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On Derived Equivalences of K3 Surfaces in Positive Characteristic

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Abstract

For an ordinary K3 surface over an algebraically closed field of positive characteristic we show that every automorphism of an ordinary K3 surface lifts to characteristic zero. Moreover, we show that the Fourier-Mukai partners of an ordinary K3 surface are in one-to-one correspondence with the Fourier-Mukai partners of the geometric generic fiber of its canonical lift. We also prove that the explicit counting formula for Fourier-Mukai partners of the K3 surfaces with Picard rank two and with discriminant equal to minus of a prime number, in terms of the class number of the prime, holds over a field of positive characteristic as well. We show that the image of the derived autoequivalence group of a K3 surface of finite height in the group of isometries of its crystalline cohomology has index at least two. Moreover, we provide an upper bound on the kernel of this natural cohomological descent map.

Further, we give an extended remark in the appendix on the possibility of an F-crystal structure on the crystalline cohomology of a K3 surface over an algebraically closed field of positive characteristic and show that the naive F-crystal structure fails in being compatible with inner product.

Zussamengfassung

Für eine gewöhnliche K3-Fläche über einem algebraisch abgeschlossenen Körper positiver Charakteristik zeigen wir, dass jeder Automorphismus einer gewöhnlichen K3-Fläche zu Charakteristik Null liftet. Darüber hinaus zeigen wir, dass die Fourier-Mukai-Partner einer gewöhnlichen K3-Fläche in einer eins-zu-eins Korrespondenz mit den Fourier-Mukai-Partnern der geometrischen generischen Faser ihres kanonischen lift sind. Wir beweisen auch, dass die explizite Zählformel für Fourier-Mukai-Partner der K3-Flächen mit Picard-Rang zwei und mit Diskriminante gleich minus einer Primzahl, bezogen auf die Klassenzahl der Primzahl, über einem Körper positiver Charakteristik gültig ist. Allgemeiner zeigen wir, dass das Bild der derivierten autoäquivalenzgruppe einer K3-Fläche, mit endlicher Höhe, in der Gruppe der Isometrien auf ihrer kristallinen Kohomologie mindestens von Index zwei ist. Außerdem geben wir eine Obergrenze für den Kern der natürlichen Abbildung von der derivierten Autoäquivalenzgruppe einer K3-Fläche zu der Gruppe der Isometrien der kristallinen Kohomologie.

Des Weiteren machen wir im Anhang eine ausführliche Bemerkung ber die Möglichkeit einer F-Kristallstruktur auf der kristallinen Kohomologie einer K3-Fläche ber einem algebraisch abgeschlossen Körper positiver Charackeristik und zeigen, dass die naive F-Kristallstruktur nicht mit dem Innenprodukt kompatibel ist.

Acknowledgement

CHAPTER 1

Introduction

The derived category of coherent sheaves on a smooth projective variety was first studied as a geometrical invariant by Mukai in the early 1980's. In case the smooth projective variety has an ample canonical or anti-canonical bundle, Bondal-Orlov [13] proved that, if two such varieties have equivalent bounded derived categories of coherent sheaves, then they are isomorphic. However, in general this is not true. The bounded derived category of coherent sheaves is not an isomorphism invariant. Mukai [54] showed that for an Abelian variety over $\mathbb{C}$, its dual has equivalent bounded derived category. Moreover, in many cases it can be shown that the dual of an Abelian variety is not birational to it, which implies that derived categories are not even birational invariants, see [31] Chapter 9. Similarly, Mukai showed in [55] that for K3 surfaces over $\mathbb{C}$, there are non-isomorphic K3 surfaces with equivalent derived categories. This led to the natural question of classifying all derived equivalent varieties.

For K3 surfaces, the case of interest to us, this was completed over $\mathbb{C}$ in late 1990's by Mukai and Orlov (see Theorem 3.8) using Hodge theory along with the Global Torelli Theorem (see Theorem 2.5). As a consequence, it was shown that there are only finitely many non-isomorphic K3 surfaces with equivalent bounded derived categories (see Proposition 3.10) and a counting formula was also proved (see Theorem 3.12). On the other hand, for K3 surfaces over a field of positive characteristic, a partial answer to the classification question was first given by Lieblich-Olsson [46] (see Theorem 3.20) in early 2010's. They showed that there are only finitely many non-isomorphic K3 surfaces with equivalent bounded derived categories. We remark here that due to unavailability of a positive characteristic version of the global Torelli Theorem for K3 surfaces of finite height, it is currently not feasible to give a complete cohomological description of derived equivalent K3 surfaces. However, a description in terms of moduli spaces was given by Lieblich-Olsson. We also point out here that the proofs of these results go via lifting to characteristic zero and thus use the Hodge theoretic description given by Mukai and Orlov. Furthermore, Lieblich-Olsson [47] also proved the derived version of the Torelli theorem (see Theorem 3.23) using the Crystalline Torelli theorem for supersingular K3 surfaces (see Theorem 2.13).

Meanwhile in 1990's another school of thought inspired by string theory in physics led Kontsevich [42] to propose the homological mirror symmetry conjecture which states that the bounded derived category $D^b(X)$ of coherent sheaves of a projective variety X is equivalent (as a triangulated category) to the bounded derived category $D^b Fuk(\check{X}, \beta)$ of the Fukaya category $Fuk(\check{X}, \beta)$ of a mirror $\check{X}$ with its symplectic structure β. Moreover, the symplectic automorphisms of $\check{X}$ induce derived autoequivalences of $D^b(X)$. This provided a natural motivation for the study of the derived autoequivalence group.

For K3 surfaces X over $\mathbb{C}$, the structure of the group of derived autoequivalences was analyzed by Ploog in [**65**], Hosono et al. in [**29**] and Huybrechts, et al. in [**33**]. They showed that the image of $Aut(D^b(X))$ under the homomorphism

$$Aut(D^b(X)) \to O_{Hodge}(\widetilde{H(X,\mathbb{Z})}),$$

where $O_{Hodge}(\widetilde{H(X,\mathbb{Z})})$ is the group of Hodge isometries of the Mukai lattice of X, has index 2. However, the kernel of this map has a description only in the special case when the Picard rank of X is 1, given by [**6**] (also see Section 2).

In this thesis, we study the above two questions in more details for the case of K3 surfaces over an algebraically closed field of positive characteristic. We collect most of the preliminary material about K3 surfaces in Chapter 2 and about derived equivalences in Chapter 3. In Chapter 4, we address the question on the group of derived autoequivalences for K3 surfaces of finite height. We show that the image of the derived autoequivalence group of a K3 surface of finite height in the group of isometries of its crystalline cohomology has index at least two (Theorem 4.29). Moreover, we provide an upper bound on the kernel of this natural cohomological descent map (Proposition 4.35). In chapter 5, we count the number of Fourier-Mukai partner for an ordinary K3 surface (Theorem 5.17) along with showing that the automorphism group lifts to characteristic 0 (Theorem 5.11). We also prove that the explicit counting formula for Fourier-Mukai partners of the K3 surfaces with Picard rank two and with discriminant equal to minus of a prime number, in terms of the class number of the prime, holds over a field of positive characteristic as well (Theorem 5.20). In Appendix 6, we define an F-crystal structure and show that this integral structure is preserved by derived equivalences but its compatibility with intersection pairing fails.

1. Conventions and Notations

For a field k of positive characteristic p, $W(k)$ will be its ring of Witt vectors. For any cohomology theory $H^*_{..}(...)$, we will denote the dimension of the cohomology groups $H^i_{..}(...)$ as $h^i_{..}(...)$. We will implicitly assume that the cardinality of $K := Frac(W(k))$ and its algebraic closure $\bar{K}$ are not bigger than that of $\mathbb{C}$, this will allow us to choose an embedding $\bar{K} \hookrightarrow \mathbb{C}$ which we will use in our arguments to transfer results from characteristic 0 to characteristic p. See also Remarks 3.18 and 3.11.

Preliminaries on K3 Surfaces

In this chapter, we start with the definition and basic properties of a K3 surface and proceed towards stating the main theorems about K3 surface over $\mathbb{C}$ as well as over any field of positive characteristic. In the next section we prove that the moduli space stable sheaves on a (relative) K3 surface is fine under specific numerical conditions, we write down the proofs in the language of gerbes and moduli stacks. The last two sections are devoted to stating the main result about lifting of K3 surface from characteristic p to characteristic 0 and non-liftability of automorphisms of supersingular K3 surfaces.

1. Basic Properties and the Torelli Theorems

We refer to [**37**] and [**4**] for details and proofs about statements in this section.

Definition 2.1. A **K3 surface** is a smooth projective geometrically integral surface X over a field k such that $\omega_X \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

Let us begin by observing the following consequences of the definition:

(1) As X is a smooth surface, one has that the cotangent bundle Ω_X is a locally free sheaf of rank 2 (see [**26**] II Theorem 8.15).
(2) Note that as for a K3 surface one has $\omega_X \cong \mathcal{O}_X$, this implies $\Omega_X \times \Omega_X \to \wedge^2 \Omega_X = \omega_X \cong \mathcal{O}_X$ which induces a non-canonical isomorphism $T_X := \Omega_X^\vee := Hom(\Omega_X, \mathcal{O}_X) \cong \Omega_X$.
(3) By definition, $h^0(X, \mathcal{O}_X) = 1$ and $h^1(X, \mathcal{O}_X) = 0$ [1] and by Serre duality [[**26**] III Corollary 7.7 and Corollary 7.12], one has $H^2(X, \mathcal{O}_X) \cong H^0(X, \omega_X)^\vee$ implying $h^2(X, \mathcal{O}_X) = 1$. Hence $\chi(X, \mathcal{O}_X) = 2$.
(4) The (algebraic) fundamental group $\pi_1(X) = 1$, i.e., is trivial for a K3 surface over a separably closed field. Indeed, let $\tilde{X} \to X$ be an irreducible étale cover of finite degree d, then $\tilde{X}$ is a smooth complete surface over k with trivial canonical bundle such that $\chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) = d\chi(X, \mathcal{O}_X) = 2d$. Note that $h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) = h^2(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 1$ as using Serre duality [[**26**] III Corollary 7.7 and Corollary 7.12] one has $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) \cong H^2(\tilde{X}, \mathcal{O}_{\tilde{X}} \otimes \omega_{\tilde{X}})$, but one has $\omega_{\tilde{X}} \cong \mathcal{O}_{\tilde{X}}$. This gives $2 - h^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 2d$ and hence $d = 1$.
(5) Note that $H^2(X, \Omega_X^2) = H^2(X, \omega_X) \cong H^2(X, \mathcal{O}_X)$ as $\omega_X \cong \mathcal{O}_X$ for a K3 surface X. Also, one has for any surface $H^0(X, \Omega_X^2) = H^0(X, \omega_X)$.
(6) Again, by Serre duality [[**26**] III Corollary 7.7 and Corollary 7.12] and $\omega_X \cong \mathcal{O}_X$, we have $H^2(X, \Omega_X) \cong H^0(X, \Omega_X)$.

1.0.1. *Hodge numbers of K3 surface.* The Hodge numbers $h^{p,q}$ are by definition the numbers $h^{p,q} := h^q(X, \Omega_X^p) = dim_k H^q(X, \Omega_X^p)$. The Hodge diamond turns out

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[1] see the notation section 1 for a clarification on the notation used.

to be the same for K3 surfaces over field of any characteristic. It looks as follows:

$$
\begin{array}{ccccccccc}
& & & h^{0,0} & & & & 1 & \\
& & h^{1,0} & & h^{0,1} & & & 0 \quad\quad 0 & \\
& h^{2,0} & & h^{1,1} & & h^{0,2} & = & 1 \quad 20 \quad 1 & \\
& & h^{2,1} & & h^{1,2} & & & 0 \quad\quad 0 & \\
& & & h^{2,2} & & & & 1 &
\end{array}
$$

Note that we have already computed the hodge numbers $h^{0,0}, h^{0,1}, h^{0,2}$ in step (3) above, $h^{2,2}, h^{2,0}$ in Step (5) above, $h^{1,2}$ in Step (6) above. We are still left to compute $h^{1,0}$ and $h^{1,1}$ as $h^{2,1} = h^{1,0}$ by Serre duality. To compute these numbers we consider the case of ground field having positive characteristic or characteristic zero separately.

Characteristic 0 **case:** If $k = \mathbb{C}$, we compute the singular cohomology group $H^1(X, \mathbb{Z})$ and show that $H^1(X, \mathbb{Z}) = 0$ for a K3 surface. Note that the long cohomology sequence of the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^\times \to 0$$

gives us the exact sequence

$$0 \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}^\times) \to H^2(X, \mathbb{Z}) \to$$
$$\to H^2(X, \mathcal{O}) \to H^2(X, \mathcal{O}^\times) \to H^3(X, \mathbb{Z}) \to 0,$$

but $H^1(X, \mathcal{O}) = 0$ so we have $H^1(X, \mathbb{Z}) = 0$, which implies $H^1(X, \mathbb{C}) = 0$. Now since the Hodge-Frölicher spectral sequence degenerates for complex K3 at E_1, we have $h^1(X, \mathbb{C}) = h^1(X, \mathcal{O}_X) + h^0(X, \Omega_X)$ (see [**4**] IV Theorem 2.7). This gives us $h^0(X, \Omega_X) = 0$. Note that this also implies that $h^0(X, T_X) = 0$, so there are no non-trivial global vector fields. Then the general case for $char(k) = 0$ follows from Lefschetz principle.

Characteristic p **case:** In this case, the result $H^0(X, T_X) = 0$ follows from a theorem of Rudakov-Shaferevich [**66**].

Now we will compute $h^{1,1}$ via the l-adic cohomology. First note that the l-adic Betti numbers for a K3 surface are $b_0(X) = b_4(X) = 1$ (follows immediately from the fact that X is a proper surface), $b_1(X) = b_3(X) = 0$, the first equality follows via Serre duality and the second one follows from the fact that $b_1(X) = 2\dim Alb(X)$, the Albanense variety of X, which is the dual of the reduced Picard scheme. The reduced Picard scheme is trivial for a K3 surface (see [**37**] Section 10.1.6 and 10.2). Using the Noether formula for surfaces, we have $12\chi(\mathcal{O}_X) = c_1(X)^2 + c_2(X)$. This gives $c_2(X) = 24$ implying $b_2(X) = 22$. Using Grothendieck-Hirzenbruch-Riemann-Roch theorem we compute

$$
\begin{aligned}
-h^{1,1} &= \chi(\Omega_X) \\
&= rank(\Omega_X).\chi(\mathcal{O}_X) + 1/2(c_1(\Omega_X).c_1(\Omega_X) - K_X)) - c_2(\Omega_X) \\
&= 4 + 0 - 24 = -20.
\end{aligned}
$$

This finishes the computation of the Hodge diamond.

Let us state the main theorems about K3 surfaces over $\mathbb{C}$, before that let us quickly recall the Hodge structure on the singular cohomology of a K3 surface.

Definition 2.2. [[**37**] Definition 3.1.1] For any free $\mathbb{Z}$-module V of finite rank or any rational vector space V, a **Hodge structure of weight** $n \in \mathbb{Z}$ on V is given by a direct sum decomposition of the corresponding complex vector space

$V_{\mathbb{C}} := V \otimes_{\mathbb{Z}} \mathbb{C}$ (resp. $V_{\mathbb{C}} := V \otimes_{\mathbb{Q}} \mathbb{C}$):

$$(1) \qquad\qquad V_{\mathbb{C}} = \oplus_{p+q=n} V^{p,q},$$

such that $\bar{V}^{p,q} = V^{p,q}$, where $\bar{V}^{p,q}$ is the complex conjugate space of $V^{p,q}$.

Examples of Hodge structures are provided by the cohomology of smooth projective varieties over $\mathbb{C}$ or, more generally, compact Kähler manifolds. For a compact Kähler manifold X the torsion free part of the singular cohomology $H^n(X, \mathbb{Z})$ comes with a natural Hodge structure of weight n given by the standard Hodge decomposition

$$(2) \qquad H^n(X, \mathbb{Z}) \otimes \mathbb{C} = H^n(X, \mathbb{C}) = \oplus_{p+q=n} H^{p,q}(X),$$

where $H^{p,q}(X)$ could either be viewed as the space of de Rham classes of bidegree (p, q) or as the Dolbeault cohomology $H^q(X, \Omega_X^p)$.

Another example is given by the **Tate Hodge structure** $\mathbb{Z}(1)$ which is the Hodge structure of weight -2 on the free $\mathbb{Z}$-module of rank one $(2\pi i)\mathbb{Z}$ (as a submodule of $\mathbb{C}$) such that $\mathbb{Z}(1)^{-1,-1}$ is one-dimensional. Similarly, one defines the **rational Tate Hodge structure** $\mathbb{Q}(1)$.

Note that one can take direct sums and tensor products of Hodge structures as well. For details see [**37**] Section 3.1.

Definition 2.3. [[**37**] Definition 3.1.6] A **polarization** of a rational Hodge structure V of weight n is a morphism of Hodge structures

$$(3) \qquad\qquad \psi : V \otimes V \to \mathbb{Q}(-n)$$

such that its $\mathbb{R}$-linear extension yields a positive definite symmetric form

$$(4) \qquad\qquad (v, w) \to \psi(v, Cw)$$

on the real part of $V^{p,q} \oplus V^{q,p}$. Here, C is the Weil operator, which acts on $V^{p,q}$ by multiplication with i^{p-q}. Then the pair (V, ψ) is called a **polarized Hodge structure**. An isomorphism $V_1 \xrightarrow{\sim} V_2$ of Hodge structures that is compatible with given polarizations ψ_1(resp. ψ_2) is called a **Hodge isometry**.

In our example of Hodge structure on the middle primitive cohomology of a smooth projective variety, the intersection pairing defines a polarization.

Let X be a K3 surface over $\mathbb{C}$ and denote by $Aut(H^2(X, \mathbb{Z}))$ the group of Hodge isomorphisms of the lattice $H^2(X, \mathbb{Z})$ which preserve the intersection pairing. We begin by defining the set

$$D_X := \{v \in H^{1,1}(X) \cap H^2(X, \mathbb{Z}) | v^2 = -2\}$$

of (-2)-classes which is used to generate the Weyl group in the sense that,

$$W_X := \{s_v \in Aut(H^2(X, \mathbb{Z})) | v \in D_X\},$$

where $s_v(x) := x + (v, x)v$ is the reflection on v. Denote by $\pm W(X)$ the group $W(X) \times <\iota>$, where $\iota := -id_{H^2(X)}$ swaps the two components C and $-C$. With this notation, the strong Torelli theorem for K3 surfaces describes the automorphism group $Aut(H^2(X, \mathbb{Z}))$ as a semidirect product.

THEOREM 2.4 (Strong Global Torelli theorem,[**4**] section VIII.11, [**37**] Theorem 7.5.3, [**65**] Theorem 2.2). *Let X be a K3 surface over $\mathbb{C}$, then*

$$(5) \qquad\qquad Aut(H^2(X, \mathbb{Z})) = \pm W(X) \ltimes Aut(X).$$

Let us also state the weaker theorem.

THEOREM 2.5 (Weak Torelli theorem, [**4**] VIII Corollary 11.2, [**37**] Theorem 7.5.3). *Two K3 surfaces X and Y over $\mathbb{C}$ are isomorphic if and only if they are Hodge isometric, i.e., we have a hodge isometry $H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$.*

Let $\Lambda = E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$ be the K3 lattice, where E_8 is the Leech lattice and U is the hyperbolic lattice and (-1) twist denotes the twist by $-Id_{8 \times 8}$ on the inner product matrix of E_8. For every K3 surface X over $\mathbb{C}$, the free $\mathbb{Z}$-group $H^2(X, \mathbb{Z})$, with the intersection pairing is isomorphic to Λ as $\mathbb{Z}$-lattices. The signature of Λ is $(3, 19)$ and it is a unimodular non-degenerate lattice with determinant $\det(\Lambda) = -1$. Now consider the associated complex vector space $\Lambda_{\mathbb{C}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ with the $\mathbb{C}$-linear extension of the inner product on Λ, denoted $(\, , \,)$, which corresponds to a homogeneous quadratic polynomial. The zero locus of the polynomial in $\mathbb{P}(\Lambda_{\mathbb{C}})$ is a quadric which is smooth because $(\, , \,)$ is non-degenerate. We denote this zero locus by P.

Definition 2.6. [Period Domain] The open subset in classical topology of P,

$$(6) \qquad D := \{x \in \mathbb{P}(\Lambda_{\mathbb{C}}) | (x)^2 = 0, \ (x, \bar{x}) > 0\} \subset P \subset \mathbb{P}(\Lambda_{\mathbb{C}}),$$

is said to be the **period domain** associated with λ and will be considered as a complex manifold.

Definition 2.7. [Moduli Functor of marked K3 surfaces] We define the **moduli functor of marked K3 surfaces** as the functor:

$$(7) \qquad \begin{aligned} \mathcal{N} : (ComplS) &\to (sets) \\ S &\mapsto \{(f : X \to S, \varphi)\}/ \sim . \end{aligned}$$

Here $(ComplS)$ denotes the category of complex spaces, $f : X \to S$ is a smooth proper family of K3 surfaces[2] with a marking $\varphi : R^2 f_* \mathbb{Z} \xrightarrow{\sim} \underline{\Lambda}$, compatible with the intersection pairing, and the equivalence relation is given by $(f : X \to S, \varphi) \sim (f' : X' \to S', \varphi')$ if there exists an isomorphism $g : X \xrightarrow{\sim} X'$ such that $f' \circ g = f$ and $\varphi' = \varphi \circ g^*$.

This moduli functor is representable by a 20-dimensional complex manifold N, but N is not Hausdorff (see [**37**] section 6.3.3 and references therein). Using the universal marking $\varphi : R^2 f_* \mathbb{Z} \xrightarrow{\sim} \underline{\Lambda}$ of the universal family $f : X \to N$ one obtains a **global period map**

$$(8) \qquad \begin{aligned} \mathcal{P} : N &\to D \subset \mathbb{P}(\Lambda_{\mathbb{C}}), \\ t &\mapsto [\varphi(H^{2,0}(X_t))] \end{aligned}$$

where t is a closed point of N.

THEOREM 2.8 (Surjectivity of period mapping). *The global period map is surjective. In other words, for any Hodge structure on Λ, which is positive definite on $(\Lambda^{2,0} \oplus \Lambda^{0,2})_{\mathbb{R}}$, there exists a K3 surface X together with a Hodge isometry $H^2(X, \mathbb{Z}) \cong \Lambda$.*

This finishes our exposition on K3 surfaces over a field of characteristic zero and in particular over $\mathbb{C}$.

For K3 surfaces over a field of positive characteristic, we start by defining the notion of height, which gives a subclass of K3 surfaces with finite height or infinite height called supersingular K3 surfaces. We will define the height of a K3 surface

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[2]Here, we use a weaker definition of K3 surfaces which allow them to be not algebraic as well: A complex K3 surface is a compact connected complex manifold X such that $\omega_X \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$. For more details we refer the reader to [**37**] Chapter 1 Section 3. This enlargement of base category is needed to prove the surjectivity of global period map.

through its F-crystal. For an introduction to Brauer group of K3 surfaces and the definition of height via the Brauer groups see [**37**] and [**50**]. Both definitions turn out to be equivalent (for example see Prop. 6.17 [**50**]).

Let k be an algebraically closed field of positive characteristic, $W(k)$ its ring of Witt vectors and $Frob_W$ the Frobenius morphism of $W(k)$ induced by the Frobenius automorphism of k. Note that $Frob_W$ is a ring homomorphism and induces an automorphism of the fraction fields $K := Frac(W(k))$, denoted as $Frob_K$. We begin by recalling the notion of F-isocrystal and F-crystals which we will use later to stratify the moduli of K3 surfaces.

Definition 2.9. [F-(iso)crystal] An **F-crystal** (M, ϕ_M) over k is a free W-module M of finite rank together with an injective $Frob_W$-linear map $\phi_M : M \to M$, that is, ϕ_M is additive, injective and satisfies

$$\phi_M(r \cdot m) = Frob_W(r) \cdot \phi_M(m) \text{ for all } r \in W(k), m \in M.$$

An **F-isocrystal** (V, ϕ_V) is a finite dimensional K-vector space V together with an injective $Frob_K$-linear map $\phi_V : V \to V$.

A **morphism** $u : (M, \phi_M) \to (N, \phi_N)$ **of F-crystals** (resp. **F-isocrystals**) is a $W(k)$-linear (resp. K-linear) map $M \to N$ such that $\phi_N \circ u = u \circ \phi_M$. An **isogeny** of F-crystals is a morphism $u : (M, \phi_M) \to (N, \phi_N)$ of F-crystals, such that the induced map $u \otimes Id_K : M \otimes_{W(k)} K \to N \otimes_{W(k)} K$ is an isomorphism of F-isocrystals.

Examples:
(1) The trivial crystal: $(W, Frob_W)$.
(2) This is the case which will be of most interest to us:
Let X be a smooth and proper variety over k. For any n, take the free $W(k)$ module M to be $H^n := H^n_{crys}(X/W(k))/torsion$ and ϕ_M to be the Frobenius F^*. The Poincaré duality induces a perfect pairing

$$\langle -, - \rangle : H^n \times H^{2dim(X)-n} \to H^{2dim(X)} \cong W$$

which satisfies the following compatibility with Frobenius

$$\langle F^*(x), F^*(y) \rangle = p^{dim(X)} Frob_W(\langle x, y \rangle),$$

where $x \in H^n$ and $y \in H^{2dim(X)-n}$. As $Frob_W$ is injective, we have that F^* is injective. Thus, (H^n, F^*) is an F-crystal. We will denote the F-isocrystal $H^n_{crys}(X/W) \otimes K$ by $H^n_{crys}(X/K)$.
(3) The F-isocrystal $K(1) := (K, Frob_K/p)$. Similarly, one has the F-isocrystal $K(n) := (K, Frob_K/p^n)$ for all $n \in \mathbb{Z}$. Moreover, for any F-isocrystal V and $n \in \mathbb{Z}$, we denote by $V(n)$ the F-isocrystal $V \otimes K(n)$.

Recall that the category of F-crystals over k up to isogeny is semi-simple and the simple objects are the F-crystals:

$$M_\alpha = ((\mathbb{Z}_p[T])/(T^s - p^r)) \otimes_{\mathbb{Z}_p} W(k), (\text{mult. by } T) \otimes Frob_W),$$

for $\alpha = r/s \in \mathbb{Q}_{\geq 0}$ and r, s non-negative coprime integers. This is a theorem of Dieudonné-Manin. Note that the rank of the F-crystal M_α is s. We call α the **slope** of the F-crystal M_α.

Definition 2.10. Let (M, ϕ) be an F-crystal over k and let

$$(M, \phi) \sim^{isogeny} \oplus_{\alpha \in \mathbb{Q}_{\geq 0}} M_\alpha^{n_\alpha}$$

be its decomposition up to isogeny. Then the elements of the set

$$\{\alpha \in \mathbb{Q}_{\geq 0} | n_\alpha \neq 0\}$$

are called the **slopes** of (M, ϕ). For every slope α of (M, ϕ), the integer $\lambda_\alpha := n_\alpha \cdot rank_W M_\alpha$ is called the **multiplicity** of the slope α.

Remark 2.11. In case (M, ϕ) is an F-crystal over a perfect field k (rather than being algebraically closed as assumed above), we define its slope and multiplicities to be that of the F-crystal $(M, \phi) \otimes_{W(k)} W(\bar{k})$, where $\bar{k}$ is an algebraic closure of k.

We still keep our assumption of k being an algebraically closed field of positive characteristic.

The above classification result of Dieudonné-Manin is more general. Any F-isocrystal V with bijective ϕ_V is isomorphic to a direct sum of F-isocrystals

$$(V_\alpha := K[T]/(T^s - p^r), (\text{mult. by } T) \otimes Frob_K),$$

for $\alpha = r/s \in \mathbb{Q}$. The dimension of V_α is s and we call α the **slope** of V_α.

Definition 2.12. [Height] The **height** of a K3 surface X over k is the sum of multiplicities of slope strictly less than 1 part of the F-crystal $H^2_{crys}(X/W)$. In other words, the dimension of the subspace of slope strictly less than one of the F-isocrystal $H^2_{crys}(X/K)$, which is $\dim(H^2_{crys}(X/K)_{[0,1)} := \oplus_{\alpha_i < 1} V_{\alpha_i}^{n_{\alpha_i}})$.

If for a K3 surface X the $\dim(H^2_{crys}(X/K)_{[0,1)}) = 0$, then we say that the height of X is infinite.

Supersingular K3 surfaces (i.e., K3 surfaces with infinite height) also have an equivalent description that their Picard rank is 22 (see [**50**] Theorem 4.8). We will be discussing more about F-crystals later in Appendix 6. We now state the Torelli theorem for supersingular K3 surfaces. This is currently the only Torelli type theorem for a class of K3 surfaces in positive characteristic, whose proof does not go via lifting to characteristic zero.

THEOREM 2.13 (Crystalline Torelli theorem Ogus [**64**], Theorem I). *If X and Y are supersingular K3 surfaces and if there exists an isomorphism*

$$(9) \qquad\qquad H^2_{crys}(X/W) \to H^2_{crys}(Y/W)$$

compatible with the Frobenius action and intersection pairing, then X and Y are isomorphic.

Remark 2.14. Note that the isomorphism (9) above is not necessarily induced by the isomorphism constructed between X and Y.

One can even determine all the isomorphisms between supersingular K3 surfaces. For that we introduce the corresponding period domain as we did over $\mathbb{C}$.

Definition 2.15. [Supersingular K3 lattice, [**64**] Definition 1.6] A **supersingular K3 lattice** is a free abelian group N of rank 22, with an even symmetric bilinear form $(\ ,\ )$ satisfying the following properties:
  (1) $disc(N \otimes \mathbb{Q}) = -1 \in \mathbb{Q}^*/\mathbb{Q}^{*2}$;
  (2) The signature of $(N \otimes \mathbb{R})$ is $(1, 21)$;
  (3) The cokernel of $(\ ,\ ) : N \to N^\vee$ is annihilated by p, where $N^\vee := Hom_{\mathbb{Z}}(N, \mathbb{Z})$.

Remark 2.16. We prefer to call this lattice supersingular K3 lattice (Ogus calls it K3 lattice) to avoid confusion with the K3 lattice introduced above. Note that the lattices are different as they have different signatures.

If N is a supersingular K3 lattice, then its discriminant, $disc(N)$, is equal to $-p^{2\sigma_0}$, with $1 \le \sigma_0 \le 10$ and it determines the lattice N up to isomorphism [**63**].

Example: The Néron-Severi group of a supersingular K3 surface is a supersingular K3 lattice.

Remark 2.17. More is true, actually every supersingular K3 lattice occurs as a Néron-Severi lattice for some supersingular K3 surface and σ_0 is called the Artin Invariant of the corresponding supersingular K3 surface.

Let us fix a supersingular K3 lattice N with $\sigma_0(N) = a$.

Definition 2.18. An N-**marking** on a K3 surface X/k is a map $\eta : N \to NS(X)$, compatible with the bilinear forms. An **isomorphism of N-marked K3 surfaces** $\theta : (X, \eta) \to (Y, \zeta)$ is an isomorphism $\theta : X \to Y$ such that $NS(\theta) \circ \zeta = \eta$.

From [**64**] 1.7, it follows that a supersingular K3 surface X/k with $\sigma_0(NS(X)) = a$ admits an N-marking η if and only if $\sigma_0(NS(X)) \leq a$. For a marked supersingular K3 surface (X, η), recall that its period has been constructed as follows in [**63**]: Composing η with the de Rham Chern character, we get a map

$$\bar{\eta} : N \otimes k \to H^2_{DR}(X/k).$$

Let $Ker_{(X,\eta)} := Ker\,\bar{\eta}$. Then from [**63**] Remark 3.16, we have $Ker_{(X,\eta)} \subseteq N_0 \otimes k := pN^\vee/pN \otimes k$ and is totally isotropic of rank a. Now define $K_{(X,\eta)} := (id_{N_0} \otimes Frob_k)^{-1}(Ker_{(X,\eta)})$. This $K_{(X,\eta)}$ is the **period** of X and determines $H^2_{crys}(X/W)$ and $NS(X)$ (see [**64**] Proposition 1.9).

Let $\Delta_N := \{\delta \in N | (\delta, \delta) = -2\}$ and

$$V_N := \{x \in N \otimes \mathbb{R} | (x, x) > 0, (x, \delta) \neq 0, \ \forall \delta \in \Delta_N\}.$$

For $\delta \in \Delta_N$, we define r_δ to be the reflection:

$$x \mapsto x + (x, \delta)\delta,$$

and R_N is defined to be the subgroup of $Aut(N)$ generated by $\{r_\delta \in \Delta_N\}$. Denote by $\pm R_N$ the subgroup generated by $-id$ and R_N. The subset V_N is open and each of its connected component meets N. The group $\pm R_N$ acts simply transitively on the set C_N of connected components on V_N (see [**64**] Proposition 1.10). Note that the ample component of $V_{NS(X)}$ corresponds to a well-defined element $\alpha_{(X,\eta)}$ of C_K.

THEOREM 2.19 (Strong Crystalline Torelli Theorem, Ogus [**64**], Theorem II"). *If (X, η) and (Y, ζ) are N-marked supersingular K3 surfaces over k and if $K_{(X,\eta)} = K_{(Y,\zeta)}$ and $\alpha_{(X,\eta)} = \alpha_{(Y,\zeta)}$, then there is a unique isomorphism $\theta : (X, \eta) \to (Y, \zeta)$.*

This gives us the following corollary:

Corollary 2.20. *If (X, η) is an N-marked K3 surface over k with period $K \subseteq N_0 \otimes k$ and ample cone $\alpha \in C_N$, let $G_K = \{g \in Aut(N) | gK = K\}$. Then $Aut(X)$ is isomorphic to the subgroup of G_K stabilizing α and G_K is the semi-direct product of $\pm R_N$ and $Aut(X)$,*

$$G_K = Aut(X) \rtimes \pm R_N.$$

For more details about the period map for supersingular K3 surfaces and proofs we refer the reader to [**50**] section 5 and [**64**], [**63**].

Next we discuss about the Moduli space of sheaves on a K3 surface as these spaces turn out to play a very important role in the theory of derived equivalences of K3 surfaces.

2. Moduli Space of Sheaves

In this section, we introduce the moduli stack of sheaves on a K3 surface and show that it's a μ_r-Gerbe under some numerical conditions. We will try to keep the exposition here characteristic independent and in case of characteristic restrictions we will mention them as necessary. Moreover, in the case of a K3 surface defined

over a field we will not assume the field to be algebraically closed and in general, for a relative K3 surface, we will work with a spectrum of a mixed characteristic discrete valuation ring as the base scheme. The main references for this section are [**45**] Section 2.3.3 and [**46**] Section 3.15. We will refer back and forth to [**1**], [**60**], [**32**], [**37**] Chap 10, in the proofs. We refer the reader to [**24**], for a comparison between the moduli stack point of view and that of more classical moduli functors.

Remark 2.21. The point of view of moduli stacks offers us the benefit that in this way it becomes more natural to generalize this theory to derived schemes and derived stacks, where the role of Artin representability theorem will be taken up by Lurie's Artin Representability theorem [**51**] and those of the Hilbert and Quot schemes by their derived versions.

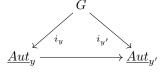We begin by defining a gerbe and then stating the theorem about their classification.

Let C be a site and let G be a sheaf of abelian groups on C. Recall that for $p : F \to C$ a stack over C and any object $x \in F$ over $X := p(x)$ we have a sheaf $\underline{Aut}_x$ over C/X.

Definition 2.22. [G-Gerbe] A **G-gerbe over** C is a stack $p : F \to C$ together with an isomorphism of sheaves of groups

$$i_x : G|_{C/p(x)} \to \underline{Aut}_x$$

for every object $x \in F$ such that the following conditions hold:

(1) For any object $Y \in C$ there exists a covering $\{Y_i \to Y\}_{i \in I}$ such that $F(Y_i)$ is non-empty for every i.
(2) For any two objects $y, y' \in F(Y)$ over the same object $Y \in C$, there exists a covering $\{f_i : Y_i \to Y\}_{i \in I}$ such that the pullback $f_i^* y$ and $f_i^* y'$ are isomorphic in $F(Y_i)$ for all $i \in I$.
(3) For every object $Y \in C$ and isomorphism $\sigma : y \to y'$ in $F(Y)$ the resulting diagram commutes:

A **morphism of G-gerbes** $(F', \{i_{x'}\}) \to (F, \{i_x\})$ is a morphism of stacks $f : F' \to F$ such that for every object $x' \in F'$ the diagram

commutes.

Example: The classifying stack of G, BG, is a G-gerbe.

Definition 2.23. A G-gerbe is called **trivial** if it is isomorphic to BG.

For proofs and details we refer the reader to Chapter 12 of [**60**]. An important fact about G-gerbes is given by the following lemma:

Lemma 2.24 ([**60**], Lemma 12.2.4). *Any morphism of G-gerbes is an isomorphism.*

Now we state the classification theorem.

THEOREM 2.25 ([**60**], Theorem 12.2.8). *The G-gerbes are classified by $H^2(C, G)$. More precisely, there exists a bijection between the sets*

$$(10) \qquad H^2(C, G) \leftrightarrow \{\text{isomorphism classes of } G\text{-gerbes}\}.$$

Before proceeding to the definition of moduli stacks of sheaves that we will be working with, let us also recall the notion of (Gieseker) semistability for coherent sheaves (for details see [**32**], section 1.2): Let X be a projective scheme over a field k. The Euler characteristic of a coherent sheaf $\mathcal{F}$ is $\chi(\mathcal{F}) = \sum (-1)^i h^i(X, \mathcal{F})$. If we fix an ample line bundle $\mathcal{O}(1)$ on X, then the Hilbert polynomial $P(\mathcal{F})$ given by $n \mapsto \chi(\mathcal{F} \otimes \mathcal{O}(n))$ can be uniquely written in the form

$$P(\mathcal{F}, n) = \sum_{i=0}^{dim(\mathcal{F})} \alpha_i(\mathcal{F}) m^i / i!,$$

with integral coefficients $\alpha_i(\mathcal{F})$. We denote by $p(\mathcal{F}, n) := P(\mathcal{F}, n)/\alpha_{dim(\mathcal{F})}(\mathcal{F})$, the **reduced Hilbert polynomial of $\mathcal{F}$**.

Definition 2.26. [Semistability] A coherent sheaf $\mathcal{F}$ of dimension d is **semistable** if $\mathcal{F}$ has no nontrivial proper subsheaves of strictly smaller dimension and for any subsheaf $\mathcal{E} \subset \mathcal{F}$, one has $p(\mathcal{E}) \leq p(\mathcal{F})$. It is called **stable** if for any proper subsheaf the inequality is strict.

Remark 2.27. The ordering on polynomials is the lexicographic ordering of the coefficients.

Definition 2.28. [Mukai vector] For a smooth projective X over k, given a perfect complex $E \in D(X)$, where $D(X)$ is the derived category of coherent sheaves on X, we define the **Mukai vector** of E to be

$$v(E) := ch(E)\sqrt{td_X} \in A^*(X)_{num,\mathbb{Q}}.$$

Here, $ch(-)$ denotes the Chern class map, td_X is the Todd genus and $A^*(X)_{num,\mathbb{Q}}$ is the numerical Chow group of X with rational coefficients.

For X a K3 surface over k, the Mukai vector of a complex is given by (see [**31**] Chapter 10):

$$v(E) = (\text{rank}(E), c_1(E), \text{rank}(E) + c_1(E)^2/2 - c_2(E)).$$

Let X be a projective scheme over k and h an ample line bundle.

Definition 2.29. [Moduli Stack] The **moduli stack of semistable sheaves**, denoted $\mathfrak{M}_h^{ss}$, is defined as follows:

$\mathfrak{M}_h^{ss} : (Sch/k) \to (\text{groupoids})$

$\qquad\qquad S \mapsto \{\mathcal{F} | \mathcal{F} \text{ an } S\text{-flat coherent sheaf on } X \times S \text{ with semistable fibers}\}.$

Similarly, the **moduli stack of stable sheaves** can be defined by replacing semistable above with stable and we denote it by $\mathfrak{M}_h^s$.

If we fix a vector $v \in A^*(X)_{num,\mathbb{Q}}$, we get an open and closed substack $\mathfrak{M}_h^{ss}(v)$ classifying semistable sheaves on X with Mukai vector v.

The following result has been proved by Lieblich [**45**], for the more general case of moduli of twisted sheaves. We follow that proof here but restricting to our case of semistable sheaves without any twisting, thereby simplifying some arguments.

THEOREM 2.30. *The stack $\mathfrak{M}_h^{ss}$ is an algebraic stack and the stack $\mathfrak{M}^{ss}(v)$ is an algebraic substack of finite type over k.*

Remark 2.31. Recall that the Mukai vector v for a sheaf on a K3 surface determines its Hilbert polynomial and its rank as well.

PROOF. The proof goes via checking the Artin's representability conditions. The plan of the proof is as follows:

(1) We show that the stack of coherent sheaves Coh_X is an algebraic stack by checking the conditions of Artin's Representability Theorem following [**1**] Tag 07Y4.

(2) We show that the stack $\mathfrak{M}_h^{ss}$ is algebraic by showing that the natural map of $\mathfrak{M}_h^{ss}$ to Coh_X is representable by open immersion.

(3) Lastly, we show that adding an additional constraint by fixing the Hilbert polynomial, gives us a substack of finite type.

For **Step 1)** we follow the proof given in [**1**] Tag 08KA.

We begin by recalling the definition of the stack of coherent sheaves Coh_X:[3]

$$Coh_X : (Sch/S) \to \{Groupoids\}$$

$$T \mapsto \{\mathcal{F} | \mathcal{F} \text{ a } T\text{-flat coherent sheaf on } X \times_S T\},$$

where we ask for $X \to S$ to be proper and flat and S to be a locally noetherian scheme.

To check that it is indeed a stack see [**1**],Tag 08W5 and 08KC. Now, we check the Artin axioms recalled along with.

(A_1) The diagonal of Coh_X, $\Delta : Coh_X \to Coh_X \times Coh_X$, is representable by algebraic space.

Let $a = (T, \mathcal{F})$ and $b = (T, \mathcal{E})$ be objects of Coh_X over a scheme T. Then from [**60**] Lemma 8.1.8, the above claim is equivalent to showing that $\underline{Isom}(\mathcal{F}, \mathcal{E})$ is an algebraic space over T and this follows from [**1**] Tag 08K9.

(A_2) The stack $p : Coh_X \to (Sch/S)_{fppf}$ is limit preserving, i.e., if for every affine scheme T over S which is a limit $T = \lim T_i$ of a directed inverse system of affine schemes T_i over S, one has an equivalence of fiber categories:

$$\text{colim } Coh_X(T_i) \to Coh_X(T).$$

This claims follows from [**1**] Tag 01ZR, where we take $X_i := X \times T_i$ and $X_T := X \times T$, which are quasi-compact and separated schemes. And we can indeed restrict to full subcategory of flat and coherent objects using for example [**1**] Tag 08K0 and 08K2.

(A_3) Consider a pushout in the category of schemes over S

$$\begin{array}{ccc} Z & \longrightarrow & Z' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y', \end{array}$$

where $Z \to Z'$ is a thickening and $Z \to Y$ is affine. Then we need to show that the functor on fiber categories:

$$Coh_X(Y') \to Coh_X(Y) \times_{Coh_X(Z)} Coh_X(Z')$$

is an equivalence.

Explicitly, we have to show that there is an equivalence between the categories:

(a) Coherent modules $\mathcal{G}'$ flat over Y'

(b) The category of triples $(\mathcal{G}, \mathcal{F}', \varphi)$, where

(i) $\mathcal{G}$ is a coherent sheaf flat over Y,

(ii) $\mathcal{F}'$ is a coherent sheaf flat over Z, and

---

[3]We do the proof in a slightly more general context so that it will be applicable in the proof of theorem 2.38, for the relative setting.

(iii) $\varphi : (X \to Y)^* \mathcal{G} \to (Z \to Z')^* \mathcal{F}$ is an isomorphism over Z.
The equivalence functor is described as:

$$\mathcal{G} \mapsto ((Y \to Y')^* \mathcal{G}, (Z' \to Y')^* \mathcal{G}, canonical map)$$

and its quasi-inverse is given by:

$$(\mathcal{G}, \mathcal{F}', \varphi) \mapsto (Y' \to Y)^* \mathcal{G} \times_{(Z \to Y')_* \mathcal{F}} (Z' \to Y')^* \mathcal{F}'$$

where $\mathcal{F} := (Z \to Z')^* \mathcal{F}'$. This assignment works follows from [**1**] Tag 08KU. This condition is the **Rim-Schlessinger condition**.

(A_4) Let k be a finite type field over S, i.e., $\mathrm{Spec}(k) \to S$ is of finite type, and let $x_0 = (\mathrm{Spec}(k), \mathcal{G}_0)$ be an object of $Coh_X(k)$, then we have to prove that the tangent space of the corresponding deformation category, $T\mathcal{F}_{Coh_X, k, x_0}$ and the infinitesimal automorphism space, $Inf(\mathcal{F}_{Coh_X, k, x_0})$ are finite dimensional k-vector spaces. Explicitly, these spaces are defined as follows:

$$T\mathcal{F}_{Coh_X, k, x_0} = \{\text{iso. classes of morphisms } x_0 \to x \text{ over } \mathrm{Spec}(k) \to \mathrm{Spec}(k[\epsilon])\}$$

and

$$Inf(\mathcal{F}_{Coh_X, k, x_0}) = \mathrm{Ker}(Aut_{\mathrm{Spec}(k[\epsilon])}(x_0') \to Aut_{\mathrm{Spec}(k)}(x_0)),$$

where x_0' is the pullback of x_0 to $\mathrm{Spec}(k[\epsilon])$.

The Rim-Schlessinger conditions formally imply that the tangent space and infinitesimal automorphism spaces are k-vector spaces, see for example [**1**] 07WU or Sernesi, [**69**] Section 2.2. To prove finite dimensionality, we reduce to the case $S = \mathrm{Spec}(k)$. Let the morphism $\mathrm{Spec}(k[\epsilon]) \to \mathrm{Spec}(k)$ be the one coming from the inclusion $k \to k[\epsilon]$. Set $X_0 := X \times_S \mathrm{Spec}(k)$ and $X_\epsilon := X \times_S \mathrm{Spec}(k[\epsilon])$. Note that the stack Coh_{X_0} is just the 2-fiber product of $Coh_X \times_S \mathrm{Spec}(k)$ as categories fibered in groupoids over $(Sch/S)_{fppf}$. Thus, by [**1**] Tag 07X2, we reduce to proving the finite-dimensionality of tangent space and the space of infinitesimal automorphisms for S, k and Coh_{X_0}. The tangent spaces of S and $\mathrm{Spec}(k)$ are finite dimensional by [**1**] Tag 07X1 and they have vanishing Inf.

Now X_ϵ is a first order thickening of X_0 flat over the first order thickening $\mathrm{Spec}(k) \to \mathrm{Spec}(k[\epsilon])$. From the definitions we see that $T\mathcal{F}_{Coh_{X_0}, k, x_0}$ is the set of liftings of $\mathcal{G}_0$ to a flat module on X_ϵ. Therefore, for example, by [**1**] Tag 08VW, we have

$$T\mathcal{F}_{Coh_{X_0}, k, x_0} = \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\mathcal{G}_0, \mathcal{G}_0),$$

and

$$Inf(\mathcal{F}_{Coh_{X_0}, k, x_0}) = \mathrm{Ext}^0_{\mathcal{O}_{X_0}}(\mathcal{G}_0, \mathcal{G}_0).$$

These spaces are finite dimensional over k as $\mathcal{G}_0$ is a sheaf over a space proper over k.

(A_5) Formal effectiveness: We need to show that the following functor is an equivalence:

$$\left\{ \begin{array}{c} \text{objects } x \text{ of } Coh_X \text{ such that } p(x) = \mathrm{Spec}(R), \\ \text{where } R \text{ is Noetherian complete local} \\ \text{with } R/\mathfrak{m} \text{ of finite type over } S \end{array} \right\} \longrightarrow \left\{ \text{formal objects of } Coh_X \right\}.$$

This follows from the Grothendieck Existence Theorem [**26**], II.9.6 and [**1**] Tag 08VP. For a definition of the formal object of a stack we refer the reader to [**1**] Tag 06H2.

(A_6) The stack Coh_X satisfies openness of versality.

To show this we use [**1**] Tag 0CYF, for which we have to show that Coh_X satisfies the following 5 conditions:

(a) The diagonal of Coh_X is representable.
   This was our axiom (A_1) above.
(b) The stack Coh_X satisfies Rim-Schlessinger condition.
   This was axiom (A_3) above.
(c) Coh_X is limit preserving.
   This was proved in Axiom (A_2) above.
(d) There exists an obstruction theory, denoted by O.
   Given an S-algebra A, M an A-module and an object x of Coh_X
   over $\mathrm{Spec}(A)$ given by $\mathcal{F}$ on X_A we set $O_x(M) = \mathrm{Ext}^2(\mathcal{F}, \mathcal{F} \otimes_A M)$
   (this will be the obstruction space) and if $A' \to A$ is a surjection
   with kernel I, then we take $o_x(A') = o(\mathcal{F}, \mathcal{F} \otimes_A I, 1) \in O_x(I) =$
   $Ext^2_{X_A}(\mathcal{F}, \mathcal{F} \otimes_A I)$ as the obstruction element. For construction of
   this element see [**1**] Tag 08MF and Tag 08VW. The construction of
   the obstruction class is functorial (see [**1**] Tag 0CYC and 0CYE).
(e) For an object x of Coh_X over $\mathrm{Spec}(A)$ and A-modules M_n, $n \geq 1$,
   we have $T_x(\prod M_n) = \prod T_x(M_n)$, and $O_x(\prod M_n) \to \prod O_x(M_n)$ is
   injective.
   We show that $T_x(\prod M_n) = \prod T_x(M_n)$, and $O_x(\prod M_n) = \prod O_x(M_n)$.
   Note that we have $T_x(M) = Ext^1_{X_A}(\mathcal{F}, \mathcal{F} \otimes M)$. Now using [**1**] Tag
   08JR, we find a perfect complex $K \in D(A)$ which computes the Ext
   groups as cohomology groups : For $i \leq 2$

$$H^i(K \otimes^{\mathbb{L}}_A M) \xrightarrow{\sim} \mathrm{Ext}^i_{X_A}(\mathcal{F}, \mathcal{F} \otimes_A M),$$

   and then using the following relation we conclude,

$$H^i(K \otimes^{\mathbb{L}}_A \prod M_n) = \prod H^i(K \otimes^{\mathbb{L}}_A M_n).$$

(A_7) The last condition is to show that $\mathcal{O}_{S,s}$ is a G-ring for all finite type points
   s of S.
   To show this we reduce to case of S being an affine scheme of a finite
   type $\mathbb{Z}$-algebra. Indeed, choose an affine open covering $S = \cup U_i$. Denote
   by Coh_{X_i} the restriction of Coh_X to $(Sch/U_i)_{fppf}$. To show that the
   stack is algebraic we only need to show that there exists a scheme W
   and a smooth surjective morphism $W \to Coh_X$ as we have already shown
   that the diagonal is representable by algebraic spaces. Thus, if we can
   find surjective smooth morphisms from schemes $W_i \to Coh_{X_i}$, we can set
   $W = \sqcup W_i$ and we obtain a surjective smooth morphism $W \to Coh_X$.
   Thus, we assume S is affine, say $S = \mathrm{Spec}(R)$. Now write $R = \mathrm{colim}(R_i)$
   as a filtered colimit with each R_i of finite type over $\mathbb{Z}$. Using [**1**] Tag
   07SK, 0851 and 08K0, we can find for some i a morphism of schemes
   $X_i \to \mathrm{Spec}(R_i)$ which is of finite presentation, separated and flat and
   whose base change to R is X. Thus, if we show Coh_{X_i} is an algebraic
   stack, we are done by base change.
   Now as R_i is a finite $\mathbb{Z}$ algebra, it is G-ring. Hence, all its local rings
   are G-rings. This checks the last condition of the Artin's Axioms and
   we have already verified all the other conditions above. This finishes the
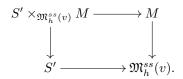   proof of Step 1.

   Proof of Step 2): We have to show that the natural map $\mathfrak{M}^{ss}_h \to Coh_X$
is representable by open immersions. To this end, using [**1**] Tag 02Y9 and Tag
01JI, we just need to show that given an object $(T, \mathcal{F})$ of Coh_X, there is an open
subscheme $U \subset T$ such that for a morphism of schemes $T' \to T$ the following are
equivalent:

(1) $T' \to T$ factors through U.

(2) the pullback $\mathcal{F}_{T'}$ of $\mathcal{F}$ by $X_{T'} \to X_T$ is semistable.

Let $U \subset T$ be the set of points $t \in T$ such that $\mathcal{F}_t$, the restriction of $\mathcal{F}$ to the fiber over t of $X_T \to T$, is semistable. From [32] Proposition 2.3.1, we note that semistability of coherent sheaves in flat families is an open property. Thus, U is open and its formation commutes with base change as in (2), we are done.

We conclude that the substack $\mathfrak{M}_h^{ss}$ is algebraic using [1] Tag 05UM.

Proof of Step 3): We need to show that fixing the Hilbert polynomial gives us a connected component of the stack $\mathfrak{M}_h^{ss}$ which is an algebraic substack of finite type. It is a connected component of $\mathfrak{M}_h^{ss}$ follows easily. In particular, this implies that it is an algebraic stack. We just have to prove that $\mathfrak{M}_h^{ss}(v)$ is of finite type [4]. First, recall the boundedness result (in characteristic 0 see [32] Theorem 3.3.7, in positive characteristic and mixed characteristic see [43]), i.e., the family of semistable sheaves with a fixed Hilbert polynomial is bounded in the sense of [32] Definition 1.7.5. Let $M \to \mathfrak{M}_h^{ss}(v)$ be a presentation of the algebraic stack. The bounded result provides us with a scheme of finite type S' over the base scheme S such that there is a sheaf $\mathcal{F}$ on $X_{S'}$ such that the fibers over closed points give all the semistable sheaves. This sheaf induces a surjective map $S' \to \mathfrak{M}_h^{ss}$. Consider the fiber product $S' \times_{\mathfrak{M}_h^{ss}(v)} M$. Its projection onto S' is smooth and surjective as we have the following cartesian diagram:

$$
\begin{array}{ccc}
S' \times_{\mathfrak{M}_h^{ss}(v)} M & \longrightarrow & M \\
\downarrow & & \downarrow \\
S' & \longrightarrow & \mathfrak{M}_h^{ss}(v).
\end{array}
$$

Hence, the map from $S' \times_{\mathfrak{M}_h^{ss}(v)} M$ to S' is surjective. Since S' is quasi-compact, there is a quasi-compact open subset U of $S' \times_{\mathfrak{M}_h^{ss}(v)} M$ surjecting onto S'. The image of U in M is quasi-compact. Let V be a quasi-compact open subscheme of M containing it. Since S' surjects onto $\mathfrak{M}_h^{ss}(v)$, so does U. Thus, $V \to \mathfrak{M}_h^{ss}(v)$ is smooth and surjective, and as V is of finite type, so $\mathfrak{M}_h^{ss}(v)$ is of finite type.

This finishes the proof. $\square$

Moreover, note that the stack $\mathfrak{M}_h^{ss}(v)$ contains an open substack of geometrically stable points (see Footnote 5) denoted $\mathfrak{M}_h^{s}(v)$. This again follows from the Step 2 above and using [32] Proposition 2.3.1. Before proceeding to show that the moduli stack $\mathfrak{M}_h^{s}(v)$ admits a coarse moduli space, we state the following result, which we will use in the proof.

Lemma 2.32 ([45], Lemma 2.3.3.3). *Let $\mathcal{X}$ be an algebraic stack, and suppose that $\mathcal{I}(\mathcal{X}) \to \mathcal{X}$ is fppf. Then the big étale sheaf $Sh(\mathcal{X})$ associated to $\mathcal{X}$ is an algebraic space and $\mathcal{X} \to Sh(\mathcal{X})$ is a coarse moduli (algebraic) space.*

Remark 2.33. The coarse moduli space in the lemma above is an algebaric space, but for the theorem below we will show that the coarse moduli space indeed turns out to be a projective scheme.

Now we prove the result on the existence of the coarse moduli space.

THEOREM 2.34. *The algebraic stack $\mathfrak{M}_h^{s}(v)$ admits a coarse moduli space.*

---

[4]The argument is adapted from Mathoverflow's post: https://mathoverflow.net/questions/126560/finite-type-artin-stack-over-mathbb-c.

PROOF. This can be proved using the standard GIT techniques as has been classically done in [**32**] or [**37**] Chapter 10. We sketch a proof along the lines of proof for moduli stack of twisted sheaves by Lieblich in [**45**], which also, in the end, uses GIT. We use Lemma 2.32 above to show we have a coarse moduli space. Let $\mathcal{X} = \mathfrak{M}_h^s(v)$. We need to show that it is faithfully flat, i.e., its surjective and flat. The surjectivity is just by the description of objects of inertia stack (see, for example, [**60**] 8.1.18). Now for flatness just note that we have the following cartesian diagram:

$$
\begin{array}{ccccc}
\mathcal{I}(\mathcal{X}) & \xleftarrow{\;\cong\;} & \mathcal{X} \times \mu_r & \longrightarrow & \mu_r \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle flat} \\
\mathcal{X} & \xleftarrow{\;id\;} & \mathcal{X} & \longrightarrow & S.
\end{array}
$$

From the construction/definition of an inertia stack we have the following cartesian square:

$$
\begin{array}{ccc}
\mathcal{I}(\mathcal{X}) & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow{\scriptstyle \Delta} \\
\mathcal{X} & \xrightarrow{\;\Delta\;} & \mathcal{X} \times \mathcal{X}.
\end{array}
$$

Thus, we just need to show that the diagonal is locally of finite presentation to show that $\mathcal{I}(\mathcal{X}) \to \mathcal{X}$ is fppf. The diagonal morphism is a morphism of locally finite presentation follows from [**1**] Tag 050Q.

Note that we have till now only shown that the coarse moduli space is an algebraic space, to show that its a projective scheme one in the end needs to use GIT theory to conclude the existence of an ample line bundle. We refer the reader to [**45**], Proposition 2.3.3.6 and the discussion above it.   □

The above theorem also implies that $\mathfrak{M}_h^s(v)$ is a μ_r-gerbe. Indeed, recall that for an algebraic stack $\mathcal{X}$, the morphism $\mathcal{X} \to Sh(\mathcal{X})$ is a μ_r-gerbe if and only if $Sh(\mathcal{X})$ is isomorphic to the final object in the topos $Sh(\mathcal{X})$ and the automorphism sheaf is isomorphic to μ_r (see, for example, [**45**] 2.1.1.12). The first condition is obvious and for the second we just need to compute the automorphism sheaf of any point in $\mathfrak{M}_h^s(v)$, but this just corresponds to finding out the automorphisms of a semistable sheaf with fixed determinant line bundle (note that the Mukai vector determines the determinant line bundle) and fixed rank r, which turns out to be μ_r. This gerbe corresponds via 2.25 to a class α_r in $H^2(X, \mu_r)$.

The Kummer exact sequence

$$0 \to \mu_r \to \mathbb{G}_m \to \mathbb{G}_m \to 0$$

induces a long exact sequence of group cohomology, giving us a map $H^2(X, \mu_r) \to H^2(X, \mathbb{G}_m)$. The image of the class α_r in $H^2(X, \mathbb{G}_m)$ gives us a corresponding $\mathbb{G}_m$-gerbe, again using 2.25. This class is the obstruction to the existence of the universal bundle on $Sh(\mathfrak{M}_h^s(v)) \times X$.

THEOREM 2.35 (Mukai-Orlov). *Let X be a K3 surface over a field k.*

(1) *Let $v \in A^*(X)_{num,\mathbb{Q}}$ be a primitive element with $v^2 = 0$ (with respect to the Mukai pairing[5]) and positive degree 0 part[6]. Then $\mathfrak{M}_h^{ss}(v)$ is non-empty.*

---

[5]The Mukai pairing is just an extension of the intersection pairing, defined as follows: let $(a_1, b_1, c_1) \in A^*(X)_{num,\mathbb{Q}}$ and $(a_2, b_2, c_2) \in A^*(X)_{num,\mathbb{Q}}$, then the Mukai pairing is $< (a_1, b_1, c_1), (a_2, b_2, c_2) >= b_2 \cdot b_1 - a_1 \cdot c_2 - a_2 \cdot c_1 \in A^2(X)_{num,\mathbb{Q}}$.

[6]The degree zero part just means the $A^0(X)_{num,\mathbb{Q}}$ term in the representation of the Mukai vector in $A^*(X)_{num,\mathbb{Q}}$.

(2) *If, in addition, there is a complex $P \in D(X)$ with Mukai vector v' such that $< v, v' >= 1$, then every semistable sheaf with Mukai vector v is locally free and geometrically stable[7], in which case $\mathfrak{M}_h^{ss}(v)$ is a μ_r-gerbe for some r, over a smooth projective surface $M_h(v)$[8] such that the associated $\mathbb{G}_m$-gerbe is trivial.*

REMARK 2.36. (1) Note that the triviality of the $\mathbb{G}_m$-gerbe is equivalent to the existence of a universal bundle over $X \times M_h(v)$, also see [**46**] Remark 3.19.

(2) See Remark 6.1.9 [**32**] for a proof that under the assumption of the above Theorem part (2), any semistable sheaf is locally free and geometrically stable.

PROOF. The non-emptiness follows from [**37**] Chapter 10 Theorem 2.7 and [**46**] Remark 3.17. For the construction of the universal bundle, one has to ,in the end, actually use GIT again. For a proof see [**37**] Chapter 10 Proposition 3.4 and [**32**] Theorem 4.6.5 (this is from where we have the numerical criteria, in particular, also see [**32**] Corollary 4.6.7.). □

2.0.2. *The relative moduli space of sheaves.* We generalize our moduli stack to the relative setting. Let X_S be a flat projective scheme over S with an ample line bundle h. (The case of $S = \mathrm{Spec}(R)$ for R a discrete valuation ring of mixed characteristic, will be of most interest to us.)

DEFINITION 2.37. [Relative Moduli Stack] The **relative moduli stack of semistable sheaves**, denoted $\mathfrak{M}_h^{ss}$, is defined as follows:

$$\mathfrak{M}_h^{ss} : (Sch/S) \to (\text{groupoids})$$
$$T \mapsto \{\mathcal{F} | \mathcal{F} \ T\text{-flat coherent sheaf on } X \times_S T \text{ with semistable fibers}\}.$$

The **relative moduli stack of stable sheaves** can be defined similarly and we denoted it by $\mathfrak{M}_h^s$.

In this section, we show the existence of the fine moduli space for the relative moduli stack, when X_R is a relative K3 surface over a mixed characteristic discrete valuation ring, under some numerical conditions. Recall that the condition of flatness is going to be always satisfied in our relative K3's case by definition as they are smooth.

This is again an algebraic stack using the argument given in Theorem 2.30. Moreover, all the results above about the moduli stack hold also for the relative stack. So, there exists a coarse moduli space (Compare from footnote 1 in [**37**] Chapter 10 or [**32**] Thm 4.3.7, the statement there is actually weaker as we do not ask for morphism of k-schemes, which is not going to be possible for mixed characteristic case. So, for the mixed characteristic case one replaces, in the GIT part of the proof, the quot functor by its relative functor, which is representable in this case as well [**57**] Theorem 5.1). Moreover, the non-emptiness results also remain valid in mixed characteristic setting and we have:

THEOREM 2.38 (Fine relative Moduli Space). *Let X_V be a relative K3 surface over a mixed characteristic discrete valuation ring V with X as a special fiber over $\mathrm{Spec}(k)$*

---

[7]A coherent sheaf $\mathcal{F}$ is **geometrically stable** if for any base field extension l/k, the pullback $\mathcal{F} \otimes_k l$ along $X_l = X \times_k \mathrm{Spec}(l) \to X$ is stable.

[8]We will denote this moduli space later as $M_X(v)$ to lay emphasis that it is the moduli space of stable sheaves over X.

(1) Let $v \in A^*(X)_{num,\mathbb{Q}}$[9] be a primitive element with $v^2 = 0$ (with respect to the Mukai pairing) and positive degree 0 part[10]. Then, $\mathfrak{M}_h^{ss}(v)$, the sub-moduli stack of $\mathfrak{M}_h^{ss}$ with fixed Mukai vector v, is non-empty.

(2) If, in addition, there is a complex $P \in D(X_V)$ with Mukai vector v' such that $< v, v' > = 1$, then every semistable sheaf with Mukai vector v is locally free and stable, in which case $\mathfrak{M}_h^{ss}(v)$ is a μ_r-gerbe for some r, over a smooth projective surface $M_h(v)$ such that the associated $\mathbb{G}_m$-gerbe is trivial.

With this we conclude our exposition on moduli stacks and spaces of sheaves and proceed to the last two sections about known results on liftings of K3 surfaces and non-liftablity of automorphisms of K3 surfaces to characteristic zero.

3. Lifting K3 Surfaces to Characteristic Zero

We state the theorem by Deligne about lifting K3 surfaces which will be used a lot in the theorems that follow.

Let X_0 be a K3 surface over a field k of characteristic $p > 0$.

DEFINITION 2.39. [Lift of a K3 surface] A **lift** of a K3 surface X_0 to characteristic 0 is a smooth projective scheme X over R, where R is a discrete valuation ring such that $R/\mathfrak{m} = k$, $K := \text{Frac}(R)$ is a field of characteristic zero, the generic fiber of X, denoted X_K, is a K3 surface and the special fiber is X_0.

THEOREM 2.40 (Deligne [**17**] Theorem 1.6, corollary 1.7, 1.8). *Let X_0 be a K3 surface over a field k algebraically closed of characteristic $p > 0$. Let L_0 be an ample line bundle on X_0. Then there exists a finite extension T of $W(k)$, the Witt ring of k, such that there exists a deformation of X_0 to a smooth proper scheme X over T and an extension of L_0 to an ample line bundle L on X.*

3.1. Hodge Filtration. Consider the situation where we have a lift of a K3 surface, i.e., let X_0 be a K3 surface over a field of characteristic $p > 0$ and X a lift over $S = \text{Spec}(R)$ as defined above. The de Rham cohomology of X/S, $H_{DR}^*(X/S)$ is equipped with a filtration induced from the Hodge to de Rham spectral sequence:

$$E_1^{i,j} = H^j(X, \Omega_{X/S}^i) \Rightarrow H_{DR}^*(X/S)$$

For a construction of this spectral sequence, see [**21**] III-0 11.2. We call this filtration on $H_{DR}^2(X/S)$ the Hodge filtration. Using the comparison isomorphism between the crystalline cohomology of the special fiber and the de Rham cohomology of X [**10**] 7.26.3,

$$H_{crys}^i(X_0/W(k)) \otimes R \cong H_{DR}^i(X/S),$$

we get a filtration on the crystalline cohomology, also called the Hodge filtration. This Hodge filtration on the crystalline cohomology depends on the choice of a lift of X_0.

4. Non-Liftable Automorphisms

In this section, we state the main results of [**19**], which complement the results in this thesis on the liftability of automorphism of a K3 surface over a field of positive characteristic to a K3 surface in characteristic 0. The reader should also compare this section with Chapter 4 section 1, where we discuss liftability of automorphisms

---

[9]Note that in the mixed characteristic setting, for any complex $E_V \in D^b(X_V)$ we define its Mukai vector to be just the Mukai vector of $E := E_V \otimes_V k$ in $A^*(X)_{num,\mathbb{Q}}$. This definition makes sense as $X_V \to V$ is flat.

[10]The degree zero part just means the $A^0(X)_{num,\mathbb{Q}}$ term in the representation of the Mukai vector in $A^*(X)_{num,\mathbb{Q}}$.

as derived autoequivalences and give the p-adic criteria for lifting an automorphism of a K3 surface to a lift of characteristic zero.

Let X be a K3 surface over an algebraically closed field k of positive characteristic $p > 0$. Let $X_R \to \operatorname{Spec}(R)$ be a relative K3 surface lifting X with $X_{\bar{K}}$ the geometric generic fiber. Then one has a **restriction homomorphism** $Aut(X_R/R) \to Aut(X/k)$. Define the subset $Aut^e(X_{\bar{K}}/\bar{K}) \subset Aut(X_{\bar{K}}/\bar{K})$ consisting of those automorphisms which lift to some relative K3 surface $X_R \to R$. Here e stands for extendable. It forms a subgroup under the group law defined after finite base change. The restriction morphism yields a **specialization homomorphism**:

(11) $$\iota : Aut^e(X_{\bar{K}}/\bar{K}) \to Aut(X/k).$$

Note that as automorphisms of X_R are in one-to-one correspondence with that of the associated formal scheme and $H^0(X, T_X) = 0$, the specialization homomorphism ι is injective.

Definition 2.41. An automorphism $f \in Aut(X/k)$ is **geometrically liftable** to characteristic 0 if it is in the image of the specialization homomorphism ι.

The geometric liftability of a subgroup $G \subset Aut(X/k)$ is defined similarly.

THEOREM 2.42 ([**19**], Theorem 5.1). *Let X be a K3 surface over k, an algebraically closed field of characteristic $p > 0$ or X be a supersingular K3 over k algebraically closed with $p > 2$. Assume that either the Picard number of X is ≥ 2 or that $Pic(X) = \mathbb{Z} \cdot H$ and $H^2 \neq 2$. Then there is a discrete valuation ring R, finite over $W(k)$, and a relative K3 surface $X_R \to \operatorname{Spec}(R)$ of X such that no subgroup $G \subset Aut(X)$, except for $G = id_X$, is geometrically liftable to X_R.*

Remark 2.43. This result shows that one can always construct a projective model $X_R \to R$ of a K3 surface X, for which almost all automorphisms are not geometrically liftable. One of the most important point of comparison of the above result with Theorem 5.11 below is in the choice of a model to which the Picard group lifts. Combining the above theorem with Theorem 5.11, shows that the projective model constructed in [**19**] is not the canonical lift for X with Picard rank ≥ 2.

4.1. The curious case of Picard rank 1. Tate conjecture implies that K3 surfaces over $\overline{\mathbb{F}}_p$ always have even Picard rank, so the rank one case does not exist for such K3 surfaces (see [**37**] Corollary 17.2.9). In general, it is still an open question whether there exists a K3 surface over any other algebraically closed field of positive characteristic with Picard rank one.

4.2. The case of supersingular K3 surfaces.

THEOREM 2.44 ([**19**], Theorem 6.4, 7.5). *There exists an automorphism on a supersingular K3 surface in characteristic 3, which has positive entropy, the logarithm of Salem number of degree 22. In particular, it does not lift to characteristic 0. Moreover, in any large characteristic, there is an automorphism of supersingular K3 which has positive entropy and does not lift to characteristic 0.*

Remark 2.45. Supersingular K3 surface have Picard rank equal to 22. On the other hand, over any algebraically closed field of characteristic 0, the maximal possible rank for Picard group of a K3 surface is 20 (see [**37**] section 1.3.3). Thus, for any supersingular K3 surface, there does not exist a Picard preserving lift. Thus, one may expect that the full automorphism group will never lift to any projective model.

CHAPTER 3

Derived Equivalences of K3 Surfaces

In this chapter we give a summary of selected results on derived equivalences of a K3 surfaces for both positive characteristic and characteristic zero. We begin by a general discussion on derived equivalences and then specialize to different characteristics.

Let X be a K3 surface over a field k and let $D^b(X)$ be the bounded derived category of coherent sheaves of X. We refer the reader to [31] for a quick introduction to derived categories and the textbooks [20], [40] for details.

Definition 3.1. Two K3 surfaces X and Y over k are said to be **derived equivalent** if there exists an exact equivalence $D^b(X) \simeq D^b(Y)$ of the derived categories as triangulated categories[1].

Definition 3.2. [Fourier-Mukai Transform] For a perfect complex $\mathcal{P} \in D^b(X \times Y)$, the **Fourier-Mukai transform** is a functor of the derived categories which is defined as follows:

$$\Phi_P : D^b(X) \to D^b(Y)$$

$$\mathcal{E} \mapsto \mathbb{R}p_{Y*}((p_X^* \mathcal{E}) \otimes^{\mathbb{L}} \mathcal{P}),$$

where p_X, p_Y are the projections from $X \times Y$ to the respective X and Y.

Remark 3.3. The boundedness of the derived categories: We restrict to the bounded derived categories as it allows us to employ cohomological methods to study derived equivalences, as explained below.

For details on the properties of Fourier-Mukai transform see [31] Chapter 5. Note that not every Fourier-Mukai transform induces an equivalence. The only general enough criteria available to check whether the Fourier Mukai transform induces a derived equivalence is by Bondol-Orlov, see for example, [37] Chapter 16 Lemma 1.4, Proposition 1.6 and Lemma 1.7. In case the Fourier-Mukai transform is an equivalence, we have the following definition:

Definition 3.4. A K3 surface Y is said to be a **Fourier Mukai partner** of X if there exists a Fourier-Mukai transform between $D^b(X)$ and $D^b(Y)$ which is an equivalence. We denote by $FM(X)$ the set of isomorphism classes of Fourier Mukai Partners of X and by $|FM(X)|$ the cardinality of the set, which is called the **Fourier Mukai number** of X.

We state here the most important result in the theory of Fourier-Mukai transforms and derived equivalences.

THEOREM 3.5 (Orlov, [31] Theorem 5.14). *Every equivalence of derived categories for smooth projective varieties is given by a Fourier Mukai transform. More*

---

[1] We don't need to start with Y being a K3 surface, this can be deduced as a consequence by the existence of an equivalence on the level of derived categories of varieties, see [31] Chapter 4 and Chapter 6 and Chapter 10 and [5] Chapter 2 for the properties preserved by derived equivalences. However, note that Orlov's Representability Theorem 3.5 is used in some proofs.

precisely, let X and Y be two smooth projective varieties and let

$$F : D^b(X) \to D^b(Y)$$

be a fully faithful exact functor. If F admits right and left adjoint functors, then there exists an object $P \in D^b(X \times Y)$ unique up to isomorphism such that F is isomorphic to Φ_P.

REMARK 3.6. This theorem allows us to restrict the collection of derived equivalences to a smaller and more manageable collection of Fourier-Mukai transforms, which will be studied via cohomological descent.

Any Fourier Mukai transform, Φ_P, descends from the level of the derived categories to various cohomological theories ($H^*_{..}(\ )$), as

$$
\begin{array}{ccc}
D^b(X) & \xrightarrow{\mathcal{E} \mapsto \mathbb{R}p_{Y*}((\mathbb{L}p_X^*\mathcal{E}) \otimes^{\mathbb{L}} P)} & D^b(Y) \\
\Big\downarrow{ch(\ )\sqrt{td_X}} & & \Big\downarrow{ch(\ )\sqrt{td_Y}} \\
H^*(X) & \xrightarrow{\alpha \mapsto p_{Y*}((p_X^*\alpha) \cdot ch(P)\sqrt{td_{X\times Y}})} & H^*(Y),
\end{array}
$$

where $ch(\ )$ is the total Chern character and td_X is the Todd genus of X. This descent provides a way to study the Fourier Mukai partners of X using cohomological methods. For details see [**31**] Section 5.2 and [**46**] Section 2.

In characteristic 0 (mostly over $\mathbb{C}$, see remark 3.18 below), we will use the singular cohomology along with p/l-adic/étale cohomology and in characteristic $p > 0$, we will use crystalline cohomology or l-adic etale cohomology. In the mixed characteristic setting, we will be frequently using a different combination of cohomologies along with their comparison theorems from p-adic Hodge theory.

REMARK 3.7. The Orlov Representability Theorem 3.5 works only for smooth projective varieties, so when we work with relative schemes we will restrict from the collection of derived equivalences and work only with the subcollection of Fourier-Mukai transforms.

1. Derived Equivalences of Complex K3 Surfaces

Over the field of complex numbers, Mukai and Orlov provide the full description of the set $FM(X)$ as:

THEOREM 3.8 (Mukai [**55**], Theorem 1.4 and Theorem 1.5, [**61**]). *Let X be a K3 surface over $\mathbb{C}$. Then the following are equivalent:*

(1) There exists a Fourier-Mukai transform $\Phi : D^b(X) \cong D^b(Y)$ with kernel $\mathcal{P}$.

(2) There exists a Hodge isometry $f : \tilde{H}^*(X, \mathbb{Z}) \to \tilde{H}^*(Y, \mathbb{Z})$, where $\tilde{H}^*(\ , \mathbb{Z})$ is the singular cohomology of the corresponding analytic space and is compared with the de Rham cohomology of the algebraic variety X which comes with a Hodge filtrations and Mukai pairing [2].

(3) There exists a Hodge isometry $f : T(X) \simeq T(Y)$ between their transcendental lattices.

(4) Y is a two dimensional fine compact moduli space of stable sheaves on X with respect to some polarization on X, i.e., $Y \cong M_X(v)$ for some Mukai vector $v \in A^*(X)_{num,\mathbb{Q}}$ [3].

---

[2]The Mukai pairing is just an extension of the intersection pairing, defined as follows: let $(a_1, b_1, c_1) \in \tilde{H}^*(X, \mathbb{Z})$ and $(a_2, b_2, c_2) \in \tilde{H}^*(X, \mathbb{Z})$, then the Mukai pairing is $< (a_1, b_1, c_1), (a_2, b_2, c_2) >= b_2 \cdot b_1 - a_1 \cdot c_2 - a_2 \cdot c_1 \in H^4(X, \mathbb{Z})$.

[3]Compare from Definition 2.28.

(5) *There is an isomorphism of Hodge structures between $H^2(M_X(v), \mathbb{Z})$ and $v^\perp/\mathbb{Z}v$ which is compatible with the cup product pairing on $H^2(M_X(v), \mathbb{Z})$ and the bilinear form on $v^\perp/\mathbb{Z}v$ induced by that on the Mukai lattice $\tilde{H}^*(X, \mathbb{Z})$.*

The following corollary provides a useful method of computing the second étale cohomology of any Fourier-Mukai partner.

Corollary 3.9 (p-adic étale cohomology version). *If X and Y are derived equivalent K3 surfaces, then there is an isomorphism between $H^2_{\acute{e}t}(M_X(v), \mathbb{Z}_p)$ and $v^\perp/\mathbb{Z}_p v$, (see footnote [4]), which is compatible with the cup product pairing on $H^2_{\acute{e}t}(M_X(v), \mathbb{Z}_p)$ and the bilinear form on $v^\perp/\mathbb{Z}_p v$ induced by that on the Mukai lattice $\tilde{H}^*(X, \mathbb{Z}_p)$, where p is a prime number and $\mathbb{Z}_p$ is the ring of p-adic integers.*

PROOF. This follows from Artin's Comparison Theorem [**22**] Tome III, Exposé 11, Théorème 4.4 between étale and singular cohomology and the theorem above. □

Proposition 3.10 ([**37**] Proposition 3.10). *Let X be a complex projective K3 surface, then X has only finitely many Fourier-Mukai partners, i.e.,*

$$|FM(X)| < \infty.$$

Remark 3.11. The above result is also true for any algebraically closed field of characteristic 0. Indeed, if X and Y are two K3 surfaces over a field K algebraically closed and characteristic 0, we have $X \cong Y \Leftrightarrow X_\mathbb{C} \cong Y_\mathbb{C}$. One way is obvious via base change and for the other direction we just need to show that every isomorphism $X_\mathbb{C} \cong Y_\mathbb{C}$ comes from an isomorphism $X \cong Y$. To define an isomorphism only finitely many equations are needed, so we can assume that the isomorphism is defined over A, a finitely generated K-algebra (take A to be the ring $K[a_1, \ldots, a_n]$, where a_i are the finitely many coefficients of the finitely many equations defining our isomorphism). Thus, we have have our isomorphism defined over an affine scheme, $X_A \cong Y_A$, where $X_A := X \times_K \operatorname{Spec}(A)$ (resp. $Y_A := Y \times_K \operatorname{Spec}(A)$). As K is algebraically closed, any closed point $t \in \operatorname{Spec}(A)$ has residue field K. Now taking a K-rational point will give us our required isomorphism.

This gives us a natural injection:

$$FM(X) \hookrightarrow FM(X_\mathbb{C})$$
$$Y \mapsto Y_\mathbb{C}.$$

Hence, we have $|FM(X)| \leq |FM(X_\mathbb{C})| < \infty$.

Let $S = NS(X)$ be the Néron-Severi lattice of X. The following theorem gives us the complete counting formula for Fourier-Mukai partners of a K3 surface.

THEOREM 3.12 (Counting formula [**30**]). *Let $\mathcal{G}(S) = \{S_1 = S, S_2, \ldots S_m\}$ be the set of isomorphism classes of lattices with same signature and discriminant as S. Then*

$$|FM(X)| = \sum_{j=1}^{m} |Aut(S_j) \backslash Aut(S_j^*/S_j)/O_{Hdg}(T(X))| < \infty.$$

Next result gives a few explicit counts of the number of Fourier-Mukai partners:

Corollary 3.13 ([**30**], Corollary 2.7). *Let X be a complex K3 surface.*

---

[4]We are abusing the notation here: The Mukai vector is now considered as an element of $H^*_{\acute{e}t}(X, \mathbb{Z}_p)$ and $v^\perp$ is the orthogonal complement of v in $H^*_{\acute{e}t}(X, \mathbb{Z}_p)$ with respect to Mukai pairing. Thus, $v^\perp$ is a $\mathbb{Z}_p$ lattice. Then we mod out this lattice by the $\mathbb{Z}_p$ module generated by v.

(1) *If the Picard rank of X is ≥ 12, then every Fourier-Mukai partner of X is isomorphic to X itself.*

(2) *If the Picard rank of X is ≥ 3 and $\det NS(X)$ is square-free, then every Fourier-Mukai partner of X is isomorphic to X itself.*

(3) *If X is an elliptic K3 surface with a section, then every Fourier-Mukai partner of X is isomorphic to X itself.*

(4) *If the Picard rank of X is 1 and $NS(X) = \mathbb{Z}H$ with $(H)^2 = 2n$, then $|FM(X)| = 2^{\omega(n)-1}$. Here, $\omega(n)$ is the function which counts the number of prime factors of $n \geq 2$, for $n = 1$ we define $\omega(1) = 1$.*

The relation with the class number $h(p)$ of $\mathbb{Q}(\zeta_p)$, for a prime p and ζ_p the pth root of unity, is:

THEOREM 3.14 ([**30**] Theorem 3.3). *Let the rank $NS(X) = 2$ for X, a K3 surface, then $\det NS(X) = -p$ for some prime p, and $|FM(X)| = (h(p)+1)/2$.*

Remark 3.15. The surjectivity of period map 2.8 along with [**37**] Corollary 14.3.1 implies that there exists a K3 with Picard rank 2 and discriminant $-p$, for each prime p (see [**30**] Remark after Theorem 3.3).

2. Derived Autoequivalence Group of Complex K3 Surfaces

In this section, we describe the known results about the derived autoequivalence group $Aut(D^b(X))$ for a K3 surface over $\mathbb{C}$. First observe that Theorem 3.8 implies that we have the following natural map of groups:

$$Aut(X) \hookrightarrow Aut(D^b(X)) \to O_{Hdg}(\tilde{H}^*(X, \mathbb{Z})).$$

We give an description of the second map:

THEOREM 3.16 ([**30**], [**65**]). *Let φ be a Hodge isometry of the Mukai lattice $\tilde{H}^*(X, \mathbb{Z})$ of a K3 surface X, i.e. $\varphi \in O_{Hdg}(\tilde{H}^*(X, \mathbb{Z}))$. Then there exists an autoequivalence*

$$(12) \qquad\qquad \Phi_E : D^b(X) \to D^b(X)$$

with $\Phi_E^H = \varphi \circ (\pm id_{H^2}) : \tilde{H}^*(X, \mathbb{Z}) \to \tilde{H}^*(X, \mathbb{Z})$. In particular, the index of image

$$(13) \qquad\qquad Aut(D^b(X)) \to O_{Hdg}(\tilde{H}^*(X, \mathbb{Z}))$$

is at most 2.

On the other hand, it has been shown that

THEOREM 3.17 ([**33**]). *The cone-inversion Hodge isometry $id_{H^0 \oplus H^4} \oplus -id_{H^2}$ on $\tilde{H}^*(X, \mathbb{Z})$ is not induced by any derived auto-equivalence. In particular, the index of image*

$$(14) \qquad\qquad Aut(D^b(X)) \to O_{Hdg}(\tilde{H}^*(X, \mathbb{Z}))$$

is exactly 2.

Remark 3.18. [[**37**] 16.4.2] The above results have been shown for K3 surfaces over $\mathbb{C}$ only but the results are valid for K3 surfaces over any algebraically closed field of characteristic 0, in the sense made precise below. The argument goes as follows: We reduce the case of $char(k) = 0$ to the case of $\mathbb{C}$. We begin by making the observation that every K3 surface X over a field k is defined over a finitely generated subfield k_0, i.e., there exists a K3 surface X_0 over k_0 such that $X := X_0 \times_{k_0} k$. Similarly, if $\Phi_P : D^b(X) \to D^b(Y)$ is a Fourier Mukai equivalence, then there exists a finitely generated field k_0 such that X, Y and P are defined over k_0. Moreover, the k_0- linear Fourier-Mukai transform induced by P_0, $\Phi_{P_0} : D^b(X_0) \to D^b(Y_0)$ will

again be a derived equivalence (use, for example, the criteria [**31**] Proposition 7.1 to check this.).

Now assume that k_0 is algebraically closed. Note that any Fourier-Mukai kernel which induces an equivalence $\Phi_{P_0} : D^b(X_0) \xrightarrow{\sim} D^b(X_0)$ is rigid, i.e. $\mathrm{Ext}^1(P_0, P_0) = 0$ (see [**37**] Proposition 16.2.1), thus any Fourier-Mukai equivalence

$$\Phi_P : D^b(X_0 \times_{k_0} k) \xrightarrow{\sim} D^b(X_0 \times_{k_0} k)$$

descends to k_0 (see for example [**37**] Lemma 17.2.2 for the case of line bundles, the general case follows similarly[5]). Hence, for a K3 surface X_0 over the algebraic closure k_0 of a finitely generated field extension of $\mathbb{Q}$ and for any choice of an embedding $k_0 \hookrightarrow \mathbb{C}$, which always exists, one has

$$Aut(D^b(X_0 \times_{k_0} k)) \cong Aut(D^b(X_0)) \cong Aut(D^b(X_0 \times_{k_0} \mathbb{C})).$$

In this sense, for K3 surfaces over algebraically closed fields k with $char(k) = 0$, the situation is identical to the case of complex K3 surfaces.

We can now write down the following exact sequence: For X a projective complex $K3$ surface one has

$$(15) \qquad 0 \to \mathrm{Ker} \to \mathrm{Aut}(D^b(X)) \to \mathrm{O}_{\mathrm{Hdg}}(\tilde{H}^*(X, \mathbb{Z}))/\{\mathfrak{i}\} \to 0,$$

where $\tilde{H}^*(X, \mathbb{Z})$ is the cohomology lattice with Mukai pairing and extended Hodge structure, and $\mathrm{O}_{\mathrm{Hdg}}(-)$ is the group of Hodge isometries, $\mathfrak{i}$ is the cone inversion isometry $\mathrm{Id}_{H^0 \oplus H^4} \oplus -\mathrm{Id}_{H^2}$.

The structure of the kernel of this map has been described only in the special case of a projective complex $K3$ surface with $\mathrm{Pic}(X) = 1$ in [**6**]. (For a discussion about the results in non-projective case see [**34**].) However, Bridgeland in [**15**] (Conjecture 1.2) has conjectured that this kernel can be described as the fundamental group of an open subset of $H^{1,1} \otimes \mathbb{C}$. Equivalently, the conjecture says that the connected component of the stability manifold (see [**14**], [**15**] for the definitions) associated to the collection of the stability conditions on $D^b(X)$ covering an open subset of $H^{1,1} \otimes \mathbb{C}$ is simply connected. The equivalence of the two formulations follows from a result of Bridgeland ([**15**] Theorem 1.1), which states that the kernel acts as the group of deck transformations of the covering of an open subset of $H^{1,1} \otimes \mathbb{C}$ by a connected component of the stability manifold. Bayer and Bridgeland [**6**] have verified the conjecture in the special cases of $\mathrm{Pic}(X) = 1$, (see [**34**] for the non-projective case).

Another formulation of the conjecture was provided by Huybrechts ([**35**] Conjecture 5.11), which is stronger in the sense that it implies finite generation of $\mathrm{Aut}(D^b(X))$. Here, the open subset considered above is replaced by an open subset of the classical period domain and the connected component of the stability manifold is replaced by the quotient stack of an open subset of the period domain by an appropriate group action, which can be proved to be a Deligne-Mumford stack.

---

[5]In the general case we sketch the proof: Use the moduli stack of simple universally gluable perfect complexes over $X_0 \times X_0/k_0$, denoted $s\mathcal{D}_{X_0 \times X_0/k_0}$, as defined in Definition 4.10. From the arguments following the definition, it is an algebraic stack which admits a coarse moduli algebraic space $sD_{X_0 \times X_0/k_0}$. Note that for any k_0 point P_0 which induces an equivalence, the local dimension of the coarse moduli space is zero as the tangent space is a subspace of $\mathrm{Ext}^1(P_0, P_0) = 0$ (see, for example, [**44**] 3.1.1 or proof of [**46**] Lemma 5.2) and the coarse moduli space is also smooth. The smoothness follows from the fact that the deformation of the complex is unobstructed (see, for example, [**1**] Tag 03ZB and Tag 02HX) in equi-characteristic case as one always has a trivial deformation. Indeed, let A be any Artinian local k-algebra, then pullback along the structure morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(k)$ gives a trivial deformation of $X \times X$ and also a trivial deformation of any complex on $X \times X$. Thus, we can repeat the argument as in [**37**] Lemma 17.2.2 as now the image of the classifying map $f : \mathrm{Spec}(A) \to sD_{X_0 \times X_0/k_0}$ is constant (In the notation of [**37**] Lemma 17.2.2).

Thus, one obtains a description of the kernel as the "stacky" fundamental group which, in particular, is finitely generated.

Remark 3.19. Note that Bridgeland defines the stability conditions for any small triangulated category, so even in the case of derived category of a $K3$ over a field of positive characteristic we can associate the stability manifold which will still be a complex manifold ([**15**], Remark 3.2).

This completes the section on derived auto-equivalences of a K3 surface over a field of characteristic 0. Next, we proceed to see the positive characteristic version of some of these results proved so far.

3. Derived Equivalences of K3 Surfaces in Positive Characteristic

In this section, we state the main results on derived equivalences of K3 surfaces over an algebraically closed field of positive characteristic known so far. For generalizations of some results to non-algebraically closed fields of positive characteristic see [**71**].

In case, $char(k) = p > 2$, Lieblich-Olsson [**46**], proved the following:

THEOREM 3.20 ([**46**], Theorem 1.1). *Let X be a K3 surface over an algebraically closed field k of positive characteristic $\neq 2$.*

(1) *If Y is a smooth projective k-scheme with $D^b(X) \cong D^b(Y)$, then Y is a K3 surface isomorphic to a fine moduli space of stable sheaves.*

(2) *There exists only finitely many smooth projective k-schemes Y with $D^b(X) \cong D^b(Y)$. If X has rank $NS(X) \geq 12$, then $D^b(X) \cong D^b(Y)$ implies that $X \cong Y$. In particular, any supersingular K3 surface is determined up to isomorphism by its derived category.*

Remark 3.21. One of the open questions is to have a cohomological criteria for derived equivalent K3 surfaces over a field of positive characteristic like we have in characteristic 0 where Hodge theory and Torelli Theorems were available. However, as there is no crystalline Torelli Theorem for non-supersingular K3 surfaces over a field of positive characteristic and the naive F-crystal (see Appendix) fails to be compatible with inner product, the description in terms of F-crystals is not yet possible. Even though one has crystalline Torelli Theorem for supersingular K3 surfaces, it is essentially not providing any more information as there are no non-trivial Fourier-Mukai partners of a supersingular K3 surface.

However, Lieblich-Olsson proved a derived Torelli Theorem using the Ogus Crystalline Torelli Theorem 2.13. First, a definition.

Definition 3.22. [Strongly filtered equivalence] We say that

$$\Phi_P : D^b(X) \to D^b(Y)$$

is **filtered** (resp. **strongly filtered**) if the Chow group realization

$$\Phi^*_{A,num,\mathbb{Q},P} : A^*(X)_{num,\mathbb{Q}} \to A^*(Y)_{num,\mathbb{Q}}$$

preserves the codimension filtration (resp. is filtered, sends $(1, 0, 0)$ to $(1, 0, 0)$, and sends the ample cone of X to plus or minus the ample cone of Y).

Recall that the codimension filtration on the Chow group,

$$A(X)^*_{num,\mathbb{Q}} = A(X)^0_{num,\mathbb{Q}} \oplus A(X)^1_{num,\mathbb{Q}} \oplus A(X)^2_{num,\mathbb{Q}},$$

is given by

$$Fil^0_X = A(X)^*_{num,\mathbb{Q}},$$
$$Fil^1_X = A(X)^1_{num,\mathbb{Q}} \oplus A(X)^2_{num,\mathbb{Q}},$$
$$Fil^2_X = A^2(X)_{num,\mathbb{Q}}.$$

Thus, $\Phi^*_{A,num,\mathbb{Q},P}$ is filtered if and only if $\Phi^*_{A,num,\mathbb{Q},P}(Fil^2_X) = Fil^2_Y$. Indeed, as for X a surface Fil^1_X is the subgroup of elements orthogonal to Fil^2_X. This implies that $\Phi^*_{A,num,\mathbb{Q},P}(Fil^1_X) = Fil^1_Y$ and Fil^0_X is always preserved. Hence, the relation.

THEOREM 3.23 (Derived Torelli, [**47**] Theorem 1.2). *If $\Phi_P : D^b(X) \to D^b(Y)$ is a strongly filtered equivalence, then there exists an isomorphism $\sigma : X \to Y$ such that the cohomological descent of the maps on the crystalline and étale cohomology induced by Φ_P and σ agree.*

Let us already show here that height of a K3 surface is a derived invariant. This will allow us to stay within a subclass of K3 surfaces while checking derived equivalences.

Lemma 3.24. *Height of a K3 surface X over an algebraically closed field of characteristic $p > 3$ is a derived invariant.*

PROOF. Recall that the height of a K3 surface X is given by the dimension of the subspace $H^2_{crys}(X/K)_{[0,1)}$ of the F-isocrystal $H^2_{crys}(X/K)$. Now note that the Frobenius acts on the one dimensional isocrystals $H^0(X/K)(-1)$ and $H^4(X/K)(1)$ (Tate twisted) as multiplication by p (see Appendix below for this computation). This implies that the slope of these F-isocrystals is exactly one. Thus, the F-isocrystal

$$H^*_{crys}(X/K) := H^0(X/K)(-1) \oplus H^2_{crys}(X/K) \oplus H^4_{crys}(X/K)(1)$$

has the same subspace of slope of dimension strictly less than one as that of the F-isocrystal $H^2_{crys}(X/K)$, i.e., $H^*_{crys}(X/K)_{[0,1)} = H^2_{crys}(X/K)_{[0,1)}$.
Note that any derived equivalence of X and Y preserves the F-isocrystal $H^*_{crys}(-/K)$, i.e., if $\Phi_P : D^b(X) \simeq D^b(Y)$ is a derived equivalence of two K3 surfaces X and Y, then the induced map on the F-isocrystals

$$\Phi^*_P : H^*_{crys}(X/K) \to H^*_{crys}(Y/K)$$

is an isometry. Thus, for the height of Y given by $\dim(H^2_{crys}(Y/K)_{[0,1)})$ we have

$$\begin{aligned}
\dim(H^2_{crys}(Y/K)_{[0,1)}) &= \dim(H^*_{crys}(Y/K)_{[0,1)}) \\
&= \dim(H^*_{crys}(X/K)_{[0,1)}) \\
&= \dim(H^2_{crys}(X/K)_{[0,1)}) = \text{height of } X
\end{aligned}$$

Hence the result. □

Remark 3.25.   (1) In characteristic 0, there is no notion of height but in this case, the Brauer group itself is a derived invariant of a K3 surface, as $Br(X) \cong Hom(T(X), \mathbb{Q}/\mathbb{Z})$, where $T(X)$ is the transcendental lattice.
(2) A related question would be: Is Picard rank a derived invariant for K3 surfaces? This is true trivially in characteristic 0 (also see [**37**] Corollary 16.2.8). The answer in characteristic p is also yes, but the proof is not direct, it goes via characteristic 0 for the finite height case using a Picard preserving lift as constructed by Leiblich-Maulik in [**48**] Corollary 4.2. And in the supersingular case it is preserved as any supersingular K3 does not have non-trivial Fourier-Mukai partners[6].

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[6]The author would like to thank Vasudevan Srinivas for pointing it out to her that there is a characteristic independent proof of the fact that the Picard rank is a derived invariant for K3 surfaces: A derived equivalence between K3 surfaces yields an isomorphism between Grothendieck groups and hence also numerical Grothendieck groups, and for a surface, the rank of the numerical Grothendieck group is two more than the Picard rank.

(3) On the other hand, the Picard lattice is not a derived invariant in any characteristic, though it trivially remains invariant in the case of K3 surfaces which do not have non-trivial Fourier-Mukai partners.

Derived Autoequivalences of K3 Surfaces in Positive Characteristic

In this chapter, we compare the deformation of an automorphism as a morphism and as a derived autoequivalence and show that for K3 surfaces these deformations are in one-to-one correspondence. Then we discuss Lieblich-Olsson's results on lifting derived autoequivalences. Then we use these lifting results to prove results on the structure of the group of derived autoequivalences of a K3 surface of finite height over a field of positive characteristic.

1. Obstruction to Lifting Derived Autoequivalences

In this section, we will recall the classical result that for a variety the infinitesimal deformation of a closed sub-variety with a vanishing $H^1(X, \mathcal{O}_X)$ as a closed subscheme is determined by the deformation of its (pushforward of) structure sheaf as a coherent sheaf on $X \times X$. We then use this result to show that on a K3 surface we can lift an automorphism as a automorphism if and only if we can lift it as a perfect complex in the derived category.

Remark 4.1. For a K3 surface this result can also be seen using [**46**] Proposition 7.1 and the p-adic criterion of lifting automorphisms on K3 surfaces [**19**] Remark 6.5.

Remark 4.2. Note that in case of varieties that have $H^1(X, \mathcal{O}_X) \neq 0$, there are more ways of deforming the automorphism as a perfect complex, which in our case is just going to be a coherent sheaf (see below for a proof): For example, an elliptic curve or any higher genus curve.

Let X be a projective variety over an algebraically closed field k of positive characteristic p, $W(k)$ its ring of Witt vectors and $\sigma : X \to X$ an automorphism of X. We put the condition of characteristic $p > 3$ as at many places we may have denominators in factors of 2 and 3, like in the definition of Chern characters for K3 surfaces, and these will become invertible in $W(k)$ due to our assumption on the characteristic.

We begin by recalling an infinitesimal deformation of X.

Definition 4.3. For any Artin local $W(k)$-algebra A with residue field k, an **infinitesimal deformation of X over A** is a proper and flat scheme X_A over A such that the following square is cartesian:

$$
\begin{array}{ccc}
X & \longrightarrow & X_A \\
\downarrow & & \downarrow \\
\mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(A).
\end{array}
$$

Remark 4.4. In case X is smooth, we ask X_A to be smooth over A as well. In this case, X_A is automatically flat over A.

Consider the following two deformation functors:

$$F_{aut} : (\text{Artin local } W(k)\text{-algebras with residue field } k) \to (Sets)$$
(16)
$$A \mapsto \{\text{Lifts of automorphism } \sigma \text{ to } A\},$$

where by lifting of automorphism σ over A we mean that there exists an infinitesimal deformation X_A of X and an automorphism $\sigma_A : X_A \to X_A$ which reduces to σ, i.e., we have the following commutative diagram:

$$
\begin{array}{ccc}
X_A & \xrightarrow{\sigma_A} & X_A \\
\uparrow & & \uparrow \\
X & \xrightarrow{\sigma} & X.
\end{array}
$$

This is the **deformation functor of an automorphism as a morphism**. Now consider the **deformation functor of an automorphism as a coherent sheaf** defined as follows:

$$F_{coh} : (\text{Artin local } W(k)\text{-algebras with residue field } k) \to (Sets)$$
(17)
$$A \mapsto \{\text{Deformations of } \mathcal{O}_{\Gamma(\sigma)} \text{ to } A\}/iso,$$

where by deformations of $\mathcal{O}_{\Gamma(\sigma)}$ to A we mean that there exists an infinitesimal deformation Y_A of $Y := X \times X$ over A and a coherent sheaf $\mathcal{F}_A$, which is a deformation of the coherent sheaf $\mathcal{O}_{\Gamma(\sigma)}$ and $\mathcal{O}_{\Gamma(\sigma)}$ is considered as a coherent sheaf on $X \times X$ via the closed embedding $\Gamma(\sigma) \hookrightarrow X \times X$. Isomorphisms are defined in the obvious way.

Remark 4.5. Note that there are more deformations of $X \times X$ than the ones of the shape $X_A \times_A X'_A$, where X_A and X'_A are deformations of X over A. From now we make a choice of this deformation (Y_A) to be $X_A \times X_A$. Also see Theorem [4.25] and compare from Theorem [4.16] and Remark [4.26] below.

Let X be a smooth projective scheme over k and for A an Artin local $W(k)$-algebra assume that there exists an infinitesimal lift of X to X_A[1]. Observe that there is a natural transformation $\eta : F_{aut} \to F_{coh}$ given by

$$\eta_A : F_{aut}(A) \longrightarrow F_{coh}(A)$$
(18)
$$(\sigma_A : X_A \to X_A) \mapsto \mathcal{O}_{\Gamma(\sigma_A)}/X_A \times X_A.$$

THEOREM 4.6. *The natural transformation* $\eta : F_{aut} \to F_{coh}$ *between the deformation functors is an isomorphism for varieties with* $H^1(X, \mathcal{O}_X) = 0$.

We provide an algebraic proof by constructing a deformation-obstruction long exact sequence connecting the two functors. The proof follows from the following more general proposition 4.8, substituting $X \times X$ for Y and taking the embedding i to be the graph of the automorphism σ. To use proposition 4.8 we need the following lemma.

Lemma 4.7 (Cf. [**27**] Lemma 24.8). *To give an infinitesimal deformation of an automorphism* $f : X \to X$ *over* X_A *it is equivalent to give an infinitesimal deformation of the graph* Γ_f *as a closed subscheme of* $X \times X$.

PROOF. To any deformation f_A of f we associate its graph Γ_{f_A}, which gives a closed subscheme of $X_A \times X_A$. It is an infinitesimal deformation of Γ_f. Conversely, given a deformation Z of Γ_f over A, the projection $p_1 : Z \hookrightarrow X_A \times_A X_A \to X_A$ gives an isomorphism after tensoring with k. From flatness (see, for example, EGA

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[1]Note that such a lift may not always exist but for the case of K3 surfaces of finite height it does, see [**48**] Corollary 4.2 and Theorem 2.40. However, for supersingular K3 surfaces, the lift does not exists over all Artin local rings but in some cases it does exist by Theorem 2.40.

IV, Corollary 17.9.5) of Z over A it follows that p_1 is an isomorphism, and so Z is the graph of $f_A = p_2 \circ p_1^{-1}$. $\square$

Proposition 4.8. *(Cf.* [**27**] *Ex 19.1) Let $i : X \hookrightarrow Y$ be a closed embedding with X integral and projective scheme of finite type over k. Then there exists a long exact sequence*
(19)
$$0 \to H^0(\mathcal{N}_X) \to \operatorname{Ext}_Y^1(\mathcal{O}_X, \mathcal{O}_X) \to H^1(\mathcal{O}_X) \to H^1(\mathcal{N}_X) \to \operatorname{Ext}_Y^2(\mathcal{O}_X, \mathcal{O}_X) \to \dots,$$
where $\mathcal{N}_X$ is the normal bundle of X.

PROOF. Consider the short exact sequence given by the closed embedding i
(20) $$0 \to I \to \mathcal{O}_Y \to i_*\mathcal{O}_X \to 0.$$

Apply the global Hom contravariant functor $\operatorname{Hom}_Y(-, i_*\mathcal{O}_X)$ to the above short exact sequence and we get the following long exact sequence from [**26**] III Proposition 6.4,

$$0 \to \operatorname{Hom}_Y(i_*\mathcal{O}_X, i_*\mathcal{O}_X) \to \operatorname{Hom}_Y(\mathcal{O}_Y, i_*\mathcal{O}_X) \to \operatorname{Hom}_Y(I, i_*\mathcal{O}_X) \to$$
$$\operatorname{Ext}_Y^1(i_*\mathcal{O}_X, i_*\mathcal{O}_X) \to \operatorname{Ext}_Y^1(\mathcal{O}_Y, i_*\mathcal{O}_X) \to \operatorname{Ext}_Y^1(I, i_*\mathcal{O}_X) \to$$
$$\operatorname{Ext}_Y^2(i_*\mathcal{O}_X, i_*\mathcal{O}_X) \to \dots.$$

Now note that we can make the following identifications
  (1) $\operatorname{Hom}_Y(i_*\mathcal{O}_X, i_*\mathcal{O}_X) \cong k$ as X is integral and projective.
  (2) $\operatorname{Hom}_Y(\mathcal{O}_Y, i_*\mathcal{O}_X) = H^0(\mathcal{O}_X) = k$ using [**26**] III Propostion 6.3 (iii), Lemma 2.10 and the fact that X is connected.
  (3) As any injective endomorphism of a field is an automorphism, we can modify the long exact sequence as follows:

$$0 \to \operatorname{Hom}_Y(I, i_*\mathcal{O}_X) \to \operatorname{Ext}_Y^1(i_*\mathcal{O}_X, i_*\mathcal{O}_X) \to \operatorname{Ext}_Y^1(\mathcal{O}_Y, i_*\mathcal{O}_X) \to \dots.$$

  (4) $\operatorname{Hom}_Y(I, i_*\mathcal{O}_X) \cong \operatorname{Hom}_X(i^*I, \mathcal{O}_X)$ using ajunction formula on page 110 of [**26**]. Moreover, using [**26**] III, Proposition 6.9, we have

$$\operatorname{Hom}_X(i^*I, \mathcal{O}_X) = \operatorname{Hom}_X(\mathcal{O}_X, \mathcal{H}om_X(i^*I, \mathcal{O}_X)),$$

and using the discussion in [**1**] Tag 01R1, we have $\mathcal{H}om_X(i^*I, \mathcal{O}_X) = \mathcal{N}_X$. Thus, putting this together with [**26**] III Proposition 6.3 (iii) and Lemma 2.10, we get
$$\operatorname{Hom}_Y(I, i_*\mathcal{O}_X) \cong H^0(\mathcal{N}_X).$$

  (5) Note that again using [**26**] III Proposition 6.3 (iii) and Lemma 2.10, we get
$$\operatorname{Ext}_Y^1(\mathcal{O}_Y, i_*\mathcal{O}_X) \cong H^1(\mathcal{O}_X).$$

  (6) Note that using the adjunction for Hom sheaves we have:
$$i_*\mathcal{N}_X = i_*\mathcal{H}om_X(i^*I, \mathcal{O}_X) \cong \mathcal{H}om_Y(I, i_*\mathcal{O}_X).$$

Thus, $H^1(\mathcal{N}_X) := H^1(X, \mathcal{N}_X) = H^1(Y, i_*\mathcal{N}_X)$ using [**26**] III Lemma 2.10. To compute $H^1(Y, i_*\mathcal{N}_X)$, we choose an injective resolution of $i_*\mathcal{O}_X$ as an $\mathcal{O}_Y$-module $0 \to \mathcal{O}_X \to \mathcal{J}^\bullet$. From [**23**] Proposition 4.1.3, we know that $\mathcal{H}om_Y(I, \mathcal{J}^i)$ are flasque sheaves and so we can compute the cohomology group using this flasque resolution. Hence,

$$H^i = \frac{Ker(\operatorname{Hom}_Y(I, \mathcal{J}^i) \to \operatorname{Hom}_Y(I, \mathcal{J}^{i+1}))}{Im(\operatorname{Hom}_Y(I, \mathcal{J}^{i-1}) \to \operatorname{Hom}_Y(I, \mathcal{J}^i))} = \operatorname{Ext}_Y^i(I, i_*\mathcal{O}_X).$$

Thus, putting all of the above observations together, we get our required long exact sequence. $\square$

PROOF OF THEOREM 4.6: Note that the obstruction spaces for the functors F_{aut} and F_{coh} are $H^1(\mathcal{N}_X)$ and $\text{Ext}^2_Y(\mathcal{O}_X, \mathcal{O}_X)$ respectively. See, for example, [**27**] Theorem 6.2, Theorem 7.3, Exercise 7.4 and Lemma 4.7 above. The same results give us the tangent spaces for the functors F_{aut} and F_{coh} and they are $H^0(\mathcal{N}_X)$ and $\text{Ext}^1_Y(\mathcal{O}_X, \mathcal{O}_X)$. Now using Proposition 4.8 along with our assumption of vanishing $H^1(X, \mathcal{O}_X)$ one has that the obstruction space of F_{aut} is a subspace of the obstruction of F_{coh} and this inclusion sends one obstruction class to the other one. Therefore, the obstruction to lifting the automorphism as a morphism vanishes if and only if the obstruction to lifting the automorphism as a sheaf vanishes. Moreover, the isomorphism of tangent spaces implies that the number of lifts in both cases is same. $\qquad\square$

This shows that for projective varieties with vanishing $H^1(X, \mathcal{O}_X)$, one doesn't have extra deformations of automorphisms as a sheaf. Note that we could still ask for deformations as a perfect complex but since the perfect complex we start with is a coherent sheaf any deformation of it as a perfect complex will also have only one non-zero coherent cohomology sheaf. Indeed, this follows from the fact that deformations cannot grow cohomology sheaves ,as if $F_A^\bullet$ is the deformation of $\mathcal{O}_X$ over A such that $H^1(F_A^\bullet) \neq 0$ (to simplify our argument we are assuming $F_A^\bullet$ is bounded above at level 1, i.e., $F_A^i = 0 \ \forall i > 1$), then we can replace this complex in the derived category by a complex like

$$\ldots \to F_A^{-1} \to Ker(F_A^0 \to F_A^1) \xrightarrow{0} H^1(F_A^\bullet) \to 0.$$

Then reducing to special fiber gives that $H^1(F_A^\bullet) \otimes_A k = 0$, but this will only happen if $H^i(F_A^\bullet) = 0$. Moreover, as we are in the derived category, we can show that the deformed perfect complex is then quasi isomorphic to a coherent sheaf. Indeed, the quotient map to the non-zero coherent cohomology sheaf provides the quasi-isomorphism. This shows that there are no extra deformations as a perfect complex as well. Hence, an automorphism σ on a projective variety X with vanishing $H^1(X, \mathcal{O}_X)$ lifts if and only if the derived equivalence it induces, $\Phi_{\mathcal{O}_{\Gamma(\sigma)}} : D^b(X) \to D^b(X)$, lifts as a Fourier-Mukai transform.

REMARK 4.9.   (1) Note that we cannot claim that the derived equivalence lifts as a derived equivalence because in the relative setting when $\mathcal{X}$ is defined over S, where S is a scheme not equal to $\text{Spec}(k)$, one does not have the Orlov Representability Theorem ([**31**], Theorem 5.14) and therefore, a priori, one cannot say that every derived equivalence comes from a Fourier-Mukai Transform. Thus, a priori, we can possibly lift more things as a derived equivalence.
(2) If we use the infinity category of perfect complexes on X, denoted by $Perf(X)$, in place of the derived category on X, we can then say that an automorphism σ on a projective variety X with vanishing $H^1(X, \mathcal{O}_X)$ lifts if and only if the equivalence $\Phi_{\mathcal{O}_{\Gamma(\sigma)}} : Perf(X) \to Perf(X)$ lifts as an autoequivalence of the infinity category of perfect complexes, as in this case we have a representability Theorem [**8**].

Now we sketch the proofs of the two theorems proved by Lieblich-Olsson which give a criteria to lifting perfect complexes. Before proceeding to that result we state the theory of moduli of complexes which will be used in proving the lifting criteria.

DEFINITION 4.10. The **stack of universally gluable relatively perfect complexes** is defined as follows. Let $Z \to S$ be a proper morphism of finite presentation between schemes. We define a category fibered in groupoids as follows:
The objects over an S-scheme $T \to S$ are the objects $E \in D(\mathcal{O}_{Z_T})$, the derived category of sheaves of $\mathcal{O}_{Z_T}$-modules on $Z_T := Z \times_S T$, such that

(1) (E is relatively perfect over T) For any quasi-compact T-scheme $U \to T$, the complex $E_U := \mathbb{L}p^*(E)$ is quasi-isomorphic to a bounded complex of quasi-coherent $\mathcal{O}_{Z_U}$-modules of finite presentation, where $p : Z_U \to Z_T$ is the natural map.

(2) (E is universally gluable) For every geometric point $s \to S$, we have that $\operatorname{Ext}^i(E_s, E_s) = 0$ for all $i < 0$.

The arrows are the obvious ones.

We will denote this stack by $\mathcal{D}_{Z/S}$.

THEOREM 4.11 ([**44**]). *The stack of universally gluable perfect complexes on a proper scheme is an algebraic stack.*

PROOF. The proof is again checking the conditions of Artin's representability Theorem ([**2**] Theorem 5.3). We refer the reader to [**44**], Theorem on page 2 or to [**1**] Tag 0DLB. $\square$

Remark 4.12. We would like to caution the reader that the Grothendieck Existence Theorem for perfect complexes proved in [**44**], Proposition 3.6.1, is only true with the assumption that $X \to S$ is proper. This gives us the formal effectiveness condition in Artin's representability Theorem.

Recall that an object $E \in \mathcal{D}_{Z/S}(T)$ over some S-scheme T is called **simple** if the map of algebraic spaces $\mathbb{G}_m \to Aut(E)$ is an isomorphism. Moreover from [**44**] 4.3.2, we get that the substack $s\mathcal{D}_{Z/S} \subset \mathcal{D}_{Z/S}$ classifying simple objects is an open substack and, in particular, $s\mathcal{D}_{Z/S}$ is an algebraic stack.

Proposition 4.13 ([**44**] Corollary 4.3.3). *There is a natural map from the classifying stack of simple objects of the stack of universally gluable perfect complexes to an algebraic space locally of finite presentation over S which realizes the stack as a $\mathbb{G}_m$-gerbe.*

PROOF. The proof again follows from the criteria 2.32 above and see the discussion after 2.34 for an argument to show the gerbe structure. $\square$

Now we fix two smooth projective varieties X and Y over a field k and define $\mathfrak{F}$ to be the groupoid of complexes $P \in D^b(X \times Y)$ such that the induced Fourier-Mukai transform is an equivalence.

Lemma 4.14 ([**46**] Lemma 5.2). *Every complex in $\mathfrak{F}$ is a simple object of $\mathcal{D}_Y(X)$. Moreover, any $P \in \mathfrak{F}$ induces a $\mu_P : X \to s\mathcal{D}_Y$ (see below for an explicit description) which on composition with $\pi : s\mathcal{D}_Y \to sD_Y$ gives an open immersion $\bar{\mu}_P : X \to sD_Y$.*

Let us explicitly write down the map $\mu_P : X \to s\mathcal{D}_Y$. As this is a morphisms of stacks, we write down what it does on the groupoid fiber over a k-scheme T:

$$\mu_P(T) : X(T) \to s\mathcal{D}_Y(T)$$
$$(f : T \to X) \mapsto \mathbb{L}f_Y^*P,$$

where $f_Y : Y \times T \to Y \times X$ is the map induced by f on the fiber product with identity on Y.

Now let $s\mathcal{D}_Y(X)^\circ$ be the groupoid of morphisms $\mu : X \to s\mathcal{D}_Y$ such that the composed map

$$X \xrightarrow{\mu} s\mathcal{D}_Y$$

with $\bar{\mu}$ down to sD_Y.

is an open immersion. With this notation, we state the following result which follows directly from the lemma above and the definition of the stack $s\mathcal{D}_Y$.

COROLLARY 4.15 ([**46**], Corollary 5.5). *The map $\mathfrak{F} \to s\mathcal{D}_Y(X)$ defined by sending P to μ_P induces a fully faithful functor of groupoids*

$$\mathfrak{F} \to s\mathcal{D}_Y^\circ.$$

Now we are ready to prove the infinitesimal lifting theorem for Fourier-Mukai kernels.

THEOREM 4.16. [2]*[Infinitesimal Lifting, [**46**] Theorem 6.3] Let X and Y be two K3 surfaces over an algebraically closed field k, and $P \in D^b(Y \times X)$ be a perfect complex inducing an equivalence $\Phi : D^b(Y) \to D^b(X)$ on the derived categories. Assume that the induced map on cohomology (see below) satisfies:*

(1) $\Phi(1, 0, 0) = (1, 0, 0)$,
(2) the induced isometry $\kappa : Pic(Y) \to Pic(X)$ sends C_Y, the ample cone of Y, isomorphically to either C_X or $-C_X$, the $(-)$ample cone of X.

Then there exists an isomorphism of infinitesimal deformation functors $\delta : Def_X \to Def_Y$ such that

(1) $\delta^{-1}(Def_{(Y,L)}) = Def_{(X,\Phi(L))}$;
(2) for each augmented Artinian W-algebra $W \to A$ and each $(X_A \to A) \in Def_X(A)$, there is an object $P_A \in D^b(\delta(X_A) \times_A X_A)$ reducing to P on $Y \times X$.

PROOF. The proof is the one given by Lieblich and Olsson, we just fill in some details.

Given an augmented Artinian W-algebra $W \to A$ and a deformation $X_A \to A$, let $s\mathcal{D}_A$ denote the stack of simple universally gluable relatively perfect complexes with Mukai vector $(0, 0, 1)$ on X_A. Similarly, we define the stack $s\mathcal{D}_k$. Note that the stack $s\mathcal{D}_A$ (resp. $s\mathcal{D}_k$) is smooth over A (resp. k). Indeed, we just need to check the infinitesimal lifting criteria of smoothness (see, for example, [**60**] Section 1.3). For this, we need to note that over a (relative) K3 surface, the obstruction to deforming a simple perfect complex is equal to the obstruction of deforming its determinant line bundle. This can be seen easily using (relative) Serre's duality and simplicity of the perfect complex along with the triviality of cotangent bundle. The determinant line bundle has a vanishing obstruction for a K3 surface.

Now by Corollary 4.15, the kernel P defines an open immersion $Y \to sD_A$ such that the fiber product $\mathbb{G}_m$-gerbe $\mathcal{Y} := Y \times_{sD_k} s\mathcal{D}_k \to Y$ is trivial (the complex P defines a section of this gerbe). Since $Y \to sD_A \otimes k$ is an open immersion and sD_A is smooth over A, we see that the open subscheme Y_A of sD_A is supported on Y and it gives a flat deformation of Y over A, carrying a $\mathbb{G}_m$-gerbe $\mathcal{Y}_A \to Y_A$ (see, for example, Knutson [**41**], Proposition 3.4 in Chapter III).

By Theorem 3.2.4 of [**7**] applied to the lisse-etale topos, a morphism of stacks $\mathcal{Y}_A \to s\mathcal{D}_A$ really does correspond to a relatively perfect universally gluable complex of $\mathcal{O}_{X \times Y}$-modules. However, the $\mathbb{G}_m$ in the inertia stack of $s\mathcal{D}_A$ acts on the universal complex by scalar multiplication. So, a map $\mathcal{Y}_A \to \mathcal{D}_A$ that is the identity on $\mathbb{G}_m$ will correspond to a complex whose action is by scalar multiplication. Now use the following results (see for example Chapter 12 of [**60**]) about complexes on the gerbe $\mathcal{Y}_A \times X_A$:

(1) the object in question is naturally given by a complex of quasi-coherent sheaves,

---

[2]The author would like to thank Martin Olsson and Max Lieblich for patiently answering her emails and questions about the proof of this theorem and Sofia Tirabassi for bringing to notice the reference [**67**] which was greatly helpful in understanding the proof.

(2) the inertial $\mathbb{G}_m$ acts by scalar multiplication on the cohomology sheaves,
(3) the category of quasi-coherent sheaves naturally breaks up as a product indexed by the characters of $\mathbb{G}_m$.

The statement on the cohomological action means that the complex we have is quasi-isomorphic to a complex of twisted quasi-coherent sheaves. Thus, let $\mathcal{P}_A$ be the perfect complex of $\mathcal{Y}_A \times X_A$-twisted sheaves corresponding to the natural inclusion $\mathcal{Y}_A \to s\mathcal{D}_A$. Let $\pi : \mathcal{Y}_A \times X_A \to Y_A \times X_A$. Now we need to untwist the complex constructed.

Claim: There is an invertible sheaf $\mathcal{L}_A$ on $\mathcal{Y}_A \times X_A$ such that the complex $P_A := \mathbb{R}\pi_*(\mathcal{P}_A \otimes \mathcal{L}_A^\vee) \in D(Y_A \times_A X_A)$ satisfies $\mathbb{L}i^* P_A \cong P \in D(Y \times X)$, where $i : Y \times X \hookrightarrow Y_A \times_A X_A$ is the natural inclusion.

Proof of Claim: Consider the complex $\mathcal{Q} := \mathcal{P}_A^\vee \otimes p_2^* \omega_{X_A}[2]$. Pulling back by the morphism $g : Y \times X \to \mathcal{Y}_A \times_A X_A$ corresponding to P yields the equality $\mathbb{L}g^* \mathcal{Q} = P^\vee$ (using the fact that P gives us a section). This implies that $\mathbb{R}(p_1)_*(\mathcal{Q})$ is a perfect complex on $\mathcal{Y}_A$ whose pullback via the section $Y \to \mathcal{Y}$ is $\Phi^{-1}(\mathcal{O}_X)$. As $\Phi(1,0,1) = (1,0,1)$, this complex has rank one and $\mathbb{L}g^* \det \mathbb{R}(p_1)_*(\mathcal{Q}) = \det \mathbb{R}(p_1)_*(P^\vee) \cong \mathcal{O}_Y$. Thus, $P \cong \mathcal{P} \otimes \det \mathbb{R}(p_1)_*(\mathcal{Q})$ and we take $\mathcal{L}_A = \det \mathbb{R}(p_1)_*(\mathcal{Q})^\vee$, finishing the proof of claim.

Now we prove that indeed our scheme Y_A works, then the above claims shows that we are done. The scheme Y_A gives a point of $Def_Y(A)$, giving the functor

$$\delta : Def_X \to Def_Y.$$

Note that using a similar construction with X and Y switched around along with the inverse kernel $P^\vee$ yields a map

$$\delta' : Def_Y \to Def_X,$$

and a lift $P_A^\vee \in D(Y_A \times \delta'(Y_A))$. Composing the two maps gives an endomorphism

$$\eta : Def_X \to Def_X,$$

and for each A-valued point of Def_X, a lift of $P^\vee \circ P$ to a complex $Q_A \in D(X_A \times \eta(X_A))$. Then using the adjunction map we get a quasi-isomorphism $\mathcal{O}_{\Delta_X} \xrightarrow{\sim} P^\vee \circ P$, therefore Q_A is a complex that reduces to the sheaf $\mathcal{O}_{\Delta_X}$ via the identification $\eta(X_A) \otimes k \cong X$. This implies that Q_A is the graph of an isomorphism $X_A \cong \eta(X_A)$, proving that $\delta' \circ \delta$ is an automorphism, hence δ is an isomorphism.

Now suppose Y_A lies in $Def_{Y,L}$, then applying $P_A^\vee$ gives a complex C_A on X_A whose determinant restricts to $\Phi(L)$ on X. It follows that X_A lies in $Def_{(X,\Phi(L))}$, as required. $\qquad\square$

THEOREM 4.17 ([**46**], Theorem 7.1). *Let k be a perfect field of characteristic $p > 0$, W be the ring of Witt vectors of k, and K be the field of fractions of W. Fix K3 surfaces X and Y over k with lifts X_W/W and Y_W/W. These lifts induce corresponding Hodge filtrations via de Rham cohomology on the crystalline cohomology of the special fibers. Denote by $F_{Hdg}^1(X) \subset H^2(X/K) \subset H^*(X/K)$ and $F_{Hdg}^1(Y) \subset H^2(Y/K) \subset H^*(Y/K)$ (similarly for $F_{Hdg}^2(-)$), where $H^*(X/K)$ and $H^*(Y/K)$ are the corresponding Mukai F-isocrystals. Suppose that $P \in D^b(X \times Y)$ is a kernel whose associated functor $\Phi : D^b(X) \to D^b(Y)$ is fully faithful. If*

$$\Phi : H^*(X/K) \to H^*(Y/K)$$

sends $F_{Hdg}^1(X)$ to $F_{Hdg}^1(Y)$ and $F_{Hdg}^2(X)$ to $F_{Hdg}^2(Y)$, then P lifts to a perfect complex $P_W \in D^b(X_W \times_W Y_W)$.

We refer the reader to [**46**] for the proof.

Remark 4.18. Note that however, this is not true infinitesimally. We have the same counterexamples as in the case of infinitesimal integral variational Hodge conjecture: take a line bundle such that $\mathcal{L}^{\otimes p} \neq \mathcal{O}_X$, then we have the Chern character of $\mathcal{L}^{\otimes p}$ is 0 as $p.ch(\mathcal{L}) = 0$, so it lies in the correct Hodge level, but it need not lift. For example: see [11] Lemma 3.10.

Remark 4.19. Note that the lifted kernel also induces an equivalence. Indeed, for a K3 surface fully faithful Fourier-Mukai functor of derived categories is an equivalence (see [31] Proposition 7.6) and so we can also lift the Fourier-Mukai kernel of the inverse equivalence. Then the composition of the equivalence we started with and its inverse will give us a lift of the identity as an derived autoequivalence. But using the fact that the $\mathrm{Ext}^1_{X \times X}(P, P) = 0$ (see Lemma 4.34) for any kernel inducing an equivalence, we get that the lift of the identity is unique and is the identity itself. Thus, the lifted Fourier-Mukai functor is an equivalence.

Corollary 4.20. *Take P to be $\mathcal{O}_{\Gamma(\sigma)}$, where $\sigma : X \to X$ is an automorphism of a K3 surface X over k. Then P lifts to an autoequivalence of $D^b(X_W)$ if and only if σ lifts to an automorphism of X_W if and only if P preserves the Hodge filtration.*

However, we see that we can still lift it as an isomorphism as follows:

Theorem 4.21 (Weak Lifting of Automorphisms). *Let $\sigma : X \to X$ be an automorphism of a K3 surface X defined over an algebraically closed field k of characteristic p. There exists a smooth projective model X_R/R, where R is a discrete valuation ring that is a finite extension of $W(k)$, with X_K its generic fiber such that there is a P_R, a perfect complex in $D^b(X_R \times Y_R)$, reducing to $\mathcal{O}_{\Gamma(\sigma)}$ on $X \times X$, where Y_R is another smooth projective model abstractly isomorphic to X_R (see Remark [4.23] below).*

Proof. We divide the proof into 3 steps:
  (1) Lifting Kernels Infinitesimally: Note that $\Phi_{\mathcal{O}_{\Gamma(\sigma)}}$ is a strongly filtered derived equivalence, i.e.,
$$\Phi^*_{\mathcal{O}_{\Gamma(\sigma)}} = \sigma^* : H^i_{crys}(X/W) \xrightarrow{\sim} H^i_{crys}(X/W)$$
  is an isomorphism which preserves the gradation of crystalline cohomology. Choose a projective lift of X to characteristic zero along with a lift of H_X. It always exists as proved by Deligne [17], i.e., a projective lift (X_V, H_{X_V}) of (X, H_X) over V a discrete valuation ring, which is a finite extension of $W(k)$, the Witt ring over k. Let $V_n := V/\mathfrak{m}^n$ for $n \geq 1$, $\mathfrak{m}$ the maximal ideal of V and let K denote the fraction field of V. Then, for each n, using the lifting criterion above, there exists a polarized lift $(X'_n, H_{X'_n})$ over V_n and a complex $P_n \in D_{Perf}(X_n \times X'_n)$ lifting $\mathcal{O}_{\Gamma(\sigma)}$.
  (2) Applying the Grothendieck Existence Theorem for perfect complexes: By the classical Grothendieck Existence Ttheorem [26], II.9.6, the polarized formal scheme $(\varprojlim X'_n, \varprojlim H_{X'_n})$ is algebraizable. So, there exists a projective lift $(X', H_{X'})$ over V that is the formal completion of $(X'_n, H_{X'_n})$. Now using the Grothendieck Existence Theorem for perfect complexes (see [44] Proposition 3.6.1 [3]) the formal limit of (P_n) is algebraizable and gives a complex $P_V \in D_{Perf}(X_V \times X'_V)$. In particular, P_V lifts $\mathcal{O}_{\Gamma(\sigma)}$ and using Nakayama's lemma, P_V induces an equivalence.
  (3) Now apply the global Torelli Theorem to show that the two models are isomorphic: For any field extension K' over K, the generic fiber complex $P_{K'} \in D^b(X_{K'} \times X'_{K'})$ induces a Fourier-Mukai equivalence $\Phi_{P_{K'}} : D(X_{K'}) \to D(X'_{K'})$. Using Bertholet-Ogus isomorphisms [11], we see

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[3]Compare from Remark 4.12.

that $\Phi_{K'}$ preserves the gradation on de Rham cohomology of $X_{K'}$. Fix an embedding of $K' \hookrightarrow \mathbb{C}$ gives us a filtered Fourier Mukai equivalence

$$\Phi_{P_{\mathbb{C}}} : D^b(X_{K'} \times \mathbb{C}) \to D^b(X'_{K'} \times \mathbb{C}),$$

which in turn induces an Hodge isometry of integral lattices:

$$H^2(X_{K'} \times \mathbb{C}, \mathbb{Z}) \xrightarrow{\sim} H^2(X'_{K'} \times \mathbb{C}, \mathbb{Z}),$$

using Theorem 3.8 and the fact that a filtered equivalence preserves the grading. This implies that $X_{K'} \times \mathbb{C} \cong X'_{K'} \times \mathbb{C}$, which after taking a finite extension V' of V gives that the generic fiber are isomorphic $X_{K'} \cong X'_{K'}$ (we abuse notation to still denote the fraction field of V' by K'). And since the polarization was lifted along, this gives actually a map of polarized K3 surfaces denoted by $f_{K'} : (X_{K'}, H_{X_{K'}}) \xrightarrow{\sim} (X'_{K'}, H_{X'_{K'}})$.

Now we can conclude that the the generic fibers are isomorphic as well by forgetting the polarization. So now we need to show that the models are isomorphic, i.e., $X_{V'} \cong X'_{V'}$, which will follow from the following proposition. $\square$

Proposition 4.22 (Matsusaka-Mumford, [**56**]). *Let X_R and Y_R be two varieties over a discrete valuation ring R with residue field k, X_K and Y_K be their generic fibers defined over K, the fraction field of R, and the special fibers X_k and Y_k be non-singular varieties. Assume that X_K, Y_K, X_k, Y_k are underlying varieties of polarized varieties, $(X_K, H_{X_K}), (Y_K, H_{Y_K}), (X_k, H_{X_k}), (Y_k, H_{Y_k})$ and that the specialization map extends to the polarized varieties. Then, for Y_k not ruled, if there is an isomorphism $f_K : (X_K, H_{X_K}) \to (Y_K, H_{Y_K})$, f_K can be extended to an isomorphism f_R of X_R and Y_R. Moreover, the graph of f_K specializes to the graph of an isomorphism f_k between (X_k, H_{X_k}) and (Y_k, H_{Y_k}).*

Remark 4.23. Note that even though the generic fibers are isomorphic which indeed implies that the models are abstractly isomorphic (via the Matsusaka-Mumford Theorem) but not as models of the special fiber as the isomorphism will not be the identity on the special fiber, just for the simple reason that we started with different polarizations on the special fibers.

Remark 4.24. This dependence on the choice of the lift X_A of X and the ability to find another lift Y_A can be seen as a reformulation of the formula stated below:

Theorem 4.25 ([**36**] Theorem on page 2). *Let P_0 be a perfect complex on a separated noetherian scheme X_0 and let $i : X_0 \hookrightarrow X$ be a closed embedding defined by an ideal I of square zero. Assume that X can be embedded into a smooth ambient space A (for example if X is quasi-projective). Then there exists a perfect complex P on X such that the derived pullback i^*P is quasi-isomorphic to P_0 if and only if*

$$0 = (id_{P_0} \otimes \kappa(X_0/X)) \circ A(P_0) \in Ext^2_{X_0}(P_0, P_0 \otimes I),$$

where $A(P_0)$ is the (truncated) Atiyah class and $\kappa(X_0/X)$ is the (truncated) Kodaira-Spencer class.

Remark 4.26. The above results can be rephrased to say that in the moduli space of lifts of $X \times X$ we cannot always deform the automorphism in the direction of $X_A \times X_A$ but can do so always in the direction of some $X_A \times Y_A$, where X_A and the automorphism determine Y_A uniquely.

Next, we discuss the structure of the derived autoequivalence group of a K3 surface of finite height.

2. The Cone Inversion Map

Let X be a K3 surface over k of finite height with $char(k) = p > 3$.

Definition 4.27. The **positive cone** $\mathcal{C}_X \subset NS(X)_{\mathbb{R}}$ is the connected component of the set $\{\alpha \in NS(X) | (\alpha)^2 > 0\}$ that contains one ample class (or equivalently, all of them).

Definition 4.28. [Cone Inversion map] Let $\mathcal{C}_X$ be the positive cone, the **cone inversion map** on the cohomology is the map that sends the positive cone $\mathcal{C}_X$ to $-\mathcal{C}_X$.

Explicitly, in characteristic 0, we define the map to be $(-id_{H^2}) \oplus id_{H^0 \oplus H^4}$: $\tilde{H}^*(X, \mathbb{Z}) \to \tilde{H}^*(X, \mathbb{Z})$, where $\tilde{H}^*(X, \mathbb{Z})$ is the Mukai lattice ([**31**], section 10.1). Note that the cone inversion map is a Hodge isometry. In characteristic $p > 3$, we define the map to be $(-id_{H^2}) \oplus id_{H^0 \oplus H^4} : H^*_{crys}(X/K) \to H^*_{crys}(X/K)$, where $H^*_{crys}(X/K)$ is the Mukai F-isocrystal (see appendix below). Note that the cone inversion map preserves the Hodge Filtration on $H^2_{crys}(X/K)$.

(In characteristic 0, the following proposition is proved in [**33**] with the Mukai F-crystal replaced with Mukai lattice.).

THEOREM 4.29. *The image of $Aut(D^b(X))$ in $Aut(\mathcal{H}^*_{crys}(X/K))$ has index at least 2, where $\mathcal{H}^*_{crys}(X/K)$ is the Mukai F-isocrystal.*

We prove the above proposition by showing that the cone inversion map on the cohomology does not come from any derived auto-equivalence. The proof is done by contradiction, we assume that such an auto-equivalence exists, then lift the kernel of the derived auto-equivalence to char 0, and then we use the results of [**33**], to get a contradiction that this does not happen.

Recall that we have the following diagram of descend to cohomology of a Fourier-Mukai transform Φ_P, for $P \in D^b(X \times Y)$:

$$
\begin{array}{ccc}
D^b(X) & \xrightarrow{\mathcal{E} \mapsto \mathbb{R}p_{Y*}(p_X^*\mathcal{E}) \otimes^{\mathbb{L}} P)} & D^b(X) \\
\downarrow{ch(\ )} & & \downarrow{ch(\ )} \\
CH^*(X) & \longrightarrow & CH^*(X) \\
\downarrow{\circ\sqrt{td_X}} & & \downarrow{\circ\sqrt{td_Y}} \\
H^*(X) & \xrightarrow{\alpha \mapsto p_{Y*}((p_X^*\alpha) \cdot ch(P)\sqrt{td_{X \times Y}})} & H^*(X),
\end{array}
$$

where $ch(-)$ is the Chern character and td_- is the Todd genus. Before proving the above theorem we state the following two lemmas which will be required for the proof.

Lemma 4.30 ([**48**], Corollary 4.2). *Any K3 surface of finite height over a perfect field k is the closed fiber of a smooth projective relative K3 surfaces $X_W \to \mathrm{Spec}(W(k))$ such that the restriction map $Pic(X_W) \to Pic(X)$ is an isomorphism. Moreover, it preserves the positive cone.*

Lemma 4.31. *For X and X_W as above and X_K the generic fiber of X_W, we have the following commutative diagram:*

$$(21) \qquad \begin{array}{ccc} NS(X) & \xrightarrow{\quad c_1 \quad} & H^2_{crys}(X/K) \\ {\scriptstyle\cong}\Big\uparrow & & \Big\uparrow{\scriptstyle\cong} \\ NS(X_W) & \xrightarrow{\quad c_1 \quad} & H^2_{DR}(X_W/W) \otimes K \\ \Big\downarrow & & \Big\Vert \\ NS(X_K) & \xrightarrow{\quad c_1 \quad} & H^2_{DR}(X_K/K). \end{array}$$

PROOF OF THEOREM 4.29. Assume that the cone inversion map is induced by a derived auto-equivalence. Then using Orlov's representability Theorem ([**61**], [**62**]), we know that this derived auto-equivalence is a Fourier-Mukai transform and we denote the kernel of the transform by $\mathcal{E}$. Since $\mathcal{E}$ induces the cone inversion map and this map preserves the Hodge filtration on the crystalline cohomology, using Theorem 4.17, we know that we can lift the perfect complex $\mathcal{E}$ to a perfect complex $\mathcal{E}_W$ in $D^b(X_W \times X_W)$, where X_W is the lift of X as in Lemma 4.30. Note that the lifted complex also induces a derived equivalence. Indeed, using Nakayama's lemma we see that the adjunction maps $\Delta_* \mathcal{O}_{X_W} \to \mathcal{E}_W \circ \mathcal{E}_W^\vee$ and $\mathcal{E}_W \circ \mathcal{E}_W^\vee \to \Delta_* \mathcal{O}_{Y_W}$ are quasi-isomorphisms. Moreover, since we have $H^*_{crys}(X/W) \cong H^*_{DR}(X_W/W)$, we know that the lifted complex induces again the cone inversion map on the cohomology. It also follows that for any field extension K'/K, the generic fiber complex $\mathcal{E}_{K'} \in D^b(X_{K'} \times_{K'} X_{K'})$ induces a Fourier Mukai equivalence $\Phi : D^b(X_{K'}) \to D^b(X_{K'})$. Choosing an embedding $K \hookrightarrow \mathbb{C}$ (see our conventions [1]) yields a Fourier-Mukai equivalence $D^b(X_K \otimes \mathbb{C}) \to D^b(X_K \otimes \mathbb{C})$ which induces the cone inversion map on $\tilde{H}^*(X, \mathbb{Z})$. This is a contradiction as in characteristic zero this does not happen, see [**33**] for a proof. $\qquad\square$

We now make an interesting observation about the kernel of the map:

Corollary 4.32. *Let X be a K3 surface over k, an algebraically closed field of positive characteristic. Then the kernel of the natural map*

$$0 \to Ker \to Aut(D^b(X)) \to Aut(H^*_{crys}(X/K))$$

lifts. More precisely, assume that X_V be a lift of X over V, a mixed characteristic discrete valuation ring with residue field k, then every derived autoequivalence in the kernel of the map above lifts as an autoequivalence of the derived category of X_V.

PROOF. This is clear as any autoequivalence in the kernel induces the identity automorphism on the cohomology which is bound to respect every Hodge filtration on the F-isocrystal and then we use Theorem 4.17. $\qquad\square$

This allows us to give at least an upper bound on the kernel as follows: Let X be a K3 surface over an algebraically closed field of characteristic $p > 2$. Choose a lift of X, denoted as X_R, such that the Picard rank of the geometric generic fiber is 1. There always exists such a lift as shown by Esnault-Oguiso.

THEOREM 4.33 (Esnault-Oguiso [**19**], Theorem 4.1). *Let X be a K3 surface defined over an algebraically closed field k of characteristic $p > 0$, where $p > 2$ if X is supersingular. Then there is a discrete valuation ring R, finite over the ring of Witt vectors $W(k)$, together with a projective model $X_R \to \mathrm{Spec}(R)$, such that the Picard rank of $X_{\bar{K}}$ is 1, where K is the fraction field of $W(k)$ and $\bar{K} \supset K$ is an algebraic closure.*

Let $\Phi_P : D^b(X) \to D^b(X)$ be a Fourier-Mukai autoequivalence induced by $P \in D^b(X \times X)$ that belong to the kernel of the natural map

$$Aut(D^b(X)) \to Aut(H^*_{crys}(X/K)).$$

We will denote the kernel of this map as Ker_X. Now using the following lemma we see that the set of infinitesimal deformations of the kernel P is a singleton set, which in turn implies that the lift of P to $X_R \times X_R$ (this was just the corollary 4.32) is unique.

Lemma 4.34. *Let X and Y be K3 surfaces over an algebraically closed field k and let $P \in D(X \times Y)$ be a complex defining the Fourier-Mukai equivalence $\Phi_P : D(X) \to D(Y)$. Then $\mathrm{Ext}^1_{X \times Y}(P, P) = 0$.*

PROOF. See [**47**], Lemma 3.7 (ii). □

Next, note that the fiber of the lift of P over the geometric generic point of R, denoted as $P_{\bar{K}}$, also belongs to the kernel of the natural map (again base changed to $\mathbb{C}$ using the embedding $\bar{K} \subset \mathbb{C}$)

$$Aut(D^b(X_{\mathbb{C}})) \to O_{Hdg}(\tilde{H}^*(X_{\mathbb{C}}, \mathbb{Z})),$$

denoted as $Ker_{X_{\mathbb{C}}}$. Indeed, this follows from the base change on cohomology and Berthelot-Ogus's isomorphism [**11**]. Let us assume that $\Phi_{P_{\mathbb{C}}}$ does not induces the identity on the singular cohomology of $X_{\mathbb{C}}$ and hence, using the following natural commutative diagram

$$
\begin{array}{ccc}
H^*(X_{\mathbb{C}}, \mathbb{C}) & \longrightarrow & H^*(X_{\mathbb{C}}, \mathbb{C}) \\
\Big\downarrow{\cong} & & \Big\downarrow{\cong} \\
H^*_{DR}(X_{\mathbb{C}}) & \longrightarrow & H^*_{DR}(X_{\mathbb{C}}),
\end{array}
$$

$\Phi_{P_{\mathbb{C}}}$ also does not induces the identity on the de Rham cohomology of $X_{\mathbb{C}}$. As the autoequivalence $\Phi_{P_{\mathbb{C}}}$ is just the base change of $\Phi_{P_{\bar{K}}}$ we see that the map induced by $\Phi_{P_{\bar{K}}}$ on the de Rham cohomology of $X_{\bar{K}}$ is not the identity. Now again $\Phi_{P_{\bar{K}}}$ comes via base change from Φ_{P_K} so it is not the identity on de Rham cohomology of X_K, now using the Berthelot-Ogus's isomorphism it does not induce the identity on the crystalline cohomology of X but this is not possible as it is a lift of an autoequivalence which induces the identity on the crystalline cohomology. This gives us the following injective map

$$Ker_X \hookrightarrow Ker_{X_{\mathbb{C}}}$$

$$\Phi_P \mapsto \Phi_{P_{\mathbb{C}}}.$$

Now, using the Picard rank 1 lift, we see that Ker_X is a subgroup of the kernel, $Ker_{X_{\mathbb{C}}}$. And this kernel has been described in [**6**] Theorem 1.4. Thus, we have shown that

Proposition 4.35. *Let X be a K3 surface over k, an algebraically closed field of characteristic $p > 3$, and $X_R \to \mathrm{Spec}(R)$ be a Picard rank one lift of X with $X_{\mathbb{C}}$ the base change to $\mathbb{C}$ of the geometric generic fiber of X_R. Here, R is mixed characteristic discrete valuation ring with residue field k. Then $Ker_X \subset Ker_{X_{\mathbb{C}}}$.*

CHAPTER 5

Counting Fourier-Mukai Partners in Positive Characteristic

In this chapter, we count the number of Fourier-Mukai partners of an ordinary K3 surface, in terms of the Fourier-Mukai partners of the geometric generic fiber of its canonical lift. Moreover, we prove that any automorphism of ordinary K3 surfaces lifts to its canonical lift. We start with comparing the Fourier-Mukai partners of a K3 surface over a field of positive characteristic with that of the geometric generic fiber of its lift to characteristic zero. Then we restrict to ordinary K3 surfaces and give a few consequences to lifting automorphisms of ordinary K3 surfaces. Moreover, we give a sufficient condition on derived autoequivalences of an ordinary K3 surface so that they lift to the canonical lift. In the last section, we show that the class number counting formula (compare from Theorem 3.14) also holds for K3 surfaces over a characteristic p field.

1. Fourier-Mukai Partners

Let X (resp. Y) be a regular proper scheme with $D^b(X)$ (resp. $D^b(Y)$) its bounded derived category. Recall that we say that Y is a **Fourier-Mukai partner** of X if there exists a perfect complex $\mathcal{P} \in D^b(X \times Y)$ such that the following map is an equivalence of derived categories:

$$\Phi_P : D^b(X) \xrightarrow{\cong} D^b(Y)$$
$$\mathcal{Q} \mapsto \mathbb{R}p_{Y*}((p_X^* \mathcal{Q}) \otimes^{\mathbb{L}} \mathcal{P}),$$

(22)

where p_X (resp. p_Y) is the projection from $X \times Y$ to X (resp. Y).

We want to count the number of Fourier-Mukai partners of a K3 surface in positive characteristic. We will do this by lifting the K3 surface to characteristic 0 and then counting the Fourier-Mukai partners of the geometric generic fibers. For this we will show that the specialization map for Fourier-Mukai partners defined below is injective and surjective:

$$\{\text{FM partners of } X_{\bar{K}}\} \to \{\text{FM partners of X}\}$$
$$M_{X_{\bar{K}}}(v) \mapsto M_X(v).$$

(23)

Here, X is a K3 surface of finite height over k an algebraically closed field of characteristic $p > 3$, $X_{\bar{K}}$ is the geometric generic fiber of X_W, which is a Picard preserving lift of X, and $M_X(v)$ (resp. $M_{X_{\bar{K}}}(v)$, $M_{X_W}(v)$) is the (fine) moduli space of stable sheaves with Mukai vector v on X (resp. $X_{\bar{K}}$, X_W). Note that from now on we will fix one such lift of X. Such a lift always exists by Lemma 4.30 for K3 surfaces of finite height. On the other hand, Theorem 3.20 shows that supersingular K3 surfaces have no nontrivial Fourier-Mukai partners, so from now we restrict to the case of K3 surfaces of finite height.

To show that the map (23) is well defined, we need the following lemma:

Lemma 5.1 ((Potentially) Good reduction). *([**47**] Theorem 5.3) Let V be a discrete valuation ring with a fraction field K, a field of characteristic 0, and residue*

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*field k of characteristic p such that there is a K3 surface X_K over K with good
reduction, then all the Fourier-Mukai partners of $X_{\bar{K}}$ have good reduction possibly
after a finite extension of K.*

PROOF. From 3.20 (1), we get that after replacing V by a finite extension Y_K
is isomorphic to a moduli space of sheaves on X_K. After replacing V by a finite
extension we may assume that we have a complex $P \in D^b(X_V \times Y_V)$ defining an
equivalence, $\Phi_P : D^b(X_K) \to D^b(Y_K)$, with $E \in D^b(Y_V \times X_V)$ the complex defining
the inverse equivalence $\Phi_E : D^b(Y_K) \to D^b(X_K)$ to Φ_P. Let $v := \Phi_E(0, 0, 1) \in
A^*(X_K)_{num,\mathbb{Q}}$ be the Mukai vector of a fiber of E at a closed point $y \in Y_K$, then v
can be written in the form

$$v = (r, [L_{X_V}], s) \in A^0(X_K)_{num,\mathbb{Q}} \oplus A^1(X_K)_{num,\mathbb{Q}} \oplus A^2(X_K)_{num,\mathbb{Q}}.$$

Then using [**46**] 8.1, we assume that r is prime to p and that L_{X_V} is very ample
though after possibly changing our choice of P, which may involve another extension
of V. Taking another extension of V, if required, we assume that v is defined over
V, and therefore by specialization also defines an element, denoted by the same
letter,

$$v = (r, [L_{X_V}], s) \in \mathbb{Z} \oplus Pic(X) \oplus \mathbb{Z}.$$

Now, note that this class has the property that r is prime to p and there exists
another class v such that $< v, v >= 1$. This, in particular, implies that v restricts to
a primitive class on the closed fiber. Fix an ample class h on X_V, and let $\mathfrak{M}_{X_V}(v)$
denote the moduli stack of semistable sheaves on X_V with Mukai vector v. By 2.38
the stack $\mathfrak{M}_{X_V}(v)$ is a μ_r-gerbe over a relative K3 surface $M_{X_V}(v)/V$, and by [**46**]
8.2, we have $Y_K \cong M_{X_V}(v)_K$. In particular, Y has potentially good reduction. $\square$

Thus for any Fourier-Mukai partner of $X_{\bar{K}}$ which is of the form $M_{X_{\bar{K}}}(v)$ is a
geometric generic fiber of $M_{X_V}(v)/V$, where V is a finite (algebraic) extension of
$W(k)$. Note that the residue field of V is still k as k is algebraically closed. Now
using functoriality of the moduli functor we note that the special fiber of $M_{X_V}(v)$
is $M_X(v)$. This is a Fourier-Mukai partner of X (see, for example, 3.20). Thus, the
map (23) is well-defined.

Proposition 5.2 (Lieblich-Olsson [46]). *The specialization map (23) above is sur-
jective.*

PROOF. From [**46**] Theorem 3.16, note that all Fourier-Mukai partners of X
are of the form $M_X(v)$. Moreover, one can always assume v to be of the form
(r, l, s) where l is the Chern class of a line bundle and r is prime to p (see [**46**],
Lemma 8.1). (Note that we take the Mukai vector here in the respective Chow
groups rather than cohomology groups). Then since we have chosen our lift X_W of
X to be Picard preserving, we can also lift the Mukai vector to (r_W, l_W, s_W), again
denoted by v, and this gives a FM partner of X_W, namely $M_{X_W}(v)$, and taking the
geometric generic fiber of it gives a Fourier-Mukai partner of $X_{\bar{K}}$. $\square$

Remark 5.3. Note that the $Pic(X_{\bar{K}}) \cong Pic(X)$, i.e., the specialization map is
an isomorphism. This is essentially due to the fact that k is algebraically closed
and every line bundle on X lifts uniquely to X_W as $Ext^1(L, L) = H^1(X, \mathcal{O}_X) = 0$
for $L \in Pic(X)$, under which the set of infinitesimal deformations of the line bundle
L is a torsor.

Remark 5.4. Note that the argument above already implies that the number of
Fourier-Mukai partners of a K3 surface over an algebraically closed field of charac-
teristic $p > 3$ is finite. This argument was given by Lieblich-Olsson in [**46**].

Injectivity: We need to show that if $M_X(v) \cong X$, then $M_{X_W}(v) \cong X_W$. For
this statement we will restrict to the case of ordinary K3 surfaces.

2. Fourier-Mukai Partners of Ordinary K3 Surfaces

We recall some results about ordinary K3 surfaces and their canonical lifts as proved by Nygaard in [**59**] and [**58**], and by Deligne-Illusie in [**18**].

Definition 5.5. [Ordinary K3 surface] A K3 surface X over a perfect field k of positive characteristic is called **ordinary** if the height of X is 1.

Proposition 5.6. *The following are equivalent:*

(1) X is an ordinary K3 surface,
(2) The height of formal Brauer group is 1,
(3) The Frobenius $F : H^2(X, \mathcal{O}_X) \to H^2(X, \mathcal{O}_X)$ is bijective.

We refer to [**59**] Lemma 1.3 for a proof of this proposition.

Let A be an Artin local ring with residue field k and let X_A/A be a lifting of the ordinary K3 surface X/k. In [**3**] Artin-Mazur showed that the enlarged Brauer group Ψ_{X_A} defines a p-divisible group on $\mathrm{Spec}(A)$ lifting Ψ_X/k.

Theorem 5.7 (Nygaard [**59**], Theorem 1.3). *Let X/k be an ordinary K3 surface. The map*

$$\{\text{Iso. classes of liftings } X_A/A\} \to \{\text{Iso. classes of liftings } G/A\}$$

defined by

$$X_A/A \mapsto \Psi_{X_A}/A$$

is a functorial isomorphism.

Recall that the enlarged Brauer group of a K3 surface fits in the following exact sequence ([**3**] Proposition IV.1.8):

$$(24) \qquad 0 \to \Psi_X^0 (= \hat{Br}_X) \to \Psi_X \to \Psi^{ét} \to 0.$$

As the height one formal groups are rigid, there is a unique lifting G_A^0 of Ψ_X^0 to A. Similarly, the étale groups are rigid as well, so there is a unique lift $G_A^{ét}$ of $\Psi_X^{ét}$ to A. This implies that if G is any lifting of Ψ_X to A, then we have an extension

$$0 \to G_A^0 \to G \to G_A^{ét} \to 0$$

lifting the extension

$$0 \to \Psi_X^0 \to \Psi_X \to \Psi_X^{ét} \to 0.$$

Therefore, the trivial extension $G = G_A^0 \times G_A^{ét}$ defines a unique lift $X_{can,A}/A$ of X/k such that $\Psi_{X_{can,A}} = G_A^0 \times G_A^{ét}$. Take $A = W_n$ and $X_n = X_{can,W_n}$, then we get a proper flat formal scheme $\{X_n\}/Spf W$.

Theorem 5.8 (Definition of **Canonical Lift**). *The formal scheme $\{X_n\}/Spf W$ is algebraizable and defines a K3 surface $X_{can}/\mathrm{Spec}(W)$.*

This theorem was proved by Nygaard in [**59**], Proposition 1.6.

One of the nice properties of the canonical lift is that it is a Picard lattice preserving lift.

Proposition 5.9 (Nygaard, [**59**], Proposition 1.8). *The canonical lift X_{can} has the property that any line bundle on X lifts uniquely to X_{can}.*

Next, we state a criteria for a lifting of an ordinary K3 surface to come from the canonical lift. This is the criteria that we will be using to determine that our lift is canonical.

Theorem 5.10 (Taelman [**70**] Theorem C). *Let $\mathcal{O}_K$ be a discrete valuation ring with perfect residue field k of characteristic p and fraction field K of characteristic 0. Let $X_{\mathcal{O}_K}$ be a projective K3 surface over $\mathcal{O}_K$ with $X_{\bar{K}}$ the geometric generic fiber and assume that $X := X_{\mathcal{O}_K} \otimes k$, the special fiber, is an ordinary K3 surface. Then the following are equivalent:*

(1) $X_{\mathcal{O}_K}$ is the base change from $W(k)$ to $\mathcal{O}_K$ of the canonical lift of X,

(2) $H^2_{et}(X_{\bar{K}}, \mathbb{Z}_p) \cong H^0 \oplus H^1(-1) \oplus H^2(-2)$ with H^i unramified $\mathbb{Z}_p[Gal_K]$-modules, free of rank 1, 20, 1 over $\mathbb{Z}_p$ respectively.

Here, the (-1) and (-2) denote Tate twists.

We now prove that the automorphisms of an ordinary K3 surface lift always to characteristic zero.

THEOREM 5.11. *Every isomorphism $\varphi : X \to Y$ of ordinary K3 surfaces over an algebraically closed field of characteristic p lifts to an isomorphism of the canonical lift of the ordinary K3's $\varphi_W : X_{can} \to Y_{can}$. In particular, every automorphism of X lifts to an automorphism of X_{can}.*

Remark 5.12. Note that the above statement is stronger than the tautological statement: If X and X' are two isomorphic ordinary K3 surfaces over a perfect field k, then their canonical lifts are isomorphic.

Remark 5.13. This statement should be compared with the result of Esnault-Oguiso [19], who constructed automorphisms which do not lift to characteristic 0, see Section 4 above.

PROOF OF THEOREM 5.11. Let $\varphi : X \to Y$ be an isomorphism of ordinary K3 surfaces. Consider the graph of this isomorphism as a coherent sheaf (or even as a perfect complex) on the product $X \times Y$, then from Theorem 4.6 the deformation of isomorphism as a morphism and as a sheaf are equivalent so we use Theorem 4.16 to construct a lifting of the isomorphism for the canonical lift X_{can} of X. As isomorphisms preserve the ample cone, the induced Fourier-Mukai transform satisfies the assumptions of Theorem 4.16. Note that the Lieblich-Olsson lifting of perfect complexes allows us to be only able to choose the lifting of X and then it constructs a unique lifting Y' of Y to which the perfect complex lifts. So, now the only remaining statement to show is that Y' is the canonical lift of Y. This follows from the criteria of canonical lift Theorem 5.10 and the observation that the isomorphism between $\varphi_{\bar{K}} : X_{can,\bar{K}} \to Y'_{\bar{K}}$ induces an isomorphism of Galois module on the second p-adic étale cohomology. This isomorphism of Galois modules provides us with the required decomposition of $H^2_{et}(Y'_{\bar{K}}, \mathbb{Z}_p)$, which shows that Y' is the canonical lift of Y. □

Remark 5.14. This gives a fixed point of the δ functor constructed by [46] (see Theorem 4.16).

Corollary 5.15. *Every isomorphism of ordinary K3 surfaces over an algebraically closed field of characteristic p preserves the Hodge filtration induced by the canonical lift. In particular, the automorphisms as well.*

PROOF. This follows from 5.11 and [19] Remark 6.5. □

THEOREM 5.16. [1] *Let X be an ordinary K3 surface, then the canonical lift of the moduli space of stable sheaves with a fixed Mukai vector is the moduli space of stable sheaves with the same Mukai vector on the canonical lift:*

(25) $$(M_X(v))_{can} \cong M_{X_{can}}(v).$$

PROOF. We use the criteria for canonical lift Theorem 5.10 to show that $M_{X_{can}}(v)$ is indeed the canonical lift of $M_X(v)$. To use the criteria, we note that

$$H^2_{\acute{e}t}(M_{X_{can}}(v)_{\bar{K}}, \mathbb{Z}_p) = v^\perp / v\mathbb{Z}_p$$
$$\subset H^0_{\acute{e}t}(X_{can,\bar{K}}, \mathbb{Z}_p) \oplus H^2_{\acute{e}t}(X_{can,\bar{K}}, \mathbb{Z}_p) \oplus H^4_{\acute{e}t}(X_{can,\bar{K}}, \mathbb{Z}_p),$$

---

[1]The author would like to thank Francios Charles for the helpful discussion which lead to completion of this proof.

where the orthogonal complement is taken with respect to the extended pairing on the étale Mukai lattice. As X_{can} is the canonical lift of X, we have the following decomposition of

$$H^2_{ét}(X_{can,\bar{K}}, \mathbb{Z}_p) = M^0_X \oplus M^1_X(-1) \oplus M^2_X(-2)$$

as Galois modules. We define the decomposition of $H^2_{ét}(M_{X_{can}}(v)_{\bar{K}}, \mathbb{Z}_p) = M^0 \oplus M^1(-1) \oplus M^2(-2)$ as Galois modules, where

$$M^0 = M^0_X$$

(26) $$M^2 = M^2_X$$

$$M^1 = H^0_{ét}(X_{can,\bar{K}}, \mathbb{Z}_p) \oplus H^4_{ét}(X_{can,\bar{K}}, \mathbb{Z}_p) \oplus (v^\perp / v\mathbb{Z}_p \cap M^1_X).$$

The last relation above holds using Corollary 3.9 and the fact that $H^0_{ét}(X_{can,\bar{K}}, \mathbb{Z}_p)$ and $H^4_{ét}(X_{can,\bar{K}}, \mathbb{Z}_p)$ are orthogonal to M^1_X. $\square$

THEOREM 5.17. *If X is an ordinary K3 surface over an algebraically closed field of char p, then the number of FM partners of X are the same as the number of Fourier-Mukai partners of the geometric generic fiber of the canonical lift of X over W.*

PROOF. From the discussion in the Chapter 5 Section 1, we see that all that is left to show is the injectivity of the specialization map on the set of Fourier-Mukai partners. That is, we need to show that if $M_X(v)$ is isomorphic to X, then the lifts of both of them are also isomorphic $X_{can} \cong M_{X_{can}}(v)$. This follows from the definition of canonical lifts and Theorem 5.16 that $M_{X_{can}}(v)$ is the canonical lift of $M_X(v)$. $\square$

Corollary 5.18. *Let X be an ordinary K3 surface over k, then the derived autoequivalences satisfying the assumptions of Theorem 4.16 lift uniquely to a derived autoequivalence of X_{can}.*

PROOF. The argument is going to be similar to the one used to show that every automorphism lifts, but now we will use the proof of Theorem 5.16. Let $\mathcal{P} \in D^b(X \times X)$ induce a derived autoequivalence on X, then, using Theorem 4.16, there exists an X'/W such that we can lift $\mathcal{P}$ to a kernel $\mathcal{P}_W \in D^b(X_{can} \times X')$. Now we need to show that X' is just X_{can}. Note that $(\mathcal{P}_W)_{\bar{K}}$ gives a derived equivalence between $D^b(X_{can,\bar{K}}) \cong D^b(X'_{\bar{K}})$, this implies that X' is isomorphic to some moduli space of stable sheaves with Mukai vector v, $M_{X_{can,\bar{K}}}(v)$. Now by functoriality of the moduli spaces, we have $M_{X_{can,\bar{K}}}(v) \cong M_{X_{can}}(v)_{\bar{K}}$ and by Theorem 5.16, we have $M_{X_{can}}(v)_{\bar{K}} \cong M_X(v)_{can,\bar{K}}$. This implies that we get the required decomposition of the second p-adic integral étale cohomology of $X'_{\bar{K}}$, which using Theorem 5.10 gives us the result. $\square$

Corollary 5.19. *Every autoequivalence of an ordinary K3 surface that satisfies the assumptions of Theorem 4.16 preserves the Hodge filtration induced by the canonical lift.*

PROOF. Follows from the corollary above and Theorem 4.17. $\square$

3. The Class Number Formula

In this last section, we give the corresponding class number formula in characteristic p to corollary 3.14.

THEOREM 5.20. *Let X be a K3 surface of finite height over an algebraically field of positive characteristic (say $q > 3$). If the Néron-Severi lattice of X has rank 2 and determinant $-p$ (p and q can also be same), then the number of Fourier-Mukai partners of X is $(h(p) + 1)/2$.*

PROOF. We lift X to characteristic 0 using the Lieblich-Maulik Picard preserving lift and then base changing to the geometric generic fiber to get $X_{\bar{K}}$. Choose an embedding of $\bar{K}$ to $\mathbb{C}$ (complex numbers) and base change to $\mathbb{C}$, to get $X_{\mathbb{C}}$. Now, from Proposition 5.2, we get that every Fourier-Mukai partner of X lifts to a Fourier-Mukai partner of $X_{\mathbb{C}}$. So, we just need to show that if any Fourier-Mukai partner, say $Y_{\mathbb{C}}$, of $X_{\mathbb{C}}$ reduces mod q to an isomorphic $K3$ surface, say Y, to X, then it is isomorphic to $X_{\mathbb{C}}$. This follows from noting that if $Y_{\mathbb{C}}$ becomes isomorphic mod q, then the Picard lattices of $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$ are isomorphic. The number of Fourier Mukai partners of $X_{\mathbb{C}}$ with isomorphic Picard lattices is given by the order of the quotient of the orthogonal group of discriminant group of $NS(X_{\mathbb{C}})$ by the Hodge isometries of the transcendental lattice (cf. Theorem 3.12), but in this case the discriminant group of $NS(X_{\mathbb{C}}) = \mathbb{Z}/p$ so the orthogonal group is just $\pm id$ and there is always $\pm id$ in the hodge isometries, so we get the quotient to be a group of order 1. Thus the result. $\qquad\square$

Remark 5.21. Note that the Picard lattice $Pic(X_K)$ and $Pic(X_{\bar{K}})$ are indeed isomorphic as after reduction we are over an algebraically closed field and the line bundles lift uniquely as Pic_X^0 is trivial for a K3 surface.

CHAPTER 6

Appendix: F-crystal on Crystalline Cohomology

In this appendix, we analyze the possibility of having a "naive" F-crystal structure on the Mukai isocrystal of a K3 surface. We begin by recalling a few results about crystalline cohomology and the action of Frobenius on it, for details we refer to [**1**] Tag 07GI and Tag 07N0, [**9**], [**11**], [**50**] Section 1.5.

Let X be a smooth and proper variety over a perfect field k of positive characteristic p. Let $W(k)$ (resp. $W_m(k)$) be the associated ring of (resp. truncated) Witt vectors with the field of fraction K. Let us denote by $Frob_k : k \to k \ x \mapsto x^p$, the Frobenius morphism of k, which induces a ring homomorphism $Frob_W : W(k) \to W(k)$, by functoriality, and there exists an additive map $V : W(k) \to W(k)$ such that $p = V \circ Frob_W = Frob_W \circ V$. Thus, $Frob_W$ is injective. For any $m > 0$, we have cohomology groups $H^*_{crys}(X/W_m(k))$. These are finitely generated $W_m(k)$-modules. Taking the inverse limit of these groups gives us the crystalline cohomology:

$$H^n_{crys}(X/W(k)) := \varprojlim H^n_{crys}(X/W_m(k)).$$

It has the following properties as a Weil cohomology theory:

(1) $H^n_{crys}(X/W(k))$ is a contravariant functor in X and the groups are finitely generated as $W(k)$-modules. Moreover, $H^n_{crys}(X/W(k))$ is 0 if $n < 0$ or $n > 2dim(X)$.

(2) Poincaré Duality: The cup-product induces a perfect pairing:

$$(27) \quad \frac{H^n_{crys}(X/W(k))}{torsion} \times \frac{H^{2dim(X)-n}_{crys}(X/W(k))}{torsion} \to H^{2dim(X)}_{crys}(X/W(k)) \cong W(k).$$

(3) $H^n_{crys}(X/W(k))$ defines an integral structure on $H^n_{crys}(X/W(k)) \otimes_{W(k)} K$.

(4) If there exists a proper lift of X to $W(k)$, that is, a smooth and proper scheme $X_W \to \mathrm{Spec}(W(k))$ such that its special fiber is isomorphic to X. Then we have, for each n,

$$H^n_{DR}(X_W/W(k)) \cong H^n_{crys}(X/W(k)).$$

(5) Consider the commutative square given by absolute Frobenius:

$$\begin{array}{ccc} X & \xrightarrow{F} & X \\ \downarrow & & \downarrow \\ k & \xrightarrow{Frob_k} & k. \end{array}$$

This, by the functoriality of the crystalline cohomology, gives us a $Frob_W$-linear endomorphism on $H^i(X/W)$ of $W(k)$-modules, denoted by F^*. Moreover, F^* is injective modulo the torsion, i.e.,

$$F^* : H^i(X/W)/torsion \to H^i(X/W)/torsion$$

is injective.

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THEOREM 6.1 (Crystalline Riemann-Roch). *Let X and Y be smooth varieties over k, a field of characteristic p, and $f : X \to Y$ be a proper map. Then the following diagram commutes:*

$$
\begin{array}{ccc}
K_0(X) & \xrightarrow{\ f_*\ } & K_0(Y) \\
\downarrow{\scriptstyle ch(\ ).td_X} & & \downarrow{\scriptstyle ch(\ ).td_Y} \\
\oplus_i H^{2i}_{crys}(X/K) & \xrightarrow{\ f_*\ } & \oplus_i H^{2i}_{crys}(Y/K),
\end{array}
$$

i.e., $ch(f_*\alpha).td_Y = f_*(ch(\alpha).td_X) \in \oplus_i H^i_{crys}(Y/K)$ for all $\alpha \in K_0(X)$, where $K_0(X)$ is the Grothendieck group of coherent sheaves on X.

Remark 6.2. The map f_* does not preserve the cohomological grading but does preserve the homological grading, i.e., if the dimensions of X and Y are n and m respectively, then we have the following commutative square:

$$
\begin{array}{ccc}
K_0(X) & \xrightarrow{\ f_*\ } & K_0(Y) \\
\downarrow{\scriptstyle ch(\ ).td_X} & & \downarrow{\scriptstyle ch(\ ).td_Y} \\
\oplus_i H^{2i}_{crys}(X/K) & \xrightarrow{\ f_*\ } & \oplus_i H^{2i+(n-m)}_{crys}(Y/K),
\end{array}
$$

and here the grading is respected. If X and Y are K3 surfaces, then $n = m = 2$ and we do not have to worry about this remark, as then the usual cohomological grading is preserved.

Next we state a few main results about the compatibility of the Frobenius action with the various relations :

Proposition 6.3 (Künneth Formula for the crystalline cohomology, [9] Chapitre 5, Théorème 4.2.1 and [38] Section 3.3). *Let X, Y be proper and smooth varieties over k. Then there is a canonical isomorphism in $D(W)$, the derived category of W modules, given as follows:*

$$
\mathbb{R}\Gamma(X/W) \otimes^{\mathbb{L}}_W \mathbb{R}\Gamma(Y/W) \cong \mathbb{R}\Gamma(X \times_k Y/W),
$$

yielding exact sequences

$$
0 \to \oplus_{p+q=n}(H^p(X/W) \otimes H^q(Y/W)) \to H^n(X \times Y/W) \to
$$
$$
\to \oplus_{p+q=n+1} Tor^W_1(H^p(X/W), H^q(Y/W)) \to 0.
$$

Remark 6.4. Note that in the case of K3 surfaces the torsion is zero, so we have the following isomorphism:

$$
\oplus_{p+q=n}(H^p(X/W) \otimes H^q(Y/W)) \xrightarrow{\sim} H^n(X \times Y/W).
$$

The action of Frobenius gives the following map:

$$
\begin{array}{ccc}
F^*H^n(X \times Y/W) & \lhook\joinrel\xrightarrow{\hspace{4cm}} & H^n(X \times Y/W) \\
\downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} \\
\oplus_{p+q=n}(F^*H^p(X/W) \otimes F^*H^q(Y/W)) & \lhook\joinrel\longrightarrow & \oplus_{p+q=n}(H^p(X/W) \otimes H^q(Y/W)).
\end{array}
$$

Proposition 6.5. *The Künneth formula is compatible with the Frobenius action in the following way:*
Let $\gamma \in H^n(X \times Y/W)$ be written (uniquely) as $\gamma = \sum \alpha_p \otimes \beta_q$, then

$$
F^*\gamma = F^*\alpha_p \otimes F^*\beta_q,
$$

where $\alpha_p \in H^p(X/W)$ and $\beta_q \in H^q(Y/W)$.

Let p_X(resp. p_Y) denote the projection $X \times Y \to X$ (resp. $X \times Y \to Y$).

Proposition 6.6. *The Frobenius has the following compatibility with the projection morphism:*

$$p_X^*(F^*(\alpha)) = F^*(p_X^*\alpha).$$

Similarly, for the other projection p_Y.

Let the denote the cup-product as follows:

$$H^i(X/W) \times H^j(X/W) \to H^{i+j}(X/W)$$

given by

$$(\alpha, \beta) \mapsto \alpha \cup \beta.$$

Proposition 6.7. *The Frobenius action is compatible with the cup-product in the following way:*

$$F^*(\alpha \cup \beta) = F^*(\alpha) \cup F^*(\beta).$$

Moreover, the Poincaré duality induces a perfect pairing as in relation [27]

$$< -, - >: \frac{H^n}{torsion} \times \frac{H^{2dim(X)-n}}{torsion} \to H^{2dim(X)} \cong W(k)$$

which satisfies the following compatibility with Frobenius:

(28) $$< F^*(x), F^*(y) >= p^{dim(X)} Frob_W(< x, y >).$$

1. F-(iso)crystals

Now let us recall the notion of F-isocrystal and F-crystals from Definition 2.9 in Chapter 2.

Definition 6.8. [F-(iso)crystal] An **F-crystal** (M, φ_M) over k is a free W-module M of finite rank together with an injective and $Frob_W$-linear map $\varphi_M : M \to M$, that is, φ_M is additive, injective and satisfies

$$\varphi_M(r \cdot m) = Frob_W(r) \cdot \varphi_M(m) \text{ for all } r \in W(k), m \in M.$$

An **F-isocrystal** (V, φ_V) is a finite dimensional K-vector space V together with an injective and $Frob_W$-linear map $\varphi_V : V \to V$.

A **morphism** $u : (M, \varphi_M) \to (N, \varphi_N)$ **of F-crystals** (resp. F-isocrystals) is a $W(k)$-linear (resp. K-linear) map $M \to N$ such that $\varphi_N \circ u = u \circ \varphi_M$. An **isogeny** of F-crystals is a morphism $u : (M, \varphi_M) \to (N, \varphi_N)$ of F-crystals, such that the induced map $u \otimes Frob_K : M \otimes_{W(k)} K \to N \otimes_{W(k)} K$ is an isomorphism of F-isocrystals.

Examples:

(1) The trivial crystal : $(W, Frob_W)$.
(2) This is the case which will be of most interest to us:
    Take the free $W(k)$ module M to be $H^n(X/W(k))/torsion$ and φ_M to be the Frobenius F^*.
(3) The isocrystal $K(1) := (K, Frob_K/p)$.

2. The Mukai F-crystal

We define an F-crystal structure on the Mukai F-isocrystal of crystalline cohomology for a K3 surface.

Let X be a K3 surface over an algebraically closed field k of characteristic $p > 3$. Let $ch = ch_{cris} : K(X) \to H^{2*}(X/K)$ be the crystalline Chern character and ch^i the $2i - th$ component of ch. Reducing to the case of a line bundle via the splitting principle, we see that the Frobenius φ_X acts in the following manner on the Chern character of a line bundle E:

$$\varphi_X(ch^i(E)) = p^i ch^i(E).$$

We normalize the Frobenius action on the F-isocrystal $H^*(X/K)$ using the Tate twist to get the *Mukai F-isocrystal* $\oplus_i H^i(X/K)(i-1)$.

We make the following observation, which shows that how the Frobenius action works on $H^4_{crys}(X/W)$. Note that for a perfect field k of characteristic p, Serre ([**68**], Thm 8, pg 43) showed that the Witt ring W(k) has p as its uniformizer. Now for $H^4_{crys}(X/K)(1)$ the action of Frobenius is given by φ_X/p. But note that $ch^2(E) = 1/2(c_1^2(E) - 2c_2(E))$, for $E \in K(X)$, where $c_i(E)$ are the Chern classes of E, and as the intersection paring is even for a K3 surface, this is integral, i.e., $ch^2(E) \in H^4(X/W)$. This along with the fact that $rank_W(H^4(X/W)) = 1$ implies that $ch^2(E) = up^n[1]$, where $u \in W^\times, p$ is the characteristic of k and $[1]$ is the generator of $H^4(X/W)$ as a $W-$module. Hence, we have

$$\begin{aligned} \varphi_X(ch^2(E)) = \varphi_X(up^n[1]) &= \sigma(up^n)\varphi_X([1]) \text{ (via semi-linearity)} \\ &= \sigma(u)p^n\varphi_X([1]) \text{ (as } \sigma \text{ is a ring map)} \\ &= p^2 \cdot ch^2(E) = p^2 up^n[1]. \end{aligned}$$

This gives us that

$$\varphi_X([1]) = u(\sigma(u))^{-1}p^2[1],$$

where $u(\sigma(u))^{-1} \in W^\times$ as σ is a ring map. Therefore, we have the Frobenius action on $H^4(X/W) \otimes K(1)$ given by $\varphi'_X([1]) = u(\sigma(u))^{-1}p[1]$. Thus, it indeed has a F-crystal inducing this F-isocrystal given by $(H^4(X/W), \varphi'_X)$. We remark that we are implicitly using the fact that $A \otimes_K K \cong A$, for any K-module A.

Note that the Mukai vector of a sheaf P in $D^b(X)$ for a K3 surface X is by definition the class

$$v(P) = ch(P)\sqrt{td(X)} = (v_0(P), v_1(P), v_2(P)) \in \mathcal{H}^*_{crys}(X/W).$$

Indeed, we have $c_1(X) = 0$ and $2 = \chi(X, \mathcal{O}_X) = td_{2,X}$, which gives us that the Todd genus $td_X = (1, 0, 2)$ and thus $\sqrt{td_X} = (1, 0, 1)$. This then implies that

$$v(P) = (rk(P), c_1(P), rk(P) + c_1^2(P)/2 - c_2(P)).$$

Note that the intersection pairing on $H^2_{crys}(X/W)$ is even, which gives us the above conclusion as $c_i(P) \in H^{2i}_{crys}(X/W)$ (see [**12**]).

Lemma 6.9. *The Mukai vector of any object $P \in D^b(X \times Y)$ is a F-crystal cohomology class.*

PROOF. (cf. [**55**]) Note that from the definition of the F-crystal structure we just need to show that $ch(P) \in H^*_{crys}(X \times Y/W)$ as the square root of the Todd genus for a K3 surface is computed as follows:

$$\sqrt{td_{X \times Y}} = p_1^*\sqrt{td_X}p_2^*\sqrt{td_Y} = p_1^*(1, 0, 1).p_2^*(1, 0, 1).$$

We write the exponential chern character as follows:

$$ch(P) = (rk(P), c_1(P), 1/2(c_1^2(P) - 2c_2(P)), ch_3(P), ch_4(P))$$

where

$$ch^3(P) = 1/6(c_1^3(P) - 3c_1c_2 + 3c_3(P))$$

and

$$ch^4(P) = 1/24(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4).$$

Note that if $char(k) \neq 2, 3$, then $2, 3$ are invertible in $W(k)$, so $ch(P) \in H^*_{crys}(X \times Y/W)$ as again we know $c_i(P) \in H^{2i}_{crys}(X \times Y/W)$. $\qquad\square$

Remark 6.10. Thus, it makes sense to talk about the descent of a Fourier-Mukai transform to the F-crystal level but note that the new Frobenius structure on $H^4(X/W)(1)$ fails to be compatible with the intersection pairing as defined in Theorem 6.7. This causes the failure of existence of an F-crystal structure on the Mukai-isocrystal and also the failure to have a cohomological criteria of derived equivalences of K3 surfaces with crystalline cohomology.

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- dass ich alle Hilfsmittel und Hilfen angegeben habe,
- dass ich auf dieser Grundlage die Arbeit selbständig verfasst habe,
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