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Rank 1 isocrystals on simply connected varieties and their moduli

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Zusammenfassung

Diese Dissertation beschäftigt sich mit der Theorie der Isokristallen, insbesondere mit zwei Vermutungen von de Jong und Deligne.

Im ersten Teil zeigen wir über log erweiterbare Isokristalle von Rang 1 auf nicht-projektiven Varietäten: Wenn die zahme Fundamentalgruppe der Varietät trivial ist, dann sind alle solche Isokristalle trivial. Dies dehnt Ergebnisse von Esnault und Shiho aus.

Ferner, stellen wir die Menge der Isokristallen von Rang 1 auf einer projektiven Kurve als Unterraum des de Rham Modulraums von Simpson und Langer dar. Wir vergleichen die ℓ-adische Kohomologie dieses Unterraums mit der ℓ-adischen Kohomologie des Modulraums, indem wir Ergebnisse aus der Theorie von Berkovich verwenden. Wir beweisen damit eine Vermutung von Deligne.

Abstract

This thesis is about the theory of isocrystals, in particular about two conjectures of de Jong and Deligne.

In the first part of the thesis we treat the case of rank 1 log extendable isocrystals on non-proper varieties: we show that if the variety has trivial tame fundamental group, then there are no non-trivial such isocrystals on it. This extends a result of Esnault and Shiho in the projective case.

Moreover, we express the set of rank 1 isocrystals on a proper curve as a subset of the de Rham moduli space, defined by Simpson and Langer. Using results from the theory of Berkovich spaces, we compare the ℓ-adic cohomology of this subspace with the ℓ-adic cohomology of the whole moduli space. This confirms a conjecture of Deligne in the rank 1 case.

Contents

Introduction

Crystals were introduced by Grothendieck in 1966 in a letter to Tate, and were so called because they are "rigid" but also "grow". They are sheaves that "grow" over PD thickenings and have to satisfy a rigidity condition. The quasi-coherent ones on an algebraic variety are equivalent to quasi-coherent sheaves with integrable connections. They are coefficients of crystalline cohomology, a Weil cohomology theory that fills the gap of ℓ-adic cohomology for ℓ equal to the characteristic of the base field.

As often happens in mathematics, one could try to understand how these objects are connected with the geometric properties of the variety on which they are defined. One example of such a question is de Jong's Conjecture, which is a crystalline version of a classical conjecture posed by Gieseker. In more detail, for a complex connected smooth projective variety X one can define the category of $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules, which is equivalent via the Riemann-Hilbert correspondence to the category of finite dimensional representations of the topological fundamental group $\pi_1^{\mathrm{top}}(X)$. By a result of Malčev [Mal40] and Grothendieck [Gro70] we know how this category relates to the étale fundamental group $\pi_1^{\text{ét}}(X)$: if $\pi_1^{\text{ét}}(X)$ is trivial, then there are no non-constant $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules. Naturally, one asks if the same holds true in positive characteristic. This conjecture was formulated by Gieseker [Gie75] and proven by Esnault and Mehta in [EM10].

In 2010 de Jong proposed a p-adic version of this conjecture: if $\pi_1^{\text{ét}}(X) = 1$, then any isocrystal $\mathcal{E} \in \mathrm{Crys}(X/W)_{\mathbb{Q}}$ is constant. This question was addressed in great detail in [ES16] and in [ES15], where various subcases were considered. There the authors prove, among other results, that for a smooth connected projective variety over an algebraically closed field of positive characteristic, the triviality of the étale fundamental group implies that any convergent isocrystal, which is filtered so that the associated graded is a sum of rank 1 convergent isocrystals, is constant [ES16, Theorem 0.1]. In particular, it is also proven that rank 1 isocrystals (not necessarily convergent) are also trivial in this case [ES15, Theorem 1.2].

In the present work we extend this result to the case of rank one log extendable isocrystals on a non-proper variety with trivial étale fundamental group. Actually we just need to assume that its abelianized tame fundamental group is trivial. More precisely, we prove the following:

Theorem (Theorem 2.6)**.** Let U be a smooth connected variety over an algebraically closed field k of positive characteristic, that admits a good compactification X, where

$X - U = \cup_{i \in I} Z_i =: Z$ is a simple strict normal crossings divisor. If $\pi_1^{\text{tame,ab}}(U) = 1$ and $\mathcal{L}$ is a rank one isocrystal on U that extends to a log isocrystal on X, equipped with the log structure coming from Z, then $\mathcal{L}$ is trivial.

As a corollary we obtain also, with the previous notations and assumptions:

Corollary (Corollary 2.9)**.** *Unipotent log extendable isocrystals on U are constant.*

On the other hand, a very natural question is whether there exists a moduli space which parametrizes such objects. Using the equivalence of the category of isocrystals on a projective variety with the category of modules with quasi-nilpotent integrable connection on a smooth lift of the variety over the Witt ring, using Grothendieck's formal function theorem, one can see that such a space can be constructed as a subspace of the de Rham moduli space of vector bundles with integrable connection. This was constructed by Simpson in [Sim94a] and [Sim94b] in characteristic zero and extended by Langer [Lan14] and others, for example [Gro16],[LP01], [BB07], in the positive characteristic case.

The second part of this thesis is about the moduli space of rank 1 isocrystals on a smooth proper curve and provides evidence for a conjecture of Deligne [Del15] in the rank 1 case. Concretely, let C_0 be a smooth projective curve over the finite field $\mathbb{F}_q$ of characteristic $p > 0$ and let C be its base change to $\overline{\mathbb{F}}_q =: k$. Denote by W the Witt ring of $\overline{\mathbb{F}}_q$, by K its field of fractions, of characteristic 0, and $\overline{K}$ an algebraic closure of it. Denote by C_W a smooth lift over $\operatorname{Spec} W$ and assume in addition that C_W admits a section $x : \operatorname{Spec} W \to C_W$.

Deligne conjectured the following in [Del15]: let M_K denote the moduli space of vector bundles of rank r on a smooth curve C, as before, endowed with an integrable connection, and respectively $M_{\overline{K}}$ the corresponding scheme on an algebraic closure $\overline{K}$ of K.

Conjecture (Conjecture 2.18 in [Del15])**.** *The cohomology of $M_{\overline{K}}$ admits an endomorphism V^*, such that for all $n \geq 1$, the number N_n of fixed points of V^* on E_r, the set of isomorphism classes of irreducible lisse ℓ'-adic sheaves of rank r on C, is given by*

$$N_n = \sum_i (-1)^i \operatorname{Tr}(V^{*n}, H^i(M_{\overline{K}})),$$

where all cohomology groups denote ℓ-adic cohomology groups, with $\ell \neq p$.

Deligne expects that there should exist an open subspace $M^0_{\overline{K}}$ inside the Berkovich analytification $M^{\text{an}}_{\overline{K}}$, which should correspond to the sublocus of isocrystals and such that it has the following two properties:

(1) the restriction morphism $H^*(M_{\overline{K}}) = H^*(M^{\text{an}}_{\overline{K}}) \to H^*(M^0_{\overline{K}})$ is an isomorphism and

(2) a crystalline interpretation of M^0 allows us to define $V = \text{Frob}^* : M^0_{\overline{K}} \to M^0_{\overline{K}}$, which induces V^* on cohomology.

In the same article, Deligne provides an example of this in the rank 1 case, [Del15, Example 2.19 and Proposition 2.20] using Pic^0 as M_0. Our goal in the second part of this thesis is to explain why this example indeed provides a comparison between the cohomology of the subset of isocrystals and that of the moduli space of rank 1 connections.

The recent work of Hongjie Yu [Yu18] actually proves Deligne's conjecture and gives explicit formulas for the number of irreducible ℓ-adic local systems fixed by the Frobenius. In our work, we mainly focus on the first part of this conjecture, specifically on the relation with the theory of isocrystals and subsequently why there should be a Frobenius on the cohomology of the moduli space.

Leitfaden

We explain here shortly our strategy for proving the afore-mentioned results. In Chapter 2 we prove Theorem 2.6: starting with a rank one isocrystal $\mathcal{L}$ on U, as in the assumptions of the theorem, we denote by $\mathscr{L}^{\log}$ the log isocrystal on X extending $\mathscr{L}$, by $L^{\log}$ a locally free lattice of it and by L the restriction of the latter to U. We first work on the value of $\mathscr{L}^{\log}$ on X. It can be seen as a log crystal on $X/\operatorname{Spec} k$ (we omit here the reference to the log structures) and can be seen as a module on the log scheme X with integrable quasi nilpotent log connection, which we denote by $(L_X^{\log}, \nabla_X^{\log})$.

In Section 2.2 we adapt the discussion of [AB05, Section 6] to our situation and obtain a residue exact sequence:

$$H^1_{crys}(X/W, \mathcal{O}^*_{X/W}) \to H^1((X,M)/W, \mathcal{O}^*_{X/W})^{\log}_{\text{crys}} \to \oplus W(k)[Z_i] \to H^2(X/W, \mathcal{O}^*_{X/W}). \tag{1}$$

We also derive from (1) that there is an injection

$$H^1((X,M)/W, \mathcal{O}^*)^{\log}_{\text{crys}} \hookrightarrow \oplus W(k)[Z_i]. \tag{2}$$

We then prove that the log crystalline Chern class $c_1^{\text{logcrys}}(L_X^{\log})$ is equal to zero and using this we conclude in Lemma 2.4, that the line bundle L_X to some power N is then of the form $\mathcal{O}_X(\sum a_i Z_i)$ with $\mathbb{Q}$-coefficients. To simplify the notation we set $E^{\log} := (L^{\log})^{\otimes N}$. By (1) we can then conclude that the map $H^1((X,M)/W, \mathcal{O}^*)^{\log}_{\text{crys}} \to H^1(X, \mathcal{O}_X^*)$ is injective, hence that the log crystal $(E_X^{\log}, \nabla_X^{\log})$ is actually "controlled" by the value of the sheaf $E_X^{\log}$.

Using this and the previous observations we get that the restriction of $(E_X^{\log}, \nabla_X^{\log})$ to U is the trivial module with the trivial connection. It then follows that we can assume $(E_X^{\log}, \nabla_X^{\log})$ to be isomorphic to $(\mathcal{O}_X^{\log}, d^{\log})$.

In Theorem 2.6 we then use a deformation argument (adapted from [ES16]) to prove that $L^{\otimes n}$ is the trivial crystal and conclude that L and therefore also $\mathcal{L}$ are trivial, using Lemmas 2.7 and 2.8.

As a corollary of this theorem and using the observations made in Section 2.2, we also obtain in Corollary 2.9 that extensions in $\mathrm{I}_{\mathrm{crys}}(X/W)^{\log}$ of the trivial log isocrystal by itself are constant and so isocrystals on the open variety whose extensions are of this form, are also constant. Therefore log extendable unipotent isocrystals on U, i.e. isocrystals that admit a filtration with the property that their associated quotients are successive extensions of the trivial isocrystal, are also constant.

Finally, we would like to point out the steps of the proof where the triviality of $\pi_1^{\mathrm{tame,ab}}(U)$. From (1.10) we obtain that then also $\pi^{\mathrm{ab}}(X) = 1$ is true. In Lemma 2.4 we use that the maximal pro-l-quotient $\pi_1^{\mathrm{ab},(l)}(U)$ is trivial for $l \neq p$ to obtain that $H^0(U, \mathcal{O}_U^*) = k^*$, while $\pi^{\mathrm{ab}}(X) = 1$ also implies $\mathrm{NS}(X) = \mathrm{Pic}(X)$, see Remark 1.26. For the observations of Section 2.2, the triviality of $\pi_1^{\mathrm{tame,ab}}(U)$ also suffices. Indeed, there we use that $H^1_{\mathrm{crys}}(X/W) = H^1_{\mathrm{crys}}(X/W, \mathcal{O}_X^*) = 0$, from [ES16, Theorem 0.1.1 and Proposition 2.9] and [ES15, Theorem 5.1]. The triviality of the abelianized tame fundamental group is indeed enough to derive these equalities. The same relations are also mainly used in Theorem 2.6 together with the fact that $\mathrm{NS}(X) = \mathrm{Pic}(X)$. To conclude Section 2.2 and for Lemma 2.7 we use that Kummer coverings of U are trivial, thus we have to assume that $\pi_1^{\mathrm{tame,ab}}(U) = 1$, since k is algebraically closed.

In Chapter 3, we consider the rank 1 case of Deligne's conjecture and use the moduli space $\mathrm{Pic}^{\nabla}(C_W)$ of line bundles of degree zero with integrable connection on C_W. This is the universal extension of $\mathrm{Pic}^0(C_W)$, see Section 3.2, and is therefore fine.

Our first goal is to characterize the subspace of rank 1 isocrystals inside this moduli space. Isocrystals can indeed be thought as vector bundles with integrable and nilpotent connection: we say that a point $[(\mathscr{L}, \nabla)]$ of $\mathrm{Pic}^{\nabla}(C_W)$ represents an isocrystal on C if the associated pair $(\mathscr{L}_{\overline{\mathbb{F}_q}}, \nabla_{\overline{\mathbb{F}_q}})$ represents an isocrystal, or equivalently if $(\mathscr{L}_{\overline{\mathbb{F}_q}}, \nabla_{\overline{\mathbb{F}_q}})$ has nilpotent p-curvature. This condition is equivalent to requiring that the characteristic polynomial of the p-curvature is zero or that $\chi([(\mathscr{L}_{\overline{\mathbb{F}_q}}, \nabla_{\overline{\mathbb{F}_q}})]) = 0$, with $\chi : \mathrm{Pic}^{\nabla}(C) \to \mathscr{A}^1(\overline{\mathbb{F}_q}) := H^0(C, \omega_C^p)$ the Hitchin map. We denote the fiber over zero of the Hitchin map by $\mathrm{Pic}^{\nabla}(C)^{\psi=0}$. By the Cartier isomorphism, [Kat70, Theorem 5.1], we have that this fiber is isomorphic to $\mathrm{Pic}^0(C^{(p)})$, which is actually isomorphic to $\mathrm{Pic}^0(C)$.

In the spirit of the above conjecture we consider in Section 3.3 the Berkovich analytification of $\mathrm{Pic}^{\nabla}(C_W)_K$ and remark that the inverse image of $\mathrm{Pic}^{\nabla}(C)^{\psi=0}$ by the reduction map

$$\mathrm{red} : (\widehat{\mathrm{Pic}^{\nabla}(C_W)})_K \to \mathrm{Pic}^{\nabla}(C)$$

is an open subset, denoted by $]\,\mathrm{Pic}^{\nabla}(C)^{\psi=0}[$. Using results of [Ber90] and [Ber96a], we show in Proposition 3.6 that

$$H^*(\mathrm{Pic}^{\nabla}(C_{\overline{K}})^{\mathrm{an}}, \mathbb{Q}_\ell) \xrightarrow{\sim} H^*(]\,\mathrm{Pic}^{\nabla}(C)^{\psi=0}[_{\overline{K}}, \mathbb{Q}_\ell)$$

and see therefore that $]\,\mathrm{Pic}^{\nabla}(C)^{\psi=0}[$ is a good candidate for being the open subset M^0 of the aforementioned conjecture.

As a last step, we prove that this subset admits a Frobenius action. To this end, we describe $]\,\mathrm{Pic}^{\nabla}(C)^{\psi=0}[$ as a subfunctor of $\mathrm{Pic}^{\nabla}(C_K)^{\mathrm{an}}$. For this, it is enough to

characterize the set
$$\operatorname{Hom}(S,]\operatorname{Pic}^{\nabla}(C)^{\psi=0}[)$$
for S an affinoid. Indeed, using results of [BL93] we obtain in Section 3.5 an isomorphism between
$$\operatorname{Hom}(S,]\operatorname{Pic}^{\nabla}(C)^{\psi=0}[)$$
and the set

$$\begin{aligned}\{(\mathscr{L},\nabla) &\text{ line bundles with integrable connection on } S\times_K C_K^{\mathrm{an}} \text{ which are}\\ &\text{isocrystals on } \mathscr{S}'\times C_W\to\mathscr{S}' \text{ where } \mathscr{S}' \text{is a formal model of } S',\\ &\text{an admissible formal blow-up of } S\}\end{aligned} \tag{3}$$

and prove that this isomorphism is well defined (independent of the choice of an admissible blow-up) and functorial.

The outline of the thesis is the following: in Chapter 1 we include some background on isocrystals, log isocrystals, the étale fundamental group and some results about the Berkovich and rigid analytification functors. We end the chapter discussing the connection of Berkovich and rigid spaces to formal schemes.

In Chapter 2 we prove de Jong's conjecture in the case of rank 1 log extendable isocrystals on a non-proper variety with abelian tame fundamental group, in Theorem 2.6.

Finally in Chapter 3 we discuss the afore-mentioned conjecture of Deligne in the rank 1 case. Section 3.2 includes a discussion about the universal extension of an abelian variety, and specifically of Pic^0, while in Section 3.3 we recall some results about the Hitchin map and discuss the construction of the subset of isocrystals. In Section 3.4 we prove a comparison between cohomology groups as expected from Deligne's conjecture and in Section 3.5 we provide a functorial interpretation of the subset of isocrystals we defined.

Chapter 1

Background

1.1 Isocrystals

In this section we provide some background on the theory of isocrystals and include some results that we use in the next two chapters of the thesis. In the rest of this section, we always denote by k an algebraically closed field of characteristic $p > 0$, by X a smooth projective variety over k and by U a smooth, non-proper variety over k. We also denote by W the Witt ring of k and set $W_n := W/p^n$.

The motivation for the theory of (iso-) crystals and crystalline cohomology comes from the Weil conjectures, actually their proof. Specifically, Grothendieck defined étale cohomology as an attempt to construct a Weil cohomology theory, in order to prove these conjectures. Later Deligne [Del74] indeed gave the proof of the analogue of Riemann hypothesis using ℓ-adic cohomology. This cohomology theory provides for any prime number $\ell \neq p$ (the characteristic of the base field), cohomology groups $\mathrm{H}^i_{\text{ét}}(X, \mathbb{Q}_\ell) = \varprojlim \mathrm{H}^i(X, \mathbb{Z}/\ell^n\mathbb{Z}) \otimes \mathbb{Q}$. This provides a Weil cohomology theory and gives us information about the rank and ℓ-torsion of the singular cohomology of a complex analytic variety. However, this is no longer true, if $\ell = p$. For example, for a g-dimensional abelian variety over k, $\mathrm{H}^i(X, \mathbb{Q}_\ell)$ has dimension $2g$, if $\ell \neq p$, but $\mathrm{H}^i(X, \mathbb{Q}_p)$ has dimension smaller that g. This motivates the need for a good p-adic cohomology theory,, which is given by crystalline cohomology. All details about crystalline cohomology and the theory of crystals can be found in the classical books [BO78] and in [Ber74]. A nice introduction to the main properties of crystals and convergent isocrystals can be found in [ES16, Preliminaries]. We sketch here shortly some of the most important results and definitions.

Definition 1.1 (Crystalline site)**.** The *crystalline site* $(X/W_n)_{\text{crys}}$ is defined as follows: the objects are given by $(U \hookrightarrow T)$, where $U \subset X$ is a Zariski open, T is a W_n-scheme and $i : U \hookrightarrow T$ is a PD thickening of U. This means that i is closed immersion of W_n schemes such that the ideal $\mathrm{Ker}(\mathcal{O}_T \to \mathcal{O}_U)$ has a PD structure δ compatible with the canonical PD structure on pW_n, which is defined by $\gamma_n(p) = p^n/n!$.

The morphisms $(U \hookrightarrow T) \to (U' \hookrightarrow T')$ are commutative diagrams

$$\begin{array}{ccc} U & \hookrightarrow & T \\ \downarrow & & \downarrow \\ U' & \hookrightarrow & T' \end{array}$$

compatible with the PD structures. The coverings are given by elements (U_i, T_i) such that $T_i \hookrightarrow T$ are open immersions and $T = \cup T_i$.

The crystalline site $(X/W)_{\text{crys}}$ is defined in the same way, where T is a W_n-scheme for some n. The *structure sheaf* $\mathcal{O}_{X/W}$ is defined by $(U \hookrightarrow T) \mapsto \Gamma(T, \mathcal{O}_T)$.

Crystals are sheaves on the crystalline site satisfying some special glueing property:

Definition 1.2 ((Iso-)crystals)**.** Defining a *sheaf* E *on the crystalline site* $(X/W)_{\text{crys}}$ is equivalent to giving for any $(U \hookrightarrow T) =: T$, a sheaf of $\mathcal{O}_T$-modules E_T in the Zariski topology of T with the following property: given a morphism $g : (U' \hookrightarrow T') \to (U \hookrightarrow T)$ in the crystalline site, there is an induced $\mathcal{O}_{T'}$-linear morphism $g^*E_T \to E_{T'}$, which is an isomorphism when $T' \to T$ is an open immersion and $U' = U \times_T T'$. A *crystal* of $\mathcal{O}_{X/W}$-modules is a sheaf on $(X/W)_{\text{crys}}$ for which the morphisms $g^*E_T \to E_{T'}$ are isomorphisms. For each element T of $(X/W)_{\text{crys}}$, we get a sheaf of $\mathcal{O}_T$-modules E_T, which is called the *value* of the crystal on T. The crystal is called *of finite presentation*, respectively *locally free*, if the sheaf E_T is of finite presentation and locally free respectively for any object T of $(X/W)_{\text{crys}}$. We denote by $\text{Crys}(X/W)$, respectively $\text{Crys}(X/W_n)$, the category of crystals of $\mathcal{O}_{X/W}$-, respectively $\mathcal{O}_{X/W_n}$-, modules of finite presentation. These categories are abelian and satisfy descent in the Zariski topology [Ber74, Proposition 1.7.6]. The category of *isocrystals*, which will be denoted by $\text{I}(X/W)$ and $\text{I}_{\text{crys}}(X/W_n)$ respectively, has the same objects as $\text{Crys}(X/W)$, $\text{Crys}(X/W_n)$ respectively, but the homomorphisms are given by

$$\text{Hom}_{\text{I}_{\text{crys}}(X/W)}(E, F) := \mathbb{Q} \otimes_W \text{Hom}_{\text{Crys}(X/W)}(E, F).$$

There is a restriction functor $\text{Crys}(X/W) \to \text{Crys}(X/W_n)$, $E \to E_n$, which is induced by the natural inclusion of sites $(X/W_n)_{\text{crys}} \hookrightarrow (X/W)_{\text{crys}}$ and moreover we have an equivalence [BO78, p. 7-22]:

$$\text{Crys}(X/W) \xrightarrow{\simeq} \varprojlim_n \text{Crys}(X/W_n). \tag{1.1}$$

If X is projective over k and $i : X \hookrightarrow \mathbb{P}^N_k$ denotes a fixed closed immersion, we have the notion of PD-envelope of $X \hookrightarrow \mathbb{P}^N_k \hookrightarrow \mathbb{P}^N_{W_n}$, as defined in [BO78, Section 3, p. 3.19, Definition 4.1], which we denote by D_n. In [BO78, Section 4], they define the notion of $\mathcal{O}_{D_n}$-modules with integrable connection with respect to the PD-derivation on D_n. The category of such objects is denoted by $\text{MIC}(D_n)$ and the subcategory of *quasi-nilpotent* ones, for the precise definition see [BO78, Definition 4.10], is denoted by $\mathrm{MIC}(D_n)^{\mathrm{qn}}$. As remarked in [Ogu04], a connection ∇ on E is called quasi-nilpotent if locally on X, every local section of E is annihilated by a power of the p-curvature (we recall the definition of the latter in (3.9)).

By [BO78, Theorem 6.6], we have that this category is equivalent to the one of crystals:

$$\mathrm{Crys}(X/W_n) \xrightarrow{\simeq} \mathrm{MIC}(D_n)^{\mathrm{qn}}; \;\; E \mapsto (E_{D_n}, \nabla_{E_{D_n}}). \tag{1.2}$$

In the case when X admits a lifting to p-adic formal scheme X_W (such a lifting exists on affine open subschemes when X is smooth over k), and if we set $X_n = X_W \otimes W_n$, evaluating at $(X \hookrightarrow X_n, \text{canonical PD structure on } p\mathcal{O}_{X_n})$ induces an equivalence, see also [ES16, p. 7]:

$$\mathrm{Crys}(X/W_n) \xrightarrow{\simeq} \mathrm{MIC}(X_n/W_n)^{\mathrm{qn}} \tag{1.3}$$

and passing to the limit:

$$\mathrm{Crys}(X/W) = \varprojlim_n \mathrm{Crys}(X/W_n) \xrightarrow{\simeq} \varprojlim_n \mathrm{MIC}(X_n/W_n)^{\mathrm{qn}} = \mathrm{MIC}(X_W/W)^{\mathrm{qn}}. \tag{1.4}$$

Crystalline cohomology is a Weil cohomology theory which takes values in the Witt ring of the base field. Crystals are coefficients for this cohomology theory. Using the equivalences described above, Berthelot [BO78, Section 7] gave a comparison between crystalline and algebraic de Rham cohomology:

Theorem 1.3 ([BO78, Theorem 7.1])**.** *Let $i : X \hookrightarrow Y$ a closed immersion of S-schemes, for S a scheme over a field k of positive characteristic, and let E be a crystal on X. If we denote by $\mathscr{E}$ the corresponding $D_{X/Y}$-module with quasi-nilpotent integrable connection, where $D_{X/Y}$ is the PD-envelope of i, then there is a canonical isomorphism*

$$\mathrm{H}^i((X/S), E)_{\mathrm{crys}} \to \mathbb{H}(X_{\mathrm{Zar}}, \mathscr{E} \otimes_{D_{X/Y}} \Omega^\bullet_{D_{X/Y}/S})$$

where $\mathscr{E} \otimes_{D_{X/Y}} \Omega^\bullet_{D/S}$ is the de Rham complex defined by the connection on $\mathscr{E}$, by considering $\mathscr{E}$ as an $\mathcal{O}_Y$-module.

We denote by $\mathrm{H}^i(X/S)_{\mathrm{crys}}$ the cohomology of the trivial crystal $\mathcal{O}_{X/S}$.

There are two special categories of isocrystals which represent connections with extra convergence properties. In [Ogu84, Definition 2.7] Ogus gave the definition of another class of isocrystals, the *convergent isocrystals*. They are crystals on the site of enlargements of X, see [Ogu84, Definition 2.1]. The convergent isocrystals coincide in turn with the *overconvergent isocrystals* on proper varieties, which were defined by Berthelot in [Ber96c]. About the theory of overconvergent isocrystals and rigid cohomology, we also refer to [Le 07].

Definition 1.4 ([Ogu84, Definition 2.1 and 2.7])**.** Let X be a W-scheme or a W-formal scheme. An *enlargement* of X/W is a pair (T, z_T), with T a flat p-adic formal W-scheme and z_T a W-morphism $T_0 \to X$, where T_0 is the reduced closed subscheme $(T_1)_{\mathrm{red}}$ of the closed subscheme T_1 of T defined by the ideal $p\mathcal{O}_T$.

A *convergent isocrystal* E on X/W is a crystal on the site of enlargements: for every enlargement (T, z_T) it is a coherent sheaf E_T of coherent $K \otimes \mathcal{O}_T$-modules, together with the expected glueing property for morphisms between enlargements.

The category of convergent isocrystals on a smooth p-adic formal W-scheme X is a full subcategory of the category of sheaves of $K \otimes \mathcal{O}_X$-modules with integrable connection relative to K, [Ogu84, Theorem 2.15].

Especially interesting in the theory of isocrystals are the objects that admit a Frobenius structure:

Definition 1.5 (F-Isocrystals [Ogu84, Definition 2.17])**.** Let X be a k-scheme, regarded as a formal W-scheme and let $F_X : X \to X$ be its absolute Frobenius endomorphism which covers the Frobenius morphism $F_W : W \to W$. A *convergent F-isocrystal* E on X/W is a convergent isocrystal on Z/W together with an isomorphism $\Phi : F_X^* E \to E$.

1.2 Log isocrystals

The main references for this section are [Kat89],[NS08],[ShI] and [ShII].

Log isocrystals are isocrystals on log schemes. We first give the definition of a log scheme: in what follows, monoids are assumed to be commutative with a unit element.

Definition 1.6 ((pre-) log structures [Kat89, 1.1 and 1.2])**.** A *pre-log structure* on a scheme X is a sheaf of monoids M endowed with a homomorphism $\alpha : M \to \mathcal{O}_X$. A pre-log structure is a *log structure* if in addition

$$\alpha^{-1}(\mathcal{O}_X^*) \simeq \mathcal{O}_X^*.$$

A scheme together with a log structure is called a *log scheme*. We can make the same definition for a p-adic formal scheme over a complete discrete valuation ring of mixed characteristic $(0, p)$. A morphism of log schemes is a pair $(f, h) : (X, M) \to (Y, N)$, with $f : X \to Y$ a morphism of schemes and $h : f^{-1}(N) \to M$ such that the following diagram commutes:

$$\begin{tikzcd} f^{-1}(N) \arrow[r, "h"] \arrow[d] & M \arrow[d] \\ f^{-1}(\mathcal{O}_Y) \arrow[r] & \mathcal{O}_X. \end{tikzcd}$$

Example 1.7. (1) Any scheme X can be endowed with the log structure $(\mathcal{O}_Y^*, \mathcal{O}_Y^* \hookrightarrow \mathcal{O}_Y)$. This is called the *trivial log structure*. Moreover, the functor (Schemes $\to$ Log schemes) is fully faithful and we can consider a scheme as a log scheme endowed with the trivial log structure, while the same holds if we replace schemes by formal schemes.

(2) We can endow any regular scheme X, which has a reduced normal crossings divisor Z, with the log structure defined by

$$M_Z := \{f \in \mathcal{O}_X | f \text{ is invertible outside } Z\} \subset \mathcal{O}_X.$$

For a monoid M, we denote by M^{gp} its Grothendieck group: it is the abelian group $\{ab^{-1}|a,b \in M\}/\sim$, where the equivalence relation is: $ab^{-1} \sim cd^{-1}$ if and only if there is an element $s \in M$ with $sad = sbc$.

Definition 1.8 (Fine and fs monoids)**.** A monoid M is called

(1) *fine* is it is finitely generated and in addition the natural homomorphism $M \to M^{\mathrm{gp}}$ is injective.

(2) *fs* if it is fine and

$$\text{if } m \in M^{\mathrm{gp}} \text{ and } m^n \in M \text{ then } m \in M.$$

Definition 1.9 (Fine and fs log schemes)**.** A log scheme (X, M, α) is called fine (respectively fs) if étale locally on X, there exists a fine (respectively fs) monoid P such that M is isomorphic to the log structure associated to the homomorphism $P_X \to \mathcal{O}_X$. This means that M is isomorphic to the push-out of the diagram

$$\begin{tikzcd} \alpha^{-1}(\mathcal{O}_X^*) \arrow[r] \arrow[d] & P \\ \mathcal{O}_X^*. & \end{tikzcd}$$

The log structure of this push-out is in general given by the map $(a, b) \mapsto \alpha(a)b$, for $a \in M, b \in \mathcal{O}_X$.

The analogue of smoothness in the log scheme setting is the following:

Definition 1.10 (Exact closed immersion, (formally) log smooth morphism [Kat89, 3.1 and 3.3])**.** Let $f : (X, M) \to (Y, N)$ be a morphism of fine log schemes. It is called an *exact closed immersion*, if the morphism $X \to Y$ is a closed immersion of schemes and the homomorphism $f^*N \to M$ is an isomorphism. If the latter is only surjective, f is called a *closed immersion.* Moreover, f is called *log smooth* if:

(1) the morphism of schemes $X \to Y$ is locally of finite presentation and

(2) if we have a commutative diagram of fine log schemes

$$\begin{tikzcd} (Z_0, P_0) \arrow[r, "a"] \arrow[d, "i"] & (X, M) \arrow[d, "f"] \\ (Z, P) \arrow[r, "b"] & (Y, N) \end{tikzcd}$$

with i an exact closed immersion, such that $J := \mathrm{Ker}(\mathcal{O}_Z \to \mathcal{O}_{Z_0})$ satisfies $J^2 = 0$, then there exists étale locally a morphism $g : (Z, P) \to (X, M)$, with $g \circ i = a$ and $f \circ f = b$.

In the case of formal log schemes, we call f formally log smooth if each $(X_n, M_n) \to (Y_n, N_n)$ is log smooth. Here, (X_n, M_n) denotes the closed subscheme defined by $p^n\mathcal{O}_X$ and M_n the pull-back of M to X_n.

An important property of fine log schemes, which we use in order to prove Theorem 2.6, is that they admit a good embedding system:

Definition 1.11 (Good embedding systems [ShII, Definition 2.2.10])**.** Assume that we have a morphism of fine log schemes

$$(X, M) \to (\mathrm{Spf}\, V, N)$$

with $X/\,\mathrm{Spf}\, V$ of finite type and V a totally ramified finite extension of W.

A *good embedding system* is a diagram

$$\begin{array}{ccc} (X^\bullet, M^\bullet) & \longrightarrow & (P^\bullet, N^\bullet) \\ \big\downarrow & & \big\downarrow \\ (X, M) & \longrightarrow & (\mathrm{Spf}\, V, N) \end{array} \tag{1.5}$$

with the following properties:

(1) $X^i \to X$ is a hyper-covering for the étale topology and $M^i (i \geq 0)$ is the inverse image of M on X^i, for all i.

(2) each (X^i, M^i) is a simplicial fine log scheme, of Zariski type, [ShII, Definition 1.1.1], and of finite type over k.

(3) each (P^i, N^i) is a simplicial fine formal log V scheme, formally log smooth and of Zariski type.

(4) $(X^i, M^i) \to (P^i, N^i)$ is a closed immersion for all i.

Remark 1.12. By [ShII, Proposition 2.2.11], given morphisms of fine log schemes $(X, M) \xrightarrow{f} (\mathrm{Spec}\, k) \xrightarrow{i} (\mathrm{Spf}\, V, N)$ where f is of finite type, i is the canonical exact closed immersion and assuming that $(\mathrm{Spf}\, V, N)$ admits a chart, there exists at least one good embedding system of (X, M) over $(\mathrm{Spf}\, V, N)$.

Example 1.13 (Addendum to Example 1.7)**.** (1) The trivial log structure defined in Example 1.7 is fs, since it is associated to the pre-log structure $\{1\}_X \to \mathcal{O}_X$.

(2) For (X, Z) as in Example 1.7(2), we can also show that the obtained log structure, which we denote by (M, α), is fs [ShI, Example 2.4.4]: for $x \in X$ a point, there exists an open affine neighborhood U of $\overline{x}$ and a regular sequence $u_1, \cdots, u_n \in \Gamma(U, \mathcal{O}_U)$, with the property that $Z \times U$ is given by $u_1 u_2 \cdots u_m = 0$, for some $m \leq n$. Here $\overline{x}$ denotes the separable closure of x according to [Kat89, Section 1]. The choice of such elements defines a map $\psi : \mathbb{N}_U^m \to M_{|U} \xrightarrow{\alpha} \mathcal{O}_U$, by $e_i \mapsto u_i$, where e_i generate $\Gamma(U, \mathbb{N}_U^m) = \mathbb{N}^m$. This is in fact a pre-log structure and its associated log structure, as defined in Definition 1.9, is in fact isomorphic to $M_{|U}$.

Last but not least, we can define the notion of log differentials for a log scheme:

Definition 1.14 (Log differentials)**.** Let $f : (X, M, \alpha) \to (Y, N, \beta)$ be a morphism of log schemes. The *module of log differentials* $\omega^{1,\log}_{(X,M)/(Y,N)}$ is defined as the quotient of

$$\Omega^1_{X/Y} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} M^{\mathrm{gp}})$$

by the $\mathcal{O}_X$-submodule locally generated by the elements:

(1) $(d\alpha(a), 0) - (0, \alpha(a) \otimes a)$, with $a \in M$.

(2) $(0, 1 \otimes a)$, where $a \in \mathrm{Im}(f^*N \to M)$.

Note that by [Kat89, p. 3.10], the module of log differentials is a locally free $\mathcal{O}_X$-module of finite type. Moreover, we can define as in the classical case the q-th exterior product of $\omega^{1,\log}_{(X,M)/(Y,N)}$ and denote it by $\omega^{q,\log}_{(X,M)/(Y,N)}$. In the following we omit the log structures from the notation, if there is no confusion, and simply write $\omega^{q,\log}_{X/Y}$.

Next, we define the notions of log crystalline site and log isocrystals and recall the equivalence of the latter category with the category of modules with log connection. These extend the analogous definitions and results of the theory of isocrystals without log structure.

Definition 1.15 (Log crystalline site)**.** Let (X, M) be a fine log scheme over k and

$$(X, M) \to (\operatorname{Spec} k, N) \hookrightarrow (\operatorname{Spf} W, N)$$

be morphisms of fine log formal schemes over W. An object of the crystalline site $((X, M)/(\operatorname{Spec} W_n, N))_{\mathrm{crys}}$, respectively $((X, M)/(\operatorname{Spec} W, N))_{\mathrm{crys}}$ is given by the 5-tuple (U, T, L, i, γ), where U is étale over X, (T, L) is a fine log scheme over $(\operatorname{Spec} W_n, N)$, respectively $(\operatorname{Spec} W_n, N)$ for some n, $i : (U, M) \to (T, L)$ is an exact closed immersion over $(\operatorname{Spec} W_n, N)$, respectively over $(\operatorname{Spec} W, N)$, and γ is a log PD structure on the ideal defining U inside T, which is compatible with the one of W_n. A morphism is defined in the obvious way, in order to satisfy all compatibilities with the log structures.

A covering is the one induced by the étale topology on T.

For simplicity we refer to such an object of the crystalline site by T and usually denote the crystalline site by $(X/W_n)^{\log}_{\mathrm{crys}}$ or $(X/W)^{\log}_{\mathrm{crys}}$. The structure sheaves $\mathcal{O}^{\log}_{X/W_n}$ and $\mathcal{O}^{\log}_{X/W}$ are defined by $\mathcal{O}^{\log}_{X/W_n}(T) := \Gamma(T, \mathcal{O}_T)$ and $\mathcal{O}^{\log}_{X/W}(T) := \Gamma(T, \mathcal{O}_T)$.

Definition 1.16 (Log Isocrystals). A *log crystal* on X/W_n is a crystal on the site $(X/W_n)^{\log}_{\mathrm{crys}}$; it is a sheaf of $\mathcal{O}^{\log}_{X/W_n}$-modules E such that, for any morphism $g : T' \to T$ in $(X/W_n)^{\log}_{\mathrm{crys}}$, the map $g^* E_T \to E_{T'}$ is an isomorphism of sheaves. The log crystal E is called *coherent*, respectively *locally free*, if the sheaf E_T, for T an object of the log crystalline site, is coherent, respectively locally free.

Similarly, we define log crystals on $(X/W)^{\log}_{\mathrm{crys}}$ and we denote by $\mathrm{Crys}(X/W_n)^{\log}$ and $\mathrm{Crys}(X/W)^{\log}$ the categories of coherent crystals on the two sites respectively. The category $\mathrm{I}_{\mathrm{crys}}(X/W)^{\log}$ of log isocrystals is defined as follows: the elements are coherent crystals on the log crystalline site and the morphisms are defined by

$$\mathrm{Hom}_{\mathrm{I}_{\mathrm{crys}}(X/W)^{\log}}(E, F) := \mathbb{Q} \otimes_W \mathrm{Hom}_{\mathrm{Crys}(X/W)^{\log}}(E, F).$$

There is a logarithmic version of the PD envelope of [BO78, Definition 4.1] given in [Kat89, Definition 5.4]:

Definition 1.17 (Log PD envelope). Let $j : (X, M) \to (Y, N)$ be a closed immersion of schemes with log structures over $(\mathrm{Spf}(W_n))$. We define the PD envelope of (X, M) in (Y, N) as follows: there is a functor from the category of pairs (i, δ), where i is an exact closed immersion of schemes with fine log structures $(X, M) \to (Y, N)$ and δ is a PD structure on the ideal of Y defining X, which is compatible with the PD structure on $\mathrm{Spf}\, W_n$, to the category of closed immersions j as before. This functor has a right adjoint, see [Kat89, Proposition 5.3], and if we denote by $(\tilde{j} : (\tilde{X}, \tilde{M}) \to (\tilde{Y}, \tilde{N}), \delta)$ the result of applying this right adjoint to j, then $(\tilde{Y}, \tilde{N})$ is called the *PD envelope of* (X, M) in (Y, N).

For its construction, as in [Kat89, (5.6)], one works étale locally and can assume that the log structure N is fine. Then there is a factorization $j = ij'$ with i étale and $j' : (X, M) \to (Z, M_Z)$ exact closed immersion, by [Kat89, 4.10(1)]. We define $(\widetilde{j} : \widetilde{X} \to D, \delta)$ to be the PD envelope in the usual sense and endow it with the inverse image of the log structure M.

Theorem 1.18 ([Kat89, Theorem 6.2]). *Let $(X, M) \hookrightarrow (Y, N)$ be a closed immersion of fine log schemes over $\mathrm{Spf}\, W$ and denote by (D_n, M_{D_n}) the complete log PD envelope of (X, M) inside $(Y_n, N_n)/(\mathrm{Spf}(W_n))$. There is an equivalence of categories*

$$\mathrm{Crys}(X/Y_n)^{\log} \simeq \mathrm{MIC}(D_n)^{\log}$$

where $\mathrm{MIC}(D_n)^{\log}$ denotes the category of coherent $\mathcal{O}_{D_n}$-modules L on D_n with integrable connection $\nabla^{\log} : L \to L \otimes_{\mathcal{O}_Y} \omega^{1,\log}_{Y/W_n}$.

As remarked in [Shi07, Remark 1.14] the above theorem also proves, if $\mathrm{MIC}(D)^{\log}$ is the category of projective systems $\{(L_n, \nabla_n)\}_n$ of objects in $\mathrm{MIC}(D_n)^{\log}$ that satisfy

$(L_{n+1}, \nabla_{n+1}) \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z} = (L_n, \nabla_n)$, that

$$\mathrm{Crys}(X/Y)^{\log} \simeq \mathrm{MIC}(D)^{\log}.$$

We also get a *log crystalline cohomology theory*:

Theorem 1.19 ([Kat89, Theorem 6.4] and [Shi07, Remark 1.14])**.** *With notations as in the previous theorem, let $u^{\log}_{X/W} : (X/Y)^{\log,}_{\mathrm{Zar}} \to X_{\mathrm{Zar}}$, E a crystal on $(X/W)^{\log}_{\mathrm{crys}}$ and L the corresponding $\mathcal{O}^{\log}_D$-module. Then the log crystalline cohomology with coefficient E is*

$$\mathrm{R}\, u^{\log}_{X/W,*}(E) \simeq L \otimes_{\mathcal{O}^{\log}_Y} \omega^{\bullet,\log}_{Y/W}.$$

In [ShI] and [ShII], Shiho also defines the categories of log convergent isocrystals and log convergent cohomology. We don't include details of these constructions here and refer to the corresponding results as we use them.

Definition 1.20 (Log extendable isocrystals)**.** We call an isocrystal $\mathscr{L}$ on an open smooth scheme U *log extendable* if there is a log isocrystal $\mathscr{L}^{\log}$ on X such that the restriction of $\mathscr{L}^{\log}$ on U is equal to $\mathscr{L}$.

Definition 1.21 (Lattices of isocrystals)**.** A crystal is said to be *p-torsion free* if multiplication by p on it is injective. A *lattice* of an isocrystal $\mathscr{L}$ on a scheme U is a p-torsion free crystal $L \in \mathrm{Crys}(U/W)$ such that $\mathscr{L} \simeq L \otimes \mathbb{Q}$.

For a given isocrystal we can find more than one lattices and it is not generally true that an isocrystal always admits a locally free lattice, i.e. a lattice L whose value on U is a locally free coherent sheaf.

However, when the isocrystal is of rank 1, it admits a locally free lattice by [ES16, Proposition 2.10]. The same is indeed true for a rank 1 log isocrystal on a proper variety X:

Lemma 1.22. *A rank 1 log isocrystal on a proper variety X admits a locally free lattice.*

Proof. The argument is the same as in [ES16, Proposition 2.10] in the local case. We can glue the local lattices as in [ES16, Lemma 2.11], because $\mathrm{H}^0(X, \mathscr{O}_X)^{\log}_{\mathrm{crys}} = \mathrm{H}^0(X, \mathscr{O}_X)_{\mathrm{crys}}$ by [AB05, Section 6]. □

1.3 Étale fundamental group

The topological fundamental group $\pi_1^{\mathrm{top}}(X)$ of a complex projective variety X classifies all covers of X and is defined in a purely topological way. An algebraic analogue of it is the étale fundamental group $\pi_1^{\text{ét}}(X, \overline{x})$ based at some geometric point $\overline{x}$, which classifies all finite étale coverings of X and is isomorphic to the profinite completion of $\pi_1^{\mathrm{top}}(X)$ (we omit the geometric point from the notation from now on). As already mentioned in the introduction, the topological fundamental group relates, via the Riemann-Hilbert correspondence, to the category of $\mathcal{O}_X$-coherent D_X-modules: the latter is equivalent to the category of finite dimensional linear representations of $\pi_1^{\text{top}}(X)$. The above two constructions were related by a result of Malčev [Mal40] and Grothendieck [Gro70]: if $\pi_1^{\text{ét}}(X) = 1$, then there are no non-constant $\mathcal{O}_X$-coherent D_X-modules.

In the positive characteristic case, Esnault and Mehta [EM10, Theorem 1.1] proved in 2010, that for a projective variety over an algebraically closed field k of positive characteristic, $\pi_1^{\text{ét}}(X) = 1$ implies that there are no non-constant $\mathcal{O}_X$-coherent D_X-modules. This was formulated as a conjecture by Gieseker [Gie75].

These results show the close relation between the fundamental group of a projective variety and the existence of D_X-modules on it and motivated in 2010 de Jong's conjecture, about the relation between simply connectedness and the existence of non-constant isocrystals, treated in [ES16].

In particular, it is shown there among other results:

Theorem 1.23 ([ES16, Theorem 0.1])**.** *Let X be a connected smooth projective variety over an algebraically closed field k of characteristic $p > 0$, with trivial étale fundamental group. Then*

(1) any convergent isocrystal, filtered so that the associated graded is a sum of rank 1 convergent isocrystals, is constant,

(2) if the maximal slope of Ω^1_X is non-positive, then any convergent isocrystal is constant. The maximal slope of a coherent torsion free module E is defined as $\deg(E)/\operatorname{rank}(E)$.

However, for a rank 1 isocrystal, the same is true even without any convergence condition:

Proposition 1.24 ([ES16, Proposition 2.10(1)])**.** *If X is as before, any rank 1 isocrystal on X is constant.*

The first part of this thesis treats de Jong's conjecture for rank 1 isocrystals on a non-proper variety over an algebraically closed field of positive characteristic. However, it is very difficult to find examples of non-proper varieties with trivial étale fundamental group; even $\mathbb{A}^1$ has a very big fundamental group, [Ray94].

For a smooth connected projective scheme, the fundamental group is finitely generated, [Sza09, Corollary 5.7.14]. This however, is false for non-proper schemes, see for example [Sza09, Theorem 4.9.5].

Therefore, when considering non-proper varieties, as is the case in the first part of this thesis, it is useful to work with the *tame fundamental group* $\pi_1^{\text{tame}}(X)$ defined in [KS10], see also [Sza09, Definition 5.7.15], whose p-part is topologically finitely generated in positive characteristic. The tame fundamental group is indeed a quotient of $\pi_1^{\text{ét}}(X)$ and is in particular a profinite group.

On the other hand, we also have the abelianized fundamental group [Sza09, Section 5]:

Definition 1.25 (Abelianized fundamental group)**.** We define the *abelianized fundamental group* $\pi_1^{\mathrm{ab}}(X, \overline{x})$ of a scheme X, as the maximal abelian profinite quotient of the fundamental group $\pi_1^{ét}(X, \overline{x})$ for some geometric point $\overline{x}$.

Remark 1.26. Let X be a smooth projective variety over k of characteristic $p > 0$. The abelianized fundamental group fits into an exact sequence [Sza09, Proposition 5.8.3]:

$$1 \to \mathcal{O}_X(X)^*/\mathcal{O}_X(X)^m \to \mathrm{Hom}(\pi_1^{\mathrm{ab}}(X), \mathbb{Z}/m\mathbb{Z}) \to \mathrm{Pic}_X^0(k)[m] \to 1 \tag{1.6}$$

where $\mathrm{Pic}_X^0(k)[m]$ denotes the m-torsion of $\mathrm{Pic}^0(X)$.

One also has a commutative diagram, [Sza09, Facts 5.8.7]:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Pic}_X^0(k) \arrow[r] \arrow[d, "m"] & \mathrm{Pic}_X^0(k) \arrow[r] \arrow[d, "m"] & \mathrm{NS}(X) \arrow[r] \arrow[d, "m"] & 0 \\
0 \arrow[r] & \mathrm{Pic}_X^0(k) \arrow[r] & \mathrm{Pic}_X^0(k) \arrow[r] & \mathrm{NS}(X) \arrow[r] & 0
\end{tikzcd} \tag{1.7}$$

with the vertical maps being the multiplication by m. This induces an exact sequence between the m-torsion subgroups:

$$0 \to \mathrm{Pic}_X^0(k)[m] \to \mathrm{Pic}_X(k)[m] \to \mathrm{NS}(X)[m] \to 0 \tag{1.8}$$

and combining this with (1.6), we get:

$$0 \to \mathrm{Pic}_X^0(k)[m] \to \mathrm{Hom}(\pi_1^{\mathrm{ab}}(X), \mathbb{Z}/m\mathbb{Z}) \to \mathrm{NS}(X)[m] \to 0. \tag{1.9}$$

When the fundamental group is abelian, we have therefore that Pic_X^0 has no torsion prime to p, and hence $\mathrm{Pic}_X^0(k)$ is trivial; indeed for an abelian variety A of dimension g, $A[l] = (\mathbb{Z}/\ell)^{2g}$, $\ell \neq p$, hence Pic_X^0 is zero dimensional and since k is algebraically closed, $\mathrm{Pic}_X^0(k)$ has to be trivial.

In Chapter 2 we will be interested in $\mathrm{Pic}^0(U)$ for U a non-proper variety, which admits a good compactification X.

Remark 1.27. One can find in [Kin13, Proof of Proposition 3.15] a discussion about this case: let U be a non-proper smooth variety over k, which admits a good compactification X such that $X \setminus U = \cup_i Z_i =: Z$ is a strict normal crossings divisor. Then one has a surjection $\mathrm{Pic}(X) \to \mathrm{Pic}(U)$ and can define $\mathrm{Pic}^0(U)$ as the image of the divisible abelian group $\mathrm{Pic}^0(X)$ inside $\mathrm{Pic}(U)$. The Néron-Severi group of X is the finitely generated group defined by $\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ and since $\mathrm{Pic}^0(U)$ is also divisible, one defines also $\mathrm{NS}(U) = \mathrm{Pic}(U)/\mathrm{Pic}^0(U)$.

Moreover, we note here the following fact regarding the tame fundamental group and its abelianization, which we use in Chapter 2:

Lemma 1.28 ([SS03, p. 10]). *Let U be an smooth scheme admitting a smooth compactification X. Then :*

$$\pi_1^{\text{tame,ab}}(U) = \pi_1^{\text{ab}}(U)(p') \oplus \pi_1^{\text{ab}}(X)(p), \tag{1.10}$$

where (p) and (p') denote the maximal pro-p and prime-to-p quotients respectively.

This relation follows from the decomposition of $\pi_1^{tame,ab}(U)$ to its pro-p and prime-to-p quotients and the fact that an abelian p-cover of U lifts to a p-cover of X. For more details we refer to [SS03, p. 10]. In particular, we note that the fundamental group is a birational invariant of a projective variety, we can conclude from this decomposition that $\pi_1^{\text{tame,ab}}(U)$ depends only on U and not on the choice of compactification.

Finally, we recall here two facts about fundamental groups, which we use later:

Proposition 1.29 ([Kin13, Proposition 3.8, Remark 3.9]). *If k is algebraically closed and U is a connected normal k-scheme of finite type such that $\pi_1^{\text{ab},(\ell)}(U)$ is trivial for some $\ell \neq char(k)$, then $\mathrm{H}^0(U, \mathcal{O}_U^*) = k^*$. Moreover, if $char(k) > 0$, $\pi_1^{\text{ab},(p)}(U) = 0$ implies $\mathrm{H}^0(U, \mathcal{O}_U) = k$.*

1.4 Berkovich and rigid analytifications

After the introduction of overconvergent isocrystals by Berthelot in [Ber96c], rigid analytic geometry became a necessary tool for the study of isocrystals. The theory of Berkovich spaces is another approach to non-archimedean geometry and provides us with spaces that are very close to rigid analytic spaces, but have a true topology, in contrast to the rigid analytic ones, which have a Grothendieck topology. In this section, we include various results that we use in the second part of this thesis. We focus mainly on the functors of analytification from algebraic schemes to the respective categories of rigid analytic and Berkovich spaces. For the basic definitions in the theory of rigid analytic geometry we refer to [BL93, Chapter 1-5]. Nice introductions to the theory of Berkovich spaces can be found in [Bak08],[Tem15],[DFN15] and of course in the original papers of Berkovich [Ber90] and [Ber93].

In this section we denote by K a complete field equipped with a non-trivial non-archimedean absolute value, R denotes the associated valuation ring, and k the residue field of R.

1.4.1 Rigid analytification

Building blocks of rigid analytic spaces are affinoid spaces, [Bos14, Section 3.2]. They are defined as maximal spectra of affinoid K-algebras, which in turn are defined as quotients of Tate algebras associated to K, for details see [Bos14]. On an affinoid K-space X, we define a Grothendieck topology $\mathscr{T}$ in the following way: the objects of Cat $\mathscr{T}$ are affinoids subdomains, morphisms are inclusions and coverings are the finite coverings by affinoid subdomains.

From this local data, we can construct a Grothendieck topology $\mathscr{T}$ on a K-space X. Objects of Cat $\mathscr{T}$ are called *admissible open subsets* of X and coverings in $\mathscr{T}$ are called *admissible coverings.*

Definition 1.30 (Rigid analytic space)**.** A rigid analytic space is a locally ringed space $(X, \mathcal{O}_X)$ equipped with a strong Grothendieck topology, [BL93, Proposition 5], which admits an admissible covering by affinoid K-spaces.

In analogy with the GAGA functor of complex algebraic geometry constructed by Serre [Ser56], there is a rigid analytification functor

$$(K-\text{schemes locally of finite type }) \to (\text{rigid}\, K-\text{spaces});\quad Z \to Z^{\text{an}}. \tag{1.11}$$

Example 1.31 ([Bos14, Section 5.4])**.** We can construct explicitly the rigid analytification of the affine space $\mathbb{A}^n_K$.

Set for $r > 0$, $T_n(r)$ the K-algebra of power series $\sum_u \alpha_u \zeta^n$, with $\zeta = (\zeta_1, \cdots, \zeta_n)$ and $\alpha_u \in K$ such that $\lim_u \alpha_u r^{|u|} = 0$. Note that the limit of elements in K is the limit in the topology on K given by its absolute value. Equivalently, we can say that $T_n(r)$ is the algebra of all power series that converge inside a closed ball of dimension n and radius r. If we also set $T_n^{(i)} = T_n(|b|^i)$, for some $b \in K$ with $|b| > 1$, we see that this is exactly the Tate algebra $K\langle b^{-i}\zeta_1, \cdots, b^{-i}\zeta_n\rangle$.

Moreover, there are natural inclusions $T^{(i)} \hookrightarrow T^{(i-1)}$, and taking the corresponding morphisms between the maximal spectra of $T^{(i)}$, we get:

$$\mathbb{B}^n = \operatorname{Sp} T_n^{(0)} \hookrightarrow \operatorname{Sp} T_n^{(1)} \hookrightarrow \cdots .$$

The rigid analytification of $\mathbb{A}^n_K$ is then defined to be the union of $\operatorname{Sp} T^{(i)}$, which can be constructed by [Bos14, Proposition 5.3.5]. We note that it is independent from the choice of b.

For general K-schemes, one can define the rigid analytification via a universal property:

Definition 1.32 (Rigid analytification)**.** The *rigid analytification* of a K-scheme $(X, \mathcal{O}_X)$ locally of finite type is a rigid analytic space $(X^{\text{an}}, \mathcal{O}_{X^{\text{an}}})$ together with a morphism of locally G-ringed K-spaces $(i, i^*) : (X^{\text{an}}, \mathcal{O}_{X^{\text{an}}}) \to (X, \mathcal{O}_X)$ which has the following universal property: any morphism of locally G-ringed K-spaces $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$, for $(Y, \mathcal{O}_Y)$ a rigid K-space, factors through (i, i^*).

If we have now a morphism of K-schemes $f : X \to S$, we obtain its analytification f^{an} of f by the universal property of analytification:

$$\begin{tikzcd} X^{\text{an}} \arrow[r, "f^{\text{an}}"] \arrow[d] & Y^{\text{an}} \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd} .$$

Hence, one indeed obtains a functor as in (1.11).

Moreover, one can show as in [Bos14, Proposition 5.4.4], that every K-scheme admits an analytification, which is constructed by gluing the analytifications of its affine parts, and that the map of sets $i : X^{\mathrm{an}} \to X$ identifies the closed points of X^{an} and X.

Example 1.33 (Example 1.31 continued [Bos14, p. 114])**.** First, we check that the definition of $\mathbb{A}_K^{n,\mathrm{an}}$ as in Example 1.31 agrees with the general definition 1.32. Recall that $\mathbb{A}_K^n = \operatorname{Spec} K[\zeta]$ and $\mathbb{A}_K^{n,\mathrm{an}} = \cup_{i=0}^{\infty} \operatorname{Sp} T_n^{(i)}$. The canonical morphisms $K[\zeta] \to \operatorname{Sp} T_n^{(i)}$ give rise to $\mathcal{O}_{\mathbb{A}^n}(\mathbb{A}^n) \to \mathcal{O}_{\mathbb{A}^{n,\mathrm{an}}}(\mathbb{A}^{n,\mathrm{an}})$, which in turn yields a morphism between locally ringed spaces by [Bos14, Lemma 5.4.2],

$$(i, i^*) : (\mathbb{A}^{n,\mathrm{an}}, \mathcal{O}_{\mathbb{A}^{n,\mathrm{an}}}) \to (\mathbb{A}^n, \mathcal{O}_{\mathbb{A}^n}).$$

To prove that this satisfies the universal property of Definition 1.32, we see that a morphism $(Z, \mathcal{O}_Z) \to (\mathbb{A}^n, \mathcal{O}_{\mathbb{A}^n})$ corresponds to a morphism $\phi : K[\zeta] \to \mathcal{O}_Z(Z)$. Choosing $i \in \mathbb{N}$ such that $\zeta_j \in K[\zeta]$ have the property $|\phi(\zeta_j)|_{\sup} \leq |c|^i$ in $\mathcal{O}_Z(Y)$, we see that the morphism ϕ extends to $T_n^{(i)}$, as needed.

Furthermore, we can show that the analytification of $\mathbb{A}_K^n$ constructed in Example 1.31 satisfies the universal property of an n-dimensional affine space: let Z be a rigid K-space. Morphisms of locally ringed G-spaces $Z \to \mathbb{A}_K^n$ can be described as:

$$\operatorname{Hom}(Z, \mathbb{A}_K^n) \xrightarrow{\sim} \operatorname{Hom}(K[\zeta_1 \cdots, \zeta_n], \mathcal{O}_Z(Z)) \xrightarrow{\sim} \mathcal{O}_Z(Z)^n.$$

However, a morphism of rigid K-spaces $Z \to \mathbb{A}_K^{n,\mathrm{an}}$, gives rise to a morphism in $\operatorname{Hom}(Z, \mathbb{A}_K^n)$, by composing with the canonical morphism $i : \mathbb{A}_K^{n,\mathrm{an}} \to \mathbb{A}_K^n$. By the universal property of analytification, we then get a bijection:

$$\operatorname{Hom}(Z, \mathbb{A}_K^{n,\mathrm{an}}) \xrightarrow{\sim} \operatorname{Hom}(Z, \mathbb{A}_K^n) \xrightarrow{\sim} \mathcal{O}_Z(Z)^n.$$

Given a coherent sheaf $\mathscr{F}$ on an algebraic variety X, one can define a coherent sheaf of $\mathcal{O}_{X^{\mathrm{an}}}$-modules $\mathscr{F}^{\mathrm{an}}$, by $\mathscr{F}^{\mathrm{an}} := i^{-1}\mathscr{F} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{O}_{X^{\mathrm{an}}}$. Moreover, there is an analogue of Serre's GAGA theorem in the rigid analytic case:

Theorem 1.34 (Rigid GAGA [Bos14, Theorem 6.3.13])**.** *If X is proper over K, then the functor $\mathscr{F} \mapsto \mathscr{F}^{\mathrm{an}}$ induces an equivalence of categories between the category of coherent $\mathcal{O}_X$-modules and the category of coherent $\mathcal{O}_{X^{\mathrm{an}}}$-modules. Moreover, for every $n \geq 0$, there exist a canonical isomorphism*

$$\mathrm{H}^n(X, \mathscr{F}) \simeq \mathrm{H}^n(X^{\mathrm{an}}, \mathscr{F}^{\mathrm{an}}).$$

In Chapter 3 we will be particularly interested in the rigid analytification of Pic^0:

Remark 1.35 ([Con06, Lemma 4.3.2])**.** Given a map between rigid spaces $X \to S$, where X has geometrically reduced and connected fibers over S, together with a section

$x \in X(S)$, we can consider the functor $\mathbf{Pic}_{X/S,x}$, which associates to any rigid analytic spaces S' over S, the set of $x_{S'}$-rigidified line bundles on $X \times_S S'$.

When the rigid spaces are rigid analytifications of algebraic varieties, this functor is actually representable: for a map $X \to S$ of algebraic K schemes with geometrically reduced and connected fibers, it is proven in [Con06, Lemma 4.3.2] that the analytification of the Picard scheme $\mathrm{Pic}_{X/S,x}$ represents the functor $\mathbf{Pic}_{X^{\mathrm{an}}/S^{\mathrm{an}},x^{\mathrm{an}}}$.

1.4.2 Berkovich analytification

Rigid analytic geometry makes it possible to define the notion of an analytic function on a non-archimedean field K, but doesn't provide a good topological space to work with; one works with a Grothendieck topology instead. Berkovich's K-analytic spaces though have a true topology, which makes working with them easier.

Building blocks of a K-analytic space are the so called K-affinoid spaces: spaces of the form $\mathscr{M}(A) =$ the space of multiplicative semi-norms on the Banach algebra A, see [DFN15, Definition 4.1.2.1]. A K-analytic space is then a topological space which admits a K-affinoid atlas on X, together with a family of subsets, called a net, see [DFN15, Definition 4.1.1.1]. We don't include here details about this definition, but we see an example:

Example 1.36. The Berkovich n-dimensional affine space $\mathbb{A}^n_{K,\mathrm{Berk}}$ is the space $\mathscr{M}(K[T_1, \cdots, T_n])$ of multiplicative semi-norms on the ring $\mathscr{A} = K[T_1, \cdots, T_n]$, whose restriction to K is bounded with respect to the Banach norm on K. Recall that every field which is complete with respect to a valuation is a commutative Banach ring. It is equipped with the weakest topology with respect to which all functions $\mathbb{A}^n \to \mathbb{R}_+$, of the form $x \to |f(x)|$, with $f \in \mathscr{A}$, are continuous.

As in the case of rigid analytic spaces, we are mainly interested in the functor which assigns a K-analytic space to an algebraic K-variety. This functor is called the *Berkovich analytification functor*.

This functor is defined as follows on affine schemes: let $X = \operatorname{Spec} K[T_1, \cdots, T_n]$. We set $X^{\mathrm{an}} := \mathscr{M}(K[T_1, \cdots, T_n])$, which is exactly the affine space of the previous example. For a scheme $\operatorname{Spec} K[T_1, \cdots, T_n]/I$, we define the analytification as the closed subset of X^{an} defined by the vanishing of $I\mathcal{O}_{X^{\mathrm{an}}}$.

The analytification functor can also be defined via a universal property: the analytification of an algebraic variety X represents the functor which assigns to a K-analytic space Y, the set of morphisms of locally ringed spaces $Y \to X$. This also means, exactly like the rigid analytic case, that there is a morphism $i : X^{\mathrm{an}} \to X$. For a coherent $\mathcal{O}_X$-module $\mathscr{F}$, we can define a coherent $\mathcal{O}_{X^{\mathrm{an}}}$-module $\mathscr{F}^{\mathrm{an}} = i^*(\mathscr{F})$.

Remark 1.37. There is a close relation between K-analytic spaces and rigid analytic K-spaces, which is explained in [Ber90, Section 3.3]. To state it, we first need to recall the following: for $\mathscr{A}$ a commutative Banach algebra over K, a point $x \in \mathscr{M}(\mathscr{A})$ gives rise to a character $\mathscr{A} \to \mathscr{H}(x)$, where $\mathscr{H}(x)$ is a field. Indeed, the multiplicative semi-norm corresponding to x extends to the fraction field of the quotient ring of $\mathscr{A}$ by its kernel and $\mathscr{H}(x)$ is then defined as the completion of this field, see [Ber90, p.4].

By [Ber90, Proposition 2.2.5], the class of strictly affinoid domains in X defines a G-topology on a separated strictly K-analytic space X and setting $X_0 = \{x \in X | [\mathscr{H}(x) : K] < \infty\}$, this has the induced G-topology and has the structure of a rigid analytic K-space. Moreover, [Ber90, Proposition 2.3.1] implies that given a sheaf $\mathscr{F}$ on X, we get an induced sheaf $\mathscr{F}_0$ on X_0 and the correspondence $\mathscr{F} \mapsto \mathscr{F}_0$ is an equivalence between the categories of coherent sheaves on X and X_0.

As explained also in [Bak08, p.58], by the correspondence $X \mapsto X_0$ we also obtain an equivalence of categories between the category of quasi-separated rigid spaces admitting a locally finite admissible covering by affinoid opens and the category of paracompact Hausdorff strictly K-analytic spaces. Moreover, the Berkovich analytification functor from algebraic K-schemes to strictly analytic K-spaces and the rigid analytification functor from algebraic K-schemes to rigid analytic spaces over K is compatible with this equivalence. This functor in fact preserves the category of locally constant sheaves and their cohomology groups, by [Ber90, Proposition 3.3.4]. Therefore, cohomological results from the theory of rigid analytic spaces are applicable to Berkovich's K-analytic spaces.

1.5 Generic fibers of formal schemes

It is possible to relate both the Berkovich and the rigid analytification of an algebraic variety with the theory of formal schemes; more specifically their generic fibers.

Berkovich spaces and generic fibers of formal schemes

Definition 1.38 (Special formal schemes)**.** Let R be a topological adic Noetherian ring whose Jacobson radical is an ideal of definition. An $\mathfrak{a}$-adic R-algebra A is called *special* if $A/\mathfrak{a}^2$ is finitely generated over R, see [Ber96a, Lemma 1.2]. A formal scheme $\mathscr{X}$ over R is called *special* if it is locally a finite union of affine formal schemes of the form $\operatorname{Spf} A$, with A an adic special algebra over R.

Let K be a non-archimedean field with a non-trivial discrete valuation, R the ring of integers of K and k its residue field. For a special R-formal scheme $\mathscr{X}$, the ringed space $(\mathscr{X}, \mathcal{O}_{\mathscr{X}}/\mathscr{I})$, where $\mathscr{I}$ is an ideal of definition of $\mathscr{X}$ that contains the maximal ideal m_R of R, is a scheme locally of finite type over k and is called the *closed fiber* of $\mathscr{X}$, denoted by $\mathscr{X}_s$. It depends on the choice of $\mathscr{I}$ but the underlying reduced scheme and the associated topos do not, see [Ber96a, p.370].

Remark 1.39 ([Ber96a, p.370])**.** If $\mathscr{Y} \subset \mathscr{X}_s$, the formal completion of $\mathscr{X}$ along $\mathscr{Y}$ is a special formal scheme over R. We denote this completion by $\mathscr{X}_{|\mathscr{Y}}$.

Berkovich defines in [Ber96a] a functor from the category of special formal R-schemes to the category of K-analytic spaces, which associates to $\mathscr{X}$ its generic fiber $\mathscr{X}_\eta$. For a formal scheme of the form $\mathscr{X} = \mathrm{Spf}\, A$, with A a special R-algebra, the generic fiber $\mathscr{X}_\eta$ is defined as the set of continuous multiplicative semi-norms on A that extend the valuation on R and have value at most 1. For an arbitrary formal scheme $\mathscr{X}$, one can take an affine covering $\mathscr{X}_i$ and glue $\mathscr{X}_{i,\eta}$. For all details we refer to [Ber96a, p.370-371].

Example 1.40. In the case when $\mathscr{X} = \mathrm{Spf}\, A$, with $A = K\{T_1, \cdots, T_n\}[[S_1, \cdots, S_m]]$, where $K\{T_1, \cdots, T_n\}$ is the algebra of restricted power series with coefficients in K. Then one defines $\mathscr{X}_\eta = E^m(0,1) \times D^n(0,1)$, where $E^m(0,1)$ and $D^n(0,1)$ are the closed and open polydiscs of radius 1 and center at zero, in $\mathbb{A}^m$ and $\mathbb{A}^n$ respectively.

Moreover, one can construct a map, called the *reduction map* $\pi : \mathscr{X}_\eta \to \mathscr{X}_s$.

Proposition 1.41 (Definition of reduction map)**.** *There is an anticontinuous map $\pi : \mathscr{X}_\eta \to \mathscr{X}_s$, called the reduction map.*

Proof. The proof of this fact can be found in [Ber96a, p. 541], but we present it here for completion. If $\mathscr{X} = \mathrm{Spf}(A)$, with A a topologically presented ring over R, i.e. $A = R\{T\}/\alpha$ with α a finitely generated ideal in $R\{T\}$, then $A \otimes_R K =: \mathscr{A}$ is a K-affinoid algebra and by definition, $\mathscr{X}_\eta$ is the space $\mathscr{M}(\mathscr{A})$. Since the image of A in $\mathscr{A}$ is contained in the subset of elements $f \in \mathscr{A}$ such that $|f(x)| \leq 1$ for all $x \in \mathscr{M}(\mathscr{A})$, by the previous discussion, an element $x \in \mathscr{M}(\mathscr{A})$ defines a character $\widetilde{A} := A/m_R A \to \widetilde{\mathscr{H}(x)}$. The kernel of this map is a prime ideal in $\widetilde{A}$, since $\widetilde{\mathscr{H}(x)}$ is a field, and we define $\pi(x) \in \mathscr{X}_s$ to be this prime ideal. This defines a map $\pi : \mathscr{X}_\eta \to \mathscr{X}_s$.

For $\mathscr{Y} \subset \mathscr{X}_s$ a closed subset defined by $(\widetilde{f_1}, \cdots, \widetilde{f_n})$ for $f_i \in A$, $\pi^{-1}(\mathscr{Y}) = \{x \in \mathscr{X}_\eta || f_i(x)| < 1, i = 1, \cdots n\}$. If $\mathscr{Y} = \mathrm{Spec}(\widetilde{A}[1/\widetilde{f}])$, $f \in A$, is instead an open subset, then $\pi^{-1}(\mathscr{Y}) = \{x \in \mathscr{X}_\eta || f_i(x)| = 1, i = 1, \cdots n\}$.

For an arbitrary $\mathscr{X}$, one can fix a finite covering $\{\mathscr{X}_i\}$ by open affine formal subschemes and the maps $\pi_i : \mathscr{X}_{i,\eta} \to \mathscr{X}_{i,s}$ induce a map $\pi : \mathscr{X}_\eta \to \mathscr{X}_s$.

From the affine case, it follows that the pre-image of a closed subset of $\mathscr{X}_s$ by π is an open in $\mathscr{X}_\eta$ and the pre-image of an open subset is instead a closed analytic subdomain of $\mathscr{X}_\eta$. □

We can describe the inverse image of a subscheme $\mathscr{Y}$ of $\mathscr{X}_s$ in more detail:

Proposition 1.42 ([Ber96a, Proposition 1.3])**.** *With notation as in the above proposition, there is a canonical isomorphism $\pi^{-1}(\mathscr{Y}) \simeq (\mathscr{X}_{|\mathscr{Y}})_\eta$.*

Remark 1.43 ([Ber96b, p.553])**.** Given now a scheme of finite type over R, one can relate to it a K-analytic space in two different ways: by associating to it the analytification of its generic fiber $(X_\eta)^{\mathrm{an}}$ and the generic fiber of its completion $(\widehat{X})_\eta$, the construction of which we summarized in this section. By construction, there is a morphism $(\widehat{X})_\eta \to (X_\eta)^{\mathrm{an}}$ and for X separated and finitely presented, this identifies $(\widehat{X})_\eta$ with a closed analytic subvariety of $(X_\eta)^{\mathrm{an}}$; in the affine case $X = \mathrm{Spec}\, A$ where A is generated by $f_1, \cdots, f_n$, $(\widehat{X})_\eta = \{x \in (X_\eta)^{\mathrm{an}} || f_i(x)| \leq 1\}$. For arbitrary varieties, one takes a finite covering by open affine subschemes and concludes in a similar way. If X

is proper, this morphism is in fact an isomorphism. This follows from the more general fact, that a proper morphism $\phi : Z \to X$ induces an isomorphism $\widehat{Z}_\eta \xrightarrow{\simeq} Z_\eta^{\mathrm{an}} \times_{X_\eta^{\mathrm{an}}} \widehat{X}_\eta$, applied to X being a point.

Formal models for rigid spaces

Just as for Berkovich spaces, we can relate rigid spaces to generic fibers of formal schemes. In this part, we include some details about this relation and some results which we use later.

Definition 1.44. With notation as in the rest of this section, an R-algebra A is called:

(1) *topologically of finite type* if $A = R\langle\xi\rangle/\mathfrak{a}$, where $R\langle\xi\rangle = R\langle\xi_1, \cdots, \xi_n\rangle$ is the R-algebra of strictly convergent power series in variables $\xi_1, \cdots, \xi_n$ and $\mathfrak{a} \subset R\langle\xi\rangle$ an ideal,

(2) *topologically of finite presentation* if a is moreover finitely generated,

(3) *admissible* if A has no m_R-torsion, where m_R is as before the maximal ideal of R.

An affine formal scheme $\operatorname{Spf} A$ is called admissible, if A is admissible and a general formal scheme X over R is called admissible if it is locally of the form $\operatorname{Spf} A$, with A an admissible R-algebra.

We can make the analogous definition for a formal scheme $\mathscr{X}$ over an R-formal scheme $\mathscr{S}$. If we denote by u a generator of the maximal ideal m_R, we assume that for $\mathscr{I} = u\mathcal{O}_S$, the ideal $\mathscr{I}\mathcal{O}_X$ is an ideal of definition of X. We call X *admissible*, if moreover $\mathcal{O}_X$ has no $\mathscr{I}$-torsion.

In order to relate formal schemes with rigid analytic spaces, we need the notion of an admissible formal blowing up.

Definition 1.45. Let $\mathscr{X}/\mathscr{S}$ an admissible formal $\mathscr{S}$-scheme with ideal of definition $\mathscr{I}\mathcal{O}_{\mathscr{X}}$ and $\mathscr{A} \subset \mathcal{O}_{\mathscr{X}}$ an open ideal. Then we define

$$\mathscr{X}' := \varinjlim_m \operatorname{Proj} \oplus_{n=0}^{\infty} (\mathscr{A}^n \otimes_{\mathcal{O}_{\mathscr{X}}} \mathcal{O}_{\mathscr{X}}/\mathscr{I}^{m+1})$$

and call the morphism of $\mathscr{S}$-formal schemes $\phi : \mathscr{X}' \to \mathscr{X}$, an *admissible formal blowing up* with respect to $\mathscr{A}$.

Among many interesting properties, see for example [BL93, Section 2], an admissible formal blowing up satisfies a universal property,[BL93, Proposition 2.1(c)]: if $\psi : \mathscr{Z} \to \mathscr{X}$ is a morphism of formal $\mathscr{S}$-schemes such that $\mathscr{A}\mathcal{O}_{\mathscr{X}}$ is invertible on $\mathscr{Z}$, there exists a unique $\mathscr{S}$-morphism $\psi' : \mathscr{Z} \to \mathscr{X}'$ such that $\psi = \phi \circ \psi'$. Note that [BL93, Lemma 2.2] gives an explicit description of an admissible blowing up in the case of an affine formal scheme.

The relation between formal schemes and rigid analytic spaces can be seen in the next theorem of Raynaud:

Theorem 1.46 ([BL93, Theorem 4.1], [Ray74])**.** *There is an equivalence of categories between:*

(1) the category of quasi-compact admissible formal schemes, localized by admissible formal blowing ups, and

(2) the category of rigid K-spaces, which are quasi-compact and quasi-separated.

In the affine case the above functor is defined as follows: to an admissible R-algebra $A = R\langle\xi\rangle/\mathfrak{a}$ we associate the affinoid K-algebra $A_{\mathrm{rig}} := A \otimes_R K = R\langle\xi\rangle/\mathfrak{a}K$. Globalizing this construction, as explained in [BL93, Section 4], we obtain a functor from admissible formal R-schemes to the category of rigid K-spaces. We denote this functor by $\mathscr{X} \mapsto \mathscr{X}_{\mathrm{rig}}$. We call $\mathscr{X}_{\mathrm{rig}}$ the *rigid analytic fiber* of $\mathscr{X}$.

In the inverse direction and in the affinoid case, this functor is given by associating to $X_K = \operatorname{Sp} A_K = \operatorname{Sp} K\langle\xi\rangle/\mathfrak{a}K$, the affine formal scheme $\mathscr{X} = \operatorname{Sp} R\langle\xi\rangle/\mathfrak{a} \cap R\langle\xi\rangle$.

We recall finally here for convenience the definition of localization of a category $\mathscr{C}$ with respect to a set of morphisms T: the localization $\mathscr{C}_T$ is a category together with a functor $\mathscr{C} \to \mathscr{C}_T$ such that, if $\mathscr{C} \to \mathscr{D}$ is a functor to another category sending all morphisms in T to isomorphisms in $\mathscr{D}$, then there is a unique functor $\mathscr{C}_T \to \mathscr{D}$ making the diagram

$$\begin{tikzcd} \mathscr{C} \arrow[r] \arrow[dr] & \mathscr{C}_T \arrow[d] \\ & \mathscr{D} \end{tikzcd} \tag{1.12}$$

commute.

Remark 1.47. In Chapter 3 we use in particular some properties of this equivalence, found in [BL93, Proof of Theorem 4.1, (a), (b) and (c)]:

(1) the above functor sends admissible blowing ups to isomorphisms: if $\phi : \mathscr{X}' \to \mathscr{X}$ is an admissible formal blowing up, then $\phi_{\mathrm{rig}} : \mathscr{X}'_{\mathrm{rig}} \to \mathscr{X}_{\mathrm{rig}}$ is an isomorphism.

(2) Two morphisms $\phi, \psi : \mathscr{Z} \to \mathscr{X}$ of formal R-schemes coincide if ϕ_{rig} and ψ_{rig} coincide.

(3) If $\phi_K : \mathscr{Z}_{\mathrm{rig}} \to \mathscr{X}_{\mathrm{rig}}$ is a morphism between the rigid analytic fibers of two formal R-schemes, then there is an admissible formal blowing up $r : \mathscr{Z}' \to \mathscr{Z}$ and a morphism $\phi_{\mathrm{rig}} : \mathscr{Z}' \to \mathscr{X}$ such that $\phi_{\mathrm{rig}} = \phi_K \circ r$.

1.6 Cohomology of Berkovich spaces

We state here some results from [Ber96b, Section 5] that we use in Chapter 3. Let X be a scheme of finite type over $\operatorname{Spec} R$, with notations as in the previous sections. We have a diagram, where X_η and X_s are the generic and closed fibers respectively:

$$\begin{tikzcd}
X_\eta \arrow[r, hook, "j"] & X & X_s \arrow[l, hook', "i"'] \\
X_{\overline{\eta}} \arrow[u] \arrow[r, hook, "\overline{j}"] & \overline{X} \arrow[u] & X_{\overline{s}}. \arrow[u] \arrow[l, hook', "\overline{i}"']
\end{tikzcd}$$

The *vanishing cycles functor* $\Psi_\eta : \widetilde{X}_{\eta,\text{ét}} \to \widetilde{X}_{\overline{s},\text{ét}}$ is then defined as $\Psi_\eta(\mathscr{F}) = \overline{i}^*(\overline{j}_*\overline{\mathscr{F}})$.

Let now $\mathscr{Y} \subset X_s$ be a subscheme of the closed fiber of $X \to \operatorname{Spec} R$. Then the formal completion of X along $\mathscr{Y}$, $\widehat{X}_{|\mathscr{Y}}$ is a special formal scheme, in the sense of [Ber96a], whose closed fiber is identified with $\mathscr{Y}$. By [Ber96a, Proposition 1.3], which we recalled in Proposition 1.41, there is a canonical isomorphism

$$(\widehat{X}_{|\mathscr{Y}})_\eta \simeq \pi^{-1}(\mathscr{Y}).$$

As recalled in Remark 1.43, we have a morphism $\widehat{X}_\eta \to X_\eta^{\text{an}}$. Moreover, there are canonical morphisms of sites $\widehat{X}_{\eta,\text{ét}} \to X^{\text{an}}_{\eta,\text{ét}} \to X_{\eta,\text{ét}}$. For details about the étale topos on an analytic space, we refer to [Ber96b, Section 3]. For a sheaf $\mathscr{F}$ on $X_{\eta,\text{ét}}$, we denote by $\mathscr{F}^{\text{an}}$ and $\widehat{\mathscr{F}}$ the respective pullbacks to $X^{\text{an}}_{\eta,\text{ét}}$ and $\widehat{X}_{\eta,\text{ét}}$, while there is also a canonical morphism of sheaves $\Psi_\eta(\mathscr{F}) \to \Psi_\eta(\widehat{\mathscr{F}})$. For this morphism more is in fact true:

Proposition 1.48 ([Ber96b, Corollary 5.3]). *If $\mathscr{F}$ is an abelian torsion sheaf on X_η, then for any $q \geq 0$, there is a canonical isomorphism*

$$R^q\Psi_\eta(\mathscr{F}) \xrightarrow{\simeq} R^q\Psi_\eta(\widehat{\mathscr{F}}).$$

Corollary 1.49 ([Ber96b, Corollary 5.4]). *If $\mathscr{X}$ is a smooth formal R-scheme and n is an integer coprime to p, then*

$$\Psi_\eta(\mathbb{Z}/n\mathbb{Z})_{\mathscr{X}_\eta} = (\mathbb{Z}/n\mathbb{Z})_{\mathscr{X}_{\overline{s}}} \quad \textit{and} \quad R^q\Psi_\eta(\mathbb{Z}/n\mathbb{Z})_{\mathscr{X}_\eta} = 0, q \geq 1.$$

In Chapter 3 we will be mainly interested in comparing the ℓ-adic cohomology of a subset $\mathscr{Y}$ of X_s and the cohomology of its inverse image by the reduction map $\pi^{-1}(\mathscr{Y})$. The following theorem of Berkovich provides us with a comparison between these cohomology groups:

Theorem 1.50 ([Ber96a, Theorem 3.1]). *Let $\mathscr{F}$ be an abelian torsion sheaf on X_η with torsion orders prime to p. Then for $\mathscr{Y} \subset X_s$ an open in the closed fiber X_s we have*

$$(\mathrm{R}^q\,\Psi_\eta\mathscr{F})_{|\mathscr{Y}} \xrightarrow{\simeq} \mathrm{R}^q\,\Psi_\eta(\widehat{\mathscr{F}_{|\mathscr{Y}}}), \quad \forall q \geq 0,$$

where $\widehat{\mathscr{F}}$ is the pullback of $\mathscr{F}$ to $\widehat{X}_\eta$.

As a corollary of this we get:

Corollary 1.51 ([Ber96a, Corollary 3.5])**.** *Let $X, \mathscr{Y}$ be as before and $\mathscr{F}$ be a constructible sheaf on X_η with torsion orders prime to p. Then there are canonical isomorphisms*

$$\mathrm{R}\Gamma(\mathscr{Y}, \mathrm{R}\Psi_\eta \mathscr{F}) \xrightarrow{\simeq} \mathrm{R}\Gamma(\pi^{-1}(\mathscr{Y}), \mathscr{F}^{\mathrm{an}}).$$

In particular:

Corollary 1.52 ([Ber96a, Corollary 3.7])**.** *With notations as before, there are isomorphisms*

$$\mathrm{H}^q(\mathscr{Y}, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\simeq} \mathrm{H}^q(\pi^{-1}(\mathscr{Y}), \mathbb{Z}/n\mathbb{Z}), \quad \forall q \geq 0, n \not| p.$$

Chapter 2

De Jong's Conjecture

In this Chapter, we address de Jong's conjecture in the case of rank 1 log extendable isocrystals, as recalled in Section 1.3. We consider the case of a non-proper variety with trivial tame fundamental group which admits a good compactification. In Section 2.2 we obtain a "residue exact sequence" from which many cohomological results easily follow and in Section 2.3 we look at the (log) crystalline Chern classes of such isocrystals. Finally, we adapt the proof of [ES16, Proposition 3.6] to prove Theorem 2.6.

2.1 Notation

Let U be a smooth connected variety over an algebraically closed field k with the property that $\pi_1^{\text{tame,ab}}(U) = 1$ and such that it admits a good compactification X, where $X - U = \cup_{i \in I} Z_i =: Z$ is a simple strict normal crossings divisor. Denote by W the Witt ring of k and by K the fraction field of W.

We endow X with the log structure associated to the strict normal crossings divisor Z and denote this log scheme by (X, M_Z) or (X, Z): if we denote the open immersion $X \setminus Z \hookrightarrow X$ by j, the inclusion

$$M_Z := \mathcal{O}_X \cap j_* \mathcal{O}^*_{X-Z} \hookrightarrow \mathcal{O}_X \tag{2.1}$$

defines a log structure on X, as already recalled in Example 1.7(2).

Since Z is a normal crossings divisor, we know by [ShI, Example 2.4.4] that M_Z is an fs, hence also fine, log structure, see 1.8, and the morphism $(X, M_Z) \to (\operatorname{Spec} k, \text{trivial log structure})$ is log smooth.

Remark 2.1. From the decomposition of Lemma 1.28, we can deduce the following: if $\pi_1^{\text{tame,ab}}(U) = 1$, then in particular $\pi_1^{\text{ab}}(U)(p') = \pi_1^{\text{ab}}(X)(p) = 1$. Since we also have a surjection $\pi_1^{\text{ab}}(U)(p') \to \pi_1^{\text{ab}}(X)(p')$, we see that $\pi_1^{\text{ab}}(X)(p')$ must be also trivial, which means that $\pi_1^{\text{ab}}(X) = 1$.

2.2 Residue exact sequence

In this section, we obtain some results about the (log) crystalline and rigid cohomology of the compactification X, by adjusting some results of [AB05, Section 6]. We investigate using these, how the value of a log crystal on the variety X controls the crystal itself.

Let $\{U_i \hookrightarrow V_i\}$ be a covering in the log crystalline site. We set $U_{ij} = U_i \cap U_j$ and denote by V_{ij} the log PD envelope of U_{ij} in V_i (or V_j) and do the same for every tuple of indices $(i_0, i_1, \cdots, i_n)$. Define $Q^r_{i_0,\ldots,i_n} := \omega^{r,log}_{V_{i_0,\ldots,i_n}} / \omega^r_{V_{i_0,\ldots,i_n}}$. We then have the following maps between the total complexes of the corresponding Čech complexes, with exact columns:

$$\begin{tikzcd}
0 \arrow[r] & \oplus\mathcal{O}^*_{V_i} \arrow[r] \arrow[d] & \oplus_{i,j}\mathcal{O}^*_{V_{ij}} \oplus_i \omega^1_{V_i} \arrow[r] \arrow[d] & \oplus_{i,j,k}\mathcal{O}^*_{V_{ijk}} \oplus_{i,j} \omega^1_{V_{ij}} \oplus_{i,j} \omega^2_{V_{ij}} \arrow[r] \arrow[d] & \ldots \\
0 \arrow[r] & \oplus\mathcal{O}^*_{V_i} \arrow[r] \arrow[d] & \oplus_{i,j}\mathcal{O}^*_{V_{ij}} \oplus_i \omega^{1,\log}_{V_i} \arrow[r] \arrow[d] & \oplus_{i,j,k}\mathcal{O}^*_{V_{ijk}} \oplus_{i,j} \omega^{1,\log}_{V_{ij}} \oplus_{i,j} \omega^{2,\log}_{V_{ij}} \arrow[r] \arrow[d] & \ldots \\
 & 0 \arrow[r] & \oplus Q^1_i \arrow[r] \arrow[d] & \oplus_{i,j} Q^1_{ij} \oplus_i Q^2_i \arrow[r] \arrow[d] & \ldots \\
 & & 0 & 0 &
\end{tikzcd} \tag{2.2}$$

and note that the third complex is the total complex of:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \oplus Q^1_i \arrow[r] \arrow[d] & \oplus Q^1_{ij} \arrow[r] \arrow[d] & \ldots \\
0 \arrow[r] \arrow[d] & \oplus Q^2_i \arrow[r] \arrow[d] & \oplus Q^2_{ij} \arrow[r] \arrow[d] & \ldots \\
0 \arrow[r] \arrow[d] & \oplus Q^3_i \arrow[r] \arrow[d] & \oplus Q^3_{ij} \arrow[r] \arrow[d] & \ldots \\
0 & \ldots & \ldots & \ldots
\end{tikzcd} \tag{2.3}$$

Taking cohomology, we obtain maps (starting from H^1)

$$\mathrm{H}^1_{\mathrm{crys}}(X/W, \mathcal{O}^*_{X/W}) \to \mathrm{H}^1((X,M)/W, \mathcal{O}^*_{X/W})^{\log}_{\mathrm{crys}} \to \mathrm{H}^1(Tot(Q_{\bullet,\bullet})) \to \mathrm{H}^2(X/W, \mathcal{O}^*_{X/W}) \tag{2.4}$$

and computing the cohomology of the complex (2.3) with spectral sequences, as in

[AB05, Lemma 6.2.1], we obtain the exact sequence

$$\begin{aligned}\mathrm{H}^1_{\mathrm{crys}}(X/W, \mathcal{O}^*_{X/W}) \to \mathrm{H}^1((X,M)/W, \mathcal{O}^*_{X/W})^{\log}_{\mathrm{crys}} \to \\ \to \mathrm{Ker}(\mathrm{H}^0(Q^1_\bullet) \to \mathrm{H}^0(Q^2_\bullet)) \to \mathrm{H}^2(X/W, \mathcal{O}^*_{X/W}).\end{aligned} \tag{2.5}$$

By the computation in [AB05, Proposition 6.3.2] we have that

$$\mathrm{Ker}(\mathrm{H}^0(Q^1_\bullet) \to \mathrm{H}^0(Q^2_\bullet)) = \mathrm{Div}_Z(X) \otimes W(k) = \oplus W(k)[Z_i], \tag{2.6}$$

so we obtain:

$$\mathrm{H}^1_{\mathrm{crys}}(X/W, \mathcal{O}^*_{X/W}) \to \mathrm{H}^1((X,M)/W, \mathcal{O}^*_{X/W})^{\log}_{\mathrm{crys}} \to \oplus W(k)[Z_i] \to \mathrm{H}^2(X/W, \mathcal{O}^*_{X/W}). \tag{2.7}$$

Moreover, in [AB05] we find the exact sequence

$$0 \to \mathrm{H}^1_{\mathrm{crys}}(X/W) \to \mathrm{H}^1((X,M)/W)^{\log}_{\mathrm{crys}} \to \oplus W(k)[Z_i] \to H^2_{\mathrm{crys}}(X/W) \tag{2.8}$$

with the last map being the crystalline Chern class map. By [ES16, Proposition 2.9(2)] and [Esn15, Theorem 4.3.1] we have that $\pi_1^{\mathrm{ab}}(X) = 1$ implies $\mathrm{H}^1_{\mathrm{crys}}(X/W) = 0$. Hence, the group $\mathrm{H}^1(X/W)^{\log}_{\mathrm{crys}}$ coincides in this case with the kernel of the crystalline Chern class map, which by [AB05, Proposition 6.3.2] is equal to $\mathrm{Div}^0_Z(X) \otimes W(k)$ and this is equal to zero in our case, as already remarked in Remark 1.26. We therefore obtain an injection $\oplus W(k)[Z_i] \hookrightarrow H^2_{\mathrm{crys}}(X/W)$.

Since X is proper, we have that $\mathrm{H}^1_{\mathrm{crys}}(X/W, \mathcal{O}^*_X) = 0$, by [ES16, Theorem 0.1.1] and [Esn15, Theorem 4.3.1]. Therefore (2.7) gives us an injection

$$\mathrm{H}^1((X,M)/W, \mathcal{O}^*)^{\log}_{\mathrm{crys}} \hookrightarrow \oplus W(k)[Z_i] \tag{2.9}$$

and combining this with the previous remarks we obtain:

$$\mathrm{H}^1((X,M)/W, \mathcal{O}^*)^{\log}_{\mathrm{crys}} \hookrightarrow \oplus W(k)[Z_i] \hookrightarrow H^2_{\mathrm{crys}}(X/W). \tag{2.10}$$

From the exact sequences above and going through the definition of c_1^{crys} and c_1^{logcrys} (see also Lemma 2.3 for the definition) we observe, as already remarked in [AB05, Proposition 6.3.2], that the following diagram commutes:

$$\begin{tikzcd} \mathrm{H}^1((X,M)/W, \mathcal{O}^*)^{\log}_{\mathrm{crys}} \arrow[r, hook] \arrow[dr] & \oplus W(k)[Z_i] \arrow[r, hook, "c_1^{\mathrm{crys}}"] & \mathrm{H}^2(X/W)_{\mathrm{crys}} \\ & \mathrm{H}^1(X, \mathcal{O}^*_X) \arrow[ur, "c_1^{\mathrm{crys}}"] & \end{tikzcd} . \tag{2.11}$$

From this we conclude that under our assumptions the projection map

$$\mathrm{H}^1((X,M)/W, \mathcal{O}^*)^{\log}_{\mathrm{crys}} \to \mathrm{H}^1(X, \mathcal{O}^*_X), \qquad (L, \nabla) \mapsto L_X \tag{2.12}$$

is in fact injective.

Remark 2.2. We can show that $\mathrm{H}^1_{rig}(U/K) = 0$ under our assumptions. For a proper scheme this is known. Observing (2.8) we conclude as before that $\mathrm{H}^1(X/W)^{\log}_{\mathrm{crys}} = 0$. Hence $\mathrm{H}^1_{rig}(U/K) \simeq \mathrm{H}^1(X/W)^{\log}_{\mathrm{crys}} \otimes \mathbb{Q} = 0$.

2.3 The value of the log extension on the compactification

Starting with a log extendable $\mathscr{L} \in \mathrm{I}_{\mathrm{crys}}(U/K)$, we take a log extension $\mathscr{L}^{\log} \in \mathrm{I}_{\mathrm{logcrys}}(X/K)$ and, by Lemma 1.22, a locally free log lattice $L^{\log}$ of $\mathscr{L}^{\log}$. We denote its restriction to U by L. In the following we denote by $L^{\log}_X \in \mathrm{Coh}(\mathscr{O}_X)$ the value of the log crystal on X.

Lemma 2.3. *With the above notations we have* $c_1^{\mathrm{logcrys}}(L^{\log}_X) = 0$.

Proof. One defines the crystalline Chern class of a locally free sheaf in analogy with [BI70]: let $\mathscr{J}^{\log}_{X/W}$ be the log PD ideal defined by the short exact sequence

$$0 \to \mathscr{J}^{\log}_{X/W} \to \mathscr{O}^{\log}_{X/W} \to i_*\mathscr{O}_X \to 0 \tag{2.13}$$

with $i : (X)_{\mathrm{Zar}} \to (X/W)^{\log}_{\mathrm{crys}}$. Consider also

$$1 \to 1 + \mathscr{J}^{\log}_{X/W} \to \mathscr{O}^{*,\log}_{X/W} \to i_*\mathscr{O}^*_X \to 1. \tag{2.14}$$

The first log crystalline Chern class is defined by taking cohomology of (2.14) and composing with the logarithm map

$$1 + \mathscr{J}^{\log}_{X/W} \to \mathscr{J}^{\log}_{X/W},\ 1 + x \mapsto \log(1 + x). \tag{2.15}$$

So we obtain $\mathrm{c}_1^{\mathrm{logcrys}} : \mathrm{H}^1(X, \mathscr{O}^*_X) \to \mathrm{H}^2(X/W, \mathscr{O}^{\log}_{X/W})^{\log}_{\mathrm{crys}}$.

Taking cohomology of (2.13) we get

$$\begin{aligned} 0 \to \mathrm{H}^0(X/W, \mathscr{J}^{\log}_{X/W})^{\log}_{\mathrm{crys}} \to \mathrm{H}^0(X/W, \mathscr{O}^{\log}_{X/W})^{\log}_{crys} \to \\ \to \mathrm{H}^0(X, \mathscr{O}_X) \to \mathrm{H}^1(X/W, \mathscr{J}^{\log}_{X/W})^{\log}_{\mathrm{crys}} \to \mathrm{H}^1(X/W, \mathscr{O}^{\log}_{X/W})^{\log}_{\mathrm{crys}}. \end{aligned} \tag{2.16}$$

By Remark 2.2 we have however that $\mathrm{H}^1(X/W, \mathscr{O}^{\log}_{X/W})^{\log}_{\mathrm{crys}} = 0$. Moreover, since X is geometrically connected we have that $\mathrm{H}^0(X/W, \mathscr{O}^{\log}_{X/W})^{\log}_{\mathrm{crys}} \simeq W$ which surjects to $\mathrm{H}^0(X, \mathscr{O}_X) \simeq k$. Hence we see that $\mathrm{H}^1(X/W, \mathscr{J}^{\log}_{X/W})^{\log}_{\mathrm{crys}} \simeq \mathrm{H}^1(X/W, 1+\mathscr{J}^{\log}_{X/W})^{\log}_{\mathrm{crys}} = 0$.

Therefore we obtain an exact sequence

$$0 \to \mathrm{H}^1(X/W, \mathscr{O}^{*,\log}_{X/W})^{\log}_{\mathrm{crys}} \to \mathrm{H}^1(X, \mathscr{O}^*_X) \to \mathrm{H}^2(X/W, \mathscr{J}^{\log}_{X/W})^{\log}_{\mathrm{crys}} \to \mathrm{H}^2(X/W, 1+\mathscr{J}^{\log}_{X/W})^{\log}_{\mathrm{crys}}. \tag{2.17}$$

By applying the map

$$\mathrm{H}^2(X/W, 1+\mathscr{I}^{\log}_{X/W})^{\log}_{\mathrm{crys}} \to \mathrm{H}^2(X/W, \mathscr{O}^{\log}_{X/W})^{\log}_{\mathrm{crys}} \tag{2.18}$$

we get

$$0 \to \mathrm{H}^1(X/W, \mathscr{O}^{*\log}_{X/W})^{\log}_{\mathrm{crys}} \to \mathrm{H}^1(X, \mathscr{O}^*_X) \xrightarrow{c_1^{\mathrm{logcrys}}} \mathrm{H}^2(X/W, \mathscr{O}^{\log}_{X/W})^{\log}_{\mathrm{crys}}. \tag{2.19}$$

Therefore, the kernel of the log crystalline Chern map is exactly the group of rank 1 log crystals, hence the value of a rank 1 log crystal on X is sent to zero by c_1^{logcrys}. □

Lemma 2.4. *The line bundle $(L_X^{\log})^{\otimes N}$ for some natural number N can be written as $\mathcal{O}_X(\sum a_i Z_i)$, with $a_i \in \mathbb{Q}$.*

Proof. As remarked in [Kin13, Proposition 3.8], we have an exact sequence:

$$0 \to \mathrm{H}^0(X, \mathcal{O}^*_X) \to \mathrm{H}^0(U, \mathcal{O}^*_U) \to \oplus\mathbb{Z}[Z_i] \to \mathrm{Pic}(X) \to \mathrm{Pic}(U) \to 0 \tag{2.20}$$

As we recalled in Proposition 1.29, we have that $\pi_1^{\mathrm{ab}}(X) = \pi_1^{\mathrm{ab}}(U) = 1$ implies $\mathrm{H}^0(U, \mathcal{O}^*_U) = k^*$ and thus $\mathrm{H}^0(X, \mathcal{O}^*_X)$ is isomorphic to $\mathrm{H}^0(U, \mathcal{O}^*_U)$.

Moreover, by Remark 1.26, this also implies that $\mathrm{Pic}(X) = \mathrm{NS}(X)$ and $\mathrm{Pic}(U) = \mathrm{NS}(U)$, and the exact sequence becomes,

$$0 \to \oplus\mathbb{Z}[Z_i] \to \mathrm{NS}(X) \to \mathrm{NS}(U) \to 0, \tag{2.21}$$

we refer to Remark 1.27 for the Néron-Severi group of U. The log crystalline Chern class can be seen as, where $\mathrm{H}^*(X/W)_{\mathrm{crys}}$ and $\mathrm{H}^*(X/W)^{\log}_{\mathrm{crys}}$ denote the cohomology groups $\mathrm{H}^*(X/W, \mathscr{O}^{\log}_{X/W})_{\mathrm{crys}}$ and $\mathrm{H}^*(X/W, \mathscr{O}^{\log}_{X/W})^{\log}_{\mathrm{crys}}$:

$$c_1^{\mathrm{logcrys}} : \mathrm{H}^1(X, \mathcal{O}^*_X) \to \mathrm{H}^2(X/W)^{\log}_{\mathrm{crys}} \otimes \mathbb{Q} \simeq \mathrm{H}^2_{rig}(U/K). \tag{2.22}$$

We observe that the following diagram commutes:

$$\begin{tikzcd} \mathrm{H}^1(X, \mathcal{O}^*_X) \arrow[r, "c_1^{\mathrm{logcrys}}"] \arrow[d, "c_1^{\mathrm{crys}}"] & \mathrm{H}^2(X/W)^{\log}_{\mathrm{crys}} \\ \mathrm{H}^2(X/W)_{\mathrm{crys}} \arrow[ur, "f^*"] & \end{tikzcd}, \tag{2.23}$$

where we denote by $f^* : \mathrm{H}^2(X/W)_{\mathrm{crys}} \to \mathrm{H}^2(X/W)^{\log}_{\mathrm{crys}}$ the homomorphism which is obtained by the exact map of topoi $f : (X/W_n)^{\log}_{\mathrm{crys}} \to (X/W_n)_{\mathrm{crys}}$ for all n and is defined as $f^*(E)((U, M_U), (T, M_T)) := E(U \subset T)$. As stated in [AB05, Section 6.3] this map is exact and commutes with global sections. Applying this to the sequence

$$1 \to 1 + \mathcal{J}_X \to \mathcal{O}^*_{X/W_n} \to \mathcal{O}^*_X \to 1 \tag{2.24}$$

we get

$$\begin{tikzcd}
1 \arrow[r] & f^*(1+\mathcal{J}_X) \arrow[r] \arrow[d] & f^*(\mathcal{O}^*_{X/W_n}) \arrow[r] \arrow[d] & f^*(\mathcal{O}^*_X) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & 1+\mathcal{J}^{\log}_X \arrow[r] & \mathcal{O}^{*,\log}_{X/W_n} \arrow[r] & \mathcal{O}^*_X \arrow[r] & 1
\end{tikzcd} \tag{2.25}$$

with exact rows and commutative squares. By taking cohomology of the first row and composing with the log map, one obtains $f^*(c_1^{\mathrm{crys}})$ whereas by taking cohomology of the second, one obtains c_1^{logcrys}.

Combining the previous observation with [Pet03, Theorem 5.23] we obtain the following commutative diagrams

$$\begin{tikzcd}
\mathrm{H}^2_{rig}(X/K) \arrow[r] & \mathrm{H}^2(X/W)_{\mathrm{crys}} \otimes \mathbb{Q} \arrow[d] \\
\mathrm{H}^1(X,\mathcal{O}^*) \arrow[u, "c_1^{rig}"] \arrow[ur, "c_1^{\mathrm{crys}}"] \arrow[r, "c_1^{\mathrm{logcrys}}"] & \mathrm{H}^2(X/W)^{\log}_{\mathrm{crys}} \otimes \mathbb{Q}
\end{tikzcd} . \tag{2.26}$$

Moreover, by [Ill79, Remark II 6.8.4], and using the fact that $\pi_1^{\mathrm{ab}}(X) = 1$ implies also that $\mathrm{Pic}(X) = \mathrm{NS}(X)$, we know that $\mathrm{H}^1(X, \mathcal{O}^*_X) \otimes K \hookrightarrow \mathrm{H}^2_{rig}(X/K)$ via the first rigid Chern class.

The line bundle $L_X^{\log}$ seen in $\mathrm{NS}(X) \otimes \mathbb{Q}$ (i.e. $L_X^{\log}$ modulo torsion) is sent to zero by c_1^{logcrys}, by Lemma 2.3.

So we have

$$\begin{tikzcd}
 & & \langle K[Z_i]\rangle_{\mathbb{Q}} \arrow[d, hook] \\
\mathrm{NS}(X) \otimes \mathbb{Q} \arrow[r, hook] & \mathrm{NS}(X) \otimes K \arrow[r, hook, "c_1^{rig}"] \arrow[dr, "j^* c_1^{rig}"'] & \mathrm{H}^2_{rig}(X/K) \arrow[d, "j"] \\
 & & \mathrm{H}^2_{rig}(U/K)
\end{tikzcd}$$

which means that $L_X^{\log}$ is contained in the kernel of the map $\mathrm{H}^2_{rig}(X/K) \to \mathrm{H}^2_{rig}(U/K)$, which is $\langle K[Z_i]\rangle_{\mathbb{Q}}$. Hence, $L_X^{\log} \in \langle K[Z_i]\rangle_{\mathbb{Q}} \cap \mathrm{NS}(X) \otimes \mathbb{Q} = \oplus \mathbb{Q}[Z_i]$.

This in turn means that $L_X^{\log}$ can be written as $\mathcal{O}_X(\sum a_i Z_i) \otimes T$, with T a torsion bundle, and therefore some power of it can be written as a formal sum of Z_i with $\mathbb{Q}$-coefficients. $\square$

Of course we could increase the value of N and obtain integral coefficients in the above decomposition. As we see in the following remark, it is actually enough to raise the bundle $(L_X^{\log})^{\otimes N}$ to a power that is prime to p and obtain the same result.

To simplify the notation set $E^{\log} := (L^{\log})^{\otimes N}$.

Remark 2.5. Recall that under our assumptions Lemma 2.4 implies that the sheaf $E^{\log} := (L^{\log})^{\otimes N}$ for some N lies in $\oplus\mathbb{Q}[Z_i]$, whereas from (2.10) it can also be written as a sum with $W(k)$ coefficients. Hence, $E_X^{\log}$ lies actually in $\oplus\mathbb{Z}_{(p)}[Z_i]$. Therefore there is $m \in \mathbb{Z} \setminus p\mathbb{Z}$ with the property that $E_X^{\otimes m} \in \oplus\mathbb{Z}[Z_i]$ is such that $E_U^{\otimes m} = \mathcal{O}_U$ and $(m,p) = 1$. This in turn defines a Kummer cover of U. However, by assumption we have that $\pi_1^{\mathrm{tame,ab}}(U) = 1$, so m can be chosen as 1 and E_X is in fact in $\oplus\mathbb{Z}[Z_i]$ and has the property that E_U is trivial (see Lemma 2.7 for detailed argument).

On the other hand, from (2.12) we see that the connection $\nabla_X^{\log}$, as well as its restriction to U can be assumed to be trivial.

We therefore have a module with log connection on X restricting to $(\mathscr{O}_U, d_U)$ on U and can take an extension of it that is isomorphic to $(\mathscr{O}_X^{\log}, d_X^{\log})$.

2.4 Triviality of the isocrystal

Theorem 2.6. *Let E a locally free lattice of a rank 1 log isocrystal $\mathcal{E}$ on (X,Z) and such that its value on (X,Z) is trivial. Then E is the trivial log crystal.*

Proof. **Step 1**: We have the following maps of fine log schemes:

$$(X,Z) \to \underline{S} \to \underline{W}(k)$$

where $\underline{S}$ is the log scheme defined by $S = \operatorname{Spec} k$ with the log structure associated to $\mathbb{N} \to k; 1 \mapsto 0$ and $\underline{W}(k)$ is the log scheme defined by $\operatorname{Spec} W(k)$ with the log structure $\mathbb{N} \to W(k); 1 \mapsto 0$.

There is at least one good embedding system for $(X,Z)/(\operatorname{Spec} k, N)$, so we assume from now on that there is an embedding $(X,Z) \hookrightarrow (P,L)$ to a smooth formal scheme over $(\operatorname{Spf} W, N)$.

Denote by D the completed log PD envelope of $(X \hookrightarrow P)$. It is the usual PD envelope equipped with the inverse image of the log structure. Denote by D_n the corresponding PD envelopes for the immersions over $P/\operatorname{Spf} W_n$.

$$\begin{tikzcd} D_n \arrow[dr] & \\ X \arrow[u] \arrow[r] \arrow[d] & P_{W_n} \arrow[dl] \\ W_n & \end{tikzcd} \quad . \tag{2.27}$$

By [Kat89, Theorem 6.2] the category $\mathrm{Crys}(X/W_n)^{\log}$ of log crystals on X/W_n is equivalent to the category of $\mathcal{O}_{\mathcal{D}_n}$-modules with integrable log quasi nilpotent connection.

Let E be a log crystal such that the restriction E_n of E_{n+1} to $\mathrm{Crys}(X/W_n)^{\log}$ is the trivial log crystal. We can define, as in [ES16, Proposition 3.6], $\mathcal{D}$ to be the set of pairs (G,ϕ) such that $G \in \mathrm{Crys}(X/W_{n+1})^{\log}$ with $\phi : \mathcal{O}_X \xrightarrow{\simeq} \overline{G} \in \mathrm{Crys}(X/W_n)^{\log}$. We claim that in this case we have an isomorphism $e : \mathcal{D} \simeq \mathrm{H}^1(X/W_1)^{\log}_{\mathrm{crys}}$. By the equivalence of log crystals and modules with log integrable quasi nilpotent connection, we can identify E_{n+1} with a $\mathcal{O}_{\mathcal{D}_{n+1}}$ module M_{n+1} with a log connection ∇_{n+1}. By assumption, there is an isomorphism

$$\phi : (\mathcal{O}_{D_n}, d) \to (M_n, \nabla_n),$$

(M_n, ∇_n) being the restriction of (M_{n+1}, ∇_{n+1}) to $\mathrm{MIC}^{\log}(D_n)^{qn} \simeq \mathrm{Crys}(X/W_n)^{\log}$.

Take an open affine cover $U = \{U_a\}$ of D_{n+1} and an isomorphism $\psi_a : \mathcal{O}_{U_a} \to M_{n+1}|_{U_a}$ lifting $\phi|_{D_n \times U_a}$. Then $\psi_a^*(\nabla_{n+1})$ defines a connection on $\mathcal{O}_{Ua}$ that is given as $d + p^n s_a$ with $s_a \in \Gamma(U_a, \omega^{1,\log}_{D_1})$. On the intersection $U_a \cap U_b =: U_{ab}$ the glueing $(\psi_a|_{U_{ab}})^{-1} \circ (\psi_b|_{U_{ab}})$ is given by $1 + p^n z_{ab}$ with $z_{ab} \in \Gamma(U_{ab}, \mathcal{O}_{D_1})$.

Because of the integrability of the connection and the compatibility of the connection with the glueing, we see as in [ES16, Proposition 3.6], that $(\{s_a\}, \{z_{ab}\})$ actually defines a 1-cocycle in $\mathrm{Tot}\,\Gamma(U, \omega^{\bullet,\log}_{D_1})$ and this defines a class in the cohomology $\mathrm{H}^1(\mathrm{Tot}\,\Gamma(U, \omega^{\bullet,\log}_{D_1})) = \mathrm{H}^1(X/W_1)^{\log}_{\mathrm{crys}}$.

Step 2: By the previous step we get:

$$\mathrm{Ker}(\mathrm{MIC}^{\log}(D_{n+1}) \to \mathrm{MIC}^{\log}(D_n)) \simeq \mathrm{H}^1(X, \omega^{\bullet,\log}_{D_1}).$$

Denoting by $K_{X/W}$ the convergent isocrystal defined by $T \mapsto K \otimes_W \Gamma(T, \mathcal{O}_T)$, we obtain the following from [ShII, Theorem 2.4.4 and Corollary 2.3.9]: for all $i \in \mathbb{N}$ there is an isomorphism

$$\mathrm{H}^i((X/W)^{\log}_{\mathrm{crys}}, K \otimes \mathcal{O}_{X/W}) \simeq \mathrm{H}^i((X/W)^{\log}_{conv}, K_{X/W}) \tag{2.28}$$

and

$$\mathrm{H}^i((X/W)^{\log}_{conv}, K_{X/W}) \simeq \mathrm{H}^i_{rig}(U/K). \tag{2.29}$$

In [ShII, page 136] Shiho defines for $\mathcal{E} = K \otimes \mathcal{F}$ in $\mathcal{I}_{\mathrm{crys}}(X/W)^{\log}$:

$$\mathrm{H}^1((X/W)^{\log}_{\mathrm{crys}}, \mathcal{E}) = \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{H}^1((X/W)^{\log}_{\mathrm{crys}}, \mathcal{F}).$$

Combining this with Remark 2.2

$$\mathrm{H}^1((X/W)^{\log}_{\mathrm{crys}}, K \otimes \mathcal{O}_{X/W}) = \mathbb{Q} \otimes \mathrm{H}^1(X/W)^{\log}_{\mathrm{crys}}, \mathcal{O}_{X/W}) = \mathrm{H}^1_{rig}(U/K) = 0.$$

From [AB05, Theorem B' and Section 2.3], we know that $\mathrm{H}^1(X/W)^{\log}_{\mathrm{crys}}$ is a free W-module, hence torsion free and therefore $\mathrm{H}^1(X/W)^{\log}_{\mathrm{crys}} = \varprojlim_n \mathrm{H}^1(X/W_n)^{\log}_{\mathrm{crys}} = 0$. Hence there is a natural number N large enough such that for all $n \geq N$, the maps $\mathrm{H}^1(X/W_n)^{\log}_{\mathrm{crys}} \to \mathrm{H}^1(X/W_1)^{\log}_{\mathrm{crys}}$ are zero.

Claim: There is a natural number $d \geq 0$ s.t. $(F^d)^* \mathrm{H}^1(X/W_1)^{\log}_{\mathrm{crys}} = 0$.

From [Mum74, Corollary on page 143] we have the decomposition of $\mathrm{H}^1(X/W_1)^{\log}_{\mathrm{crys}}$ as a direct sum of the part on which Frobenius acts as an isomorphism and the part on which it acts nilpotently. Denote the former by $\mathrm{H}^1((X/W_1)^{\log}_{\mathrm{crys}})_{ss}$. We have the following maps:

$$\begin{tikzcd}
\mathrm{H}^0(X, \Omega_X^{\geq 1}) \arrow[r, hook] \arrow[d, hook] & \mathrm{H}^1_{\mathrm{crys}}(X/k) \arrow[r] \arrow[d] & \mathrm{H}^1(X, \mathcal{O}_X) \arrow[d] \\
\mathrm{H}^0(X, \omega_X^{\geq 1, \log}) \arrow[r, hook] & \mathrm{H}^1(X/W_1)^{\log}_{\mathrm{crys}} \arrow[r] & \mathrm{H}^1(X, \mathcal{O}_X)
\end{tikzcd}.$$

Since F^* acts by zero on the image of $\mathrm{H}^0(X, \Omega^1_X)$ in $\mathrm{H}^1_{\mathrm{crys}}(X/k)$, it acts by zero on the image of $\mathrm{H}^0(X, \omega_X^{1,\log})$ in $\mathrm{H}^1(X/W_1)^{\log}_{\mathrm{crys}}$ and therefore $\mathrm{H}^1((X/W_1)^{\log}_{\mathrm{crys}})_{ss} \subset \mathrm{H}^1(X, \mathcal{O}_X)_{ss} = 0$, the latter equality being true because:

$$\mathrm{H}^1(X, \mathcal{O}_X)_{ss} = \mathrm{H}^1(X, \mathcal{O}_X)^{F=1} \otimes_{\mathbb{F}_p} k = \mathrm{H}^1_{\text{ét}}(X, \mathbb{F}_p) \otimes k = \mathrm{Hom}(\pi_1^{\mathrm{ab}}(X), \mathbb{F}_p) \otimes k = 0.$$

So, there exists some $d \in \mathbb{N}$ s.t. $(F^d)^* \mathrm{H}^1(X/W_1)^{\log}_{\mathrm{crys}} = 0$. □

By the same computations as in Step 1, one can actually have:

$$\mathrm{Ker}\left(\mathrm{MIC}^{\log}(D_{n+m}) \to \mathrm{MIC}^{\log}(D_n)\right) \simeq \mathrm{H}^1(X, \omega^{\bullet,\log}_{D_m}), \text{ for all } 1 \leq m \leq n.$$

Assume now that we have E as in the assumption of the theorem. Then $E_2 \in \mathrm{Ker}\left(\mathrm{MIC}^{\log}(D_2) \to \mathrm{MIC}^{\log}(D_1)\right)$, and this defines a class $e(E_2) \in \mathrm{H}^1(X/W_1)^{\log}_{\mathrm{crys}}$. By the definition of d above, we have $(F^d)^* e(E_2) = e((F^*)^d E)_2 = 0$. Hence, by Step 1, $((F^*)^d E)_2$ is trivial. Repeating this we find $r \in \mathbb{N}$ such that $((F^r)^* E)_N = (F^r)^* E_N =: E'_N$ is trivial.

We also see that the image of E'_{2N} via the restriction $\mathrm{H}^1(X, \omega^{\bullet,\log}_{D_N}) \to \mathrm{H}^1(X, \omega^{\bullet,\log}_{D_1})$, which is equal to E'_{N+1} is trivial.

Continuing like this, we can show that E'_n is trivial for all $n \geq N$, therefore E' is trivial. The functor F^* is fully faithful restricted to locally free crystals, [Ogu94, Example 7.3.4] so we conclude that E is itself trivial.

□

This proves that, with notations as in the previous sections, $E = (L^{\log})^{\otimes N}$ is trivial and therefore that a power of the rank one crystal L is trivial. Using the following two lemmas, we conclude that L itself is trivial as well as the rank 1 isocrystal we started with.

Lemma 2.7. *With the previous assumptions, if $L^{\otimes n} = 1$ with $(n, p) = 1$, then L is trivial.*

Proof. Since $L^{\otimes n}$ is trivial, we have $L_U^{\otimes n} \simeq \mathcal{O}_U$ with $(n,p)=1$, hence

$$\underline{\mathrm{Spec}}\left(\bigoplus_{i=0}^{n-1} L_U^i\right) \to U$$

is a Kummer cover.

Since $\pi_1^{\mathrm{tame,ab}}(U) = 1$ this cover has to be trivial, since this group parametrizes exactly Kummer covers of U, so $n=1$ and $L_U \simeq \mathcal{O}_U$. Then we can choose a locally free extension L_X that is trivial. Applying the previous theorem to this, we get the desired result. □

Again with the previous notation we have the following:

Lemma 2.8. *If $L^{\otimes p} = 1$, L is trivial.*

Proof. If $L^{\otimes p} = 1$ then as coherent sheaves $L_U^{\otimes p} \simeq \mathcal{O}_U$, thus $M_X := (F^*L)_U \simeq \mathcal{O}_U$. We choose a locally free extension M_X that is the trivial log crystal and apply Theorem 2.6 (F^*L is also log extendable if L is). We get that F^*L is trivial. But the functor $F^* : \mathrm{Crys}(U/W) \to \mathrm{Crys}(U/W)$ restricted to locally free crystals is fully faithful [Ogu94, Example 7.3.4]. Hence L itself is trivial. □

Corollary 2.9. *Extensions of the trivial object by itself in $\mathrm{I}_{\mathrm{crys}}(X/W)^{\log}$ are trivial. In particular, log extendable unipotent isocrystals on U are constant.*

Proof. The group of log isocrystals $\mathcal{E}$ on $(X,M)/W$ for which the sequence

$$0 \to \mathcal{O}_{X/W} \otimes \mathbb{Q} \to \mathcal{E} \to \mathcal{O}_{X/W} \otimes \mathbb{Q} \to 0 \tag{2.30}$$

is exact, is precisely $\mathrm{Ext}^1(\mathcal{O}_{X/W} \otimes \mathbb{Q}, \mathcal{O}_{X/W} \otimes \mathbb{Q}) = \mathrm{H}^1((X,M)/W, \mathcal{O}_{X/W} \otimes \mathbb{Q})^{\log}_{\mathrm{crys}}$. However,

$$\mathrm{H}^1((X,M)/W, \mathcal{O}_{X/W} \otimes \mathbb{Q})^{\log}_{\mathrm{crys}} = \mathrm{H}^1((X,M)/W, \mathcal{O}_{X/W})^{\log}_{\mathrm{crys}} \otimes \mathbb{Q} \simeq \mathrm{H}^i_{rig}(U/K) = 0. \tag{2.31}$$

Let now $\mathcal{E}$ be a unipotent isocrystal on U, i.e. an isocrystal that admits a filtration whose associated graded quotients are extensions of the unit isocrystal by itself. If $\mathcal{E}$ is log extendable, then so are its associated graded quotients, which we denote by $\mathcal{E}_i$. They fit into exact sequences

$$0 \to \mathcal{O}_{X/W} \otimes \mathbb{Q} \to \mathcal{E}_i \to \mathcal{O}_{X/W} \otimes \mathbb{Q} \to 0 \tag{2.32}$$

in $\mathrm{I}_{\mathrm{crys}}(X/W)^{\log}$. Then we also have exact sequences in $\mathrm{Crys}(U/W)_{\mathbb{Q}}$

$$0 \to \mathcal{O}_{U/W} \otimes \mathbb{Q} \to \mathcal{E}_i \to \mathcal{O}_{U/W} \otimes \mathbb{Q} \to 0. \tag{2.33}$$

Indeed, (2.32) being exact means that for all log PD thickenings $((V, M_V), (T, M_T))$ in the log crystalline site of X/W, we have

$$0 \to \mathcal{O}_T \otimes \mathbb{Q} \to \mathcal{E}_{i,T} \to \mathcal{O}_T \otimes \mathbb{Q} \to 0. \tag{2.34}$$

The log PD thickenings $((V, M_V), (T, M_T))$ restrict to PD thickenings (V, T) on X and they in turn define PD thickenings $(V \cap U, T)$ on U. Every open of U is an open in X, thus we have the above exact sequence in $\mathrm{Crys}(U/W)_{\mathbb{Q}}$. So $\mathcal{E}_i$ is itself an extension of the trivial isocrystal on U by itself and its extension to X is trivial. Applying the above theorem to this, we obtain that the $\mathcal{E}_i$ are trivial on $\mathrm{Crys}(U/W)_{\mathbb{Q}}$. $\square$

Chapter 3

Moduli space of rank 1 isocrystals

In this Chapter we address the rank 1 case of Deligne's Conjecture, see Section 3.3.3. We start by recalling some facts about the universal extension of Pic^0 of a proper variety in Section 3.2. We then identify the subset defined by isocrystals inside this universal extension in Section 3.3 and compare in Section 3.4 the cohomologies of these spaces in the Berkovich analytic setting. Finally, we discuss why there is a Frobenius action on these cohomology groups, in Section 3.5.

3.1 Notation

Let C_0 be a smooth projective curve over $\mathbb{F}_q$, where $q = p^n$ for some prime number p. We denote by $\overline{\mathbb{F}_q}$ an algebraic closure of $\mathbb{F}_q$, and denote by C the curve defined by

$$\begin{tikzcd} C \arrow[r] \arrow[d] & C_0 \arrow[d] \\ \operatorname{Spec} \overline{\mathbb{F}_q} \arrow[r] & \operatorname{Spec} \mathbb{F}_q \end{tikzcd}$$

Denote by W the Witt ring of $\overline{\mathbb{F}_q}$ and by K its field of fractions, of characteristic 0, $\overline{K}$ an algebraic closure of it and by C_W a smooth lift over $\operatorname{Spec} W$.

$$\begin{tikzcd} C \arrow[r] \arrow[d, "proper{,}smooth"] & C_W \arrow[d] & C_K \arrow[l] \arrow[d] \\ \operatorname{Spec} \overline{\mathbb{F}_q} \arrow[r, hook] & \operatorname{Spec} W & \operatorname{Spec} K. \arrow[l] \end{tikzcd} \tag{3.1}$$

3.2 Moduli of line bundles with integrable connection

Let X be an S scheme, with S a noetherian scheme over W, together with a section $x : S \to X$. In general, we can define a functor

$$\mathrm{Pic}^{\#}_{X/S} : (S - \mathrm{Sch}) \to (\mathrm{Groups}) \tag{3.2}$$

which associates to an S-scheme T, the group of isomorphism classes of line bundles $\mathscr{L}$ on $T \times_S X$ endowed with an integrable connection $\nabla : \mathscr{L} \to \mathscr{L} \otimes \mathrm{pr}_X^* \Omega^1_{X/S}$, such that $\mathscr{L}_{|x \times S} \simeq \mathscr{O}_S$, where $\mathrm{pr}_X : X \times_S T \to X$ is the projection. We denote by $\mathrm{Pic}^{\#}(X/S)$ the group of isomorphism classes of such pairs on X/S. As remarked in [Mes73, (2.5.3)], there is an identification

$$\mathrm{Pic}^{\#}(X/S) = \mathrm{H}^1(X, \mathscr{O}_X \xrightarrow{\mathrm{dlog}} \Omega^1_{X/S} \to \Omega^2_{X/S} \to \cdots)$$

and then if $X \times_S T =: X_T$ for an S- scheme T admits a section over T, we also have [Mes73, (2.6.4)]

$$\mathrm{Pic}^{\#}_{X/S}(T) = \mathrm{Coker}(\mathrm{Pic}(T) \xrightarrow{f^*} \mathrm{Pic}^{\#}(X \times_S T/T)). \tag{3.3}$$

In characteristic zero, this functor can be immediately seen to be representable by a quasi-projective scheme: in [Bos13, Sections 2.3.3, 2.3.5] it is actually remarked that for bundles of any rank r, this functor is the same as the representation functor $R_{\mathrm{DR}}(X, x, r)$, which is defined in [Sim94b, p. 55 and Theorem 6.13]. There, it is also proven that this functor is representable.

We consider here the subfunctor of degree 0 line bundles as above, with integrable connection on the curve X/S, and denote it by $\mathrm{Pic}^{\nabla}_{X/S}$. In this case we have that this functor is representable also in positive characteristic, since it is the universal vector extension of $\mathrm{Pic}^0_{X/S}$, as was first proven in [Mes73, Proposition 2.8.1] and [MM74, Theorem 2.6 and 3.2.3]. In more detail:

Definition 3.1. The universal vector extension of an abelian scheme A over S is an abelian scheme E over S sitting in an exact sequence of fppf sheaves

$$0 \to V \to E \to A \to 0, \tag{3.4}$$

where V is a vector group over S, with the following universal property: for any other abelian scheme E' and vector group V' sitting in an exact sequence of fppf sheaves of abelian groups

$$0 \to V' \to E' \to A \to 0 \tag{3.5}$$

there is an $\mathscr{O}_S$-linear morphism of abelian groups $\psi : V \to V'$ such that (3.5) is isomorphic to the pushout of (3.4) along ψ.

Remark 3.2. [MM74, (2.6)] Assume we have an extension

$$0 \to \mathbb{G}_m \to E \to A \to 0$$

of an abelian variety A over a scheme S by $\mathbb{G}_m$. By [MM74, (2.2.1)] there is an exact sequence:

$$0 \to \omega_A \to \mathbb{E}_A \to \underline{\mathrm{Ext}^1}(A, \mathbb{G}_m) \to 0, \tag{3.6}$$

where ω_A denotes as in [MM74] and [Mes73] the module of invariant differentials of A. In [MM74, (2.6)] it is proven however that, since $\mathrm{Ext}^1(A, \mathbb{G}_m)$ is isomorphic to the dual abelian variety A^*, $\mathbb{E}_A$ is representable by a smooth S -group scheme and (3.6) gives the universal extension of A^*.

In the case of a relative curve $C_W/\operatorname{Spec} W$, which admits a section, the same is true: as remarked in [Kat14, Section 3] we have an exact sequence

$$0 \to \mathrm{H}^0(C, \Omega^1_{C_W/W}) \to \mathrm{Pic}^\nabla(C_W) \to \mathrm{Pic}^0(C_W) \to 0. \tag{3.7}$$

Indeed, every line bundle $\mathscr{L}$ on C_W over $\operatorname{Spec} W$ which is fiber by fiber of degree 0 admits an S- linear connection, [MM74, p. 46]. Moreover, if we have two connections $(\mathscr{L}, \nabla_1)$ and $(\mathscr{L}, \nabla_2)$, the difference of ∇_1 and ∇_2 is an element $\omega \in \mathrm{H}^0(C, \Omega^1_{C_W/W})$.

In this case however, the dual abelian scheme A^* of the above Remark is isomorphic to the Jacobian $\mathrm{Pic}^0(C_W)$ and we get, by [MM74, Theorem 3.2.3] that (3.7) is the universal extension of $\mathrm{Pic}^0(C_W)$. All details about this case are presented in [BK09, Appendix] and recalled in [Lau96, Section 2].

We can see the same in a concrete example in the classical case:

Example 3.3. [[Mes73, (3.0)]] If X is a non-singular, connected curve over $\mathbb{C}$, we denote by J its Jacobian and choose a canonical (Abel-Jacobi) map $X \to J$. This map induces a map between the corresponding exact sequences on X and J, obtained by [Mes73, (2.6.4)]:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^0(\Omega^1_X) & \longrightarrow & \mathrm{Pic}^\#(X) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & \mathrm{H}^1(\Omega^1_X) \\ & & \simeq\uparrow & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathrm{H}^0(\Omega^1_J) & \longrightarrow & \mathrm{Pic}^\#(J) & \longrightarrow & \mathrm{Pic}(J) & \longrightarrow & \mathrm{H}^2(\tau(\Omega^\bullet_J)) \end{array} \tag{3.8}$$

where $\tau(\Omega^\bullet_J)$ is the complex $\Omega^1_{X/S} \to \Omega^2_{X/S} \to \cdots$ with $\Omega^1_{X/S}$ is in degree 1. Note that we use here that the global 1-forms on X and J are all closed, since X is a curve and J is smooth and projective. An element in the image of $\mathrm{Pic}^\# \to \mathrm{Pic}$ has to be a torsion element in $H^2(X, \mathbb{Z})$ and $\mathrm{H}^2(J, \mathbb{Z})$ respectively. Hence, since $\mathrm{H}^2(X, \mathbb{Z})$ and $\mathrm{H}^2(J, \mathbb{Z})$ are torsion free, its Chern class is zero and therefore element has to be inside Pic^0. Moreover, we have an isomorphism $\mathrm{Pic}^0(J) \xrightarrow{\simeq} \mathrm{Pic}^0(X)$. Hence, $\mathrm{Pic}^\#(X) \xrightarrow{\simeq} \mathrm{Pic}^\#(J)$, which is the universal extension of $\mathrm{Pic}^0(X)$, by the abelian variety case, see [Mes73, Proposition 2.9.2].

3.3 The subset of isocrystals.

As the previous section shows, we get a fine moduli space of line bundles with integrable connection, as described above, represented by a W-group scheme $\mathrm{Pic}^{\nabla}(C_W/\operatorname{Spec} W)$, which we denote by $\mathrm{Pic}^{\nabla}(C_W)$ for simplicity. We then have $\mathrm{Pic}^{\nabla}(C_W) \otimes K \simeq \mathrm{Pic}^{\nabla}(C_K)$ and $\mathrm{Pic}^{\nabla}(C_W) \otimes \overline{\mathbb{F}_q} \simeq \mathrm{Pic}^{\nabla}(C)$ for its generic and closed fibers, as well as $\mathrm{Pic}^{\nabla}(C_W) \otimes \overline{K} \simeq \mathrm{Pic}^{\nabla}(C_{\overline{K}})$.

3.3.1 Hitchin map

In the following let X be a smooth scheme over S, which is a scheme over a field of characteristic $p > 0$. We denote the absolute Frobenius of S by $F_S : S \to S$ (i.e. the p-th power mapping on $\mathcal{O}_S$) and by $F_{X/S} : X \to X^{(p)}$ the relative Frobenius, which is defined by the cartesian diagram:

$$\begin{tikzcd} X \arrow[r, "F_{X/S}"] \arrow[dr] & X^{(p)} \arrow[r] \arrow[d] & X \arrow[d] \\ & S \arrow[r, "F_S"] & S. \end{tikzcd}$$

We denote by $\mathrm{MIC}(X/S)$ the abelian category of $\mathcal{O}_X$-modules with integrable connection on X/S. Given an element $(\mathscr{E}, \nabla)$ of $\mathrm{MIC}(X/S)$, we denote by $\mathscr{E}^{\nabla}$ the kernel of ∇.

For an element $(\mathscr{E}, \nabla)$ of $\mathrm{MIC}(X/S)$, Katz defines in [Kat70, Section 5] its p-curvature by

$$\mathrm{Der}(X/S) \to \mathrm{End}_S(\mathscr{E}); \;\; D \mapsto (\nabla(D))^p - \nabla(D^p). \tag{3.9}$$

The morphism $\psi(\nabla)$ is p-linear [Kat70, Proposition 5.2]: it is additive and $\psi(\nabla)(fD) = f^p\psi(\nabla)(D)$, for f and D local sections of $\mathscr{O}_X$ and $\mathrm{Der}(X/S)$ over an open subset of S. Using p-linearity, we can consider the p-curvature as a global section in $\mathrm{H}^0(X, \mathrm{End}(\mathscr{E}) \otimes F^*_{X/S}\Omega^1_{X^{(p)}/S})$ and in fact, $\psi(\nabla)$ lies in the kernel of the connection on $\mathrm{End}(\mathscr{E}) \otimes F^*_{X/S}\Omega^1_{X^{(p)}/S}$, which is induced by the canonical connection on $F^*_{X/S}\Omega^1_{X^{(p)}/S}$ and the connection ∇^{End}, which is induced by ∇.

Cartier's theorem provides a characterization of connections that have zero p-curvature.

Theorem 3.4 (Cartier, Theorem 5.1 in [Kat70])**.** *With notations as before, there is an equivalence of categories between the category of quasi-coherent sheaves on $X^{(p)}$ and the full subcategory of $\mathrm{MIC}(X/S)$ consisting of objects $(\mathscr{E}, \nabla)$ whose p-curvature is equal to zero.*

More explicitly, the equivalence is given in the following way: given a quasi-coherent sheaf $\mathscr{F}$ on $X^{(p)}$, there is a unique integrable S-connection ∇_{can} on $F^*_{X/S}(\mathscr{F})$, which has p-curvature zero and is such that $\mathscr{F} \simeq (F^*_{X/S}(\mathscr{F}))^{\nabla_{\mathrm{can}}}$.

Conversely, given $(\mathscr{E}, \nabla) \in \mathrm{MIC}(X/S)$ *of zero p-curvature,* $\mathscr{E}^{\nabla}$ *is a quasi-coherent sheaf on* $X^{(p)}$.

We now focus in particular on the case $S = \operatorname{Spec}\overline{\mathbb{F}_q}$. We denote by $\psi(\nabla)$ the corresponding section in $\mathrm{H}^0(X, \mathrm{End}(\mathscr{E}) \otimes F^*_{X/S}\Omega^1_{X^{(p)}/\overline{\mathbb{F}_q}})$ and note that by [Kat70, Proposition 5.2.3], the p-curvature of any connection (E, ∇) is flat under the tensor product of the canonical connection on $F^*_{X/S}\Omega^1_{X^{(p)}}$ and the one of $\mathrm{End}(\nabla)$ on $\mathrm{End}(E)$.

The Hitchin map, [Hit87] and [Lan14, p.12], is then defined by mapping an integrable connection to the characteristic polynomial of its p-curvature, which by the above discussion will have coefficients in the global symmetric forms on $X^{(p)}$:

$$\chi(\psi(\nabla)) := \det(-\psi(\nabla) + \mu\,\mathrm{Id}) = \mu^r - a_1\mu^{r-1} + \cdots + (-1)^r a_r$$

with $a_i \in \mathrm{H}^0(X^{(p)}, \mathrm{Sym}^i(\Omega^1_{X^{(p)}})))$ and $r = \operatorname{rank}\mathscr{E}$.

In the case of a line bundle with connection on the curve C, we get a map, see for example [LP01, Proposition 3.2],[BB07, Section 4], [Gro16, Definitions 3.12 and 3.16]:

$$\chi(\psi(\nabla)) : \mathrm{Pic}^{\nabla}(C) \to \mathscr{A}^1 := \mathrm{H}^0(C^{(p)}, \Omega^1_{C^{(p)}}),$$

which is is known to be proper, [Lan14, Theorem 3.8].

3.3.2 Isocrystals

On the other hand, we say that a point $[(\mathscr{L}, \nabla)]$ of $\mathrm{Pic}^{\nabla}(C_W)$ represents an isocrystal on C if the associated pair $(\mathscr{L}_{\overline{\mathbb{F}_q}}, \nabla_{\overline{\mathbb{F}_q}})$ has nilpotent p-curvature. This condition is equivalent to requiring that the characteristic polynomial of the p-curvature is zero or that $\chi([(\mathscr{L}_{\overline{\mathbb{F}_q}}, \nabla_{\overline{\mathbb{F}_q}})]) = 0$, by definition of the Hitchin map. Equivalently $(\mathscr{L}_{\overline{\mathbb{F}_q}}, \nabla_{\overline{\mathbb{F}_q}})$ is in the fiber above zero of the Hitchin map, which we denote by $\mathrm{Pic}^{\nabla}(C)^{\psi=0}$. By the Cartier isomorphism, [Kat70, Theorem 5.1], we have that this fiber is isomorphic to $\mathrm{Pic}^0(C^{(p)})$.

Since we are working on the perfect field $\overline{\mathbb{F}_q}$, we have that the Frobenius on it is an isomorphism. Then also $\mathrm{Pic}^0(C^{(p)})$ and $\mathrm{Pic}^0(C)$ are isomorphic as schemes. If we worked over $\mathbb{F}_p$, they would even be isomorphic as $\mathbb{F}_p$-schemes.

3.3.3 A conjecture of Deligne

In [Del15, Section 2.17] Deligne considers the following situation: let M_K, and respectively $M_{\overline{K}}$ denote the moduli space of vector bundles of rank r on a smooth curve C_K, respectively $C_{\overline{K}}$, endowed with an integrable connection. The space $M_{\overline{K}}$ is obtained by M_K by extension of scalars. Denote as well by E_r the set of isomorphism classes of $\overline{\mathbb{Q}}_\ell$ - irreducible lisse sheaves of rank r on C. For a fixed embedding $\iota : \overline{K} \to \mathbb{C}$, denote by Σ the Riemann surface obtained by $C_{\overline{K}}$, by extending the scalars to $\mathbb{C}$. The space $M_{\mathbb{C}}$ corresponding to E_r is induced by $M_{\overline{K}}$ by extension of scalars and the ℓ'-adic cohomology, for $\ell' \neq p$ a prime number, of $M_{\overline{K}}$ is isomorphic to the cohomology of

$M_{\mathbb{C}}$ with $\mathbb{Q}_{\ell'}$ coefficients. (Note that all cohomology groups below will denote ℓ'-adic cohomology groups, denoted by $\mathrm{H}^*(-)$.)

Conjecture ([Del15, Conjecture 2.18]). *The cohomology of $M_{\overline{K}}$ admits an endomorphism V^*, such that for all $n \geq 1$, the number N_n of fixed points of V^* on E_r is given by*

$$N_n = \sum_i (-1)^i \operatorname{Tr}(V^{*n}, \mathrm{H}^i(M_{\overline{K}})).$$

In general, we do not have a Frobenius action on the moduli space of vector bundles with integrable connection. Indeed, Deligne expects that there should exist an open subspace $M^0_{\overline{K}}$ inside the Berkovich analytification of $M_{\overline{K}}$, which corresponds to the sublocus of isocrystals and such that it has the following two properties:

(1) the restriction morphism $\mathrm{H}^*(M_{\overline{K}}) = \mathrm{H}^*(M^{\mathrm{an}}_{\overline{K}}) \to \mathrm{H}^*(M^0_{\overline{K}})$ is an isomorphism and

(2) a crystalline interpretation of M^0 allows us to define $V = \mathrm{Frob}^* : M^0_{\overline{K}} \to M^0_{\overline{K}}$, which induces V^* on cohomology.

Then the number of fixed points of the action of V^n would be given by

$$N_n := \sum (-1)^n \operatorname{Tr}(V^{*,n}, \mathrm{H}^i(M_{\overline{K}})). \tag{3.10}$$

The conjectured morphism $V : M^0_{\overline{K}} \to M^0_{\overline{K}}$ should send $M^0_{\overline{K}}$ into a proper open subset of $M^0_{\overline{K}}$, and one can take ordinary cohomology, instead of cohomology with compact support in the Lefschetz Trace formula.

Deligne provided in [Del15, Example 2.19] an example for this conjecture in the rank 1 case, without explaining why the properties (1) and (2) from above are fulfilled.

Example 3.5 ([Del15, Example 2.19 and Proposition 2.20]). Denote the absolute Frobenius of C_0 by F_{C_0} and that of $\mathrm{Pic}^0(C_0)$ by $F_{\mathrm{Pic}^0(C_0)}$. Pullback by the absolute Frobenius of C_0 defines $F^*_{C_0}$, an endomorphism of $\mathrm{Pic}^0(C_0)$ for which [Del15, (2.2.2)]:

$$F^*_{C_0} \circ F_{\mathrm{Pic}^0(C_0)} = F_{\mathrm{Pic}^0(C_0)} \circ F^*_{C_0} = q. \tag{3.11}$$

Because of this we denote $F^*_{C_0}$ by V (Verschiebung).

Extending scalars to the algebraic closure $\overline{\mathbb{F}_q}$, we obtain the corresponding endomorphisms F_C on C: F_C maps a point of C with affine coordinates $(x_1, \dots, x_n)$ to the point with coordinates $(x_1^q, \dots, x_n^q)$. We call this the *Frobenius endomorphism* of C. The fixed points of the action of this map are exactly the $\mathbb{F}_q$ points of C, and are computed by the Lefschetz Trace formula, given below in (3.17).

The morphism F_C induces an endomorphism on cohomology:

$$\mathrm{H}^i(C, \mathbb{Q}_\ell) \to \mathrm{H}^i(C, \mathbb{Q}_\ell)$$

In the rank 1 case, the moduli space $M_{\overline{K}}$ is the same as $\mathrm{Pic}^{\nabla}(C_{\overline{K}})$, which, as already discussed in Section 3.2, is the universal extension of $\mathrm{Pic}^0(C_{\overline{K}})$. By homotopy invariance, it is true that

$$\mathrm{H}^*(\mathrm{Pic}^0(C_{\overline{K}})) \xrightarrow{\simeq} \mathrm{H}^*(\mathrm{Pic}^{\nabla}(C_{\overline{K}})) \tag{3.12}$$

while we also have a natural isomorphism by smooth base change

$$\mathrm{H}^*(\mathrm{Pic}^0(C)) \xrightarrow{\simeq} \mathrm{H}^*(\mathrm{Pic}^0(C_{\overline{K}})). \tag{3.13}$$

On $\overline{\mathbb{F}}_q$, we have an endomorphism $V : \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(C)$ induced by functoriality from the Frobenius endomorphism of C and a pullback on cohomology:

$$V^* : \mathrm{H}^*(\mathrm{Pic}^0(C)) \longrightarrow \mathrm{H}^*(\mathrm{Pic}^0(C)) \tag{3.14}$$

and because of (3.12) and (3.13) this defines

$$V^* : \mathrm{H}^* \mathrm{Pic}^{\nabla}(C_{\overline{K}}) \longrightarrow \mathrm{H}^* \mathrm{Pic}^{\nabla}(C_{\overline{K}}). \tag{3.15}$$

In [Del15, Proposition 2.20], Deligne further computes the fixed points of this action for $n = 1$ in (3.10). Note that one can reduce to this case, taking an extension of scalars from $\mathbb{F}_q$ to $\mathbb{F}_{q^n}$. As also explained in [Del15, (6.1)], the number of fixed points by the Frobenius correspond to the points fixed by V on E_1, and as we see later by our crystalline interpretation, this number should be the number of F-isocrystals defined over $\mathbb{F}_q$.

The Frobenius endomorphism $F_{\mathrm{Pic}^0(C)}$ and V on $\mathrm{Pic}^0(C)$ are transpose to each other, since $\mathrm{Pic}^0(C)$ is auto-dual, and we therefore obtain, [Del15, Proposition 2.20]:

$$\mathrm{Tr}(V^*, \mathrm{H}^i(\mathrm{Pic}^0(C))) = \mathrm{Tr}(F^*_{\mathrm{Pic}^0(C)}, \mathrm{H}^i(\mathrm{Pic}^0(C))) \tag{3.16}$$

and

$$\sum (-1)^i \,\mathrm{Tr}(V^*, \mathrm{H}^i(\mathrm{Pic}^0(C))) = |\,\mathrm{Pic}^0(C_0)(\mathbb{F}_q)|. \tag{3.17}$$

Our goal in this chapter is mainly to explain Deligne's example and provide a crystalline interpretation of it. We revisit this at the end of this chapter, in Example 3.10. In the spirit of the conjecture, we are considering the Berkovich analytification of $\mathrm{Pic}^{\nabla}(C_W)_K$ and its base change to $\overline{K}$, $\mathrm{Pic}^{\nabla}(C_W)_{\overline{K}}$. As recalled in Proposition 1.41, we have the anti-continuous reduction map

$$\pi : (\widehat{\mathrm{Pic}^{\nabla}(C_W)})_K \to \mathrm{Pic}^{\nabla}(C) \tag{3.18}$$

where $(\widehat{\mathrm{Pic}^{\nabla}(C_W)})_K \hookrightarrow \mathrm{Pic}^{\nabla}(C)^{\mathrm{an}}_K$, which would be an isomorphism if $\mathrm{Pic}^{\nabla}(C)_K$ was proper, see Remark 1.43. We denote the inverse image of $\mathrm{Pic}^{\nabla}(C)^{\psi=0}$, which is also called the *tube* of $\mathrm{Pic}^{\nabla}(C)^{\psi=0}$ inside the analytification, by $]\,\mathrm{Pic}^{\nabla}(C)^{\psi=0}[$. It is an open subset of $\mathrm{Pic}^{\nabla}(C)^{\mathrm{an}}_K$ and is actually isomorphic to

$$(\widehat{\mathrm{Pic}^{\nabla}(C_W)}_{\mathrm{Pic}^{\nabla}(C)^{\psi=0}})_K,$$

the generic fiber of the completion of $\mathrm{Pic}^{\nabla}(C_W)$ along the closed subset $\mathrm{Pic}^{\nabla}(C)^{\psi=0}$, see 1.42.

3.4 Comparison of cohomology groups.

In order to verify the aforementioned conjecture of Deligne, we have to compare the cohomology groups of $M^0_{\overline{K}}$ and $M_{\overline{K}}$ of Conjecture 3.3.3, which in our case are $]\mathrm{Pic}^{\nabla}(C)^{\psi=0}[_{\overline{K}}$ and $\mathrm{Pic}^{\nabla}(C)^{\mathrm{an}}_{\overline{K}}$. As before, cohomology here means ℓ-adic cohomology.

We can use the results recalled in Section 1.6 in order to prove the following comparison:

Proposition 3.6. *There is an isomorphism of cohomology groups*

$$\mathrm{H}^*(\mathrm{Pic}^{\nabla}(C_{\overline{K}})^{\mathrm{an}}, \mathbb{Q}_\ell) \simeq \mathrm{H}^*(]\mathrm{Pic}^{\nabla}(C)^{\psi=0}[_{\overline{K}}, \mathbb{Q}_\ell)$$

Proof. By Corollary 1.52 we have

$$\mathrm{H}^*(]\mathrm{Pic}^{\nabla}(C)^{\psi=0}[, \mathbb{Q}_\ell) \simeq \mathrm{H}^*(\mathrm{Pic}^{\nabla}(C)^{\psi=0}, \mathbb{Q}_\ell) \tag{3.19}$$

but by the smooth base change theorem, see [Mil08, Theorem 20.1]

$$\mathrm{H}^*(]\mathrm{Pic}^{\nabla}(C)^{\psi=0}[, \mathbb{Q}_\ell) \simeq \mathrm{H}^*(]\mathrm{Pic}^{\nabla}(C)^{\psi=0}[_{\overline{K}}, \mathbb{Q}_\ell) \tag{3.20}$$

and as explained in Section 3.3,

$$\mathrm{H}^*(\mathrm{Pic}^{\nabla}(C)^{\psi=0}, \mathbb{Q}_\ell) \simeq \mathrm{H}^*(\mathrm{Pic}^0(C^{(p)}), \mathbb{Q}_\ell) \simeq \mathrm{H}^*(\mathrm{Pic}^0(C), \mathbb{Q}_\ell). \tag{3.21}$$

By Theorem 20.5 of Milne [Mil08] we have that, since $\mathrm{Pic}^0(C_W)$ is proper over $\mathrm{Spec}\, W$, it is also true that

$$\mathrm{H}^*(\mathrm{Pic}^0(C), \mathbb{Q}_\ell) \simeq \mathrm{H}^*(\mathrm{Pic}^0(C)_{\overline{K}}, \mathbb{Q}_\ell). \tag{3.22}$$

However, since $\mathrm{Pic}^{\nabla}(C)_{\overline{K}}$ is the universal extension of $\mathrm{Pic}^0(C)_{\overline{K}}$, by Section 3.2, we have

$$\mathrm{H}^*(\mathrm{Pic}^0(C)_{\overline{K}}, \mathbb{Q}_\ell) \simeq \mathrm{H}^*(\mathrm{Pic}^{\nabla}(C_{\overline{K}}), \mathbb{Q}_\ell) \tag{3.23}$$

Finally, the latter cohomology group is isomorphic to $\mathrm{H}^*(\mathrm{Pic}^{\nabla}(C_{\overline{K}})^{\mathrm{an}}, \mathbb{Q}_\ell)$, by [Ber93, Corollary 7.5.4]. $\square$

3.5 Frobenius action

The goal of this section is to show that there is a Frobenius action on $]\operatorname{Pic}^\nabla(C)^{\psi=0}[$, which induces a Frobenius pullback morphism on cohomology. In general, if we have an S-scheme M which represents a functor G, then to define a morphism $M \to M$, it is enough to define such a morphism $G(S') \to G(S')$ for all S-schemes S'.

With this in mind, we obtain in this section a moduli interpretation of $]\operatorname{Pic}^\nabla(C)^{\psi=0}[$ and this yields, because of its nature, a natural Frobenius action.

3.5.1 Functorial interpretation

We are interested in the functorial interpretation of the subset of isocrystals inside the analytic space $\operatorname{Pic}^\nabla(C)^{\mathrm{an}}_K$. We consider here the rigid analytification. Indeed, the Berkovich and rigid analytifications of an algebraic variety are compatible constructions, as explained in Remark 1.37.

From now on, $(-)^{\mathrm{an}}$ denotes the rigid analytification functor and $(-)_K$ denotes the rigid analytic generic fiber.

By Remark 1.35, we can define the functor $\mathbf{Pic}^\nabla_{C^{\mathrm{an}}_K/K,x^{\mathrm{an}}}$ which is represented by the analytification of $\operatorname{Pic}^\nabla(C_K)$. Our goal is therefore to characterize $]\operatorname{Pic}^\nabla(C)^{\psi=0}[$ as a subfunctor of $\mathbf{Pic}^\nabla_{C^{\mathrm{an}}_K/K}$. For this it is enough to describe the set

$$\operatorname{Hom}(S,]\operatorname{Pic}^\nabla(C)^{\psi=0}[)$$

for a rigid analytic space S. This will be the set of line bundles with connection on $S \times_K C^{\mathrm{an}}_K \to S$, as before, with some extra property. Hence, it is enough to assume S is an affinoid and equal to $\operatorname{Sp}(A) \otimes K = \operatorname{Sp} A_K$, for some W-algebra A.

A morphism of rigid analytic spaces

$$S = \operatorname{Sp} A_K \xrightarrow{\phi_K}]\operatorname{Pic}^\nabla(C)^{\psi=0}[= (\widehat{\operatorname{Pic}^\nabla(C_W)}_{\operatorname{Pic}^\nabla(C)^{\psi=0}})_K \tag{3.24}$$

extends as recalled in Remark 1.47 to a morphism of formal schemes

$$\mathscr{S}' \xrightarrow{\phi'} \widehat{\operatorname{Pic}^\nabla(C_W)}_{\operatorname{Pic}^\nabla(C)^{\psi=0}} \tag{3.25}$$

where $\mathscr{S}' \to \mathscr{S} = \operatorname{Spf}(A)$ is an admissible formal blow-up. For $C_W \to \operatorname{Spf} W$ the formal lift of C, having a morphism as in (3.25) means that the morphism $\mathscr{S}' \to \operatorname{Pic}^\nabla(C_W)$ factors as

$$\begin{tikzcd}
\mathscr{S}' \arrow[rr] \arrow[dr] & & \operatorname{Pic}^\nabla(C_W) \\
& \widehat{\operatorname{Pic}^\nabla(C_W)}_{\operatorname{Pic}^\nabla(C)^{\psi=0}} \arrow[ur] &
\end{tikzcd} \tag{3.26}$$

which in turns yields

$$
\begin{tikzcd}
S'_{\overline{\mathbb{F}_q}} \arrow[rr] \arrow[dr] & & \mathrm{Pic}^{\nabla}(C) \\
& \mathrm{Pic}^{\nabla}(C)^{\psi=0} \arrow[ur] &
\end{tikzcd}
\tag{3.27}
$$

This is equivalent to saying that the pair $(\mathscr{L}, \nabla)$ defined by (3.25) is actually an isocrystal on $\mathscr{S}' \times C_W \to \mathscr{S}'$.

Remark 3.7. This construction is independent of the choice of the admissible blow-up: suppose there is another admissible blow-up $\mathscr{S}''$ of $\mathscr{S}$ such that the morphism (3.24) lifts to a morphism of formal schemes

$$\mathscr{S}'' \xrightarrow{\phi''} \widehat{\mathrm{Pic}^{\nabla}(C_W)}_{\mathrm{Pic}^{\nabla}(C)^{\psi=0}}. \tag{3.28}$$

Then as in Remark 1.47 we have that ϕ' and ϕ'' coincide, since the rigid analytic generic fibers $\mathscr{S}'_K$ and $\mathscr{S}''_K$ coincide; they are isomorphic to S.

On the other hand, if we start with a pair of a line bundle with connection on $S \times C_K^{\mathrm{an}} \to S$ for an affinoid S, which is an isocrystal on $\mathscr{S}' \times C_W \to \mathscr{S}'$, for $\mathscr{S}' \to \mathscr{S}$ an admissible blow-up of $\mathscr{S}$, this means by definition that we have a morphism as in (3.26). Taking the associated map between the rigid generic fibers, we obtain a morphism

$$S' \to]\,\mathrm{Pic}^{\nabla}(C)^{\psi=0}[.$$

By [BL93, (a) in p.307], recalled in Remark 1.47, there is an isomorphism between the rigid generic fiber of $\mathscr{S}$ and $\mathscr{S}'$, which means that this morphism is an element of $\mathrm{Hom}(S,]\,\mathrm{Pic}^{\nabla}(C)^{\psi=0}[)$. Note that the category of isocrystals on a formal lift of a scheme depends only on its reduction, because of the nilpotence condition.

By definition of taking the extension of a morphism between rigid spaces to a morphism of the associated formal models and that of taking the rigid generic fiber of a morphism of formal spaces, these two constructions are inverse to each other.

Proposition 3.8. *We obtain therefore an isomorphism between the set*

$$\mathrm{Hom}(S,]\,\mathrm{Pic}^{\nabla}(C)^{\psi=0}[) \tag{3.29}$$

and the set

$$\{(\mathscr{L}, \nabla) \text{ line bundles with integrable connection on } S \times C_K^{\mathrm{an}} \text{ which are isocrystals on } \mathscr{S}' \times C_W \to \mathscr{S}', \text{ for } S' \text{ an admissible formal blow-up of } S\} \tag{3.30}$$

We denote the latter by $]\,\mathrm{Pic}^{\nabla}(C)^{\psi=0}[(S)$.

Remark 3.9. Note also that this isomorphism is functorial: if $S \to T$ is a morphism of affinoid spaces, we have a map $]\operatorname{Pic}^{\nabla}(C)^{\psi=0}[(S) \to]\operatorname{Pic}^{\nabla}(C)^{\psi=0}[(T)$. ([Ber96c] : the category of overconvergent isocrystals on (X, S) is functorial on (X, S)).

3.5.2 Frobenius

In order to define a Frobenius pull-back morphism on $]\operatorname{Pic}^{\nabla}(C)^{\psi=0}[$, it is enough to define it on $]\operatorname{Pic}^{\nabla}(C)^{\psi=0}[(S)$. Because of the previous discussion, given a $(\mathscr{L}, \nabla) =: L$ in (3.30), and choosing a lift ϕ of the relative Frobenius of $S' \times C \to S'$ to $\mathscr{S}' \times C_W \to C_W$, we have a well-defined isocrystal $\phi^* L$ on $\mathscr{S}' \times C_W$.

Example 3.10 ([Del15, Example 2.19])**.** In light of our interpretation of the subset $]\operatorname{Pic}^{\nabla}(C)^{\psi=0}[$, we can understand why Deligne used the cohomology of Pic^0 for the comparison on cohomology and why indeed this example gives an affirmative answer to his conjecture in rank 1. By his computation, we thus obtain the number of fixed points of the Frobenius action, which correspond to the isocrystals with Frobenius structure, recalled in 1.5.

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Selbständigkeitserklärung

Ich versichere, diese Dissertation selbständig verfasst und alle verwendeten Hilfsmittel sowie Hilfen angegeben zu haben. Ferner habe ich diese Arbeit nicht in einem früheren Promotionsverfahren eingereicht.

Berlin, den 26.07.2018
Efstathia Katsigianni