reaction-Diffusion Equations. *Proc. of the Roy. Soc. Edinburgh*, pages 45 – 55, (1985).

[67] C. Rocha. Examples of attractors in scalar reaction-diffusion equations. *Journal of Differential Equations*, pages 178–195, (1988).

[68] C. Rocha. Properties of the Attractor of a Scalar Parabolic PDE. *J. Dyn. Diff. Eq.*, pages 575–591, (1991).

[69] C. Rocha. On the Singular Problem for the Scalar Parabolic Equation with Variable Diffusion. *Journal of Mathematical Analysis and Applications*, pages 413–428, (1994).

[70] C. Rocha and J. Pimentel. Noncompact global attractors for scalar reaction-diffusion equations. *São Paulo Journal of Mathematical Sciences*, pages 299–310, (2015).

[71] K. P. Rybakowski. On the Homotopy Index for Infinite-Dimensional Semiflows. *Transactions of the American Mathematical Society*, (1982).

[72] A. Saldaña and T. Weth. Asymptotic axial symmetry of solutions of parabolic equations in bounded radial domains. *Journal of Evolution Equations*, pages 697 –712, (2012).

[73] I. Schneider. An introduction to the control triple method for partial differential equations. *To appear*, (2017).

[74] B. Schutz. A First Course in General Relativity, 2nd Edition. *Oxford University Press*, (2009).

[75] J. Serrin. A symmetry problem in potential theory. *Archive for Rational Mechanics and Analysis*, pages 304–318, (1971).

[76] J. Sharples. Local existence of quasispherical space-time initial data. *Journal of Mathematical Physics*, (2005).

[77] A. Shoshitaishvili. Bifurcations of topological type at singular points of parametrized vector fields. *Functional Analysis and Its Applications*, pages 169–170, (1971).

[78] F. Siebel. Simulation of axisymmetric flows in the characteristic formulation of general relativity. *Master Thesis*, (2002).

[79] B. Smith. Black hole initial data with a horizon of prescribed geometry. *General Relativity and Gravitation*, pages 1013–1024, 2009.

[80] J. Smoller. Shock waves and reaction-diffusion equations. *Springer-Verlag*, 1983.

[81] C. Sturm. Sur une classe d'équations à différences partielles. *J. Math. Pures. Appl. I*, pages 373–444, (1836).

[82] M. Tabata. A finite difference approach to the number of peaks of solutions for semilinear parabolic problems. *Journal of the Mathematical Society of Japan*, pages 171–192, (1980).

[83] H. Uecker, D. Wetzel, and J. D. M. Rademacher. pde2path-a matlab package for continuation and bifurcation in 2d elliptic systems. *Numerical Mathematics: Theory, Methods and Applications 7.01*, pages 58–106, (2014).

[84] N. Uraltseva, O. Ladyzhenskaya, and V. A. Solonnikov. *Linear and Quasi-linear Equations of Parabolic Type*. American Mathematical Society, (1968).

[85] R. M. Wald. *General Relativity*. University Of Chicago Press, (1984).

[86] M. Wolfrum. A Sequence of Order Relations: Encoding Heteroclinic Connections in Scalar Parabolic PDE. *Journal of Differential Equations*, pages 56 – 78, (2002).

[87] T. I. Zelenyak. Stabilization of solutions of boundary value problems for a second order parabolic equation with one space variable. *Journal of Differential Equations*, pages 17 – 22, (1968).

SELBSSTÄNDIGKEITSERKLÄRUNG

Hiermit bestätige Ich, Phillipo Lappicy, dass Ich die vorliegende Dissertation mit dem Thema

Einstein constraints: A dynamical approach

selbständig angefertigt und nur die genannten Quellen und Hilfen verwendet habe. Die Arbeit ist erstmalig und nur an der Freien Univertät Berlin eingereicht worden.

Berlin, den 28. März 2017

ABSTRACT

The Einstein constraint equations describe the space of initial data for the evolution equations, dictating how space should curve within spacetime. Under certain assumptions, the constraints reduce to a scalar quasilinear parabolic equation on the sphere, and nonlinearity being the prescribed scalar curvature of space. We focus on self-similar solutions of Schwarzschild type, which describe the space of initial data of certain black holes, for example.

The first main result gives a detailed study of the axially symmetric solutions, since the domain is now one dimensional and nodal properties can be used to describe certain asymptotics of the rescaled self-similar solutions. Such asymptotics describe the possible metrics arising at an event horizon of a black hole, depending on the metric inside the horizon. Those are described by Sturm attractors. In particular, we compute an example for a prescribed scalar curvature.

The second main result state a symmetrization property of certain metrics in the event horizon, namely, how the symmetry of the spherical domain can influence the symmetry of solutions.

ZUSAMMENFASSUNG

Die Einsteinschen Zwangsgleichungen charakterisieren die Menge der Anfangsdaten der Einsteinschen Evolutionsgleichungen. Diese beschreiben, wie sich der Raum innerhalb der Raumzeit krümmt. Unter gewissen Annahmen reduzieren sich die Zwangsbedingungen auf eine einzige skalare, quasilineare parabolische Gleichung auf der Sphäre und einer, durch die vorgeschriebene skalare Krümmung des Raumes, gegebenen Nichtlinearität. In dieser Doktorarbeit konzentrieren wir uns auf selbstähnliche Lösungen von Schwarzschild, welche zum Beispiel die Anfangsdaten von Schwarzen Löcher beschreiben.

Das erste Hauptresultat ist eine detaillierte Untersuchung von axialsymmetrischen Lösungen, da sich diese Lösungen durch nodale Eigenschaften analysieren lassen, zum Beispiel um bestimmte Asymptotiken der reskalierten, selbstähnlichen Lsungen zu erhalten. Die Asymptotiken korrespondieren zu möglichen Metriken, die an einem Ereignishorizont eines Schwarzen Lochs, abhängig von der Metrik innerhalb des Horizonts, entstehen. Dabei knnen die mglichen Metriken durch Sturm-Attraktoren charakterisiert werden. Wir zeigen dies insbesondere an einem Bespiel einer bestimmten Nichtlinearität.

Das zweite Hauptergebnis ist eine Symmetrierungseigenschaft von bestimmten Metriken im Ereignishorizont, also insbesondere wie die Symmetrie der Sphäre die Symmetrie der Lösungen beeinflusst.