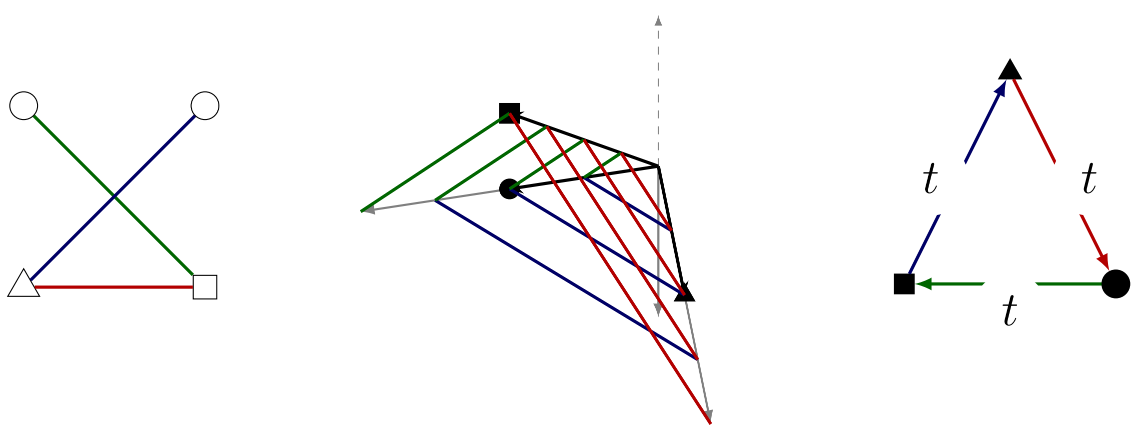
Rigidity of toric varieties associated to bipartite graphs

İrem Portakal

Dissertation



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Die vorliegende Dissertation wurde von Prof. Dr. Klaus Altmann betreut.

Selbständigkeitserklärung

Hiermit versichere ich, dass ich alle Hilfsmittel und Hilfen angeben habe und auf dieser Grundlage die Arbeit selbständig verfasst habe. Meine Arbeit ist nicht schon einmal in einem früheren Promotionsverfahren eingereicht worden.

İrem Portakal, Berlin, 5. Februar 2018.

1. Gutachter: Prof. Dr. Klaus Altmann
2. Gutachter: Prof. Dr. Jan Arthur Christophersen

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Abstract

One can associate to a bipartite graph a so-called edge ring and its spectrum is an affine normal toric variety. We first characterize the faces of the (edge) cone associated to this toric variety in terms of certain independent sets of the bipartite graph. Then, we give first examples of rigid toric varieties associated to bipartite graphs. We show their rigidity combinatorially, to wit, purely in terms of graphs. In the next chapters, we determine the two and three-dimensional faces of the edge cone. With this information, we show that these toric varieties are smooth in codimension two and the non-simplicial three-dimensional faces are generated by exactly four extremal rays. In the latter case, we get non rigid toric varieties. Lastly, we study torus actions on matrix Schubert varieties. In the toric case, we present a classification for their rigidity.

Dedicated to my beloved grandfather M.Sc. Fikret Acar Çalışal

Contents

Remarks on Notation

In this thesis, to distinguish the lattices, for an element in N, we use the parenthesis $(-)$ and for an element in M, we use square brackets $[-]$. A canonical basis element in N is denoted by e_i and a canonical basis element in M is denoted by e^i.

The labels of the vertices of a bipartite graph $G \subseteq K_{m,n}$ are $U_1 = \{1, \ldots, m\}$ and $U_2 = \{m+1, \ldots, m+n\}$. In order to avoid ambiguity, we draw circles for the vertices in U_1 and squares for the vertices in U_2. However, in Chapter 6, we label the vertices on the both disjoint sets starting from 1, for the consistency of our construction.

During the investigation of the faces of an edge cone σ_G of a connected bipartite graph G, we use two different parentheses. Suppose that we investigate the d-dimensional faces of σ_G. We write a t-tuple of the first independent set in

- curly parenthesis as $\{A^1, \ldots, A^t\}$, if it does not form a d-dimensional face.
- normal parenthesis as $(A^1, \ldots, A^t)$, if it does form a d-dimensional face.

Here, we have $t \geq d$ and the relation between these numbers is studied in Theorem 3.2.2 and in Corollary 3.2.3. Moreover the term " a d-dimensional face " means the same as "a d-face".

In Chapter 4, we study the connected bipartite graphs with exactly one two-sided first independent set. We notate this first independent set as $A \in \mathcal{I}_G^{(1)}$. However, in Chapter 5 and Chapter 6, while we study more general cases with more two-sided first independent sets, we notate A as a one-sided first independent set $U_1 \backslash \{\bullet\}$ and C as a two-sided first independent set.

Chapter 1

Introduction

Let G be a simple graph. We denote its vertex set as $V(G)$ and its edge set as $E(G)$. One defines the *edge ring* associated to G as

$$\mathrm{Edr}(G) := \mathbb{C}[t_i t_j | (i,j) \in E(G), i, j \in V(G)].$$

Consider the ring morphism

$$\begin{array}{rcl} \mathbb{C}[x_e \mid e \in E(G)] & \to & \mathrm{Edr}(G) \\ x_e & \mapsto & t_i t_j \end{array}$$

where $e = (i,j) \in E(G)$. The kernel I_G of this morphism is called the *edge ideal*. The associated toric variety to the graph G is denoted by $\mathrm{TV}(G) := \mathrm{Spec}(\mathbb{C}[x_e \mid e \in E(G)]/I_G)$. The edge ring $\mathrm{Edr}(G)$ is an integrally closed domain and hence $\mathrm{TV}(G)$ is a normal variety. We study the first order deformations of this normal toric variety, more precisely we search certain criteria for the bipartite graph G such that the first order deformations of $\mathrm{TV}(G)$ are all trivial, equivalently $\mathrm{TV}(G)$ is *rigid*.

The first attempt on this topic has been done in [BHL15]. In this paper, one considers the connected bipartite graph $G \subsetneq K_{n,n}$ with one edge removal from the complete bipartite graph $K_{n,n}$. It has been proven that $\mathrm{TV}(G)$ is rigid for $n \geq 4$ and $\mathrm{TV}(G)$ is not rigid for $n = 3$. The proof is done by some algebraical techniques which we do not use in this thesis. In the end of their introduction, the authors emphasize that "it remains a challenging problem to classify all rigid bipartite graphs". We follow intrinsic geometrical techniques which utilize the properties of the bipartite graph G and dive into this challenging problem.

Many aspects of the infinitesimal deformations of toric varieties have been studied by K. Altmann in [Alt00]. In that paper, it has been shown that the first order deformations of affine normal toric varieties are multi-graded. The homogeneous pieces are given by a so-called *deformation degree* $R \in M$. One considers the crosscut picture, which is

$[R=1] := \{a \in N_{\mathbb{Q}} \mid \langle R, a\rangle = 1\}$ intersected with the associated cone of the variety. For the homogenous piece $T^1_X(-R)$ of the vector space of first order deformations of the toric variety X, one examines the two-dimensional faces of this crosscut and how these two-dimensional faces are connected to each other. We follow this technique for our investigation on the rigidity of toric varieties associated to bipartite graphs.

The first example of a rigid singularity is the Segre cone over $\mathbb{P}^r \times \mathbb{P}^1$ in $\mathbb{P}^{2r+1}$ $(r \geq 1)$ which has been introduced by Grauert and Kerner in [GK64]. We will observe that this is in fact the toric variety associated to the complete bipartite graph $K_{r+1,2}$. One of the other well-known rigid varieties are introduced by Schlessinger in [Sch71], which are isolated quotient singularities with dimension bigger than three. In this thesis, we provide new families of rigid varieties (not necessarily isolated singularities), which can be expressed in terms of graph theory language.

For this, we first describe the associated cone to the toric variety $\mathrm{TV}(G)$. We call the cone $\sigma_G^\vee$ the *(dual) edge cone*, where $\mathrm{TV}(G) = \mathrm{Spec}(\mathbb{C}[\sigma_G^\vee \cap M])$. The description for the extremal ray generators of the edge cone σ_G has been studied by C.H. Valencia and R.H. Villarreal in [VV05]. We present a different description for the extremal rays of the edge cone. We consider a so-called *first independent set* $A \subsetneq V(G)$. We define an induced subgraph $\mathrm{G}\{A\}$ associated to this first independent set. By using this language, we moreover determine explicitly the faces of σ_G.

Main Result 1 (Theorem 3.1.13, Theorem 3.2.2)**.** *Let t and d be positive integers with $t \geq d$. There exists a one-to-one correspondence between t-tuples of first independent sets $(A^1, \ldots, A^t)$ and d-dimensional faces of σ_G where $\bigcap_{i\in[t]} \mathrm{G}\{A^i\}$ has $d+1$ connected components.*

This result allows us to study the first order deformations by Altmann's technique. We denote the disjoint sets of a bipartite graph by U_1 and U_2. We first consider the connected bipartite graphs $G \subset K_{m,n}$ where we remove all the edges between two vertex sets $A_1 \subsetneq U_1$ and $A_2 \subsetneq U_2$. For the case where $|A_1| = 1$ and $|A_2| = 1$, we recover the result in [BHL15] without the assumption of $m = n$.

Main Result 2 (Theorem 4.3.3)**.** *Let $G \subsetneq K_{m,n}$ be a connected bipartite graph constructed by removing all edges between two vertex sets $A_1 \subsetneq U_1$ and $A_2 \subsetneq U_2$. Then*

1. $\mathrm{TV}(G)$ *is not rigid, if $|A_1| = 1$ and $|A_2| = n-2$ or if $|A_1| = m-2$ and $|A_2| = 1$.*
2. $\mathrm{TV}(G)$ *is rigid, otherwise.*

In particular, we prove the rigidity of the toric variety $\mathrm{TV}(K_{m,n})$ in terms of graphs. This is the classical result of the rigidity of the cone of the Segre embedding $\mathbb{P}^m \times \mathbb{P}^n \hookrightarrow \mathbb{P}^{(m+1)(n+1)-1}$. For a more general classification of rigid toric varieties arising from bipartite graphs, we steer our investigation to necessary and sufficient conditions for two and three-dimensional faces of the edge cone. The precise calculations give us the following results.

Main Result 3 (Theorem 5.1.5, Lemma 5.3.1)**.** *The affine normal toric variety* $\mathrm{TV}(G)$ *is smooth in codimension 2. The non-simplicial three-dimensional faces of the edge cone are spanned by exactly four extremal rays.*

In the case where the edge cone σ_G has a non-simplicial three-dimensional face, we prove that $\mathrm{TV}(G)$ is not rigid.

Main Result 4 (Theorem 5.3.2)**.** *If the edge cone* σ_G *has a non-simplicial three-dimensional face, then* $\mathrm{TV}(G)$ *is not rigid. Moreover, these cases can be explicitly described in terms of graphs as in Section 5.2.*

Next, we focus our investigation in matrix Schubert varieties. These varieties appear in Fulton's paper from when he was studying the degeneracy loci of flagged vector bundles in [Ful92]. These varieties are normal and admit an effective torus action. It turns out that the weights of this torus action can be found by examining the torus action of the toric variety $\mathrm{TV}(G)$ for a bipartite graph $G \subseteq K_{m,n}$. In the case of toric matrix Schubert varieties, we classify the rigid toric matrix Schubert varieties.

Main Result 5 (Theorem 6.3.7)**.** *Let* $\pi \in S_n$ *be a permutation and* $\overline{X_\pi} \cong Y_\pi \times \mathbb{C}^q$ *be a matrix Schubert variety. Assume that the affine normal variety* $Y_\pi := \mathrm{TV}(\sigma_\pi)$ *is toric. Then* Y_π *is rigid if and only if the three-dimensional faces of* σ_π *are all simplicial.*

Moreover, in Corollary 6.3.8, we reformulate this result in terms of the so-called Rothe diagram of π. We also examine the cases explicitly where there exist non-simplicial three-dimensional faces of σ_π in Lemma 6.3.3 and in Lemma 6.3.5.

We now give an overview of the structure of this thesis. In the second chapter, we present a brief overview on edge ideals, toric varieties, and deformation theory. We also repeat the material for the deformations of toric varieties in [Alt00] without proofs, therefore making this thesis as self-contained as possible. In Chapter 3, we first present the description for the facets of the edge cone σ_G from [BHL15]. Then we develop an equivalent description for the extremal rays of an edge cone. Our version has the advantage of finding an explicit description for a face of an edge cone as in Main Result 1. This description allows us to reformulate the rigidity question in terms of graphs. Hence, we can study the deformations of the toric variety $\mathrm{TV}(G)$ combinatorially. In Chapter 4, we apply the techniques from Chapter 3 to certain connected bipartite graphs. For this, we first characterize the two and three-dimensional faces. For rigidity, we look more closely at the pairs of extremal rays not forming a two-dimensional face and non-simplicial three-dimensional faces. Finally, we arrive to Main Result 2 in which we present certain rigid toric variety families. In Chapter 5, we study the two and three-dimensional faces of the edge cone σ_G for any connected bipartite graph $G \subseteq K_{m,n}$. We determine the necessary and sufficient conditions for the first independent sets to form these faces and deduce Main Result 3. We precisely describe the cases where σ_G has non-simplicial three-dimensional faces and prove that $\mathrm{TV}(G)$ is not rigid as stated in Main Result 4. Hence we narrow our investigation to edge cones with only simplicial three-dimensional faces. In this case, analogously to Chapter 4, we study the non 2-face pairs and non 3-face triples of extremal rays of an edge cone. We next consider the matrix Schubert varieties and their effective torus action. These varieties can been seen as T-varieties and it turns out that the torus action can be understood in terms of graphs. In Chapter 6, we prove that there are no complexity-one matrix Schubert T-varieties. In the toric case, we arrive to Main Result 5. In the end of this chapter, we present the future work aspects in the topic of Kazhdan-Lusztig varieties. Our aim is first to classify complexity-one T-variety ones by using directed graphs and then work on their deformations. Throughout this thesis, many examples have been checked by the software `Polymake` [GJ00] and the computer algebra system `Singular` [DGPS15]. In Chapter 7, we present the function which receives the dual edge cone and outputs the information about rigidity of the associated toric variety. In particular it draws the representative picture of the crosscut $Q(R)$ for any given deformation degree $R \in M$.

Chapter 2

Preliminaries

In this chapter, we introduce three different topics converging under the main investigation of this thesis. We begin with toric geometry. Then we introduce the tools to deform affine normal toric varieties developed in [Alt00]. In the last section, we introduce the edge ideals which are the binomial ideals constructed by graphs. In the case where one considers simple graphs, we observe that the semigroup associated to the edge cone is saturated. Therefore the associated toric variety is normal.

2.1 Toric Geometry

In this section, we recall basic constructions and facts on toric varieties from [CLS10] and [Ful93]. Let $T \cong (\mathbb{C}^*)^n$ be an algebraic torus. We denote the characters of T by M and the one-parameter subgroups of T by N. The groups N and M are free abelian groups of rank n and therefore they are lattices. There is a natural bilinear pairing which is the usual dot product

$$\langle \bullet, \bullet \rangle \colon M \times N \to \mathbb{Z}.$$

Let $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ and $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$ be the corresponding vector spaces to the lattices N and M. Let $\sigma \subseteq N_{\mathbb{Q}}$ be a convex rational polyhedral cone, i.e. $\sigma = \mathrm{Cone}(S)$ for some finite set $S \subseteq N$. The dual cone $\sigma^\vee$ is defined as

$$\{m \in M_{\mathbb{Q}} \mid \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma\}$$

We define the affine toric variety as $\mathrm{TV}(\sigma) := \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$. The semigroup $\sigma^\vee \cap M$ is finitely generated. We are interested in the faces of the cone σ in order to examine the deformations of $\mathrm{TV}(\sigma)$ combinatorially. We will see this in more detail in Section 2.2.

Definition 2.1.1. Let $m \in M$ be a lattice element. The hyperplane $\mathcal{H}_m$ is defined as the set $\{n \in N_{\mathbb{Q}} \mid \langle m, n \rangle = 0\}$. A *face* τ of a cone σ is $\tau = \mathcal{H}_m \cap \sigma$ for some $m \in \sigma^\vee$. A face τ different to the cone σ itself is called a proper face. We write it as $\tau \prec \sigma$.

Definition 2.1.2. Let $\tau \preceq \sigma \subseteq N_{\mathbb{Q}}$ be a face. We define

$$\tau^{\perp} = \{m \in M_{\mathbb{Q}} \mid \langle m, a \rangle = 0, \text{ for all } a \in \tau\}$$

$$\tau^{*} = \{m \in \sigma^{\vee} \mid \langle m, a \rangle = 0, \text{ for all } a \in \tau\} = \sigma^{\vee} \cap \tau^{\perp}$$

We call $\tau^{*} \preceq \sigma^{\vee}$ the dual face of τ.

Note that every face of a convex polyhedral cone σ is again convex polyhedral. Also, every proper face $\tau \prec \sigma$ is the intersection of the facets of σ containing τ. Furthermore, one has $\dim(\tau) + \dim(\tau^{*}) = \dim(N_{\mathbb{Q}})$.

Throughout this thesis, we work on affine normal toric varieties. We see that the normality of a toric variety has a nice combinatorial interpretation.

Definition/Proposition 2.1.3. Let $\sigma \subseteq N_{\mathbb{Q}}$ be a polyhedral cone. A rational polyhedral cone is called *strongly convex* if and only if one of the following equivalent statements holds:

- $\{0\}$ is a face of σ.
- σ contains no positive-dimensional subspace of $N_{\mathbb{Q}}$.
- $\dim(\sigma^{\vee}) = n$.
- $\sigma \cap (-\sigma) = \{0\}$.

Proposition 2.1.4. *Let* $\mathrm{TV}(\sigma)$ *be an affine toric variety. The following statements are equivalent.*

1. $\mathrm{TV}(\sigma)$ *is normal.*
2. *The cone* $\sigma \subseteq N_{\mathbb{Q}}$ *is a strongly convex rational polyhedral cone.*

One can determine the smoothness of the toric variety again in terms of its associated cone.

Definition 2.1.5. Let $\sigma \subseteq N_{\mathbb{Q}}$ be a strongly convex polyhedral cone.

1. σ is called smooth if its minimal generators form a part of a $\mathbb{Z}$-basis of N.
2. σ is called simplicial if its minimal generators are linearly independent over $\mathbb{Q}$.

Proposition 2.1.6. $\mathrm{TV}(\sigma)$ *is smooth (an orbifold, i.e. has only finite quotient singularities) if and only if* $\sigma \subseteq N_{\mathbb{Q}}$ *is smooth (simplicial).*

Definition 2.1.7. Let $\sigma \subseteq N_{\mathbb{Q}}$ be a strongly convex rational polyhedral cone. The unique minimal generator set of the semigroup $\sigma^{\vee} \cap M$ is called the *Hilbert Basis* of $\sigma^{\vee}$.

The Hilbert Basis of $\sigma^\vee$ translates to be the minimal generator set of $\mathbb{C}[\sigma^\vee \cap M]$ as a $\mathbb{C}$-algebra. Let $H = \{h_1, \ldots, h_N\}$ be the Hilbert Basis of $\sigma^\vee$. We write the following surjective morphism

$$\begin{array}{rccc} \varphi: & \mathbb{C}[x_1, \ldots, x_N] & \longrightarrow & \mathbb{C}[\sigma^\vee \cap M] \\ & x_i & \mapsto & x^{h_i} \end{array}.$$

Definition 2.1.8. The kernel of the map φ is called the *toric ideal*.

2.2 Deformation Theory

A deformation of an affine algebraic variety X_0 is a flat map $\pi\colon \mathcal{X} \longrightarrow S$ with $0 \in S$ such that $\pi^{-1}(0) = X_0$, i.e. we have the following commutative diagram.

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\ 0 \arrow[r, hook] & S \end{tikzcd}$$

The variety $\mathcal{X}$ is called the total space and S is called the base space of the deformation. Let $\pi\colon \mathcal{X} \longrightarrow S$ and $\pi'\colon \mathcal{X}' \longrightarrow S$ be two deformations of X_0. We say that two deformations are isomorphic if there exists a map $\phi : \mathcal{X} \longrightarrow \mathcal{X}'$ over S inducing the identity on X_0. Let S be an Artin ring. For an affine algebraic variety X_0, one has a contravariant functor Def_{X_0} such that $\mathrm{Def}_{X_0}(S)$ is the set of deformations of X_0 over S modulo isomorphisms.

Definition 2.2.1. The map π is called a first order deformation of X_0 if $S = \mathrm{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2))$. We set $T^1_{X_0} := \mathrm{Def}_{X_0}(\mathbb{C}[\epsilon]/(\epsilon^2))$.

The variety X_0 is called *rigid* if $T^1_{X_0} = 0$. This implies that a rigid variety X_0 has no non-trivial infinitesimal deformations. This means that every deformation $\pi \in \mathrm{Def}_{X_0}(S)$ over a Artin ring S is trivial i.e. isomorphic to the trivial deformation $X_0 \times S \longrightarrow S$.

From now on, let X_0 be an affine normal toric variety. We refer to the techniques which are developed in [Alt00] in order to investigate the $\mathbb{C}$-vector space $T^1_{X_0}$. The deformation space $T^1_{X_0}$ is multigraded by the lattice elements of M, i.e. $T^1_{X_0} = \bigoplus_{R \in M} T^1_{X_0}(-R)$. We first set some definitions in order to define the homogeneous part $T^1_{X_0}(-R)$. Then, we will introduce the formula for $T^1_{X_0}$, if X_0 is smooth in codimension 2.

Let us call $R \in M$ a deformation degree and let $\sigma \subseteq N$ be generated by the extremal ray generators $a_1, \ldots, a_n$. We consider the following affine space

$$[R = 1] := \{a \in N_\mathbb{Q} \mid \langle R, a\rangle = 1\} \subseteq N_\mathbb{Q}.$$

We define the crosscut of σ in degree R as the polyhedron $Q(R) := \sigma \cap [R = 1]$ in the assigned vector space $[R = 0]$. It has the cone of unbounded directions $Q(R)^\infty = \sigma \cap [R = 0]$ and the compact part $Q(R)^c$ of $Q(R)$ is generated by the vertices $\overline{a_i} = a_i/\langle R, a_i\rangle$ where $\langle R, a_i\rangle \geq 1$. Note that $\overline{a_i}$ is a lattice vertex in $Q(R)$ if $\langle R, a_i\rangle = 1$.

Definition 2.2.2. (i) Let $d^1, \dots, d^N \in R^\perp \subset N_{\mathbb{Q}}$ be the compact edges of $Q(R)$. The vector $\bar{\epsilon} \in \{0, \pm 1\}^N$ is called a sign vector assigned to each two-dimensional compact face ϵ of $Q(R)$ defined as

$$\overline{\epsilon_i} = \begin{cases} \pm 1, & \text{if } d^i \text{ is an edge of } \epsilon \\ 0 \end{cases}$$

such that $\sum_{i\in[N]} \overline{\epsilon_i} d^i = 0$, i.e the oriented edges $\overline{\epsilon_i} d^i$ form a cycle along the edges of ϵ.

(ii) For every deformation degree $R \in M$, the related vector space is defined as

$$V(R) = \{\bar{t} = (t_1, \dots, t_N) | \sum_{i\in[N]} t_i \overline{\epsilon_i} d^i = 0, \text{ for every compact 2-face } \epsilon \preceq Q(R)\}.$$

In particular, another way of understanding this vector space $V(R)$ is to investigate the Minkowski decompositions of positive multiples of $Q(R)$. By a Minkowski decomposition of a polyhedron P, we mean the investigation of polyhedra P_i such that the Minkowski sum

$$\sum_i P_i := \{ \sum_i p_i \mid p_i \in P_i \}$$

equals to P. Then, the points of the rational polyhedral cone $V(R) \cap \mathbb{R}^N_{\geq 0}$ correspond to the Minkowski summands of positive dilations of $Q(R)$. This approach can be found in [Alt00] and [Alt97].

Example 1. Let us consider the cone over a double pyramid P over a triangle in $N \cong \mathbb{Z}^4$ with extremal ray generators $a_1 = (1, 0, 0, 0)$, $a_2 = (1, 0, 0, 1)$, $a_3 = (1, 1, 1, -1)$, $a_4 = (0, 1, 0, 0)$, $a_5 = (0, 0, 1, 0)$. For the deformation degree $R = [0, 1, 1, 0] \in M$, we obtain the compact part $Q(R)^c$ as a two-dimensional face generated by $\overline{a_3}$, $\overline{a_4}$, and $\overline{a_5}$. We assign the sign vector $\bar{\epsilon} = (1, 1, 1)$ to this two-dimensional face and we obtain the elements of $V(R)$ as $\bar{t} = (t, t, t)$.

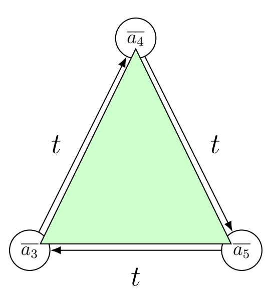


Figure 2.1: The compact part of the crosscut $Q(R)$ and the vector space $V(R)$.

Theorem 2.2.3 (Corollary 2.7, [Alt00]). *If the affine normal toric variety X_0 is smooth in codimension 2, then $T^1_{X_0}(-R)$ is contained in $V_{\mathbb{C}}(R)/\mathbb{C}(\underline{1})$. Moreover, it is built by those $\bar{t}$'s satisfying $t_{ij} = t_{jk}$ where $\overline{a_j}$ is a non-lattice common vertex in $Q(R)$ of the edges $d^{ij} = \overline{\overline{a_i}\ \overline{a_j}}$ and $d^{jk} = \overline{\overline{a_j}\ \overline{a_k}}$.*

Remark 1. The following two cases in Figure 2.2 will appear often when we study the classification of rigid toric varieties. Hence we would like to look more closely at the vector space $V(R)$ and how the previous result may apply to these situtations.

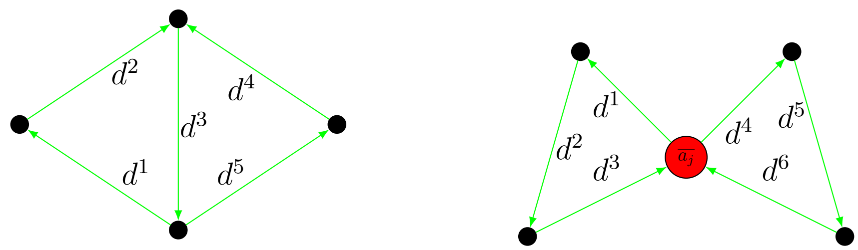


Figure 2.2: Compact 2-faces sharing an edge or a non-lattice vertex in $Q(R)$

• Let $\epsilon^1, \epsilon^2 \preceq Q(R)$ be the compact 2-faces sharing the edge d^3. We choose the sign vectors as $\overline{\epsilon^1} = (1, 1, 1, 0, 0)$ and $\overline{\epsilon^2} = (0, 0, 1, 1, 1)$. Suppose that $\bar{t} = (t_1, t_2, t_3, t_4, t_5) \in V(R)$. As in Example 1, we observe that $t_1 = t_2 = t_3$ for the 2-face ϵ^1 and $t_3 = t_4 = t_5$ for the 2-face ϵ^2.
• Let $\epsilon^1, \epsilon^2 \preceq Q(R)$ be the compact 2-faces connected by the vertex $\overline{a_j}$. As in the previous case we obtain that $t_1 = t_2 = t_3$ and $t_4 = t_5 = t_6$. By Theorem 2.2.3, if a_j is a non-lattice vertex, then we obtain $t_3 = t_4$. We note also that there are pairs of extremal rays which do not form two dimensional faces. We refer to this as "non 2-faces".

These two cases are sometimes mentioned as "t is transfered by an edge or a vertex" during the investigation of the skeleton of $Q(R)$.

In general, if the toric variety X_0 is not smooth in codimension 2, then the homogeneous piece $T^1_{X_0}(-R)$ consists of elements of $V(R) \oplus W(R)/\mathbb{C}(\underline{1}, \underline{1})$ satisfying certain conditions. Here the vector space $W(R)$ is equal to $\mathbb{R}^{\#(\text{non-lattice vertices of Q(R)})}$. One can always find a deformation degree $R \in M$ such that the crosscut $Q(R)$ has non-lattice vertices. In this case, we obtain that $T^1_{X_0}(-R) \neq 0$, i.e. X_0 is not rigid.

The first intuition after Theorem 2.2.3 and Example 1 is to think that X_0 is rigid if and only if all three-dimensional faces of $\sigma \subseteq N_{\mathbb{Q}}$ are simplicial. Although we will see such examples through the thesis, this statement is not true in general.

Example 2. Let us consider $\text{Cone}(P) \subseteq N_{\mathbb{Q}}$ from Example 1. We observe that the three-dimensional faces of $\text{Cone}(P)$ are all generated by three extremal rays. However for the deformation degree $R = [1, 0, 0, 0] \in M$, the compact part $Q(R)^c$ consists of the compact edges $\overline{\overline{a_1}\ \overline{a_3}}$ and $\overline{\overline{a_1}\ \overline{a_2}}$. Since $\overline{a_1}$ is a lattice vertex in $Q(R)$, one obtains that $T^1(-R) \neq 0$.

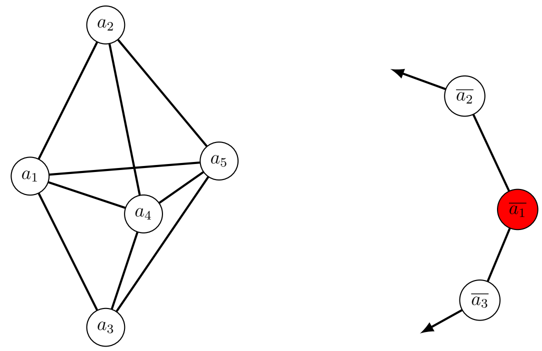


Figure 2.3: The double tetrahedron P and a crosscut picture $Q(R)$ of $\mathrm{Cone}(P)$.

2.3 Edge Ideals

Let G be a finite connected simple graph with d vertices and with n edges. We denote its vertex set by $V(G)$ and its the edge set by $E(G)$. Let $a_i = (u_i, v_i) \in E(G)$ be an edge with two endpoints $u_i, v_i \in V(G)$ and let $t^{a_i} := t_{u_i} t_{v_i} \in \mathbb{C}[t_1, \dots, t_d]$. We define

$$\mathrm{Edr}(G) := \mathbb{C}[t^{a_1}, \dots, t^{a_n}]$$

to be the *edge ring* associated to G. Consider the morphism

$$\begin{array}{cccc} \varphi : & \mathbb{C}[x_1, \dots, x_n] & \longrightarrow & \mathrm{Edr}(G) \\ & x_i & \mapsto & t^{a_i} \end{array}.$$

The kernel I_G of this map is called the (toric) *edge ideal* and $\mathrm{TV}(G) := \mathrm{Spec}(\mathbb{C}[x_1, \dots, x_n]/I_G)$ is the associated affine toric variety. We call $\sigma_G^\vee$ the *(dual) edge cone*, where $\mathrm{TV}(G) = \mathrm{TV}(\sigma_G)$.

Let $\Gamma := (a_{i_1}, \dots, a_{i_{2q}})$ be an even closed walk. We define the binomial $f_\Gamma \in \mathbb{C}[x_1, \dots, x_n]$ by $f_\Gamma = \Pi_{k=1}^q x_{i_{2k-1}} - \Pi_{k=1}^q x_{i_{2k}}$. Let $f_\Gamma^+ = \Pi_{k=1}^q x_{i_{2k-1}}$ and $f_\Gamma^- = \Pi_{k=1}^q x_{i_{2k}}$. We say that an even closed walk Γ is primitive if there is no other even closed walk Γ' with $\Gamma' \neq \Gamma$ such that $f_{\Gamma'}^+ | f_\Gamma^+$ and $f_{\Gamma'}^- | f_\Gamma^-$.

Theorem 2.3.1 ([HO99], Lemma 3.1)**.** *The edge ideal I_G is generated by the binomials f_Γ where Γ is a primitive even closed walk.*

Let $\Gamma = (a_1, \dots, a_c)$ be a cycle. A chord of the cycle Γ is an edge between two vertices of the cycle which is not a part of the cycle Γ. Note that any cycle in a bipartite graph has a pair length.

Corollary 2.3.2. *Let $G \subseteq K_{m,n}$ be a connected bipartite graph. The edge ideal I_G is generated by the binomials f_Γ where Γ is a cycle without a chord.*

Proof. Suppose that $\Gamma = (a_1, \dots, a_c)$ is a cycle with a chord a_{c+1}. Then there exist two cycle $\Gamma' = (a_1, \dots, a_k, a_{c+1})$ and $\Gamma'' = (a_{k+1}, \dots, a_c, a_{c+1})$ where k is an odd integer. One then obtains $f_\Gamma = (x_{k+2} \dots x_{c-1}) f_{\Gamma'} - (x_2 \dots x_{k-1}) f_{\Gamma''}$. This concludes the proof. □

Throughout this thesis, we focus on the bipartite case. Let $G \subseteq K_{m,n}$ be a connected bipartite graph. We denote its disjoint sets as U_1 and U_2. Let e^i denote the canonical basis of $\mathbb{Z}^m \times 0$ and f^j denote the canonical basis of $0 \times \mathbb{Z}^n$. By construction of the edge ideal, one obtains that the dual dual edge cone $\sigma_G^\vee$ is generated by the ray generators $e^i + f^j \in \mathbb{Z}^{m+n}$, for $(i,j) \in E(G)$. If G is not a tree, then the generators of the dual edge cone $\sigma_G^\vee$ in $\mathbb{Q}^{m+n}$ are linearly dependent. The relations are formed by the cycles of G. If G is a tree, $\sigma_G^\vee$ has $m+n-1$ generators. In both cases, the dual cone $\sigma_G^\vee$ is not a full dimensional cone in the vector space $\mathbb{Q}^{m+n}$. Equivalently, the edge cone $\sigma_G \subseteq \mathbb{Q}^n$ is not strongly convex.

Proposition 2.3.3. *Let $G \subseteq K_{m,n}$ be a connected bipartite graph. Then the dimension of the dual edge cone $\sigma_G^\vee$ is $m+n-1$.*

Proof. Let A_G be the (incidence) matrix whose columns are the ray generators of $\sigma_G^\vee$. Suppose that $x \in \mathbb{Q}^{m+n}$ is an element of $\operatorname{coker}(A_G)$. Then $x_i + x_j = 0$ whenever there is a path from vertex i to vertex j. Since G is connected, we obtain that the corank of A_G is at most one. However the rows of A_G are linearly dependent and therefore the rank of A_G is smaller than or equal to $m+n-1$. It follows that $\dim \sigma_G^\vee = m+n-1$. $\square$

We calculate $(\sigma_G^\vee)^\perp$ as

$$\{a \in \mathbb{Q}^{m+n} \mid \langle b, a\rangle = 0 \text{ for all } b \in \sigma_G^\vee\} = \langle (\sum_{i=1}^m e_i - \sum_{j=1}^n f_j)\rangle$$

The one-dimensional subspace $(\sigma_G^\vee)^\perp$ is the minimal face of $\sigma_G \subseteq \mathbb{Q}^{m+n}$. We denote it by $\overline{(1,-1)}$. Hence we consider the cone $\sigma_G/\overline{(1,-1)} \subseteq \mathbb{Q}^{m+n}/\overline{(1,-1)}$ which is a strongly convex polyhedral cone. Therefore we set the lattices we use for the edge and dual edge cone as follows:

$$N := \mathbb{Z}^{m+n}/\overline{(1,-1)} \text{ and } M := \mathbb{Z}^{m+n} \cap \overline{(1,-1)}^\perp.$$

By Definition 2.1.3, we note that the affine toric variety $\mathrm{TV}(G) := \mathrm{TV}(\sigma_G)$ is normal.

If $G = K_{m+1,n+1}$ is the complete bipartite graph, then $\mathrm{TV}(K_{m+1,n+1})$ is the affine cone over a Segre variety which is the image of the embedding $\mathbb{P}^m \times \mathbb{P}^n \longrightarrow \mathbb{P}^{(m+1)(n+1)-1}$. The length of a primitive cycle without a chord in $K_{m,n}$ is four and hence by Corollary 2.3.2, the edge ideal $I_{K_{m,n}}$ is generated by quadratic binomials. For $i \in [m]$ and $j \in \{m+1, \ldots, m+n\}$, let $M \in M_{m\times n}$ be the matrix whose (i,j)th entry is $t_i t_j$. Then the edge ideal $I_{K_{m,n}}$ can be seen as the 2×2 minors of M and therefore $\mathrm{TV}(K_{m,n})$ is a determinantal singularity. It is a famous result by Thom, Grauert-Kerner and Schlessinger as in [KL71] that the affine cone over the Segre embedding is rigid whenever $m \geq 1$ and $n \geq 2$. We prove this classical result purely combinatorially in Section 4.1.

Chapter 3

Graph Theoretical Construction of Toric Varieties associated to Bipartite Graphs

By the combinatorial study of deformations of toric varieties in Section 2.2, one observes that we first need to investigate the two and three-dimensional faces of σ_G to study the rigidity of $\mathrm{TV}(G)$. In this chapter, we introduce a combinatorial technique to represent the faces of σ_G in terms of certain induced subgraphs of G. We reformulate the question in terms of Graph Theory language and examine the rigidity by using these tools.

In this thesis, our investigation is on connected bipartite graphs $G \subseteq K_{m,n}$ with disjoint sets U_1 and U_2. We shall emphasise that the connectivity assumption should not be taken as a strong assumption. To illustrate that, we assume for a moment that $G = G_1 \sqcup G_2 \subsetneq K_{m,n}$ is not connected. Then one calculates the edge cones $\sigma_{G_1}^\vee \subseteq M_\mathbb{Q}^1$ and $\sigma_{G_2}^\vee \subseteq M_\mathbb{Q}^2$ for the connected components G_1 and G_2. Then $\sigma_G^\vee = \sigma_{G_1}^\vee + \sigma_{G_2}^\vee \subseteq M_\mathbb{Q}^1 \oplus M_\mathbb{Q}^2$. Hence, the associated toric variety is simply $\mathrm{TV}(G) = \mathrm{TV}(G_1) \times \mathrm{TV}(G_2)$. If one of these toric varieties is not rigid, then $\mathrm{TV}(G)$ is also not rigid. If every connected component of G yields a rigid associated toric variety, then $\mathrm{TV}(G)$ is rigid. This argument gives us the opportunity to study only connected graphs.

3.1 Description of the extremal rays of an edge cone

We start with some definitions from Graph Theory. Although these definitions hold for an arbitrary abstract graph G, we preserve our assumption of G being connected and bipartite.

Definition 3.1.1.

1. A nonempty subset A of $V(G)$ is called an *independent set* if it contains no adjacent vertices.

2. The *neighbor set* of $A \subseteq V(G)$ is defined as

$$N(A) := \{v \in V(G) \mid v \text{ is adjacent to some vertex in } A\}.$$

3. The *supporting hyperplane* of the dual edge cone $\sigma_G^\vee \subseteq M_\mathbb{Q}$ associated to an independent set $\emptyset \neq A$ is defined as

$$H_A := \{x \in M_\mathbb{Q} \mid \sum_{v_i \in A} x_i = \sum_{v_i \in N(A)} x_i\}.$$

Note that since no pair of vertices of an independent set A is adjacent, we obtain that $A \cap N(A) = \emptyset$.

Definition 3.1.2.

1. A subgraph of G with the same vertex set as G is called a *spanning subgraph* (or *full subgraph*) of G.

2. Let $S \subset V(G)$ be a subset of the vertex set of G. The *induced subgraph* of S is defined as the subgraph of G formed from the vertices of S and all of the edges connecting pairs of these vertices. We denote it as $\mathrm{G}[S]$.

In the next proposition, it is shown that every facet of $\sigma_G^\vee$ can be constructed by an independent set satisfying certain conditions. We will interpret this result and give a brief one-to-one description for the extremal ray generators of σ_G.

Proposition 3.1.3. [[VV05], *Proposition* 4.1, 4.6] *Let $A \neq U_i$ be an independent set. Then $H_A \cap \sigma_G^\vee$ is a proper face of $\sigma_G^\vee$. In particular, if $A \subsetneq U_1$, then $H_A \cap \sigma_G^\vee$ is a facet of $\sigma_G^\vee$ if and only if $\mathrm{G}[A \sqcup N(A)]$ and $\mathrm{G}[(U_1 \backslash A) \sqcup (U_2 \backslash N(A))]$ are connected and their union is a spanning subgraph of G. Furthermore, any facet of $\sigma_G^\vee$ has the form $H_A \cap \sigma_G^\vee$ for some $A \subsetneq U_i$, $i = 1$ or $i = 2$.*

Example 3. Let $G \subsetneq K_{2,2}$ be the connected bipartite graph with disjoint sets $U_1 = \{1, 2\}$ and $U_2 = \{3, 4\}$ and with the edge set $E(G) = E(K_{2,2}) \backslash (1, 3)$. Recall that we have the edge cone σ_G in $N_\mathbb{Q} \cong \mathbb{Q}^4/(1, 1, -1-1) \cong \mathbb{Q}^3$ and the dual edge cone $\sigma_G^\vee$ in $M_\mathbb{Q} \cong \mathbb{Q}^4 \cap$

$(1,1,-1,-1)^\perp \cong \mathbb{Q}^3$. By Proposition 3.1.3, the independent sets inducing the facets of $\sigma_G^\vee$ are those colored in yellow. Here, we do not consider the independent set $\{3\}$, since we have $H_{\{3\}} \cap \sigma_G^\vee = H_{\{1\}} \cap \sigma_G^\vee$. This is explained further in Remark 2. The blue color represents the induced subgraph $\mathrm{G}[(U_1 \backslash A) \sqcup (U_2 \backslash N(A))]$ and the black color represents the induced subgraph $\mathrm{G}[A \sqcup N(A)]$. The graphs are labeled by their associated facets of $\sigma_G^\vee$.

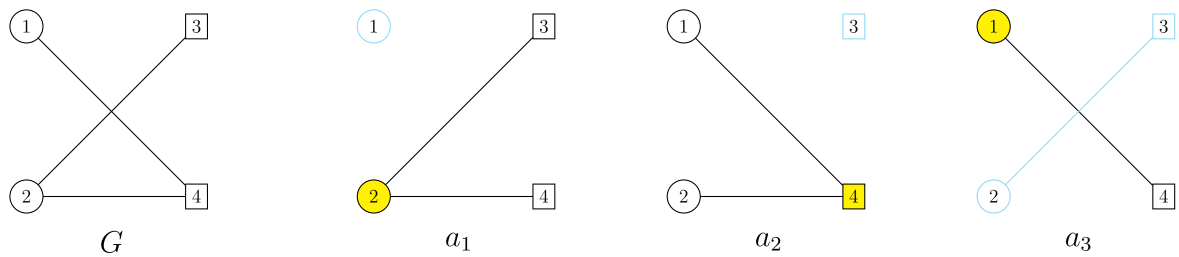


Figure 3.1: The represention of the extremal rays of the edge cone of a connected bipartite graph.

The three-dimensional cone $\sigma_G^\vee \subset M_\mathbb{R}$ is generated by the extremal rays $[1,0,0,1]$, $[0,1,1,0]$ and $[0,1,0,1]$. Let us calculate the extremal ray generators of the facet a_1 of $\sigma_G^\vee$ given by the independent set $A_1 = \{2\}$. Equivalently, we calculate the extremal ray a_1^* of σ_G. The supporting hyperplane associated to A_1 is $H_{A_1} = \{\overline{x} \in M_\mathbb{R} \mid x_2 = x_3 + x_4\}$. Therefore the facet a_1 is generated by $[0,1,1,0]$ and $[0,1,0,1]$. In the same way, one obtains that a_2 is generated by $[1,0,0,1]$ and $[0,1,0,1]$ and a_3 is generated by $[1,0,0,1]$ and $[0,1,1,0]$. Moreover we obtain $a_1^* = e_1$, $a_2^* = e_3$, and $a_3^* = e_2 - e_3$.

Remark 2. We are not interested in the disjoint sets U_1 and U_2 as independent sets. For instance, in the previous example, if we consider the independent set $U_1 = \{1,2\}$, then we obtain $\mathrm{G}[\{1,2\} \sqcup N(\{1,2\})] = G$, i.e. $H_{\{1,2\}} \cap \sigma_G^\vee = \sigma_G^\vee$. Furthermore, one might suspect that all faces are induced by independent sets. However, this is unfortunately not true. Let us consider the one-dimensional face $a_1 \cap a_2 = \langle [0,1,0,1] \rangle \prec \sigma_G^\vee$. It is represented by the edge $(2,4) \in E(G)$, but there exists no independent set A such that $\langle [0,1,0,1] \rangle = H_A \cap \sigma_G^\vee$. It is because one has that $H_{U_2} \cap \sigma_G^\vee = \sigma_G^\vee$ and $H_{\{1\}} \cap \sigma_G^\vee = H_{\{3\}} \cap \sigma_G^\vee = a_3$

We now introduce more definitions for our upgraded description of the facets of $\sigma_G^\vee$. It will be crucial for us also to give a characterization for the lower dimensional faces of $\sigma_G^\vee$. Furthermore, distinguishing between one and two-sided cases will provide us with some advantages during our examination of the faces of σ_G. In particular, we will see the argument unifying these two types of independent sets in Remark 6.

Definition 3.1.4. An independent set A is called a *maximal independent set* if there is no other independent set containing it. We say that an independent set is *one-sided* if it is contained either in U_1 or in U_2. In a similar way, $A = A_1 \sqcup A_2$ is called a *two-sided independent set* if $\emptyset \neq A_1 \subset U_1$ and $\emptyset \neq A_2 \subset U_2$.

While Proposition 3.1.3 puts its focus on the independent sets satisfying certain conditions, we put our focus on presenting a one-to-one relation between the extremal ray generators and special independent sets. Note that the next two statements hold for any two-sided maximal independent set, i.e. it does not have to produce a facet.

Proposition 3.1.5. *Let $A = A_1 \sqcup A_2$ be a two-sided maximal independent set. Then, one has $N(A_2) = U_1 \backslash A_1$ and $A_2 = U_2 \backslash N(A_1)$.*

Proof. Let $x \in N(A_2)$. By definition there exists a vertex $y \in A_2$ such that $(x, y) \in E(G)$. Since A is an independent set, x can not be in A_1. Conversely, let $x \in U_1 \backslash A_1$. Since G is connected, there exists a vertex $y \in U_2$ such that $(x, y) \in E(G)$. Suppose that $x \notin N(A_2)$. This means that for any $a_2 \in A_2$, $(x, a_2) \notin E(G)$. This implies that $x \in A_1$ by maximality of the independent set A, a contradiction. The other equality follows similarly. $\square$

Remark 3. Let A be a two-sided maximal independent set. By the equalities from Proposition 3.1.5, we observe that

$$\mathrm{G}[A_1 \sqcup N(A_1)] = \mathrm{G}[(U_1 \backslash N(A_2)) \sqcup (U_2 \backslash A_2)]$$

$$\mathrm{G}[(U_1 \backslash A_1) \sqcup (U_2 \backslash N(A_1))] = \mathrm{G}[A_2 \sqcup N(A_2)]$$

hold. In particular, the union of the induced subgraphs $\mathrm{G}[A_1 \sqcup N(A_1)]$ and $\mathrm{G}[(U_1 \backslash A_1) \cup (U_2 \backslash N(A_1))]$ is a spanning subgraph.

Now, we would like to characterize the independent sets resulting a facet of $\sigma_G^\vee$. We deduce precise conditions on an independent set A. Let us start with two-sided independent sets.

Definition 3.1.6. Let $G[[A]]$ be the subgraph of G associated to the independent set A defined as

$$\begin{cases} G[A \sqcup N(A)] \sqcup G[(U_1 \backslash A) \sqcup (U_2 \backslash N(A))], \text{if } A \subseteq U_1 \text{ is one-sided.} \\ G[A \sqcup N(A)] \sqcup G[(U_2 \backslash A) \sqcup (U_1 \backslash N(A))], \text{if } A \subseteq U_2 \text{ is one-sided.} \\ G[A_1 \sqcup N(A_1)] \sqcup G[A_2 \sqcup N(A_2)], \text{if } A = A_1 \sqcup A_2 \text{ is two-sided.} \end{cases}$$

We define the *associated bipartite subgraph* $\mathrm{G}\{A\} \subseteq G$ to the independent set A as the spanning subgraph $G[[A]] \sqcup \big(V(G) \backslash V(G[[A]])\big)$.

Example 4. Let $G \subsetneq K_{2,2}$ be the connected bipartite graph from Example 3. We observe that $\{1\} \sqcup \{3\}$ is a two-sided maximal independent set and the associated subgraph $\mathrm{G}\{\{1\} \sqcup \{3\}\}$ is the fourth bipartite graph in Figure 3.1. Likewise, the second and third graphs are the associated subgraphs $\mathrm{G}\{\{2\}\}$ and $\mathrm{G}\{\{4\}\}$ to the one-sided independent sets $\{2\} \subset U_1$ and $\{4\} \subset U_2$. Moreover, we have $G = \mathrm{G}\{\{1, 2\}\} = \mathrm{G}\{\{3, 4\}\}$.

Lemma 3.1.7. *If $A = A_1 \sqcup A_2$ is a two-sided independent set and if $H_{A_1} \cap \sigma_G^\vee$ is a facet of $\sigma_G^\vee$, then there exists a maximal two-sided independent set $A' = A_1 \sqcup A_2'$ for some vertex set $A_2' \supseteq A_2$.*

Proof. Assume that the two-sided maximal independent set $A' = A_1 \sqcup A_2'$ is not maximal, i.e. there exists a vertex set $A_1' \supset A_1$ such that $A'' = A_1' \sqcup A_2'$ is a maximal two-sided independent set. Let $v \in A_1' \backslash A_1$ be a vertex. By Proposition 3.1.3, since $H_{A_1} \cap \sigma_G^\vee$ is a facet of $\sigma_G^\vee$, the induced subgraph $\mathrm{G}[(U_1 \backslash A_1) \sqcup (U_2 \backslash N(A_1))] = \mathrm{G}[(U_1 \backslash A_1) \sqcup A_2']$ must be connected. However v is an isolated vertex in $\mathrm{G}[(U_1 \backslash A_1) \sqcup U_2 \backslash N(A_1)]$ which contradicts with the connectedness assumption. $\square$

Remark 4. We observe that there is a symmetry for the supporting hyperplanes for a two-sided maximal independent set $A = A_1 \sqcup A_2$. Recall that the supporting hyperplane associated to a one-sided independent set $A_i \subseteq U_i$ is defined as

$$H_{A_i} = \{x \in M_{\mathbb{Q}} \mid \sum_{v_i \in A_i} x_i = \sum_{v_i \in N(A_i)} x_i\}.$$

Assume that $x \in H_{A_1}$. By the previous definition and since $M_{\mathbb{Q}} \cong \mathbb{Q}^{m+n}/\overline{(1,-1)}^{\perp}$, it follows that $(H_{A_1} \cap \sigma_G^\vee)^* = (H_{A_2} \cap \sigma_G^\vee)^* \subseteq N_{\mathbb{Q}}$, hence $H_{A_1} \cap \sigma_G^\vee = H_{A_2} \cap \sigma_G^\vee$. Therefore it is enough to consider only one component A_i of the maximal two-sided independent set $A = A_1 \sqcup A_2$ for the associated supporting hyperplane.

Now, we examine the one-sided independent sets resulting a facet of $\sigma_G^\vee$.

Lemma 3.1.8. *Let A be a one-sided independent set not contained in any two-sided independent set. Then $N(A)$ is equal to one of the disjoint sets of G. If $H_A \cap \sigma_G^\vee$ is a facet of $\sigma_G^\vee$, then $A = U_i \backslash \{u_i\}$ for some $u_i \in U_i$. Moreover, one obtains the following equality $H_A \cap \sigma_G^\vee = H_{e_i} \cap \sigma_G^\vee$.*

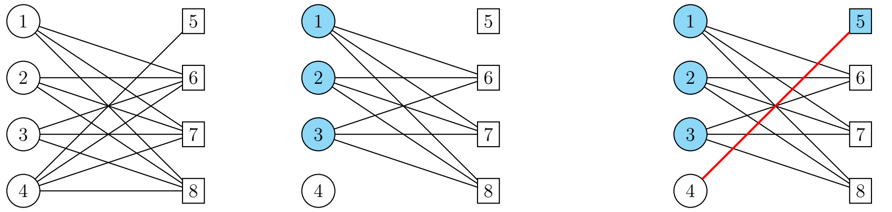
Proof. Let $A \subsetneq U_1$ be a one-sided independent set. Suppose that $N(A) = U_2$, then the induced subgraph $\mathrm{G}[(U_1 \backslash A) \sqcup (U_2 \backslash N(A))]$ consists of isolated vertices. Thus this induced subgraph is connected if and only if $|A| = m-1$. Suppose that $N(A) \neq U_2$, then $A \sqcup (U_2 \backslash N(A))$ is a two-sided independent set containing A. Hence, if A is a one-sided independent set not contained in any two-sided independent set, then $|A| = m-1$ and $N(A) = U_2$. The supporting hyperplane H_A associated to A is

$$\{x \in M_{\mathbb{Q}} \mid x_1 + ... + \widehat{x_i} + ...x_m = x_{m+1} + ... + x_{m+n}\}.$$

Since the chosen lattice $N = \mathbb{Z}^{m+n}/\overline{(1,-1)}$, we obtain the equality $H_A \cap \sigma_G^\vee = H_{e_i} \cap \sigma_G^\vee$. $\square$

Example 5. Let $G \subsetneq K_{4,4}$ be the connected bipartite graph with the edge set $E(G) = E(K_{4,4}) \backslash \{(1,5),(2,5),(3,5)\}$. We consider the one-sided independent set $A = \{1,2,3\}$.

Since $N(A) = \{6, 7, 8\} \subsetneq U_2$, it is contained in a two-sided independent set $\{1, 2, 3, 5\}$. We observe in the figure below that this two-sided independent set forms a facet τ of $\sigma_G^\vee$ and it is maximal. Therefore, one obtains that $\tau = H_{\{1,2,3\}} \cap \sigma_G^\vee = H_{\{5\}} \cap \sigma_G^\vee$.



G $\qquad$ $\mathrm{G}[A \cup N(A)]$ $\qquad$ $\mathrm{G}[A \sqcup N(A)] \sqcup \mathrm{G}[\{5\} \sqcup N(\{5\})]$

Moreover, the independent sets of form $U_i \backslash \{\bullet\}$ other than A give the remaining facets of $\sigma_G^\vee$. Here $\{\bullet\}$ stands for a single vertex in U_i.

Remark 5. As noted before, the one-sided independent set A from Lemma 3.1.8 cannot be the whole disjoint set U_i. The supporting hyperplane $H_A \cap \sigma_G^\vee$ is then equal to the cone $\sigma_G^\vee$. Also, similarly as in the case of two-sided maximal independent sets, if $A = U_i \backslash \{u_i\}$ is a one-sided independent set, then the union of the induced subgraphs $\mathrm{G}[A_i \sqcup N(A_i)]$ and $\mathrm{G}[(U_i \backslash A_i) \cup (U_j \backslash N(A_j))] = u_i$ is a spanning subgraph of G.

Let us collect the independent sets of G that we obtained in Lemma 3.1.7 and Lemma 3.1.8 in a set:

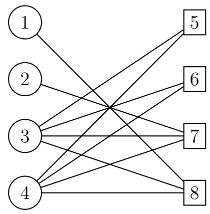
$$\mathcal{I}_G^{(*)} := \{\text{Two-sided maximal independent sets}\} \sqcup \{\text{One-sided independent sets } U_i \backslash \{\bullet\} \text{ not contained in any two-sided maximal independent set}\}$$

To put it succinctly, we present the following theorem.

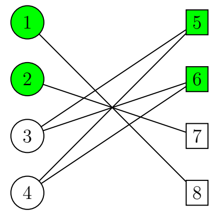
Theorem 3.1.9. *If $H_{A_1} \cap \sigma_G^\vee$ is a facet of $\sigma_G^\vee$, then there exists an independent set $A = A_1 \sqcup A_2 \in \mathcal{I}_G^{(*)}$.*

By Remark 3 and Remark 5, the condition in Proposition 3.1.3 about $\mathrm{G}\{A\}$ being a spanning subgraph where $A \in \mathcal{I}_G^{(*)}$ can be dropped. However, the induced subgraphs $\mathrm{G}[A_1 \sqcup N(A_1)]$ and $\mathrm{G}[A_2 \sqcup N(A_2)]$ might not be connected. In the next example, we observe that $\mathcal{I}_G^{(*)}$ is a necessary but not sufficient condition to form a facet. This remark will be useful for us once we start describing the lower dimensional faces of $\sigma_G^\vee$.

Example 6. Let $G \subsetneq K_{4,4}$ as in the figure below. Consider the two-sided independent set $A = A_1 \sqcup A_2 = \{1,2\} \sqcup \{5,6\}$. We see that $N(A_1) = \{7,8\}$ and $N(A_2) = \{3,4\}$. One can observe that although $A = \{1,2,5,6\}$ is a maximal two-sided independent set, the induced subgraph $G[A_1 \sqcup N(A_1)]$ is not a connected graph.



G



$\mathrm{G}[A_1 \sqcup N(A_1)]$

In the next proposition, we examine the case where $\mathrm{G}\{A\}$ has more than two connected components.

Proposition 3.1.10. *Let $A = A_1 \sqcup A_2 \in \mathcal{I}_G^{(*)}$ be an independent set. Suppose that the induced subgraph $G[A_1 \sqcup N(A_1)]$ consists of d connected bipartite graphs G_i with vertex sets $X_i \subseteq A_1$ and $N(X_i) \subseteq N(A_1)$ and the induced subgraph $G[(U_1 \backslash A_1) \sqcup (U_2 \backslash N(A_1))]$ is connected. Then, for each $i \in [d]$ there exist two-sided maximal independent sets $X_i \sqcup (A_2 \sqcup \bigsqcup_{j \neq i} N(X_j))$ forming facets of $\sigma_G^{\vee}$.*

Proof. We have two cases to examine:

(i) Let $A_1 = U_1 \backslash \{u\}$. We obtain the two-sided maximal independent sets $X_i \sqcup (\bigsqcup_{j \neq i} N(X_j))$. Since G is connected, for each $i \in [d]$, there exists a vertex $x_i \in N(X_i) \subseteq N(A_1)$ such that $(u, x_i) \in E(G)$. The associated subgraphs $\mathrm{G}\{X_i \sqcup (\bigsqcup_{j \neq i} N(X_j))\}$ have therefore two connected components. Thus, these maximal independent sets form facets of $\sigma_G^{\vee}$.

(ii) Let $A = A_1 \sqcup A_2$ be a two-sided maximal independent set. We obtain again new two-sided maximal independent sets $X_i \sqcup (A_2 \sqcup \bigsqcup_{j \neq i} N(X_j))$. Since G is connected, $N(N(A_2)) \supset A_2 \sqcup \bigcup_{i \in [k]} x_i$, where $x_i \in N(X_i)$. Therefore, the associated subgraphs $\mathrm{G}\{X_i \sqcup (A_2 \sqcup \bigsqcup_{j \neq i} N(X_j))\}$ have two connected components. Thus, these maximal independent sets form facets of $\sigma_G^{\vee}$.

In particular, if $A_2 \neq \emptyset$, one can state the proposition symmetrically with $G[A_2 \sqcup N(A_2)]$ having d connected components and $\mathrm{G}[A_1 \sqcup N(A_1)]$ being connected.

$\square$

Example 7. Consider the graph $G \subsetneq K_{4,4}$ from Example 6 and the maximal two-sided independent set $A = \{1,2,5,6\}$. The induced subgraph $\mathrm{G}[A_2 \sqcup N(A_2)]$ is connected. The two connected bipartite graphs of $\mathrm{G}[A_1 \sqcup N(A_1)]$ have the vertex sets $X_1 \sqcup N(X_1) := \{1\} \sqcup \{8\}$ and $X_2 \sqcup N(X_2) := \{2\} \sqcup \{7\}$. Hence, we obtain the following first independent sets

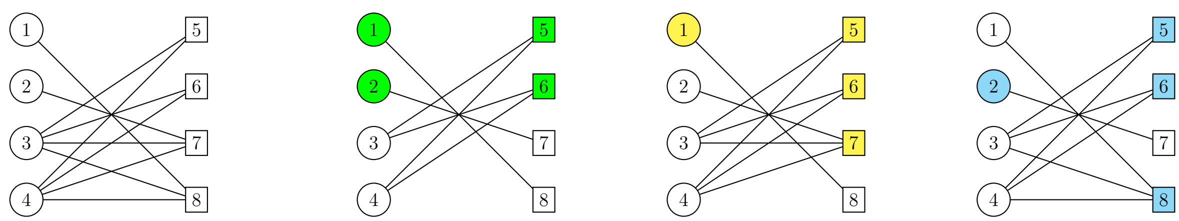
$$X_1 \sqcup A_2 \sqcup N(X_2) = \{1, 5, 6, 7\}$$

$$X_2 \sqcup A_2 \sqcup N(X_1) = \{2, 5, 6, 8\}$$

With the motivation of Proposition 3.1.10, in order to give a sufficient condition on an independent set to form a facet, we present the following definition which is just another way of saying that $G[A_1 \sqcup N(A_1)]$ and $G[A_2 \sqcup N(A_2)]$ are connected.

Definition 3.1.11. $A \in \mathcal{I}_G^{(*)}$ is called *indecomposable* if $\mathrm{G}\{A\}$ has two connected components.

Example 8. We consider the same graph $G \subsetneq K_{4,4}$ from Example 6. Then $A = \{1, 2, 5, 6\}$, $A' = \{1, 5, 6, 7\}$, and $A'' = \{2, 5, 6, 8\}$ are two-sided maximal independent sets of G. We see in the figure below that $\mathrm{G}\{A\}$ has three connected components, therefore A is decomposable and does not form a facet. By Proposition 3.1.10, there exist two-sided maximal independent sets A' and A'' forming a facet. In particular, we will observe in Example 10 that $(H_{A_1} \cap \sigma_G^\vee)^*$ is actually a two-dimensional face of σ_G.



G $\qquad$ $\mathrm{G}\{A\}$ $\qquad$ $\mathrm{G}\{A'\}$ $\qquad$ $\mathrm{G}\{A''\}$

Definition 3.1.12. We define the *first independent sets* of G as the indecomposable elements of $\mathcal{I}_G^{(*)}$. We denote the set of first independent sets by $\mathcal{I}_G^{(1)}$.

Remark that if a one-sided independent set $U_i \backslash \{\bullet\}$ is indecomposable, then it is not contained in any two-sided maximal independent set. In Section 2.1, we have seen that there is a one-to-one correspondence between the facets of $\sigma_G^\vee$ and the extremal rays of σ_G. The face $\tau \preceq \sigma_G^\vee$ is a facet of $\sigma_G^\vee$ if and only if $\tau^* := \tau^\perp \cap \sigma_G$ is an extremal ray of σ_G.

Theorem 3.1.13. *There is a one-to-one correspondence between the set of extremal generators of the cone σ_G and the first independent set $\mathcal{I}_G^{(1)}$ of G. In particular, the map is given* *as*

$$\begin{array}{rcl} \pi : \mathcal{I}_G^{(1)} & \longrightarrow & \sigma_G^{(1)} \\ A & \mapsto & \mathfrak{a} := (H_{A_1} \cap \sigma_G^{\vee})^* \end{array}$$

Proof. By Lemma 3.1.7 and Lemma 3.1.8, the map is surjective. Suppose that we have the equality $(H_{A_1} \cap \sigma_G^{\vee})^* = (H_{B_1} \cap \sigma_G^{\vee})^*$. We can do this without loss of generality by Remark 4. Since $\sigma_G^{\vee}$ is full dimensional, one must have that $H_{A_1} = H_{B_1}$. Then, the graphs $\mathrm{G}\{A\}$ and $\mathrm{G}\{B\}$ are the same. This implies that they are either both one-sided or either both two-sided. If they are both one-sided, then $A = B$. Let both of them be two-sided and assume that we have $A_1 = N(B_2)$ and $B_2 = N(A_1)$. This implies that $N(N(B_2)) = B_2$ and $N(N(A_1)) = A_1$. This means that G is not connected. Therefore, we get $A = B$. □

Proposition 3.1.14. *The generators of the cone $\sigma_G^{\vee}$ for a bipartite graph G form the Hilbert Basis of $\sigma_G^{\vee}$.*

Proof. See [[VV05], Lemma 3.10]. The notation $\mathbb{R}_+\mathcal{A}$ used in this paper is $\sigma_G^{\vee} \subseteq M \otimes_{\mathbb{Z}} \mathbb{R}$ in our context. Also, $\mathbb{N}\mathcal{A}$ stands for the semigroup generated by the generators of $\sigma_G^{\vee}$ with nonnegative integer coefficients. □

Definition 3.1.15. The *degree (valency) sequence* of a graph $G \subseteq K_{m,n}$ is the $(m+n)$-tuple of the degrees (valencies) of its vertices. Let $A \in \mathcal{I}_G^{(1)}$ be a first independent set. We denote the degree sequence of the associated subgraph $\mathrm{G}\{A\}$ by $\mathcal{V}\mathfrak{al}_A \in \sigma_G^{\vee} \cap M$.

Theorem 3.1.16. *Let $A \in \mathcal{I}_G^{(1)}$ be a first independent set. Then, the extremal ray generators of the facet $\mathfrak{a}^*$ formed by A are exactly the extremal ray generators of $\sigma_{\mathrm{G}\{A\}}^{\vee}$. Moreover, one obtains that $\mathfrak{a} = (H_{A_i} \cap \sigma_G^{\vee})^* = \mathcal{H}_{\mathcal{V}\mathfrak{al}_A} \cap \sigma_G$.*

Proof. Let $\mathfrak{a}^* = H_{A_1} \cap \sigma_G^{\vee} \prec \sigma_G^{\vee}$ be the facet associated to the first independent set A. Since the extremal rays of $\sigma_G^{\vee}$ form the Hilbert Basis by Proposition 3.1.14, the facet $\mathfrak{a}^*$ is generated by the extremal rays of $\sigma_{G'}^{\vee}$, where G' is a subgraph of G. By the definition of the supported hyperplane H_{A_1}, the extremal rays of $\sigma_{\mathrm{G}\{A_1\}}^{\vee}$ are in the set of extremal ray generators of $\mathfrak{a}^*$. If A is two-sided, then $\sigma_{\mathrm{G}\{A_2\}}^{\vee}$ is also included in $\mathfrak{a}^*$. These are the only extremal ray generators of $\mathfrak{a}^*$. To show this, we examine the edges in $E(G)\backslash E(G\{A\})$ in two cases:

- If $A = U_1\backslash\{u_i\}$ is one-sided, then for $j \in [m]$, $e^i + f^j \in M$ is not in the generator set of $\mathfrak{a}^*$.
- If $A = A_1 \sqcup A_2$ is two-sided, then the remaining rays $e^i + f^j$ for $i \in N(A_2)$ and $j \in N(A_1)$ with $(i, j) \in E(G)$ are not in the generator set of $\mathfrak{a}^*$.

By construction, $\mathcal{V}\mathfrak{al}_A \in \sigma_G^\vee \cap M$. We have $\mathfrak{a} = \mathcal{H}_{\mathcal{V}\mathfrak{al}_A} \cap \sigma_G$ if and only if $\mathcal{V}\mathfrak{al}_A \in \mathrm{Relint}(\mathfrak{a}^*)$. Since we chose $\mathcal{V}\mathfrak{al}_A \in \sigma_G^\vee$ to be the sum of the generators of the facet $\mathfrak{a}^*$, we obtain $\mathcal{V}\mathfrak{al}_A \in \mathrm{Relint}(\mathfrak{a}^*)$.

□

Note that the degree sequence $\mathcal{V}\mathfrak{al}_A \in \mathrm{Relint}(a^*)$ defining the extremal ray $\mathfrak{a}$ is not unique. One can see it more precisely in the following example.

Example 9. Consider the first independent set $A' = \{1, 5, 6, 7\}$ of $G \subsetneq K_{4,4}$ from Example 6. We have that $\mathcal{V}\mathfrak{al}_A = [1, 1, 3, 3, 2, 2, 3, 1] \in \mathrm{Relint}(\mathfrak{a}^*)$ and hence by Theorem 3.1.16 $\mathfrak{a} = \mathcal{H}_{\mathcal{V}\mathfrak{al}_A} \cap \sigma_G$. However the degree sequence $[1, 1, 2, 2, 1, 1, 3, 1] \in \mathrm{Relint}(\mathfrak{a}^*)$ also gives the extremal ray $\mathfrak{a}$.

3.2 Description of the faces of an edge cone

In this section, we introduce the technique to find the faces of σ_G by using the induced subgraphs $G\{A\}$ that we presented in the previous section.

Lemma 3.2.1. *Let $G \subseteq K_{m,n}$ be a bipartite graph with k connected components. Then $\dim(\sigma_G) = m + n - 1$ and $\dim(\sigma_G^\vee) = m + n - k$.*

Proof. Recall that by Proposition 2.3.3, if G is connected, then the rank of the incidence matrix A_G is $m+n-1$. Suppose that G has d connected components G_i. Then the incidence matrix A_G is

$$\begin{bmatrix} A_{G_1} & 0 & 0 & \dots & 0 \\ 0 & A_{G_2} & 0 & \dots & 0 \\ \vdots & 0 & \ddots & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_{G_d} \end{bmatrix}$$

Therefore the rank of A_G, i.e. dimension of the dual edge cone is $m + n - d$. Furthermore, since $\sigma_G^\vee$ contains no linear subspace, the edge cone $\sigma_G \subseteq N_{\mathbb{Q}}$ is full dimensional and hence $\dim(\sigma_G) = m + n - 1$. □

Theorem 3.2.2. *Let $I \subseteq \mathcal{I}_G^{(1)}$ be a subset of d first independent sets and let π be the bijection from Theorem 3.1.13. The extremal ray generators $\pi(I)$ form a face of dimension d if and only if the dimension of the dual edge cone of the graph $\mathrm{G}[I] := \bigcap_{A\in I} \mathrm{G}\{A\}$ is $m+n-d-1$, i.e. $\mathrm{G}[I]$ has $d+1$ connected components. In particular, the face can be written as $\mathcal{H}_{\mathcal{V}\mathfrak{al}_I} \cap \sigma_G$ where $\mathcal{V}\mathfrak{al}_I$ is the degree sequence of the graph $\mathrm{G}[I]$.*

Proof. By Theorem 3.1.13, if $A \in I$, then the associated facet $a^* \preceq \sigma_G^\vee$ is generated by the extremal ray generators of $\sigma_{\mathrm{G}\{A\}}^\vee$. Hence, intersecting these induced subgraphs $\mathrm{G}\{A\}$ is equivalent to intersecting the extremal ray generators of the facets. This intersection forms a face τ of σ_G (and therefore a face of $\sigma_G^\vee$) since we have:

$$\tau^* = \bigcap_{\mathfrak{a}\in\pi(I)} \mathfrak{a}^* = \bigcap_{\mathfrak{a}\in\pi(I)} (\mathcal{H}_{\mathcal{V}\mathfrak{al}_A} \cap \sigma_G)^* = (\mathcal{H}_{\mathcal{V}\mathfrak{al}_I} \cap \sigma_G)^*$$

where $\mathcal{V}\mathfrak{al}_I \in \mathrm{Relint}(\sigma_{\mathrm{G}[I]}) \subsetneq \sigma_G^\vee$. By Lemma 3.2.1, $\dim(\sigma_G^\vee) = \dim(\sigma_G) = m+n-1$. Thus, the dimension of τ is d if and only if the dimension of τ^* is $m+n-d-1$. Hence, this means that the dimension of the cone $\sigma_{\mathrm{G}[I]}^\vee$ is $m+n-d-1$.

□

Corollary 3.2.3. *Let $\tau := \mathcal{H}_{\mathcal{V}\mathfrak{al}_I} \cap \sigma_G \preceq \sigma_G$ be a face of dimension d which is given by the intersection of subgraphs formed by a subset $I \subsetneq \mathcal{I}_G^{(1)}$ not necessarily of d elements. If $G[I] \subset G\{A'\}$ for some $A' \in \mathcal{I}_G^{(1)} \backslash I$, then the associated extremal ray generator $\mathfrak{a}'$ is also included in the generators of the face τ.*

Proof. It follows from Theorem 3.2.2 by dropping the condition of I consisting of d elements.

□

Remark 6 (Extremal rays of σ_G)**.** If $A \in \mathcal{I}_G^{(1)}$, then the dual edge cone of $\mathrm{G}\{A\}$ is $m+n-2$ dimensional, i.e. $\mathrm{G}\{A\}$ has two connected components. Let $\tau \prec \sigma_G^\vee$ be a facet generated by the extremal rays of the dual edge cone $\sigma_{G'}^\vee$ where G' is a spanning subgraph of G with two connected components. Two types of first independent sets (one-sided and two-sided) arise as follows:

- The subgraph G' has exactly one isolated vertex and one connected graph without any isolated vertices. Then $G' = \mathrm{G}\{A\}$ where A is one-sided.

- The subgraph G' has two connected components without any isolated vertices. Then $G' = \mathrm{G}\{A\}$ where A is two-sided.

This argument unifies the two types (one-sided and two-sided) of first independent sets. However we keep them as they help with calculations in the next chapters.

Proposition 3.2.4. *The maximal independent sets of Proposition 3.1.10 form a d-dimensional face $\tau \preceq \sigma_G$. Moreover $\tau = (H_{A_1} \cap \sigma_G^\vee)^*$.*

Proof. Let C^i denote the two-sided maximal independent sets $X_i \sqcup (A_2 \sqcup \bigsqcup_{j\neq i} N(X_j))$ for $i \in [d]$. By Theorem 3.2.2, the dual edge cone of the intersection subgraph $\bigcap \mathrm{G}\{C^i\}$ is $m+n-d-1$. Furthermore, since $\bigcap \mathrm{G}\{C^i\} = \mathrm{G}\{A\}$, one obtains

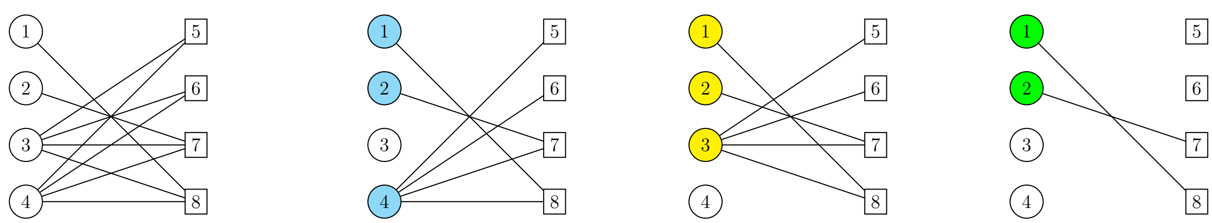
$$\langle \mathfrak{c}^1, \ldots, \mathfrak{c}^d \rangle = ((H_{C_1^1} \cap \sigma_G^\vee) \cap \ldots \cap (H_{C_1^d} \cap \sigma_G^\vee))^* = (H_{A_1} \cap \sigma_G^\vee)^*.$$

$\square$

Proposition 3.2.5. *Let A be an independent set of $V(G)$. Then $\tau = \mathcal{H}_{\mathcal{V}\mathfrak{al}_A} \cap \sigma_G$ is a d-dimensional face of σ_G where $m + n - d - 1 = \dim(\sigma^{\vee}_{\mathrm{G}\{A\}})$.*

Proof. It follows from Proposition 3.1.3 and Theorem 3.2.2. $\square$

Example 10. We examine the two and three-dimensional faces of σ_G for $G \subsetneq K_{4,4}$ from Example 6. We use the notation from Theorem 3.1.13. The edge cone σ_G is generated by the extremal ray generators $e_1, e_2, e_3, e_4, f_1, f_2, \mathfrak{a}', \mathfrak{a}''$. From the figure in Example 8, we observe that $\mathrm{G}\{A\} = \mathrm{G}\{A'\} \cap \mathrm{G}\{A''\}$, thus $(H_{A_1} \cap \sigma^{\vee}_G)^*$ is a two-dimensional face generated by $\mathfrak{a}'$ and $\mathfrak{a}''$. Furthermore, we see that the intersection of the associated subgraphs $\mathrm{G}\{A'\} \cap \mathrm{G}\{A''\}$ with another associated subgraph to an extremal ray of σ_G has four connected components. The only pair of extremal rays which does not span a two-dimensional face of σ_G is $\{e_3, e_4\}$. One can infer this in Figure 3.2 below: The intersection $\mathrm{G}\{U_1 \backslash \{3\}\} \cap \mathrm{G}\{U_1 \backslash \{4\}\}$ has the edge set consisting of only two edges $(1, 8)$ and $(2, 7)$. This implies that any triple of extremal ray generators containing $\{e_3, e_4\}$ does not span a three-dimensional face of σ_G. In particular, by Proposition 3.2.5, for the independent set $\{1, 2\}$, we obtain a five-dimensional face of σ_G, since $\mathrm{G}\{\{1, 2\}\}$ has six connected components as seen in the figure. Lastly, a computation on the intersection of associated subgraphs shows that any triple not containing both e_3 and e_4 spans a three-dimensional face of σ_G.



G $\qquad$ $\mathrm{G}\{U_1 \backslash \{3\}\}$ $\qquad$ $\mathrm{G}\{U_1 \backslash \{4\}\}$ $\qquad$ $\mathrm{G}\{\{1, 2\}\}$

Figure 3.2: Studying faces of the edge cone via intersecting associated subgraphs to first independent sets.

Chapter 4

First Examples of Rigid Affine Toric Varieties

In [BHL15], it has been proven that the affine toric varieties $\mathrm{TV}(G)$ are rigid when $G \subsetneq K_{m,n}$ is obtained by an edge removal from $K_{m,n}$ with $m = n \geq 4$. We give a proof of this result with our methods presented in Chapter 3 without the assumption that $m = n$. We also generalize this result to multiple edge removals and present certain rigid affine toric variety families. In the case of complete bipartite graphs, we observe that the toric variety is isomorphic to the cone over a Segre embedding. We examine their rigidity alternatively by using our methods.

Label the vertices in U_1 with $\{1, \ldots, m\}$ and the vertices in U_2 with $\{m+1, \ldots, m+n\}$. Recall that the lattices are $N \cong \mathbb{Z}^{m+n}/\overline{(1,-1)}$ and $M \cong \mathbb{Z}^{m+n} \cap \overline{(1,-1)}^{\perp}$. We utilize the bijections from Theorem 3.1.13 and Theorem 3.2.2. The first one is the map between the first independent sets of G and the extremal ray generators of σ_G. The second one is between N-tuples of the first independent sets of G and the d-dimensional faces of σ_G.

$$\begin{array}{rcl} \pi\colon \mathcal{I}_G^{(1)} & \longrightarrow & \sigma_G^{(1)} \\ A & \mapsto & \mathfrak{a} = (H_{A_1} \cap \sigma_G^{\vee})^* \end{array}$$

$$\begin{array}{rcl} \mathcal{I}_G^{(1)} \times \ldots \times \mathcal{I}_G^{(1)} & \longrightarrow & \sigma_G^{(d)} \\ I = (A^1, \ldots, A^N) & \mapsto & (\mathfrak{a}^1, \ldots, \mathfrak{a}^N) = \mathcal{H}_{\mathcal{V}\mathfrak{al}_I} \cap \sigma_G \end{array}$$

where $\mathcal{V}\mathfrak{al}_I$ is the degree sequence of the intersection subgraph $\bigcap_{i\in[N]} \mathrm{G}\{A^i\}$ and $d+1$ is the number of connected components of $\bigcap_{i\in[N]} \mathrm{G}\{A^i\}$.

4.1 Complete bipartite graphs

First of all, we study the rigidity of the toric variety $\mathrm{TV}(K_{m,n})$ for a complete bipartite graph $K_{m,n}$. It is the easiest configuration and a nice illustration for the application of our method from Chapter 3. In particular, the cone $\sigma^{\vee}_{K_{m,n}}$ is the Segre cone over the embedding $\mathbb{P}^m \times \mathbb{P}^n \hookrightarrow \mathbb{P}^{(m+1)(n+1)-1}$. As we have seen in Section 2.3, $\mathrm{TV}(K_{m,n})$ is a determinantal singularity and rigid. We prove its rigidity by using bipartite graphs.

Proposition 4.1.1. *The edge cone $\sigma_{K_{m,n}} \subseteq N_{\mathbb{Q}}$ is generated by $e_1, \ldots, e_m, f_1, \ldots, f_n$.*

Proof. The complete bipartite graph has no edge removals, therefore it has no two-sided first independent set. The associated subgraph $\mathrm{G}\{U_i \backslash \{u\}\}$ is connected for each $u \in U_i$ and $i = 1, 2$. $\square$

These generators are extremal ray generators if $m \neq 1$ and $n \neq 1$. If $m = 1$ or $n = 1$, the extremal ray generators are $f_1, \ldots, f_n$ and $e_1, \ldots, e_m$ respectively. In these cases, $\mathrm{TV}(K_{m,n})$ is smooth and hence rigid.

Proposition 4.1.2. *The two-dimensional faces of $\sigma_{K_{m,n}}$ are*

(1) *all pairs except (e_1, e_2), if $m = 2$ and $n \geq 3$.*

(2) *all pairs of extremal rays, if $m \geq 3$ and $n \geq 3$.*

The three-dimensional faces of $\sigma_{K_{m,n}}$ are

(1) *all triples of extremal rays not containing both e_1 and e_2, if $m = 2$, $n \geq 4$.*

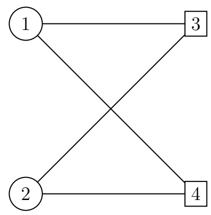
(2) *all triples of extremal rays except (e_1, e_2, e_3), if $m = 3$ and $n \geq 4$.*

(3) *all triples of extremal rays, if $m \geq 4$ and $n \geq 4$.*

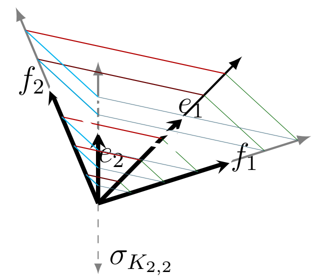
Proof. For the two-dimensional faces, if $m = 2$, then intersection subgraph $\mathrm{G}\{U_1 \backslash \{1\}\} \cap \mathrm{G}\{U_1 \backslash \{2\}\}$ consists of $n + 2$ isolated vertices. Hence, if $n = 1$, then (e_1, e_2) spans the edge cone of $K_{2,1}$ and in particular a 2-face. Otherwise, (e_1, e_2) does not span a 2-face. If $m \neq 2$ and $n \neq 2$, then the intersection of two associated bipartite graphs of any two first independent sets of $K_{m,n}$ has three connected components with two isolated vertices.

For the three-dimensional faces, if $m = 2$, a tuple containing both e_1 and e_2 forms a 3-face if and only if $n = 2$. However, in this case, $\sigma_{K_{2,2}}$ is three-dimensional and generated by e_1, e_2, f_1, f_2. If $m = 3$, $\mathrm{G}\{U_1 \backslash \{1\}\} \cap \mathrm{G}\{U_1 \backslash \{2\}\} \cap \mathrm{G}\{U_1 \backslash \{3\}\}$ consists of $n + 3$ isolated vertices. Hence, if $n = 1$, then (e_1, e_2, e_3) spans the edge cone of $K_{3,1}$ and in particular a 3-face. Otherwise (e_1, e_2, e_3) does not span a 3-face. If $m \geq 4$ and $n \geq 4$, then the intersection of three associated bipartite graphs of any three first independent sets of $K_{m,n}$ has four connected components with three isolated vertices. $\square$

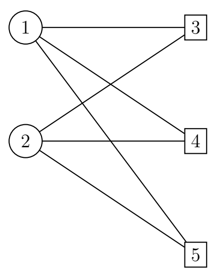
Example 11. Let us calculate the small examples $K_{2,2}$, $K_{2,3}$, and $K_{3,3}$ which are excluded in Proposition 4.1.2. As seen in the proof above, the three-dimensional edge cone $\sigma_{K_{2,2}}$ is generated by the extremal rays e_1, e_2, f_1, f_2 where (e_1, e_2) and (f_1, f_2) do not span a 2-face. Next, consider the complete bipartite graph $K_{2,3}$. We see that the intersection graphs $\mathrm{G}\{U_1 \backslash \{1\}\} \cap \mathrm{G}\{U_1 \backslash \{2\}\}$ and $\mathrm{G}\{U_2 \backslash \{3\}\} \cap \mathrm{G}\{U_2 \backslash \{4\}\} \cap \mathrm{G}\{U_2 \backslash \{5\}\}$ have five isolated vertices and therefore (e_1, e_2) does not span a 2-face and (f_1, f_2, f_3) does not span a 3-face. In the figure below, the cone over the double tetrahedron P is $\sigma_{K_{2,3}}$.



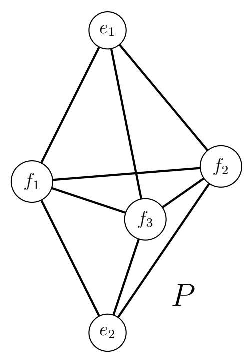
$K_{2,2}$



$\sigma_{K_{2,2}}$



$K_{2,3}$



P

Finally, consider the complete bipartite graph $K_{3,3}$. Similar to the calculation on $K_{2,3}$, we observe that (e_1, e_2, e_3) and (f_1, f_2, f_3) do not span 3-faces. Any other triple of extremal ray generators spans a 3-face.

Now, we would like to apply the deformation theory techniques which we introduced in Section 2.2. We recall the setting and the statement from Section 2.2. Let $R \in M$ be a deformation degree and consider the affine space $[R = 1] := \{a \in N_{\mathbb{Q}} \mid \langle R, a \rangle = 1\} \subseteq N_{\mathbb{Q}}$. We define the crosscut of σ_G in degree R as the polyhedron $Q(R) := \sigma_G \cap [R = 1]$. Let $d^1, \ldots, d^N$ be the compact edges of $Q(R)$ and let $\bar{\epsilon} \in \{0, \pm 1\}^N$ be the sign vector assigned to each two-dimensional compact face ϵ of $Q(R)$. For every deformation degree $R \in M$, the related vector space is defined as

$$V(R) = \{(t_1, \ldots, t_N) | \sum_{i \in [N]} t_i \overline{\epsilon_i} d^i = 0, \text{ for every compact 2-face } \epsilon \preceq Q(R)\}.$$

Corollary 4.1.3. *[Corollary 2.7, [Alt00]] If the affine normal toric variety X is smooth in codimension 2, then $T^1_X(-R)$ is contained in $V_{\mathbb{C}}(R)/\mathbb{C}(\overline{1})$. Moreover, it is built by those $\overline{t}$'s satisfying $t_{ij} = t_{jk}$ where $\overline{a_j}$ is a non-lattice common vertex in $Q(R)$ of the edges $d^{ij} = \overline{\overline{a_i}\ \overline{a_j}}$ and $d^{jk} = \overline{\overline{a_j}\ \overline{a_k}}$.*

Note that since a two-dimensional face of $\sigma_{K_{m,n}}$ is a pair of the canonical basis elements of $\mathbb{Z}^{m+n}$, $\mathrm{TV}(K_{m,n})$ is smooth in codimension 2.

Example 12. We examine the rigidity of toric varieties associated to complete bipartite graphs from Example 11. The three-dimensional edge cone $\sigma_{K_{2,2}} \subsetneq N_{\mathbb{Q}}$ is generated by the extremal rays $\langle e_1, e_2, f_1, e_1 + e_2 - f_1 \rangle$. For $R = [1,1,1,1] \in M$, the vertices of $Q(R)$ are all lattice vertices. This implies that $T^1_{\mathrm{TV}(K_{2,2})}(-R) \neq 0$. Next, let us consider the edge cone $\sigma_{K_{2,3}}$. It does not have any non-simplicial 3-face. It suffices to check the cases where the non 2-face pair (e_1, e_2) or non 3-face triple (f_1, f_2, f_3) appears in the crosscut. Suppose that $\overline{f_1}, \overline{f_2}, \overline{f_3}$ are vertices in $Q(R)$, for a deformation degree $R = [R_1, \ldots, R_5] \in M$. Then we obtain that $R_1 + R_2 \geq 3$. This means that there exists a non-lattice vertex $\overline{e_i} \in Q(R)$. Now suppose that $\overline{e_1}$ and $\overline{e_2}$ are vertices in $Q(R)$. Then we have that $R_3 + R_4 + R_5 \geq 2$ and thus there exists a non-lattice vertex f_j or there exist two lattice vertices f_k and f_l in $Q(R)$. In Figure 4.1, these cases and their vector space $V(R)$ are illustrated.

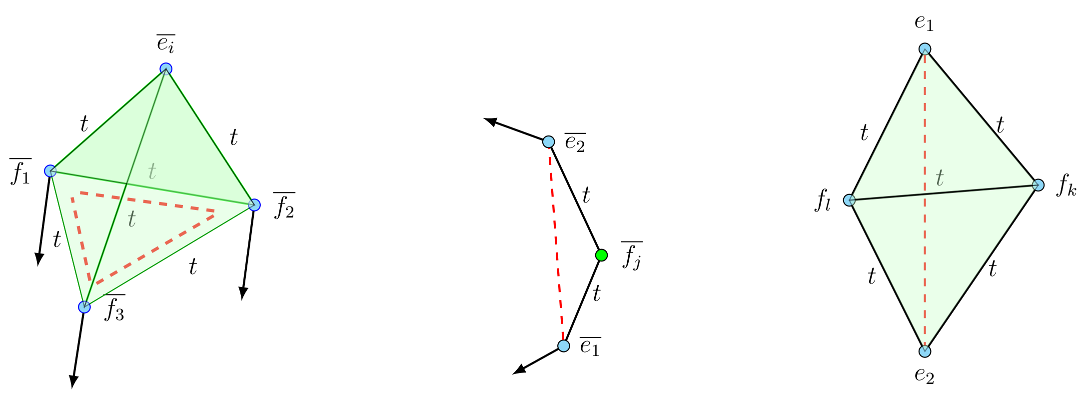


Figure 4.1: Some crosscut pictures of the edge cone $\sigma_{K_{2,3}}$

Finally, we consider the edge cone of $K_{3,3}$. Similar to $\sigma_{K_{2,3}}$, if $\overline{f_1}$, $\overline{f_2}$ and $\overline{f_3}$ are vertices in $Q(R)$, then there exists a non-lattice vertex e_i in $Q(R)$. The same follows symmetrically for the vertices $\overline{e_1}$, $\overline{e_2}$ and $\overline{e_3}$.

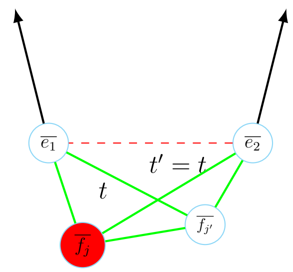
Theorem 4.1.4. $\mathrm{TV}(K_{m,n})$ *is rigid except for* $m = n = 2$.

Proof. It remains to prove three cases:

[$m = 2$ and $n \geq 4$]: The 2-faces are all pairs except (e_1, e_2) and the 3-faces are all triples which do not contain both e_1 and e_2. Assume that there exists a deformation degree $R \in M$ such that $\overline{e_1}$ and $\overline{e_2}$ are vertices in $Q(R)$ and $\overline{f_j}$ is a lattice vertex in $Q(R)$ for some $j \in [n]$. Then we obtain that

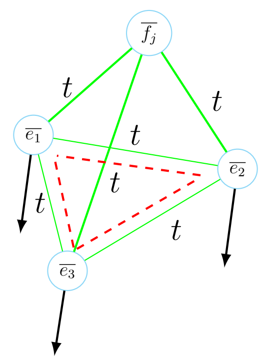
$$R_3 + \ldots + R_{j+1} + R_{j+3} + \ldots + R_{n+2} \geq 1.$$

Thus there exists a vertex $\overline{f_{j'}} \in Q(R)$ with $j' \neq j$. Hence we conclude that $T^1_{K_{m,n}}(-R) = 0$, since $(e_1, f_j, f_{j'})$ and $(e_2, f_j, f_{j'})$ are 3-faces.



The 2-faces are colored in green. The red vertex is a lattice vertex in $Q(R)$

[$m = 3$ and $n \geq 4$]: The 2-faces are all pairs and the 3-faces are all triples except (e_1, e_2, e_3). We just need to check the case where the non 3-face (e_1, e_2, e_3) appears in $Q(R)$. In this case, we obtain that $\sum_{i=4}^{n+3} R_i \geq 3$. This implies that there exists a vertex $\overline{f_j}$ for some $j \in [n]$. Thus t is transfered by the 2-faces (f_j, e_1), (f_j, e_2) and (f_j, e_3).



The dashed red area means that $\{e_1, e_2, e_3\}$ do not span a 3-face.

[$m \geq 4$ and $n \geq 4$]: All pairs are 2-faces and all triples are 3-faces. Hence the associated toric variety is rigid. $\square$

Example 13. This result has been checked for $K_{2,3}$ with `Polymake`'s Fulton application which uses `Singular`. The script can be found in Section 7.2.2.

Remark 7. The edge cone of $K_{2,3}$ has a similar shape to the cone in Example 2. While the toric variety in this example is not rigid, $\mathrm{TV}(K_{2,3})$ is rigid. Hence, we emphasise that it is important to calculate all cross-cuts $Q(R)$ for each deformation degree $R \in M$.

4.2 Bipartite graphs with one edge removal

In this section, we investigate the affine normal toric variety $\mathrm{TV}(G)$ where G is the connected bipartite graph with one edge removal from the complete bipartite graph, i.e $E(G) = E(K_{m,n})\backslash\{e\}$ for some $e \in E(K_{m,n})$. Due to symmetry, we may assume that the removed edge is $(1, m+1)$. Since we are studying the connected bipartite graphs, while we investigate the bipartite graphs with edge removals, we omit the graph with $m = 1$ or $n = 1$. In these cases we obtain an edge cone isomorphic to the edge cone of some complete bipartite graph. We examined this case in Section 4.1.

Let $A \in \mathcal{I}_G^{(1)}$ be a two-sided first independent set. Since one has that $\sum_{i\in[m]} e_i = \sum_{j\in[n]} f_j$ in N, one obtains the symmetrical expression for the associated extremal ray generator

$$\pi(A) = \mathfrak{a} = \sum_{i\in N(A_2)} e_i - \sum_{A_2} f_1 = \sum_{j\in N(A_1)} f_j - \sum_{i\in A_1} e_i.$$

where π is the bijective map from Theorem 3.1.13.

In this section, we study the rigidity of toric varieties associated to connected bipartite graphs with one edge removal, however the next two propositions examine a more general case. More precisely, we investigate the connected bipartite graphs where we remove all the edges from $K_{m,n}$ between two vertex sets $A_1 \subsetneq U_1$ and $A_2 \subsetneq U_2$. For Proposition 4.2.1 and Proposition 4.2.2, let $G \subsetneq K_{m,n}$ be a connected bipartite graph and $A = A_1 \sqcup A_2 \in \mathcal{I}_G^{(1)}$ be the its only two-sided first independent set.

Proposition 4.2.1. *The edge cone σ_G is generated by $e_1, \ldots, e_m, f_1, \ldots, f_n, \mathfrak{a}$.*

Proof. We know by the proof of Lemma 3.2.1 that the canonical basis elements of $\mathbb{Z}^{m+n}$ are all in the set of generators of σ_G. Recall the definition of the supporting hyperplane $H_{A_1} = \{x \in M_\mathbb{Q} \mid \sum_{i\in A_1} x_i = \sum_{i\in N(A_1)} x_i\}$. Hence we have $(H_{A_1} \cap \sigma_G^\vee)^* = (H_{A_2} \cap \sigma_G^\vee)^* = \mathfrak{a}$. □

Proposition 4.2.2. *The three-dimensional faces of σ_G are simplicial if and only if $|A_1| \neq 1$ and $|A_2| \neq n-2$ or $|A_1| \neq m-2$ and $|A_2| \neq 1$.*

Proof. Assume that $\tau \preceq \sigma_G$ is a non-simplicial face. Then there exists a non 2-face pair from the generators of τ. Now, we study the intersection of two bipartite graphs associated to two first independent sets which has four connected components. Consider first the non 2-face pair (e_1, f_1). Since e_1 and f_1 are extremal ray generators, one cannot have $|A_1| = m-1$ or $|A_2| = n-1$. The intersection of the associated bipartite graphs cannot have four isolated vertices, otherwise e_1 or f_1 is not an extremal ray generator. The only other possibility is that the intersection has three isolated vertices. This is possible if and only if $A_1 = U_1\backslash\{1,2\}$ and $A_2 = \{m+1\}$ or $A_1 = \{1\}$ and $A_2 = U_2\backslash\{1,2\}$. In these cases, we obtain that $\tau = \langle e_1, e_2, f_1, \mathfrak{a}\rangle$ or $\tau = \langle e_1, f_1, f_2, \mathfrak{a}\rangle$ respectively. It remains to consider the non 2-face pair $(e_1, \mathfrak{a})$. We just covered the case where $|A_1| = \{1\}$. Assume that $\{1\} \in N(A_2)$. In this case the intersection has four components if and only if $N(A_2) = \{1\}$. However, this is impossible since $U_1\backslash\{1\} \in \mathcal{I}_G^{(1)}$.

□

Remark 8. The generators of σ_G as in Proposition 4.2.1 are not necessarily the extremal ray generators. The independent set $U_i\backslash\{\bullet\}$ is decomposable if it is contained in the maximal independent set A. This means that $|A_1| = m-1$ or $|A_2| = n-1$. In these cases $U_i\backslash\{\bullet\}$ is not a first independent set.

One could suspect that for rigidity of $\mathrm{TV}(G)$, it is enough that all 3-faces of σ_G are simplicial. We have seen in Example 2 that it is not true in general. However, as soon as the cone has a non-simplicial three-dimensional face, the possibility to obtain a non-rigid toric variety is very high. The trick is that there exists a deformation degree $R \in M$ such that the crosscut $Q(R)$ consists only of this non-simplicial face. We explain this argument for general affine normal toric varieties in the next proposition.

Proposition 4.2.3. *Let* $\mathrm{TV}(\sigma)$ *be an affine normal toric variety. Assume that* τ *is a face of* σ *and* $\mathrm{TV}(\tau)$ *is not rigid. Then* $\mathrm{TV}(\sigma)$ *is also not rigid.*

Proof. Let $m \in \sigma^\vee$ and let $\tau = \mathcal{H}_m \cap \sigma$ be a face of σ. Since $\mathrm{TV}(\tau)$ is not rigid, there exists a deformation degree $R \in M$ such that $T^1_{\mathrm{TV}(\tau)}(-R) \neq 0$. Let us set another deformation degree $R' = R - k.m \in M$ for some positive integer $k \gg 0$. Since $-m \in R$ evaluates negative on $\sigma\backslash\tau$, we obtain that the compact part of $Q(R')$ consists of the face τ. Therefore $T^1_{\mathrm{TV}(\sigma)}(-R') = T^1_{\mathrm{TV}(\tau)}(-R) \neq 0$.

□

Now, we go back to our investigation for the graph $G \subsetneq K_{m,n}$ with one edge removal. Unless otherwise stated, $G \subsetneq K_{m,n}$ is the connected bipartite graph with one edge removal throughout this section. In this case, $\mathfrak{a} = \sum_{i\neq 1} e_i - f_1 = \sum_{j \neq 1} f_j - e_1$. By Proposition 4.2.1 and Remark 8, let us write down the extremal ray generators explicitly:

- $\sigma_G = \langle e_1, f_1, \ldots, f_n, \mathfrak{a}\rangle$, if $m = 2$ and $n \geq 3$.
- $\sigma_G = \langle e_1, \ldots, e_m, f_1, \ldots, f_n, \mathfrak{a}\rangle$, if $m \geq 3$ and $n \geq 3$.

The case $m = 2$ and $n = 2$ is depicted in the following example.

Example 14. Consider the following graph $G \subset K_{2,2}$ where $E(G) = E(K_{2,2})\backslash(1,3)$. The three-dimensional cone $\sigma_G \subseteq N_{\mathbb{Q}}$ is generated by $e_1 = (1,0,0,0)$, $f_1 = (0,0,1,0)$ and $\mathfrak{a} = e_2 - f_1 = (0,1,-1,0)$ in N. Remark that $f_2 = \mathfrak{a} + e_1$ and $e_2 = \mathfrak{a} + f_1$ are not one of the extremal rays. The two-dimensional faces are all the pairs of extremal rays.

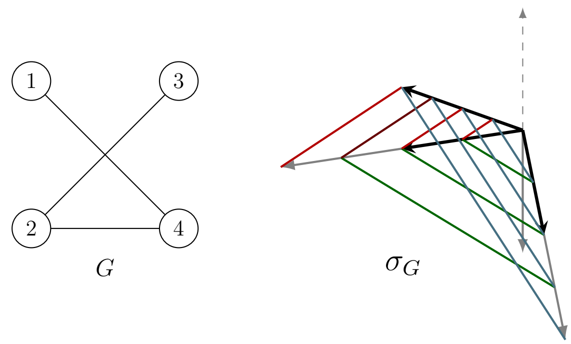


Figure 4.2: The connected bipartite graph G and its associated strongly convex rational polyhedral cone σ_G.

We study now the two and three-dimensional faces of G in the next two propositions.

Proposition 4.2.4. *The two-dimensional faces of σ_G are generated by all pairs of the extremal ray generators except*

1. $(\mathfrak{a}, e_1)$*, if* $m = 2$ *and* $n \geq 3$*.*

2. $(\mathfrak{a}, e_1)$ *and* $(\mathfrak{a}, f_1)$*, if* $m \geq 3$ *and* $n \geq 3$*.*

Proof. Let us first consider the pair $(\mathfrak{a}, e_1)$. By Theorem 3.2.2, this pair spans a 2-face if and only if the intersection $\mathrm{G}\{A\} \cap \mathrm{G}\{U_1 \backslash \{1\}\}$ has three connected components. It is only possible if $n = 2$. Similarly, $(\mathfrak{a}, f_1)$ spans a 2-face if and only if $m = 2$. In the other cases, the intersection of the associated subgraph $\mathrm{G}\{A\}$ with another graph associated to a one-sided first independent set has three connected components with one isolated vertex. The intersection of two graphs associated to two one-sided first independent sets has again three connected components with two isolated vertices.

□

Proposition 4.2.5. *Assume that* $n \neq 2$*. Then the triples containing both* $\mathfrak{a}$ *and* e_1 *are not 3-faces. Furthermore,*

1. *If* $m = 3$*, then* $(f_1, e_2, e_3, \mathfrak{a})$ *is a 3-face and the triple* (e_1, e_2, e_3) *is not a 3-face.*

2. *If* $m = 4$*, then* (e_2, e_3, e_4) *is not a 3-face.*

Proof. The intersection $\mathrm{G}\{A\} \cap \mathrm{G}\{U_1 \backslash \{1\}\}$ has four components if and only if $n = 3$. In this case the 4-tuple $(e_1, f_2, f_3, \mathfrak{a})$ is a 3-face. If $n \neq 3$, then this intersection has more than five connected components. By Corollary 3.2.3, no n-tuple containing both $\mathfrak{a}$ and e_1 is a 3-face. If $m = 3$, similarly to the case where $n = 3$, $(f_1, e_2, e_3, \mathfrak{a})$ is a 3-face. The intersection $\mathrm{G}\{U_1 \backslash \{1\}\} \cap \mathrm{G}\{U_1 \backslash \{2\}\} \cap \mathrm{G}\{U_1 \backslash \{3\}\}$ consists of $n + 3$ isolated vertices and therefore (e_1, e_2, e_3) is not a 3-face. If $m = 4$, the intersection $\mathrm{G}\{U_1 \backslash \{2\}\} \cap \mathrm{G}\{U_1 \backslash \{3\}\} \cap \mathrm{G}\{U_1 \backslash \{4\}\}$ has more than five connected components. Hence the triple do not span a 3-face. □

In Chapter 5, we will prove that $\mathrm{TV}(G)$ is smooth in codimension 2 for any connected bipartite graph $G \subseteq K_{m,n}$, For this, we first present the characterization of two-dimensional faces of G. Nevertheless, we prove this claim for our graphs with one edge removal.

Lemma 4.2.6. $\mathrm{TV}(G)$ *is smooth in codimension 2.*

Proof. We need to show that the extremal generators of two-dimensional faces form a part of a $\mathbb{Z}$-basis of N. The statement is clear for the canonical basis pairs. For $i \in [m]\backslash\{1\}$, the pair $\{\mathfrak{a}, e_i\}$ is a subset of the basis $\{\mathfrak{a}, e_1, \ldots, \widehat{e_k}, \ldots, e_m, f_2, \ldots f_{n-1}\}$ where $e_k \neq e_i$. Similarly, for $j \in [n]\backslash\{1\}$, $\{\mathfrak{a}, f_j\}$ is a subset of the basis $\{\mathfrak{a}, e_1, \ldots, e_{m-1}, f_1, \ldots, \widehat{f_{k'}}, \ldots, f_n\}$ where $f_{k'} \neq f_j$.

□

The following result is an alternative proof to the result in [BHL15] which has been mentioned in the introduction.

Theorem 4.2.7. $\mathrm{TV}(G)$ *is rigid for* $m, n \geq 4$.

Proof. By Proposition 4.2.2, the three-dimensional faces of σ_G are all simplicial generated by three ray generators. If the compact 2-faces are all connected by a common edge or by a common non-lattice vertex, the sign vector $\bar{t}$ is transferred by this common edge or vertex. Hence by Corollary 4.1.3, we obtain $T^1_{\mathrm{TV}(G)}(-R)$=0 for all $R \in M$.

In the crosscuts $Q(R)$ where the non 2-faces $(\bar{\mathfrak{a}}, e_1)$ and $(\bar{\mathfrak{a}}, f_1)$ do not appear, $T^1(-R) = 0$ holds. Therefore, we need to consider the cases where these non 2-faces appear in the crosscut. Suppose that there exist two compact edges connected by a common lattice vertex in $Q(R)$ for some deformation degree $R \in M$. Then there must exist four non 2-face pairs which is impossible. In Figure 4.3, the first illustration represents this case. In particular, all triples in this illustration span 3-faces.

Suppose now that the 2-faces $(\overline{e_1}, \overline{e_i})$ and $(\bar{\mathfrak{a}}, \overline{e_i})$ connected by the lattice vertex e_i in $Q(R)$ for some deformation degree $R \in M$. If there exists another extremal ray of σ_G other than f_1 which is a lattice or non-lattice vertex in $Q(R)$, then $T^1(-R) = 0$ holds. We would like show that there exists no such deformation degree $R \in M$. This means no such $R = [R_1, \ldots, R_{m+n}] \in M$ such that it evaluates zero or negative with the extremal rays other than the triple $(e_1, e_i, \mathfrak{a})$:

$$\begin{cases} \langle R, e_1 \rangle \geq 1 \\ \langle R, \mathfrak{a} \rangle \geq 1 \\ \langle R, e_i \rangle = 1, \text{for some } i \in [m+n]\backslash\{1\} \\ \langle R, e_j \rangle \leq 0, \forall j \in [m+n]\backslash\{1, i, m+1\} \end{cases} \implies \begin{cases} R_1 \geq 1 \\ R_1 + \ldots R_m - R_{m+1} \geq 2 \\ R_i = 1, \text{for some } i \in [m+n]\backslash\{1\} \\ R_j \leq 0, \forall j \in [m+n]\backslash\{1, i, m+1\} \\ R_1 + \ldots + R_m - R_{m+1} - \ldots - R_{m+n} = 0 \end{cases}$$

If $i \in [m]\backslash\{1\}$, then $R_{m+n} = R_1 + \dots R_m - R_{m+1} - \dots - R_{m+n-1} \geq 2$ which is a contradiction. Hence, there exists a $i' \in [m+n]\backslash\{i, m+1\}$ such that $R_{i'} \geq 1$. Similarly, if $i \in \{m+2, \dots, m+n\}$, then $R_m = R_{m+1} + \dots + R_{m+n} - R_1 - \dots - R_m \geq R_{m+1} + 1$. It is only possible if $R_{m+1} \leq 1$. However since $R_{m+2} + \dots + R_{m+n} - R_1 \geq 1$, $R_1 \geq 1$ and $R_{m+i} = 1$ for some $i \in [n]\backslash\{1\}$, there exists $i' \in [n]\backslash\{1, i\}$ such that $R_{i'} \geq 1$. One can see a representation of these cases in Figure 4.3. The dashed red lines represent non 2-faces and the red vertices are lattice vertices in $Q(R)$. This concludes our proof as $T^1(R) = 0, \forall R \in M$.

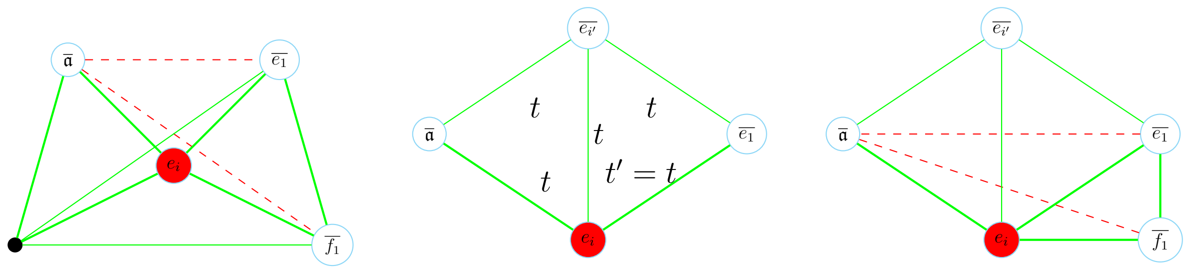


Figure 4.3: The illustration of the proof for the rigidity of $\mathrm{TV}(G)$ where $G \subsetneq K_{m,n}$ is the connected bipartite graph with one edge removal from $K_{m,n}$ with $m, n \geq 4$.

$\square$

It remains to treat the cases with small m or n.

Lemma 4.2.8. $\mathrm{TV}(G)$ *is rigid for* $m = 2$, $n \geq 2$ *and* $n = 2$ *and* $m \geq 2$.

Proof. Recall that σ_G is generated by its extremal ray generators as $\sigma_G = \langle e_1, f_1, \dots f_n, \mathfrak{a} \rangle$. By Proposition 4.2.4, $(\mathfrak{a}, e_1)$ does not span a 2-face. If $n = 2$, then $\mathrm{TV}(G)$ is smooth. If $n = 3$, by Proposition 4.2.5, the 3-faces are all triples except (f_1, f_2, f_3) and except the ones containing both $\mathfrak{a}$ and e_1. It is a cone over a double tetrahedron and, as in the proof of the rigidity of $\mathrm{TV}(K_{2,3})$, we conclude that $\mathrm{TV}(G)$ is rigid. If $n \geq 3$, we mimic the proof of Theorem 4.2.7. $\square$

Example 15. For the graph $G \subset K_{2,2}$ from Example 3, $\mathrm{TV}(G) \cong \mathbb{C}^3$ is rigid. By ignoring the fact that the variety is smooth, one can show the rigidity again using Corollary 4.1.3.

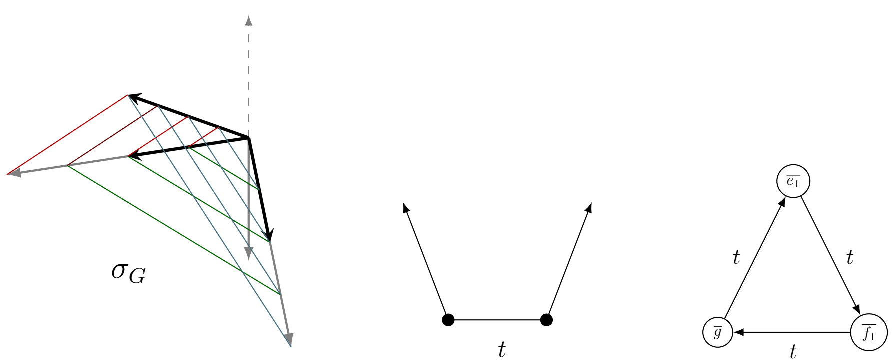


Figure 4.4: Two possible cross-cuts $Q(R)$ of the edge cone σ_G

In the following proposition, we use the argument from Proposition 4.2.3 to show non-rigidity.

Proposition 4.2.9. *The affine normal toric variety* $\mathrm{TV}(G)$ *is not rigid for* $m = 3$, $n \geq 3$ *and for* $n = 3$, $m \geq 3$.

Proof. By Proposition 4.2.5, $\tau := (e_2, e_3, f_1, \mathfrak{a})$ is a non-simplicial 3-face. The vector space $T^1_{\mathrm{TV}(\tau)}(-R) \neq 0$, for $R = [0, 1, 1, 1, 1, 1, 0, \ldots, 0] \in M$. In particular, we choose $m = [n - 1, 0, 0, 0, 1, \ldots, 1] \in \sigma_G^\vee$ and $R' = R - 2m \in M$. Thus we conclude that $T^1_{\mathrm{TV}(G)}(-R') \neq 0$. $\square$

Example 16. Let $G \subset K_{3,3}$ be the bipartite graph with one edge removal. The edge cone is generated by the extremal rays $\langle e_1, e_2, e_3, f_1, f_2, f_3, e_2 + e_3 - f_1 \rangle$. The three-dimensional face $\tau := \langle e_2 + e_3 - f_1, e_2, e_3, f_1 \rangle$ is non-simplicial. Let $R = [-4, 1, 1, 1, -1, -2] \in M$. Then $Q(R)$ consists of the compact 2-face with lattice vertices coming from the rays of τ in an four dimensional ambient space. Therefore the homogeneous piece $T^1(-R)$ is not zero.

4.3 Bipartite graphs with multiple edge removals

In this section, we would like to examine a general case with more edge removals. We consider two vertex sets $A_1 \subsetneq U_1$ and $A_2 \subsetneq U_2$ of the complete bipartite graph $K_{m,n}$ and we remove all the edges between these two sets. This means that we obtain a two-sided first independent set $A := A_1 \sqcup A_2 \in \mathcal{I}_G^{(1)}$. Without loss of generality, we assume that $A_1 = \{1, \ldots, t_1\}$ and $A_2 = \{m + 1, \ldots, m + t_2\}$. Therefore $\pi(A) = \mathfrak{a} = \sum_{i > t_1} e_1 - \sum_{j \leq t_2} f_j$.

Example 17. Let us consider the edge removals from the vertex 3 of the complete bipartite graph $K_{2,2}$. We already studied these cases; for zero, one and two edge removals, we obtained non-rigid, rigid and rigid varieties respectively.

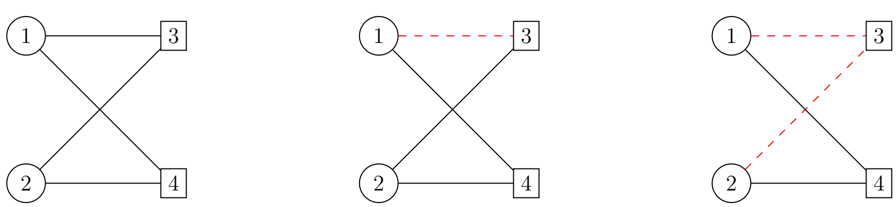


Figure 4.5: The bipartite graphs with multiple edge removals

First, we examine some facts about the two-dimensional faces of the edge cone σ_G.

Proposition 4.3.1. *Let* $G \subset K_{m,n}$ *be a connected bipartite graph with exactly one two-sided first independent set* $A \in \mathcal{I}_G^{(1)}$.

1. *The pair* (f_{n-1}, f_n) *does not span a two-dimensional face if and only if* $|A_2| = n - 2$. *Moreover,* (f_{n-1}, f_n) *is not contained in any simplicial three-dimensional face.*

2. *The pair* $(\mathfrak{a}, e_1)$ *does not span a two-dimensional face if and only if* $|A_1| = 1$ *and* $|A_2| \neq n-1$*, Moreover, no simplicial three-dimensional face contains both* $\mathfrak{a}$ *and* e_1*.*

3. *If* $|A_1| = 1$ *and* $|A_2| = n-2$*, then* $\mathrm{TV}(G)$ *is not rigid.*

Proof. By Proposition 4.2.2, we obtain the non-simplicial 3-faces $(\mathfrak{a}, e_1, f_{n-1}, f_n)$ in the third case. It results to a non-rigid toric variety $\mathrm{TV}(G)$ as in the proof of Proposition 4.2.3. For the first case, consider the intersection subgraph $\mathrm{G}\{U_2\backslash\{m+n-1\}\}\cap\mathrm{G}\{U_2\backslash\{m+n\}\}$. Since we have exactly one two-sided first independent set, the only possibility that this intersection subgraph has more than three connected components is that $|A_2| = n-2$. In this case the intersection subgraph has $|A_1|+2$ isolated vertices and one connected component consisting of the induced subgraph $\mathrm{G}[A_2 \sqcup N(A_2)]$. Let us now consider the intersection subgraph $\mathrm{G}\{A\}\cap\mathrm{G}\{U_1\backslash\{1\}\}$. Similarly, the only possibility that this intersection subgraph has more than three connected components is that $|A_1| = 1$. In this case, it has $|N(A_1)|+1$ isolated vertices and one connected component consisting of the induced subgraph $\mathrm{G}[A_2 \sqcup N(A_2)]$. In particular, if $|N(A_1)| = 1$, $(\mathfrak{a}, e_1)$ spans a 2-face. $\square$

Note that the cases where $|A_2| = 1$ and $|A_1| = m-2$ can be studied symmetrically. In the next proposition, we examine the three-dimensional faces of σ_G. These statements can also be studied symmetrically.

Proposition 4.3.2. *Let* $G \subset K_{m,n}$ *be a connected bipartite graph with exactly one two-sided first independent set* $A \in \mathcal{I}_G^{(1)}$

1. *The triple* (f_{n-2}, f_{n-1}, f_n) *does not span a three-dimensional face if and only if* $|A_2| = n-3$*.*

2. *The triple* $(\mathfrak{a}, e_1, e_2)$ *does not span a three-dimensional face if and only if* $|A_1| = 2$ *and* $|N(A_1)| \neq 1$*.*

Proof. For the first case, the intersection subgraph $\mathrm{G}\{U_2\backslash\{m+n-2\}\}\cap\mathrm{G}\{U_2\backslash\{m+n-1\}\}\cap\mathrm{G}\{U_2\backslash\{m+n\}\}$ has more than four components if and only if $|A_2| = n-3$. More precisely, one has $|A_1|+3$ isolated vertices and one connected component $\mathrm{G}[A_2]$. For the second case, intersection subgraph $\mathrm{G}\{A\}\cap\mathrm{G}\{U_1\backslash\{1\}\}\cap\mathrm{G}\{U_1\backslash\{2\}\}$ has more than four components if and only if $|A_1| = 2$. In this case, there are $|N(A_2)|+2$ isolated vertices and the connected component $\mathrm{G}[A_2]$. In particular, if $|N(A_1)| = 1$, $(\mathfrak{a}, e_1, e_2)$ spans a 3-face. $\square$

Remark 9. We covered all the types of non 2-faces and 3-faces in Proposition 4.3.1 and Proposition 4.3.2. By Proposition 4.2.2, a triple of type (e_i, f_j, f_k) does not span a 3-face if and only if $|A_1| = 1$ and $|A_2| = n-2$. A triple of type $(\mathfrak{a}, e_i, f_j)$ does not span a 3-face if and only if $|A_1| = 1$ or $|A_2| = 1$.

As we mentioned in the previous section, for a connected bipartite graph $G \subseteq K_{m,n}$, the toric variety $\mathrm{TV}(G)$ is smooth in codimension 2. We prove this in Theorem 5.1.5. In particular, for the case where we consider multiple edge removals, we can show this analogously to the proof of Lemma 4.2.6.

Theorem 4.3.3. *Let $G \subsetneq K_{m,n}$ be a connected bipartite graph with exactly one two-sided first independent set $A \in \mathcal{I}_G^{(1)}$. Then*

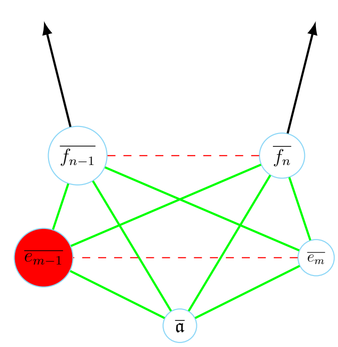
1. *$\mathrm{TV}(G)$ is not rigid, if $|A_1| = 1$ and $|A_2| = n-2$ or if $|A_1| = m-2$ and $|A_2| = 1$.*
2. *$\mathrm{TV}(G)$ is rigid, otherwise.*

Proof. The first case follows from Proposition 4.3.1. For the other cases, we first study the non 2-faces and 3-faces from the previous two propositions separately in the crosscut picture. Then we examine the intersecting cases and prove that there exists no deformation degree $R \in M$ such that $T^1_{\mathrm{TV}(G)}(R) \neq 0$. First of all, note that there can be no cases such as two 2-faces connected by a common lattice vertex in $Q(R)$. This is because, it would mean that there exist four non 2-faces and this is impossible as shown in the first drawing of Figure 4.3.

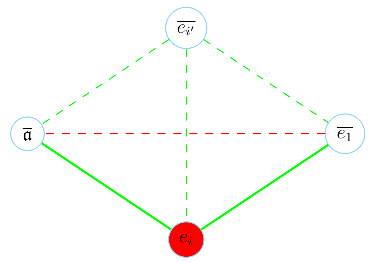
• Assume that $|A_2| = n-2$. We consider the non 2-face (f_{n-1}, f_n) in $Q(R)$. This means that $R_{n-1} \geq 1$ and $R_n \geq 1$. This implies that there exists $i \in [m]$ such that $\langle R, e_i \rangle \geq 1$ and (e_i, f_{n-1}) and (e_i, f_n) are 2-faces.

Suppose that R evaluates zero or negative on all other extremal rays except e_i, f_{n-1} and f_n. Then $\overline{e_i}$ cannot be a lattice vertex in $Q(R)$. If e_i is not an extremal ray, then $\overline{\mathfrak{a}}$ is not a lattice vertex in $Q(R)$ and $(\mathfrak{a}, f_{n-1})$ and $(\mathfrak{a}, f_{n-1})$ are 2-faces.

Suppose that there exists another $i' \in [m]\backslash\{i\}$ such that $R_{i'} \geq 1$. If $\overline{e_i}$ and $\overline{e_{i'}}$ are not lattice vertices , we are done. If at least one of them is a lattice vertex, then we check if $(e_i, e_{i'})$ spans a 2-face. If it does span a 2-face, then we obtain the 3-faces $(e_i, e_{i'}, f_{n-1})$ and $(e_i, e_{i'}, f_n)$. If it does not span a 2-face, then $|A_1| = n-2$ and let $e_i = e_{n-1}$ and $e_{i'} = e_n$. In that case, $\overline{\mathfrak{a}}$ is a non-lattice vertex and we obtain the 3-faces $(\mathfrak{a}, e_i, f_j)$ where $i \in \{m-1, m\}$ and $j \in \{n-1, n\}$ as in the figure below.



• Assume that $|A_1| = 1$. For the non 2-face $(\mathfrak{a}, e_1)$, we refer to the proof of Theorem 4.2.7. We conclude that there exists no $R \in M$ that evaluates on the extremal rays $\mathfrak{a}$ and e_1 positive, hence e_i is a lattice vertex in $Q(R)$ where $i \in [m+n]\backslash\{1\}$ and R evaluates on all the other extremal rays evaluates negative or zero. Therefore, there exists $i' \in [m+n]\backslash\{1, i\}$ such that $R_{i'} \geq 1$. Now, we must check if $(e_{i'}, e_i)$, $(e_{i'}, e_1)$ and $(e_{i'}, \mathfrak{a})$ are 2-faces.



1. If $|A_2| = 1$, then this is the case which we studied in Section 4.2. In particular, we have shown that if $|A_2| = 1 = n - 2$, i.e. if $n = 3$, then $\mathrm{TV}(G)$ is not rigid. For the other cases where $n \neq 3$, $\mathrm{TV}(G)$ is rigid.

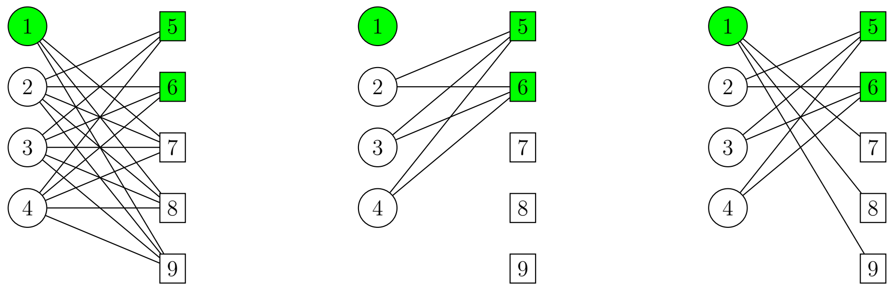
2. If $|A_1| = m-2$, then (e_2, e_3) does not span a 2-face. However (e_1, e_3) and $(e_3, \mathfrak{a})$ do span 2-faces. Furthermore we have $R_4 + \ldots + R_{n+4} \geq 3$. This means that there exists $j \in [n]$ such that $\langle R, f_j \rangle \geq 1$. The ray f_j is an extremal ray, generator, otherwise $(\mathfrak{a}, e_1)$ spans a 2-face. Therefore, $(\mathfrak{a}, f_j)$, (e_1, f_j), (e_2, f_j) and (e_3, f_j) span 2-faces. Additionally $(\mathfrak{a}, e_2, f_j)$, (e_1, e_2, f_j), $(\mathfrak{a}, e_3, f_j)$ and (e_1, e_3, f_j) span 3-faces.

• Assume that $|A_2| = n - 3$. For the non 3-face (f_{n-2}, f_{n-1}, f_n), we refer to case **4** of the proof of Theorem 4.1.4. There is a small detail here that one needs to pay attention to. If $|A_1| = n - 1$, then e_m is not an extremal ray generator of σ_G. The deformation degree $R \in M$ with $R_m = R_{m+n-2} + R_{m+n-1} + R_{m+n}$ evaluates positive on $\mathfrak{a} \in \sigma_G^{(1)}$. The triples $(\mathfrak{a}, f_j, f_k)$ are 3-faces where $j, k \in \{n-1, n-2, n\}$.

• Assume that $|A_1| = 2$. For the non 3-face $(\mathfrak{a}, e_1, e_2)$, we have $R_1 \geq 1$, $R_2 \geq 1$ and $R_3 + \ldots + R_m - R_{m+1} \ldots - R_{m+t} \geq 1$. This implies that $R_{m+t+1} + \ldots + R_{m+n} \geq 3$ where $t = |A_2|$. Then there exists $j \in N(A_1)$ such that $R_{m+j} \geq 1$. Note that if f_j is not an extremal ray generator then $\{\mathfrak{a}, e_1, e_2\}$ is a 3-face. Otherwise, (e_1, e_2, f_j) is always a 3-face. The pair (a, f_j) is not a 2-face if and only if $j \in A_2$ and $|A_2| = 1$, which is impossible.

□

Example 18. Let $G \subset K_{4,5}$ be a connected bipartite graph constructed by removing two edges connected to the vertex $\{1\}$ in U_1. This means that there exists a two-sided first independent set $A = A_1 \sqcup A_2 \in \mathcal{I}_G^{(1)}$ with $|A_1| = 1$ and $|A_2| = 2$. In Figure 4.6, the second graph is the intersection subgraph associated to the extremal ray set (f_3, f_4, f_5) and the third graph is $\mathrm{G}\{A\}$. We observe that the second graph has five connected components, hence (f_3, f_4, f_5) does not span a three-dimensional face. Furthermore, this intersection subgraph is equal to the intersection $\mathrm{G}\{A\} \cap \mathrm{G}\{U_1 \backslash \{1\}\}$. Therefore $(e_1, \mathfrak{a})$ does not span a two-dimensional face. Furthermore, $(\mathfrak{a}, f_1, f_2)$ does not span a three-dimensional face. Let us for example consider the crosscut $Q(R)$ for $R = [2, 0, 0, 0, 0, -1, 1, 1, 1] \in M$ as in the figure below. Except from the triples $(\mathfrak{a}, e_1, f_3)$ and (f_3, f_4, f_5), all triples in this figure span 3-faces. Therefore $T^1_{\mathrm{TV}(G)}(-R) = 0$.



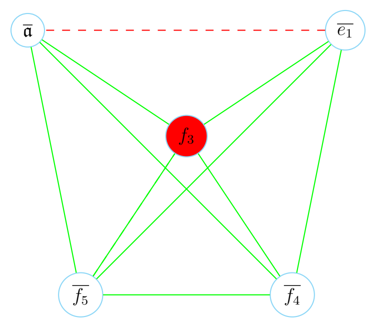


Figure 4.6: Examining the rigidity through bipartite graphs.

Chapter 5

On the rigidity of Toric Varieties arising from Bipartite Graphs

This chapter provides a detailed exposition of two and three-dimensional faces of the edge cone σ_G for a connected bipartite graph G. Our methods use mostly basics from Graph Theory, therefore we obtain an intrinsic technique. Using these methods, we prove that the toric variety $\mathrm{TV}(G)$ is smooth in codimension 2. Next, we prove that the non-simplicial three-dimensional faces of an edge cone are generated exactly by four extremal ray generators. We conclude that the toric varieties associated to the edge cones having non-simplicial three-dimensional faces are not rigid. Moreover, we characterize the bipartite graphs whose edge cones have only simplicial three-dimensional faces.

5.1 The two-dimensional faces of the edge cone

In the previous chapters, we started with the study of bipartite graphs which were obtained by removing edges between two vertex sets $A_1 \subsetneq U_1$ and $A_2 \subsetneq U_2$ of a complete bipartite graph. In these cases we obtained exactly one two-sided first independent set. However, for the general case, the face structure of the edge cone σ_G becomes complicated and can be hard to keep track of. We will see this in detail in this section while we investigate all possible cases of pairs of first independent sets.

Recall that Theorem 3.1.13 gives a one to one correspondence between first independent sets $\mathcal{I}_G^{(1)}$ of G and the extremal rays of the edge cone σ_G as below:

$$\begin{array}{rcl} \pi : \mathcal{I}_G^{(1)} & \longrightarrow & \sigma_G^{(1)} \\ I & \mapsto & \mathfrak{i} = (H_{I_1} \cap \sigma_G^{\vee})^* \end{array}$$

For the rest of this chapter, we label the first independent sets $\mathcal{I}_G^{(1)}$ as in three types: $A = U_1 \backslash \{a\}$, $B = U_2 \backslash \{b\}$ and the two-sided maximal independent set $C = C_1 \sqcup C_2$. Our purpose is to find necessary and sufficient graph theoretical conditions for the pairs of extremal rays to span a two-dimensional face of σ_G. We will also use these results to prove that $\mathrm{TV}(G)$ is smooth in codimension 2.

We introduce the notation for the tuples of first independent sets forming d-dimensional faces analogously to $\mathcal{I}_G^{(1)}$ as in the following definition.

Definition 5.1.1. A tuple from the first independent set $\mathcal{I}_G^{(1)}$ is said to form a d-dimensional face, if their associated tuple of extremal ray generators of σ_G under the map π of Theorem 3.1.13 forms a d-dimensional face of σ_G. We denote the set of these tuples by $\mathcal{I}_G^{(d)}$.

The pairs of type $(A, A'), (A, B), (A, C)$

The next two propositions follow naturally by the results from Lemma 3.2.1 and Theorem 3.2.2. They are presented here nevertheless for the sake of completeness of our investigation. The main idea of the proofs is to use Theorem 3.2.2 in which we have proven that the faces of σ_G are obtained by intersecting graphs associated to first independent sets. After intersecting the associated subgraphs, we detect the dimension of the face by using Lemma 3.2.1.

Proposition 5.1.2. *Let $A = U_1 \backslash \{a\}$, $A' = U_1 \backslash \{a'\}$ and $B = U_2 \backslash \{b\}$ be first independent sets.*

(1) $(A, A') \in \mathcal{I}_G^{(2)}$ *if and only if* $\mathrm{G}\{A \cap A'\}$ *is connected.*

(2) $(A, B) \in \mathcal{I}_G^{(2)}$ *if and only if* $\mathrm{G}[A \sqcup B]$ *is connected.*

Proof. The pair $(A, A') \in \mathcal{I}_G^{(2)}$ if and only if the dual edge cone of associated intersection subgraph $\mathrm{G}\{A\} \cap \mathrm{G}\{A'\}$ is of dimension $m + n - 3$. Equivalently, the intersection subgraph $\mathrm{G}[(A, A')]$ has three connected components. Since a and a' are isolated vertices of the intersection subgraph, $\mathrm{G}\{A \cap A'\}$ must be connected in order to obtain $(A, A') \in \mathcal{I}_G^{(2)}$. Similarly, $\mathrm{G}[A \sqcup B]$ must be connected.

□

Example 19. Let us consider the bipartite graph $G \subset K_{5,4}$ as in Figure 5.1. We observe the existence of two two-sided first independent sets $C = \{3\} \sqcup \{6, 7\}$ and $C' = \{1, 2\} \sqcup \{8, 9\}$. Let $A = U_1 \backslash \{4\}$ and $A' = U_1 \backslash \{5\}$. Since $\mathrm{G}\{A \cap A'\}$ has four connected components, $(A, A') \notin \mathcal{I}_G^{(2)}$. In particular, we obtain that $(A, A', C, C') \in \mathcal{I}_G^{(3)}$.

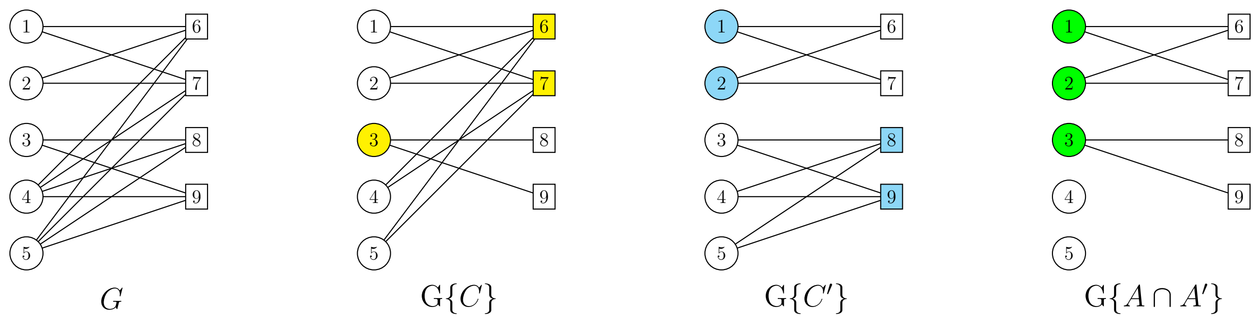


Figure 5.1: A case where two first independent sets do not form a 2-face of σ_G.

Proposition 5.1.3. *Let $A = U_1 \backslash \{a\}$ and $C = C_1 \sqcup C_2$ be first independent sets. One has $(A, C) \in \mathcal{I}_G^{(2)}$ if and only if one of the three following conditions is satisfied*

(1) *$A \cap C_1 = \emptyset$ and $C_2 = U_2 \backslash \{\bullet\}$.*

(2) *$C_1 \subsetneq A$ and $\mathrm{G}[C_2 \sqcup N(C_2) \backslash \{a\}]$ is connected.*

(3) *$N(C_2) \subsetneq A$ and $\mathrm{G}[C_1 \backslash \{a\} \sqcup N(C_1)]$ is connected.*

Proof. Assume $A \cap C_1 = \emptyset$, i.e. $C_1 = \{a\}$. Then the graph $\mathrm{G}\{A\} \cap \mathrm{G}\{C\}$ has the isolated vertex set $C_1 \sqcup N(C_1)$. In this case, $(A, C) \in \mathcal{I}_G^{(2)}$ if and only if $C_2 = U_2 \backslash \{b\}$ for some vertex $b \in U_2$. This implies in particular that $U_2 \backslash \{b\} \notin \mathcal{I}_G^{(1)}$. Now let us consider the case where $A \cap C_1 \neq \emptyset$. Since $A = U_1 \backslash \{a\}$, it is either $C_1 \subsetneq A$ or $N(C_2) \subsetneq A$. We prove (2), the case (3) follows symmetrically. We require the intersection subgraph $\mathrm{G}[A] \cap \mathrm{G}[C]$ to have three connected components. Since it consists of $\mathrm{G}[C_1] \sqcup (\mathrm{G}[A] \cap \mathrm{G}[C_2])$, and a is an isolated vertex, $\mathrm{G}[C_2 \sqcup N(C_2) \backslash \{a\}]$ must be connected.

□

Example 20. Let us consider the first independent sets $A = U_1 \backslash \{3\}$, $A' = U_1 \backslash \{4\}$, and $C = \{3\} \sqcup \{6, 7\} \in \mathcal{I}_G^{(1)}$ from Example 19. Since we have that $A \cap C_1 = \emptyset$ and $C_2 = U_2 \backslash \{8, 9\}$, $(A, C) \notin \mathcal{I}_G^{(2)}$. On the other hand $(A', C) \in \mathcal{I}_G^{(2)}$, since $\{3\} \subset A'$ and the induced subgraph $\mathrm{G}[\{6, 7\} \sqcup \{1, 2, 5\}]$ is connected.

The pairs of type (C, C')

Before presenting the conditions for the two-dimensional faces of σ_G, we would like to consider the possible pairs of two-sided first independent sets, which we denote by $C = C_1 \sqcup C_2$ and $C' = C'_1 \sqcup C'_2$. Suppose that $C_1 \subsetneq C'_1$. Then $C_1 \sqcup C_2 \cup C'_2$ is also a two-sided independent set strictly containing C, unless $C_2' \subsetneq C_2$. By the maximality condition on C and C', it is impossible that $C_1 = C_1'$ or $C_2 = C_2'$. By the connectivity assumption on G, it is impossible that $C_1 \cup C_1' = U_1$ and $C_2 \cup C_2' = U_2$. Consequently, under the conditions where $C_1 \neq C_1'$ or $C_2 \neq C_2'$, and $C \cup C' \neq U_1 \sqcup U_2$, one obtains five types of pairs of (C, C'):

Type (i): $C_1 \subsetneq C_1'$ and $C_2' \subsetneq C_2$.
Type (ii): $C_1 \cap C_1' = \emptyset$ and $C_2 \cap C_2' = \emptyset$.
Type (iii): $C_1 \cap C_1' \neq \emptyset$ and $C_2 \cap C_2' = \emptyset$.
Type (iv): $C_1 \cap C_1' = \emptyset$ and $C_2 \cap C_2' \neq \emptyset$.
Type (v): $C_1 \cap C_1' \neq C_1 \neq C_1' \neq \emptyset$ and $C_2 \cap C_2' \neq C_2 \neq C_2' \neq \emptyset$.

We investigate the 2-face conditions by following these types in the following Lemma.

Lemma 5.1.4. *Let C and C' be two-sided first independent sets with $C = C_1 \sqcup C_2$ and $C' = C_1' \sqcup C_2'$. Then $(C, C') \in \mathcal{I}_G^{(2)}$ if and only if it is one of the following types:*

(1) *$C_1 \subsetneq C_1'$ and $C_2' \subsetneq C_2$, where $G[(C_1' \backslash C_1) \sqcup (C_2 \backslash C_2')]$ is connected.*

(2) *$C_1 \sqcup C_1' = U_1 \backslash \{\bullet\}$ and $C_2 \sqcup C_2' = U_2$ or $C_2 \sqcup C_2' = U_2 \backslash \{\bullet\}$ and $C_1 \sqcup C_1' = U_1$.*

(3) *$C_1 \cap C_1' \neq \emptyset$ and $C_2 \cap C_2' = \emptyset$, where $G\{C_1 \cap C_1'\}$ is connected with $N(C_1 \cap C_1') = U_2 \backslash (C_2 \sqcup C_2')$ and $C_1 \cup C_1' = U_1$.*

(4) *$C_1 \cap C_1' = \emptyset$ and $C_2 \cap C_2' \neq \emptyset$, where $G\{C_2 \cap C_2'\}$ is connected with $N(C_2 \cap C_2') = U_1 \backslash (C_1 \sqcup C_1')$ and $C_2 \cup C_2' = U_2$.*

Proof. The main idea of the proof is again to use Theorem 3.2.2 in which we have proven that the faces of σ_G are obtained by intersecting the associated subgraphs to first independent sets. This means that the pair (C, C') forms a 2-face of σ_G if and only if the dual edge cone of associated intersection subgraph $\mathrm{G}\{C\} \cap \mathrm{G}\{C'\}$ is of dimension $m+n-3$. By Lemma 3.2.1, this is equivalent to the fact that the intersection subgraph $\mathrm{G}\{C\} \cap \mathrm{G}\{C'\}$ has three connected components. We would like to divide the proof into five types which we introduced just before the statement of this Lemma. For the related intersection subgraph $\mathrm{G}\{C\} \cap \mathrm{G}\{C'\}$, we must calculate four intersections:

$\mathbf{G_1}$= $\mathrm{G}[(C_1 \cap C_1') \sqcup (U_2 \backslash (C_2 \cup C_2'))]$
$\mathbf{G_2}$= $\mathrm{G}[(C_2 \cap C_2') \sqcup (U_1 \backslash (C_1 \cup C_1'))]$
$\mathbf{G_3}$= $\mathrm{G}[(C_1 \backslash C_1') \sqcup (C_2' \backslash C_2)]$
$\mathbf{G_4}$= $\mathrm{G}[(C_1' \backslash C_1) \sqcup (C_2 \backslash C_2')]$

And we have that $\mathrm{G}\{C\} \cap \mathrm{G}\{C'\} = \mathbf{G_1} \sqcup \mathbf{G_2} \sqcup \mathbf{G_3} \sqcup \mathbf{G_4}$.

Type (i): $(C_1 \subsetneq C_1'$ and $C_2' \subsetneq C_2)$. One obtains two connected subgraphs $\mathbf{G_1}=\mathrm{G}\{C_1\}$ and $\mathbf{G_2}=\mathrm{G}\{C_2'\}$. The graph $\mathbf{G_3}$ is empty, since $C_1 \backslash C_1' = \emptyset$ and $C_2' \backslash C_2 = \emptyset$. The subgraph $\mathbf{G_4}$ is not empty. Assume that $\mathbf{G_4}$ has an isolated vertex $u \in C_1' \backslash C_1$. Then $C_1 \sqcup \{x\} \sqcup C_2$ is an independent set. This contradicts the fact that C is maximal. Similarly, there exists no isolated vertex in $C_2 \backslash C_2'$ of the subgraph $\mathbf{G_4}$, otherwise C' is not maximal. However it is possible that $\mathrm{G}[(C_1' \backslash C_1) \sqcup (C_2 \backslash C_2')]$ has $k \geq 2$ connected components with vertex sets $X_i \subsetneq C_1' \backslash C_1$ and $Y_i \subsetneq C_2 \backslash C_2'$, for $i \in [k]$. This means in particular that for $I \subsetneq [k]$, there exist first independent sets of form $\mathcal{C}^I := (C_1 \sqcup \bigsqcup_{i\in I} X_i) \sqcup (C_2 \backslash \bigsqcup_{i \in I} Y_i)$. We examine this case in Lemma 5.2.3.

Type (ii): $(C_1 \cap C_1' = \emptyset$ and $C_2 \cap C_2' = \emptyset.)$ The subgraphs $\mathbf{G_1}$ and $\mathbf{G_2}$ are empty. Since $C_1' \subseteq U_1 \backslash C_1 = N(C_2)$ and $C_2' \subseteq U_2 \backslash C_2 = N(C_1)$, we obtain that $\mathbf{G_3}=\mathrm{G}[C_1 \sqcup C_2']$ and $\mathbf{G_4}=\mathrm{G}[C_2 \sqcup C_1']$. Since we cannot have that $C_1 \sqcup C_1' \neq U_1$ and $C_2 \sqcup C_2' \neq U_2$, there must exist exactly one isolated vertex v such that $\mathrm{G}\{C\} \cap \mathrm{G}\{C'\} = \mathbf{G_3} \sqcup \mathbf{G_4} \sqcup \{\mathbf{v}\}$. For if not, $\mathrm{G}\{C\} \cap \mathrm{G}\{C'\}$ has more than three connected components. Let us suppose for the moment $\{v\} = U_1 \backslash (C_1 \sqcup C_1')$. Then $\mathbf{G_3} = \mathrm{G}\{C_1\}$ and $\mathbf{G_4} = \mathrm{G}\{C_2\}$ are connected and therefore $(C, C') \in \mathcal{I}_G^{(2)}$. It follows similarly if $v \in U_2 \backslash (C_2 \sqcup C_2')$. Note that in these cases, $U_i \backslash \{v\} \notin \mathcal{I}_G^{(1)}$.

Type (iii): $(C_1 \cap C_1' \neq \emptyset$ and $C_2 \cap C_2' = \emptyset.)$ The subgraph $\mathbf{G_2}$ is empty. Assume that $C_1 \cup C_1' \neq U_1$. Then the intersection subgraph $\mathrm{G}\{C\} \cap \mathrm{G}\{C'\}$ do not contain $U_1 \backslash (C_1 \cup C_1')$ as a vertex set. This implies that one must have $C_1 \cup C_1' = U_1$ for otherwise $\mathrm{G}\{C\} \cap \mathrm{G}\{C'\}$ has at least four connected components. The subgraphs $\mathbf{G_3}=\mathrm{G}[(C_1 \backslash C_1') \sqcup C_2']$ and $\mathbf{G_4}=\mathrm{G}[(C_1' \backslash C_1) \sqcup C_2]$ are connected subgraphs. Consequently, $(C, C') \in \mathcal{I}_G^{(2)}$ if and only if $\mathrm{G}[C_1 \cap C_1' \sqcup U_2 \backslash (C_2 \sqcup C_2')]$ is connected and $C_1 \cup C_1' = U_1$. Type (iv) $(C_1 \cap C_1' = \emptyset$ and $C_2 \cap C_2' \neq \emptyset)$ follows similarly to Type (iii).

Type (v): $(C_1 \cap C_1' \neq C_1 \neq C_1' \neq \emptyset$ and $C_2 \cap C_2' \neq C_2 \neq C_2' \neq \emptyset.)$ Assume that $C_1 \cup C_1' = U_1$. Then $C_2 \cap C_2' \neq \emptyset$ is an isolated vertex set of G. The same holds for the assumption $C_2 \cup C_2' = U_2$. This means that we must have $C_1 \cup C_1' \neq U_1$ and $C_2 \cup C_2' \neq U_2$. But, this implies that $\mathrm{G}\{C\} \cap \mathrm{G}\{C'\}$ has at least four connected components. This follows because for $i \in [4]$ none of the subgraphs $\mathbf{G_i}$ is empty by the assumption.

□

Example 21. Let us consider the first independent sets $C = \{3\} \sqcup \{6, 7\}$ and $C' = \{1, 2\} \sqcup \{8, 9\}$ from Example 19. The pair (C, C') is of Type (ii). But we observe that $C_1 \sqcup C_1' = U_1 \backslash \{4, 5\}$. Hence $(C, C') \notin \mathcal{I}_G^{(2)}$.

Now, we utilize the information from Lemma 5.1.4 in order to give a concise proof for the next theorem. Let us label the vertices of U_1 with the set $[m]$ and the vertices of U_2 with the set $\{m + 1, \ldots, m + n\}$.

Theorem 5.1.5. *Let $G \subseteq K_{m,n}$ be a connected bipartite graph. Then* $\mathrm{TV}(G)$ *is smooth in codimension 2.*

Proof. Recall that $N = \mathbb{Z}^{m+n}/\overline{(1,-1)} \cong \mathbb{Z}^{m+n-1}$. Let $A, B, C \in \mathcal{I}_G^{(1)}$ be types of first independent sets as before. The pairs of one-sided first independent sets are the pairs of the canonical basis of $\mathbb{Z}^{m+n}$. The extremal rays of σ_G associated to two-sided first independent sets are in form of $\mathfrak{c} = \sum_{i \in U_1 \setminus C_1} e_i - \sum_{m+j \in C_2} f_j \in N$. Consider now the pairs of type $(A, C) \in \mathcal{I}_G^{(2)}$. The set $\{e_1, \dots, \hat{e_i}, \dots, e_m, f_1, \dots, \hat{f_j}, \dots, f_n\}$ for any $i \in N(C_2)$ different than a and $m+j \in N(C_1)$ extends the extremal ray $\mathfrak{c}$ to a $\mathbb{Z}$-basis of N. Note that if $i \in N(C_2)$ is unique, then $A \setminus \{i\} \notin \mathcal{I}_G^{(1)}$. It follows the same for a unique $m+j \in N(C_2)$. We now consider the pair of two extremal rays $\{\mathfrak{c}, \mathfrak{c}'\}$ associated to two-sided first independent sets C and C'. By Lemma 5.1.4, there are four cases we should consider:

- For the case (1), the set $\{e_1, \dots, \hat{e_i}, \dots, \hat{e_{i'}}, \dots, e_m, f_1, \dots, \hat{f_j}, \dots, f_n\}$ for some $i \in C_1' \setminus C_1$ and $i' \in N(C_2')$, and $m+j \in N(C_1)$,
- for the case (2), the set $\{e_1, \dots, \hat{e_i}, \dots, \hat{e_{i'}}, \dots, e_m, f_1, \dots, \hat{f_j}, \dots, f_n\}$ for some $i \in C_1'$ and $i' \in C_1$, and $m+j \in C_2$,
- for the case (3), the set $\{e_1, \dots, \hat{e_i}, \dots, \hat{e_{i'}}, \dots, e_m, f_1, \dots, \hat{f_j}, \dots, f_n\}$, for some $i \in C_1 \setminus C_1'$ and $i' \in C_1' \setminus C_1$, and $m+j \in N(C_1) \cap N(C_1')$,
- for the case (4), the set $\{e_1, \dots, \hat{e_i}, \dots, e_m, f_1, \dots, \hat{f_j}, \dots, \hat{f_{j'}}, \dots, f_n\}$, for some $i \in N(C_2) \cap N(C_2')$ and $m+j \in C_2 \setminus C_2'$ and $m+j' \in C_2' \setminus C_2$

extends the pair $\{c, c'\}$ to a $\mathbb{Z}$-basis of N.

□

Since the toric variety $\mathrm{TV}(G)$ is smooth in codimension 2, we can apply now Theorem 2.2.3 to pursue our investigation on the rigidity of $\mathrm{TV}(G)$. As we observed Chapter 4, once we have a non-simplicial three-dimensional face of σ_G, we conclude that $\mathrm{TV}(G)$ is not rigid. Hence, we first investigate the non-simplicial three-dimensional faces of σ_G. For this, we study the cases where a pair of first independent sets do not form a two-dimensional face although they are contained in a three-dimensional face.

5.2 The three-dimensional faces of the edge cone

Let $\tau \preceq \sigma_G$ be a non-simplicial three-dimensional face. Then there exists a pair of extremal ray generators of τ which does not form a two-dimensional face. Therefore, we treat the pairs of first independent sets which do not form a two-dimensional face and which are contained in the set of extremal ray generators of a three dimensional face. By using Corollary 3.2.3 and the 2-face conditions from Section 5.1, we conclude that non-simplicial three-dimensional faces of σ_G are generated exactly by four extremal ray generators.

Lemma 5.2.1. *Let $A = U_1 \setminus \{a\} \in \mathcal{I}_G^{(1)}$ and $A' = U_1 \setminus \{a'\} \in \mathcal{I}_G^{(1)}$. Assume that $\{\mathfrak{a}, \mathfrak{a}'\}$ forms part of the extremal ray generators of a three-dimensional face of σ_G.*

(1) *If* $(A, A') \in \mathcal{I}_G^{(2)}$*, then the three-dimensional face is simplicial.*

(2) *If* $(A, A') \notin \mathcal{I}_G^{(2)}$*, then either*

  (i) $(A, A', B, C) \in \mathcal{I}_G^{(3)}$*, where* $B = U_2 \backslash \{b\} \in \mathcal{I}_G^{(1)}$ *and* $C = (A \cap A') \sqcup \{b\} \in \mathcal{I}_G^{(1)}$ *or*

  (ii) $(A, A', C, C') \in \mathcal{I}_G^{(3)}$*, where* $C_1 \sqcup C_1' = A \cap A'$ *and* $C_2 \sqcup C_2' = U_2$*.*

Proof. For (1), $G\{A \cap A'\}$ has three connected components. Let $B = U_2 \backslash \{b\}$. We first investigate the intersection subgraph $\mathrm{G}\{A\} \cap \mathrm{G}\{A'\} \cap \mathrm{G}\{B\}$. By assumption, the dimension of its dual edge cone must be $m + n - 4$. Therefore, the intersection subgraph has four connected components with three isolated vertices a, a' and b. Hence $(A, B), (A', B) \in \mathcal{I}_G^{(2)}$. The fact that $(A, A', A'') \in \mathcal{I}_G^{(3)}$ can be similarly obtained. Let $C \in \mathcal{I}_G^{(1)}$. We next investigate the intersection subgraph $\mathrm{G}\{A\} \cap \mathrm{G}\{A'\} \cap \mathrm{G}\{C\}$. It has by assumption four connected components with two isolated vertices, a and a'. Hence $\mathrm{G}\{A\} \cap \mathrm{G}\{C\}$ and $\mathrm{G}\{A'\} \cap \mathrm{G}\{C\}$ have three connected components, i.e. $(A, C), (A', C) \in \mathcal{I}_G^{(2)}$. Therefore $(A, A', C) \in \mathcal{I}_G^{(3)}$.

For (2), since $(A, A') \notin \mathcal{I}_G^{(1)}$, the dual edge cone of $\mathrm{G}\{A\} \cap \mathrm{G}\{A'\}$ has dimension strictly less than $m + n - 3$. By assumption, our purpose is to investigate the dual edge cone of dimension $m + n - 4$, i.e. the graph $\mathrm{G}\{A\} \cap \mathrm{G}\{A'\}$ with four connected components. Since $a, a' \in U_1$ are isolated vertices of this graph, the proof of (2) falls naturally into two parts:

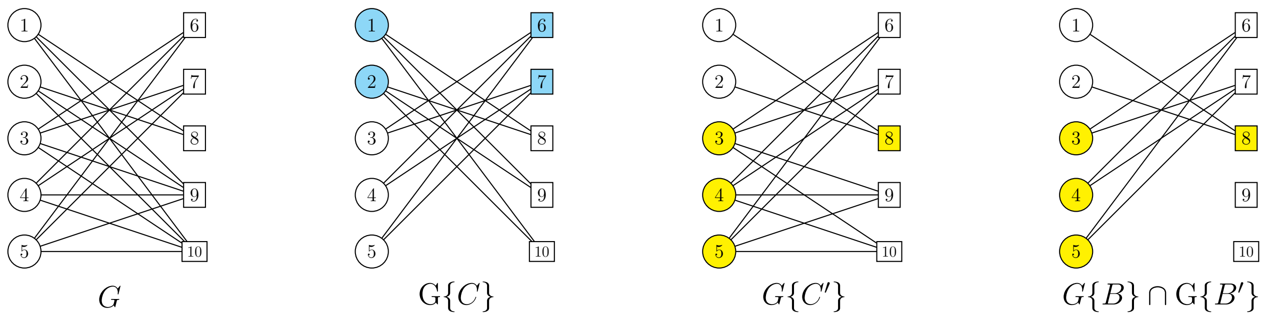
(i) $\mathrm{G}\{A\} \cap \mathrm{G}\{A'\}$ has an isolated vertex $b \in U_2$ and $\mathrm{G}[(A \cap A') \sqcup (U_2 \backslash \{b\})]$ is connected. Since G, $\mathrm{G}[A \sqcup N(A)]$, and $\mathrm{G}[A' \sqcup N(A')]$ are connected, we obtain that $C := (A \cap A') \sqcup \{b\} \in \mathcal{I}_G^{(1)}$ and $B := U_2 \backslash \{b\} \in \mathcal{I}_G^{(1)}$, Also, since $\mathrm{G}[(A \cap A') \sqcup (U_2 \backslash \{v\})]$ is connected, then $\mathrm{G}[U_2 \sqcup B]$ is connected, i.e. $B := U_2 \backslash \{b\} \in \mathcal{I}_G^{(1)}$. We observe in particular that $(A, B), (A', B), (A, C), (A', C) \in \mathcal{I}_G^{(2)}$. Hence, we obtain $(A, A', B, C) \in \mathcal{I}_G^{(3)}$. In particular, in the case where $G = K_{2,2}$, the first independent set $C = \{b\}$ and therefore we obtain the edge cone $\sigma_{K_{2,2}}$ as the non-simplicial 3-face.

(ii) $\mathrm{G}[(A \cap A') \sqcup U_2]$ has two connected components with no isolated vertices. Let us denote the vertex sets as $X_1 \sqcup X_2 = A \cap A'$ and $Y_1, \sqcup Y_2 \subsetneq U_2$ where $G[X_1 \sqcup Y_1]$ and $G[X_2 \sqcup Y_2]$ are connected. Since $\mathrm{G}\{A\}$ and $\mathrm{G}\{A'\}$ have two connected components, there exist edges $(a, y_1), (a', y_2), (a, y_2), (a', y_1) \in E(G)$ for some vertices $y_1 \in Y_1$ and $y_2 \in Y_2$. Thus $\mathcal{C} := X_1 \sqcup Y_2 \in \mathcal{I}_G^{(1)}$ and $\mathcal{C}' := X_2 \sqcup Y_1 \in \mathcal{I}_G^{(1)}$. By Lemma 5.1.4 (2), we know that $(C, C') \notin \mathcal{I}_G^{(2)}$ and by Lemma 5.1.3 (2), we know that $(A, C), (A, C'), (A', C), (A', C') \in \mathcal{I}_G^{(2)}$. Hence, we obtain that $(A, A', C, C') \in \mathcal{I}_G^{(3)}$.

$\square$

Remark 10. The classification of the three-dimensional faces of σ_G containing both $\mathfrak{b}$ and $\mathfrak{b}'$ can be deduced analogously by Lemma 5.2.1.

Example 22. An example of the case from Lemma 5.2.1 (2)(i) has been studied in Theorem 4.3.3 (1). We consider now the case from Lemma 5.2.1 (2)(ii) symmetrically for a pair $(B, B') \notin \mathcal{I}_G^{(2)}$. Let $G \subsetneq K_{5,5}$ be the bipartite graph as in the figure below. We see that there exist two two-sided first independent sets $C := \{1,2\} \sqcup \{6,7\}$ and $C' := \{3,4,5\} \sqcup \{8\}$. Let $B = U_1 \backslash \{9\} \in \mathcal{I}_G^{(1)}$ and $B' = U_1 \backslash \{10\} \in \mathcal{I}_G^{(1)}$. We observe that the intersection subgraph $\mathrm{G}\{B\} \cap \mathrm{G}\{B'\}$ has four connected components, hence $(B, B') \notin \mathcal{I}_G^{(2)}$ and $\{\mathfrak{b}, \mathfrak{b}'\}$ forms part of the extremal ray generators of a three-dimensional face of σ_G. Since $C_1 \sqcup C_1' = U_1$ and $C_2 \sqcup C_2' = B \cap B'$, we obtain that $(B, B', C, C') \in \mathcal{I}_G^{(3)}$.



Lemma 5.2.2. *Let $A = U_1 \backslash \{a\} \in \mathcal{I}_G^{(1)}$ and $B = U_2 \backslash \{b\} \in \mathcal{I}_G^{(1)}$. Assume that $\{\mathfrak{a}, \mathfrak{b}\}$ forms part of the extremal generators of a three-dimensional face of σ_G.*

(1) *If $(A, B) \in \mathcal{I}_G^{(2)}$ then the three-dimensional face is either*

  (i) *the non-simplicial one from Lemma 5.2.1 (2)(i) or*

  (ii) *$(A, B, C, C') \in \mathcal{I}_G^{(3)}$, with $C_1 \backslash C_1' = \{a\}$ and $C_2' \backslash C_2 = \{b\}$ or $C_1' \backslash C_1 = \{a\}$ and $C_2 \backslash C_2' = \{b\}$ or*

  (iii) *simplicial.*

(2) *If $(A, B) \notin \mathcal{I}_G^{(2)}$, then $(A, B, C, C') \in \mathcal{I}_G^{(3)}$, where $C_1 \sqcup C_1' = A$ and $C_2 \sqcup C_2' = B$.*

Proof. For (i), the intersection subgraph $\mathrm{G}\{A\} \cap \mathrm{G}\{B\}$ has three connected components with two isolated vertices a and b. Analysis similar to the proof of Lemma 5.2.1 (1) shows that $(A, A', B) \in \mathcal{I}_G^{(3)}$ and $(A, B, B') \in \mathcal{I}_G^{(3)}$. We investigate now the intersection $\mathrm{G}\{A\} \cap \mathrm{G}\{B\} \cap \mathrm{G}\{C\}$. If $\{a\} = C_1$ and $b \in N(C_1)$ with $N(C_1) \geq 3$, then we have that $(A, B, C) \in \mathcal{I}_G^{(3)}$ unless $\{b\} \sqcup C_1 \backslash \{a\}$ is an independent set. In this case, we obtain a first independent set $C' \in \mathcal{I}_G^{(1)}$ with $C_1 \backslash C_1' = \{a\}$ and $C_2' \backslash C_2' = \{b\}$. If $N(C_1) = 2$, this gives rise to the case (2) (ii) from Lemma 5.2.1 where $(A, A', B, C) \in \mathcal{I}_G^{(3)}$. In the other cases similar to proof of Lemma 5.2.1, we obtain that $(A, B, C) \in \mathcal{I}_G^{(3)}$.

For (ii), the intersection $\mathrm{G}\{A\} \cap \mathrm{G}\{B\}$ has of four connected components. This intersection subgraph cannot have four isolated vertices, because this means that we have that $G \subseteq K_{2,2}$. We studied these cases in Example 3 and in Theorem 4.1.4. Assume that the intersection subgraph has three isolated vertices $\{a, a', b\}$ and one connected component. This means that $a' \cap U_2 \backslash \{b\}$ is an independent set. But this contradicts the fact that $B \in \mathcal{I}_G^{(1)}$. The case with three isolated vertices $\{a, b, b'\}$ is similarly impossible, because $A \in \mathcal{I}_G^{(1)}$. Assume lastly that the intersection has two isolated vertices $\{a, b\}$ and two connected graphs with vertex sets $X_1 \sqcup X_2 = A$ and $Y_1 \sqcup Y_2 = B$. Since $\mathrm{G}[A \sqcup N(A)]$ and $\mathrm{G}[B \sqcup N(B)]$ are connected, we obtain that $C := X_1 \sqcup Y_2$ and $C' := X_2 \sqcup Y_1$ of Type (ii) and $(C, C') \notin \mathcal{I}_G^{(2)}$. We conclude that $(A, C), (A, C'), (B, C), (B, C') \in \mathcal{I}_G^{(2)}$ and $(A, B, C, C') \in \mathcal{I}_G^{(3)}$.

□

The calculation of an intersection of subgraphs associated to three two-sided independent sets can easily become heavily combinatorial. Therefore, by using Lemma 5.1.4, we would like to eliminate some cases of these two-sided independent sets resulting in a non-rigid toric variety. This will simplify the calculations for three-dimensional faces in Lemma 5.2.4.

Lemma 5.2.3. *Let $C = C_1 \sqcup C_2 \in \mathcal{I}_G^{(1)}$ and $C' = C_1' \sqcup C_2' \in \mathcal{I}_G^{(1)}$. If $(C, C') \notin \mathcal{I}_G^{(2)}$ is of Type* (i)*, then* $\mathrm{TV}(G)$ *is not rigid.*

Proof. Recall that (C, C') of Type (i) means that $C_1 \subsetneq C_1'$ and $C_2' \subsetneq C_2$. By Lemma 5.1.4 (1), we infer that if $(C, C') \notin \mathcal{I}_G^{(2)}$, then $\mathrm{G}[(C_1' \backslash C_1) \sqcup (C_2 \backslash C_2')]$ has $k \geq 2$ connected components without isolated vertices. Denote the vertex sets $X_i \subsetneq C_1' \backslash C_1$ and $Y_i \subsetneq C_2 \backslash C_2'$, for $i \in [k]$. Since $C \in \mathcal{I}_G^{(1)}$, we know that $\mathrm{G}[C_2 \sqcup N(C_2)]$ is connected. Thus, for each $i \in [k]$, we obtain that $N(Y_i) = X_i \sqcup Z_i$ where $Z_i \subseteq N(C_2')$. We can use the connectivity argument of $\mathrm{G}[C_1' \sqcup N(C_1')]$ symmetrically for each neighborhood vertex set $N(X_i)$. This implies that for a subset $I \subsetneq [k]$, there exist first independent sets of form

$$\mathcal{C}^I := (C_1 \sqcup \bigsqcup_{i \in I} X_i) \sqcup (C_2 \backslash \bigsqcup_{i \in I} Y_i).$$

Now let $i, j \in [k]$ and consider the pair $(\mathcal{C}^i, \mathcal{C}^j) \notin \mathcal{I}_G^{(2)}$ of Type (v). We calculate the intersection subgraph $\mathrm{G}\{\mathcal{C}^i\} \cap \mathrm{G}\{\mathcal{C}^j\}$ as

$$\mathrm{G}[C_1 \sqcup N(C_1)] \sqcup \mathrm{G}[C_2 \backslash (Y_i \sqcup Y_j) \sqcup U_1 \backslash (C_1 \sqcup X_i \sqcup X_j)] \sqcup \mathrm{G}[X_i \sqcup Y_i] \sqcup \mathrm{G}[X_j \sqcup Y_j]$$

and conclude that it has four connected components. This means that $\{\mathfrak{c}^i, \mathfrak{c}^j\}$ is contained in the extremal generator set of a 3-face of σ_G. By Corollary 3.2.3, we search for first independent sets such that the intersection subgraph $\mathrm{G}\{\mathcal{C}^i\} \cap \mathrm{G}\{\mathcal{C}^j\}$ is a subgraph of their associated subgraph. We observe that $\mathrm{G}\{C\}$ and $\mathrm{G}\{\mathcal{C}^{i,j}\}$ satisfy this condition. Moreover $(C, \mathcal{C}^i), (C, \mathcal{C}^j), (\mathcal{C}^i, \mathcal{C}^{i,j}), (\mathcal{C}^j, \mathcal{C}^{i,j}) \in \mathcal{I}_G^{(2)}$ of Type (i). Hence we obtain the non-simplicial 3-face $(C, \mathcal{C}^i, \mathcal{C}^j, \mathcal{C}^{i,j}) \in \mathcal{I}_G^{(3)}$. Let $\alpha \in N(C_2')$ and $\beta \in N(C_1)$ be two vertices and let $R = e_\alpha + f_\beta \in M$ be a deformation degree. Since the associated extremal rays to the tuple

$(C, \mathcal{C}^i, \mathcal{C}^j, \mathcal{C}^{i,j})$ are all lattice vertices in $Q(R)$, by Proposition 4.2.3, we conclude that $\mathrm{TV}(G)$ is not rigid. $\square$

Proposition 5.2.4. *Let $C = C_1 \sqcup C_2 \in \mathcal{I}_G^{(1)}$ and $C' = C'_1 \sqcup C'_2 \in \mathcal{I}_G^{(1)}$. Assume that $(C, C') \notin \mathcal{I}_G^{(2)}$ and $\{\mathfrak{c}, \mathfrak{c}'\}$ forms part of the extremal generators of a three-dimensional face of σ_G.*

(1) *If (C, C') is of Type* (ii)*, then one obtains the three-dimensional face either from Lemma 5.2.1* (2)(ii) *or from Lemma 5.2.2* (2)*.*

(2) *If (C, C') is of Type* (iii)*, then one obtains either one of the following*

  (i) *$(A, C, C', C'') \in \mathcal{I}_G^{(3)}$, where $C'' = (C_1 \cap C'_1) \sqcup (C_2 \sqcup C'_2) \in \mathcal{I}_G^{(1)}$ and $A = C_1 \cup C'_1 \in \mathcal{I}_G^{(1)}$.*

  (ii) *$(C, C', \mathcal{C}, \mathcal{C}') \in \mathcal{I}_G^{(3)}$, where $C_1 \cup C'_1 = U_1$, $\mathcal{C}_1 \sqcup \mathcal{C}'_1 = C_1 \cap C'_1$, $\mathcal{C}_2 \cap \mathcal{C}'_2 = C_2 \sqcup C'_2$, and $\mathcal{C}_2 \sqcup \mathcal{C}'_2 = U_2$.*

  (iii) *$(B, C, C', C'') \in \mathcal{I}_G^{(3)}$, where $C_1 \cup C'_1 = U_1$, $C'' = (C_1 \cap C'_1) \sqcup (C_2 \sqcup C'_2) \sqcup \{b\} \in \mathcal{I}_G^{(1)}$, and $B = U_2 \backslash \{b\} \in \mathcal{I}_G^{(1)}$.*

(3) *If (C, C') is of Type* (v)*, then there exist the first independent sets $\mathcal{C} := (C_1 \cap C'_1) \sqcup (C_2 \cup C'_2) \in \mathcal{I}_G^{(1)}$, $\mathcal{C}' := (C_1 \cup C_1) \sqcup (C_2 \cap C'_2) \in \mathcal{I}_G^{(1)}$ and one obtains that $(C, C', \mathcal{C}, \mathcal{C}') \in \mathcal{I}_G^{(3)}$.*

Proof. By the assumption, the intersection subgraph $\mathrm{G}\{C\} \cap \mathrm{G}\{C'\}$ has four connected components.
(1) The intersection subgraph $\mathrm{G}\{C\} \cap \mathrm{G}\{C'\}$ has the following isolated vertices:

$$(N(C_2) \cap N(C'_2)) \sqcup (N(C_1) \sqcup N(C'_1)).$$

The number of isolated vertices can be at most two. If there is exactly one isolated vertex, we concluded in Lemma 5.1.4 that $(C, C') \in \mathcal{I}_G^{(2)}$. Hence, we conclude that there are two isolated vertices. Assume that $N(C_2) \cap N(C'_2) = \{a, a'\}$ and $C_2 \sqcup C'_2 = U_2$. Since $\mathrm{G}[C_1 \sqcup N(C_1)]$ and $\mathrm{G}[C'_1 \sqcup N(C'_1)]$ are connected, we have that $A, A' \in \mathcal{I}_G^{(1)}$. We observe that $(A, A') \notin \mathcal{I}_G^{(2)}$ and therefore it is the case that we examined in Lemma 5.2.1(2)(ii). Assume now that $N(C_2) \cap N(C'_2) = \{a\}$ and $N(C_1) \cap N(C'_1) = \{b\}$. Similarly to the previous investigation, we have that $A, B \in \mathcal{I}_G^{(1)}$ and it is the case that we examined in Lemma 5.2.2 (2).

(2) It is impossible that $C_2 \sqcup C'_2 = U_2$, because then $C_1 \cap C'_1$ is a set of isolated vertices in G. We also conclude that $U_1 \backslash (C_1 \cup C'_1)$ has at most one vertex. Assume first that $C_1 \cup C'_1 = U_1$. In the intersection subgraph $\mathrm{G}\{C\} \cap \mathrm{G}\{C'\}$, there cannot be isolated vertices in $C_1 \cap C'_1$, because this implies that these are isolated vertices in G. Since $\mathrm{G}[C_2 \sqcup N(C_2)]$ and $\mathrm{G}[C'_2 \sqcup N(C'_2)]$ are connected, there are two possibilities for the subgraph $\mathrm{G}[(C_1 \cap C'_1) \sqcup (N(C_1) \sqcup N(C'_1))]$:

• The subgraph $\mathrm{G}[(C_1 \cap C_1') \sqcup U_2 \backslash (C_2 \sqcup C_2' \sqcup \{b\})]$ is connected. This implies that there exist first independent sets $C'' := (C_1 \cap C_1') \sqcup C_2 \sqcup C_2' \sqcup \{b\}$ and $B = U_2 \backslash \{b\}$. Moreover $(C, C'') \in \mathcal{I}_G^{(2)}$ and $(C', C'') \in \mathcal{I}_G^{(2)}$ are of Type (i) and $(B, C'') \notin \mathcal{I}_G^{(2)}$.

• The subgraph has two connected components and no isolated vertices. Let us denote their vertex sets as $X_i \subsetneq C_1 \cap C_1'$ and $Y_i \subsetneq U_2 \backslash \{C_2 \sqcup C_2'\}$. Then there exist two first independent sets:

$$\mathcal{C} := X_1 \sqcup C_2 \sqcup C_2' \sqcup Y_2 \in \mathcal{I}_G^{(1)}$$

$$\mathcal{C}' := Y_1 \sqcup C_2 \sqcup C_2' \sqcup Y_1 \in \mathcal{I}_G^{(1)}$$

We observe that $(C, \mathcal{C}), (C, \mathcal{C}'), (C', \mathcal{C}), (C', \mathcal{C}') \in \mathcal{I}_G^{(2)}$ of Type (i). In particular, $(\mathcal{C}, \mathcal{C}') \notin \mathcal{I}_G^{(2)}$ of Type (iv).

Assume now that $U_1 \backslash C_1 \cup C_1' = \{a\}$. Then the subgraphs $\mathbf{G_1}$, $\mathbf{G_3}$, and $\mathbf{G_4}$ must be connected. Moreover, there exist two first independent sets $C'' := (C_1 \cap C_1') \sqcup C_2 \sqcup C_2' \in \mathcal{I}_G^{(1)}$ and $A = U_1 \backslash \{a\}$. We observe that $(A, C), (A, C') \in \mathcal{I}_G^{(2)}$ and the pairs $(C, C''), (C', C'') \in \mathcal{I}_G^{(2)}$ are of Type (i).

(3) One cannot have that $C_1 \cup C_1' = U_1$ or $C_2 \cup C_2' = U_2$, because otherweise G has isolated vertices. Also, the subgraph $\mathbf{G_i}$ must be connected for each $i \in [4]$. We thus observe that there exist two first independent sets

$$\mathcal{C} := (C_1 \cap C_1') \sqcup (C_2 \cup C_2') \in \mathcal{I}_G^{(1)}$$

$$\mathcal{C}' := (C_1 \cup C_1') \sqcup (C_2 \cap C_2') \in \mathcal{I}_G^{(1)}$$

of Type (i). Moreover we have that $(C, \mathcal{C}), (C', \mathcal{C}), (C, \mathcal{C}'), (C', \mathcal{C}') \in \mathcal{I}_G^{(2)}$, but $(C, C') \notin \mathcal{I}_G^{(2)}$. □

Example 23. We observe in Example 22 that $(C, C) \notin \mathcal{I}_G^{(2)}$ is of Type (ii) and we know that $(B, B) \notin \mathcal{I}_G^{(2)}$. Therefore we obtain the case from Lemma 5.2.1 (2)(ii).

Consider the pair $(B, C) \in \mathcal{I}_G^{(2)}$ such that $\{\mathfrak{b}, \mathfrak{c}\}$ forms part of the extremal ray generators of a three-dimensional face. We covered all possible triples of form (A, B, C) and (B, B, C) in Lemma 5.2.1 and Lemma 5.2.2. For the triples of form (B, C, C'), we finished studying the cases where $(C, C') \notin \mathcal{I}_G^{(2)}$. We are left with the task of determining the cases where $(C, C') \in \mathcal{I}_G^{(2)}$.

Lemma 5.2.5. *Let* $B = U_2\backslash\{b\} \in \mathcal{I}_G^{(1)}$*,* $C = C_1 \sqcup C_2 \in \mathcal{I}_G^{(1)}$ *and* $C' = C_1' \sqcup C_2' \in \mathcal{I}_G^{(1)}$*. Assume that* $(B, C) \in \mathcal{I}_G^{(2)}$*,* $(C, C') \in \mathcal{I}_G^{(2)}$ *and* $\{\mathfrak{b}, \mathfrak{c}, \mathfrak{c}'\}$ *forms part of the extremal generators of a three-dimensional face of* σ_G*. Then the three-dimensional face is either*

(1) $(A, B, C, C') \in \mathcal{I}_G^{(3)}$ *from Lemma 5.2.2* (1) (ii) *or*

(2) *simplicial.*

Proof. Consider the intersection $\mathrm{G}\{C\} \cap \mathrm{G}\{C'\}$. If (C, C') is of Type (i), without loss of generality let us assume that $C_1' \subsetneq C_1$ and $C_2 \subsetneq C_2'$. For each type of $(C, C') \in \mathcal{I}_G^{(2)}$, the induced subgraph $\mathrm{G}[C_2 \sqcup N(C_2)]$ is not empty. If $b \in C_2$, then we obtain that $(B, C') \in \mathcal{I}_G^{(2)}$. For the rest, we divide the proof into the four types of the pair $(C, C') \in \mathcal{I}_G^{(2)}$:

Type (i): Let $b \in C_2'\backslash C_2$. The triple $(B, C, C') \notin \mathcal{I}_G^{(2)}$ if and only if $C_1\backslash C_1' = \{a\}$. This is the case from Lemma 5.2.2 (1)(ii). Let $b \in N(C_1')$. We conclude that $\mathrm{G}[C_1' \sqcup N(C_1')\backslash\{b\}]$ is connected and therefore $(B, C') \in \mathcal{I}_G^{(2)}$.

Type (ii): Let $b \in C_2'$. Then $\mathrm{G}[C_2'\backslash\{b\} \sqcup N(C_2')]$ is connected, since otherwise $B \notin \mathcal{I}_G^{(1)}$. Hence $(B, C') \in \mathcal{I}_G^{(2)}$. Note that we cannot have that $C_2 \sqcup C_2' \sqcup \{b\} = U_2$, since otherwise $B \in \mathcal{I}_G^{(1)}$.

Type (iii): Let $b \in C_2'$. Then $\mathrm{G}[(C_2'\backslash\{b\}) \sqcup (U_1\backslash C_1')]$ is connected and therefore $(B, C') \in \mathcal{I}_G^{(2)}$. If $b \in U_2\backslash(C_2 \sqcup C_2')$, we conclude similarly that $(B, C, C') \in \mathcal{I}_G^{(3)}$. Note that as in the case of Type (ii), $C_2 \sqcup C_2' \sqcup \{b\} \neq U_2$.

Type (iv): Let $b \in C_2'\backslash C_2$. Since $B \in \mathcal{I}_G^{(1)}$, the induced subgraph $\mathrm{G}[C_2'\backslash\{b\} \sqcup N(C_2')]$ is connected. Hence $(B, C') \in \mathcal{I}_G^{(2)}$.

□

Corollary 5.2.6. *Let* $B = U_2\backslash\{b\} \in \mathcal{I}_G^{(1)}$ *and* $C = C_1 \sqcup C_2 \in \mathcal{I}_G^{(1)}$*. Assume that* $(B, C) \notin \mathcal{I}_G^{(2)}$ *and* $\{\mathfrak{b}, \mathfrak{c}\}$ *forms part of the extremal generators of a three-dimensional face of* σ_G*. Then one obtains the non-simplicial three-dimensional face in Lemma 5.2.1* (2)(i) *or in Lemma 5.2.2 or in Proposition 5.2.4* (2)(i) *and* (iii)*.*

Proof. We only need to show that there exists no three-dimensional face containing the extremal rays $\{\mathfrak{b}, \mathfrak{c}, \mathfrak{c}'\}$ where $(C, C') \in \mathcal{I}_G^{(2)}$ and $(B, C') \notin \mathcal{I}_G^{(2)}$. Consider the intersection $\mathrm{G}\{B\} \cap \mathrm{G}\{C\}$ which has four connected components. Since we want to have another $\mathfrak{c}'$ in the generator set, we have two possibilities:

• If $b \in C_2$, there exist two first independent sets $\mathcal{C}^1$ and $\mathcal{C}^2$ such that $C_1 \cup \mathcal{C}_1^1 \cup \mathcal{C}_1^2 = U_1$ and $\mathcal{C}_2^1 \sqcup \mathcal{C}_2^2 \sqcup \{b\} = C_2$.

• If $b \in N(C_1)$, there exist two first independent sets $\mathcal{C}^1$ and $\mathcal{C}^2$ such that $\mathcal{C}_1^1 \sqcup \mathcal{C}_1^2 = C_1$ and $C_2 \cup \mathcal{C}_2^1 \cup \mathcal{C}_2^2 \sqcup \{b\} = N(C_1)$.

However, these have been examined in Proposition 5.2.4 (2)(i) and (iii).

$\square$

Finally, we want to characterize the triples $(C, C', C'') \in \mathcal{I}_G^{(3)}$. The next result, follows by the recent calculations.

Corollary 5.2.7. *Let C, C' and C'' be three first independent sets of G. Assume that $(C, C', C'') \in \mathcal{I}_G^{(3)}$ forms a three-dimensional face of σ_G. Then its two-dimensional faces are one of the following type:*

- $((\mathrm{i}), x, x)$, $x \in \{(\mathrm{i}), (\mathrm{ii}), (\mathrm{iii}), (\mathrm{iv})\}$.
- $((\mathrm{i}), (\mathrm{ii}), (\mathrm{iii}))$, $((\mathrm{i}), (\mathrm{ii}), (\mathrm{iv}))$, $((\mathrm{i}), (\mathrm{iii}), (\mathrm{iv}))$.

5.3 Non-rigidity for toric varieties with non-simplicial three-dimensional faces

This section is intended to compile all possible non-simplicial three-dimensional faces of σ_G. In these cases, we will show that $\mathrm{TV}(G)$ is not rigid. After that, we are reduced to proving the rigidity for the toric varieties whose edge cone σ_G admits only simplicial three-dimensional faces. We classified this type of edge cones explicitly in Section 5.2.

Theorem 5.3.1. *Let $G \subseteq K_{m,n}$ be a connected bipartite graph and let $\tau \preceq \sigma_G$ be a three-dimensional non-simplicial face of the edge cone σ_G. Then τ is spanned by four extremal rays.*

Proof. It follows by Lemma 5.2.1 (2)(i), (ii), Lemma 5.2.2 (1) (ii) and (2), Lemma 5.2.3, and Proposition 5.2.4. $\square$

Theorem 5.3.2. *Let $G \subseteq K_{m,n}$ be a connected bipartite graph. Assume that the edge cone σ_G admits a three-dimensional non-simplicial face. Then $\mathrm{TV}(G)$ is not rigid.*

Proof. We are reduced to examine the non-simplicial 3-faces from Lemma 5.2.1 (2)(i), (ii), Lemma 5.2.2 (2), and Proposition 5.2.4. For each case, by Proposition 4.2.3, it is sufficient to show that there exists a deformation degree $R \in M$ such that the associated extremal rays are lattice vertices in R. We find such deformation degrees as following:
Lemma 5.2.1
(2) (i): $e_a + e_{a'} + f_b + f_{b'}$, where $b \neq b'$.
(2) (ii): $e_a + e_{a'} + f_b + f_{b'}$, where $b \in C_2$ and $b' \in C_2'$.
Lemma 5.2.2
(1) (ii): $e_a + e_{a'} + f_b + f_{b'}$, where $b \in N(C_2)$ and $b' \in N(C_1')$.
(2): $e_a + f_b$.
Proposition 5.2.4
(2) (i): $e_a + f_b$, where $b \in U_2 \backslash (C_2 \sqcup C_2')$.

(2) (ii): $e_a + e_{a'} + f_b + f_{b'}$, where $a \in N(C_2')$, $a' \in C_1'$, $b \in \mathcal{C}_2 \backslash (C_2 \cup C_2')$, and $b' \in \mathcal{C}' \backslash (C_1 \cup C_1')$.
(2) (iii): $e_a + e_{a'} + f_b + f_{b'}$, where $a \in N(C_2')$, $a' \in N(C_2)$, and $b' \in U_2 \backslash C''$.
(3): $e_a + f_b$, where $a \in N(\mathcal{C}_2')$ and $b \in N(\mathcal{C}_1)$.

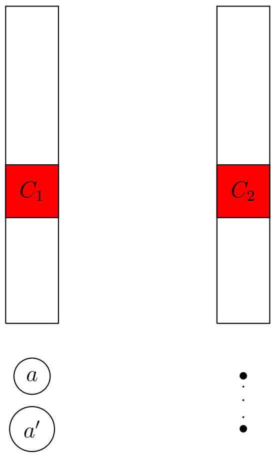
$\square$

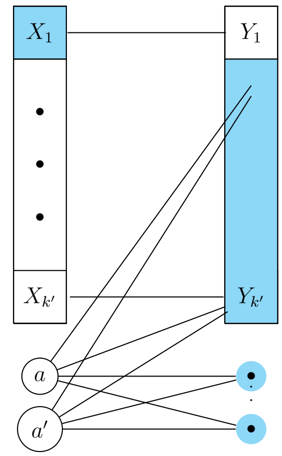
5.4 The pairs of first independent sets not spanning a two-dimensional face

Our technique which utilizes subgraphs associated to first independent sets sheds some new light on the rigidity of a toric variety $\mathrm{TV}(G)$. Given $G \subseteq K_{m,n}$ a connected bipartite graph with its first independent sets $\mathcal{I}_G^{(1)}$, one can study the rigidity of its associated toric variety $\mathrm{TV}(G)$ by using the information from Section 5.2. As we have seen in several examples of Chapter 4, we start with establishing if any of the three-dimensional faces of σ_G is non-simplicial. If there exists such three-dimensional face, by Theorem 5.3.2, we conclude that $\mathrm{TV}(G)$ is not rigid. If there exists no such three-dimensional face, we determine its two and three-dimensional faces and we focus on its non 2-faces pairs and non 3-faces triples. In Chapter 6, we illustrate these steps for a more general case, i.e. the edge cones with more than just one two-sided first independent sets.

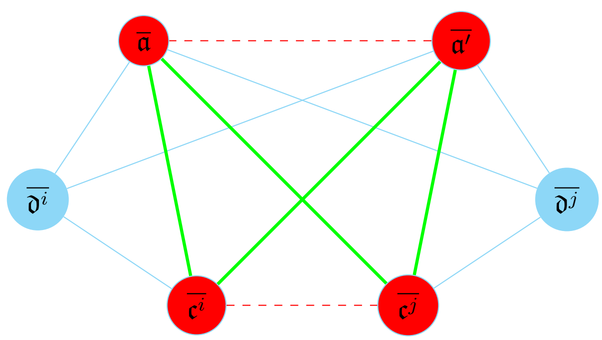
However, in the general setting, the complexity of the bipartite graph might be unpredictable. We explain the challenge about the classification of rigid toric varieties associated to bipartite graphs in the next example.

Example 24. Let $G \subsetneq K_{m,n}$ be a connected bipartite graph and let $A = U_1 \backslash \{a\}$ and $A' = U_1 \backslash \{a'\}$ be two first independent sets. Assume that $(A, A') \notin \mathcal{I}_G^{(1)}$ and the edge cone σ_G does not have any non-simplicial three-dimensional face. By Proposition 5.1.2 and Lemma 5.2.1, the induced subgraph $\mathrm{G}[(U_1 \backslash \{a, a'\}) \sqcup U_2]$ has k connected components where $k \geq 3$. If this induced subgraph has isolated vertices, say the set $Y \subsetneq U_2$ as in the first figure, then we obtain the maximal independent set $(A \cap A') \sqcup Y$. This maximal independent set is not a first independent set, unless $\mathrm{G}[A \cap A' \sqcup (U_2 \backslash Y)]$ is connected. However, even if this induced subgraph is connected, there might exist another first independent set, say $C \in \mathcal{I}_G^{(1)}$ with $C_1 \subsetneq A \cap A'$ and $C_2 \subsetneq U_2 \backslash Y$. This possibility makes the investigation iterative and hard to control.





Another possibility is that $k' \geq 2$. In this case, there exist disjoint vertex sets $X_i \subsetneq A \cap A'$ and $Y_i \subsetneq U_2 \backslash Y$ where $\mathrm{G}[X_i \sqcup Y_i]$ is connected as illustrated in the second figure. Since $\mathrm{G}\{A\}$ and $\mathrm{G}\{A'\}$ have two connected components, we obtain the first independent sets $C^i := X_i \sqcup (U_2 \backslash (X_i \sqcup Y))$. A pair (C^i, C^j) is of Type (iv) and does not form a 2-face. Let $R = e_a + e_{a'} - e_{x_i} - e_{x_j} \in M$ where $x_i \in X_i$ and $x_j \in X_j$ and we consider the crosscut $Q(R)$. Although $\mathrm{G}[X_i \sqcup Y_i]$ is connected, as in the previous situation there might exists an independent set D with $D_1^i \subsetneq X_i$, $D_2^i \subsetneq Y_i$ and $x_i \in D_1^i$. Moreover there might exist first independents set $\mathcal{D}^i := C_1 \sqcup C_2 \sqcup \mathcal{C}_2^i$. We observe that $(\mathcal{D}^i, C^i)$ of Type (i) forms a 2-face, otherwise by Lemma 5.2.3, σ_G has non-simplicial three-dimensional faces. However $(\mathcal{D}^i, C^j)$ is of Type (iv) and does not form a 2-face. Furthermore, there cannot exist any first independent set containing both X_i and Y_i. Hence we obtain that $T^1(-R) \neq 0$ for this possibility. However, for rigidity, one needs to examine all non 2-face pairs, e.g. $(\mathcal{D}^i, C^j) \notin \mathcal{I}_G^{(2)}$.



We observe that as long as we know more information about the bipartite graph G, it is more probable that we are able to determine the rigidity of $\mathrm{TV}(G)$. In the following chapter, we will study the edge cones associated to so-called toric matrix Schubert varieties. After examining their face structure, we are able to classify the rigid toric matrix Schubert varieties.

Chapter 6

Applications of the combinatorial technique to matrix Schubert varieties

In the previous chapters, we have reformulated the question of rigidity into Graph Theory. In this chapter, we show how these techniques give rise to a better understanding of matrix Schubert varieties. In particular, we observe that the bipartite graphs appear naturally when one investigates the dimension of the effective torus action on matrix Schubert varieties. In the case of toric matrix Schubert varieties, we give a complete classification of rigid toric matrix Schubert varieties. This chapter ends with possible research directions in the topic of Kazhdan-Lusztig varieties.

6.1 Preliminaries

In this section, we set up the notation and terminology for Matrix Schubert Varieties. The matrix Schubert varieties first appeared in [Ful92] while Fulton was studying the degeneracy loci of flagged vector bundles. Let M_n be the set of $n \times n$ matrices over $\mathbb{C}$. Let GL_n denote the invertible $n \times n$ matrices and B_- denote the invertible lower triangular matrices. The matrix Schubert variety $\overline{X_\pi} \subset M_n$ is in fact related to Schubert variety X_π in the flag manifold GL_n/B_-. We are mainly interested in matrix Schubert varieties for their effective torus actions and deformations. The statements presented in this section can be found in [Ful92] and [KM05].

Let $\pi \in S_n$ be a permutation. We denote its *permutation matrix* as $\pi \in M_n$ as well and define it as follows:

$$\pi_{(i,j)} = \begin{cases} 1, & \text{if } \pi(j) = i \\ 0, & \text{otherwise.} \end{cases}$$

Now, let us denote B_+ as the invertible upper triangular $n \times n$ matrices. Let $M_- \in B_-$ and $M_+ \in B_+$. The product $B_- \times B_+$ acts from left on M_n and it is defined as:

$$\begin{array}{rcl}(B_{\text{-}} \times B_+) \times M_n & \longrightarrow & M_n \\ ((M_{\text{-}}, M_+), M) & \mapsto & M_{\text{-}} M M_+^{-1}\end{array}$$

Definition 6.1.1. Let $M_{(a,b)} \in M_{a\times b}$ be the matrix on the upper left corner submatrix of $M \in M_n$, where $1 \leq a \leq n$ and $1 \leq b \leq n$. The rank function of M is defined by $r_M(a,b) := \text{rank}(M_{(a,b)})$.

Note that the multiplication of a matrix $M \in M_n$ on the left with $M_{\text{-}}$ corresponds to the downwards row operations and multiplication of M on the right with M_+ corresponds to the rightward column operations. Hence, we observe that $M \in B_{\text{-}}\pi B_+$ if and only if $r_M(a,b) = r_\pi(a,b)$ for all $(a,b) \in [n] \times [n]$.

Definition 6.1.2. The Zariski closure of the orbit $\overline{X_\pi} := \overline{B_{\text{-}}\pi B_+}$ inside M_n is called *the matrix Schubert variety* of π.

Rothe developed a combinatorial technique for visualizing permutations in 1800's.

Definition 6.1.3. The *Rothe diagram of* π *is defined as* $D(\pi) = \{(\pi_j, i) : i < j, \pi_i > \pi_j\}$.

One can draw the diagram in the following way: We consider the permutation matrix π. We cross out the south and east entries of each 1 of the matrix. The remaining entries represents the Rothe diagram.

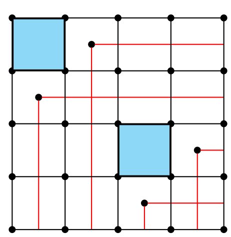


Figure 6.1: The Rothe Diagram of $(12)(34) \in S_4$.

Theorem 6.1.4 ([Ful92], Proposition 3.3)**.** *The matrix Schubert variety* $\overline{X_\pi}$ *is an affine variety of dimension* $n^2 - |D(\pi)|$. *It can be defined as a scheme by the equations* $r_M(a,b) \leq r_\pi(a,b)$ *for all* $(a,b) \in [n] \times [n]$.

Definition 6.1.5. The connected part containing the box $(1,1)$ in the diagram is called the dominant piece $\text{dom}(\pi)$. The set consisting of south-east corners of $D(\pi)$ is called the essential set $\text{Ess}(\pi)$. We define $NW(\pi)$ as the union of north-west boxes of each box in $D(\pi)$ and let $L(\pi) := NW(\pi) - \text{dom}(\pi)$ and $L'(\pi) := L(\pi) - D(\pi)$.

In Figure 6.2 below, one observes the examples of these definitions for the permutation $(12)(34) \in S_4$.

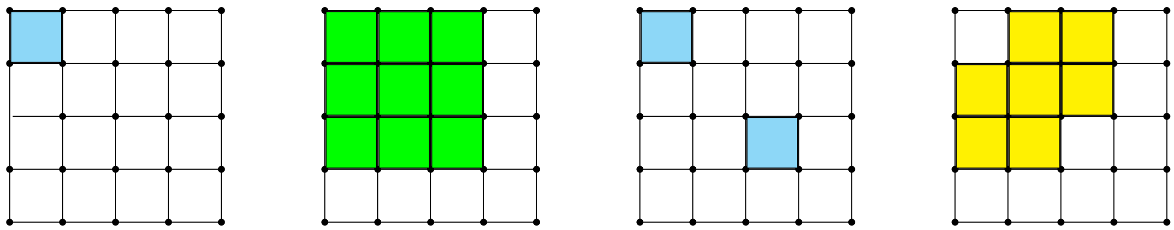


Figure 6.2: The representations of $\mathrm{dom}(\pi)$, $NW(\pi)$, $\mathrm{Ess}(\pi)$, and $L'(\pi)$.

One obtains a better description of the matrix Schubert variety by using the essential set of π.

Theorem 6.1.6 ([Ful92], Lemma 3.10)**.** *The ideal, which defines the matrix Schubert variety $\overline{X_\pi}$, is generated by the equations $r_M(a,b) \leq r_\pi(a,b)$ for all $(a,b) \in Ess(\pi)$.*

6.2 Torus action on matrix Schubert varieties

Let us define V_π as the projection of the matrix Schubert variety $\overline{X_\pi} \subseteq M_n$ onto the entries which are not north-west of any entry of $D(\pi)$. By Theorem 6.1.6, these entries are free in $\overline{X_\pi}$, therefore they are isomorphic to $\mathbb{C}^q$ where q is equal to $n^2 - |NW(\pi)|$. Also, we define Y_π as the projection onto the entries of $L(\pi)$. Note that one obtains $(a,b) \in \mathrm{dom}(\pi)$ if and only if $r_\pi(a,b) = 0$. Hence, $\overline{X_\pi} = Y_\pi \times V_\pi$ holds. In particular, by Theorem 6.1.4,

$$\dim(Y_\pi) = n^2 - |D(\pi)| - n^2 - |NW(\pi)| = |NW(\pi)| - |D(\pi)| = |L'(\pi)|.$$

In this section, our investigation is on the torus action on Y_π. This question has been first studied by Escobar and Mészáros in [EM16]. In this paper, all toric varieties Y_π have been characterized. We would like to study the torus action on Y_π in terms of graphs and determine the complexity of the *T-variety* Y_π. T-varieties are normal varieties with effective torus action having not necessarily a dense torus orbit. They can be considered as the generalization of toric varieties with respect to the dimension of their torus action. For more details about T-varieties, we refer to [AIPSV12].

Definition 6.2.1. An affine normal variety X is called a *T-variety* of complexity d if it admits an effective T torus action with $\dim(X) - \dim(T) = d$.

Example 25. The toric varieties are T-varieties of complexity zero.

The matrix Schubert varieties are normal varieties (see [KM05], Theorem 2.4.3.). The action of $B_- \times B_+$ on $\overline{X_\pi}$ restricts to the action of $T^n \times T^n$, where $T^n \cong (\mathbb{C}^*)^n$ is diagonal matrix of size $n \times n$. This action of the torus $(\mathbb{C}^*)^{2n}$ is not effective, because

$$\begin{array}{rcl}(\mathbb{C}^*)^{2n} \times \overline{X_\pi} & \longrightarrow & \overline{X_\pi} \\ (a.I_n, a.I_n).M & = & M\end{array}$$

where $I_n \in M_n$ is the identity matrix. Therefore we investigate the stabilizer $\mathrm{Stab}((\mathbb{C}^*)^{2n})$ of this torus action and consider the action of the quotient $T := (\mathbb{C}^*)^{2n}/\,\mathrm{Stab}((\mathbb{C}^*)^{2n})$ on the matrix Schubert variety $\overline{X_\pi}$. Our purpose is to investigate the dimension of the effective action of T on Y_π in terms of bipartite graphs.

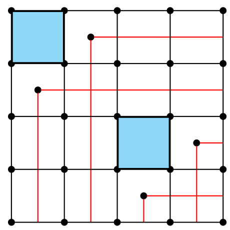
For this investigation, we follow the arguments in [EM16]. Let p be a general point in Y_π. Then $\overline{(\mathbb{C}^*)^{2n}.p}$ is the affine toric variety associated to the so-called $(\mathbb{C}^*)^{2n}$-moment cone of Y_π. We denote it by $\Phi(Y_\pi)$ and it is generated by the images under the so-called moment map of $(\mathbb{C}^*)^{2n}$- fixed points of Y_π. One obtains that $\dim(\Phi(Y_\pi)) = \dim(\overline{(\mathbb{C}^*)^{2n}.p})$. Since $\overline{(\mathbb{C}^*)^{2n}.p}$ and Y_π are both irreducible, we examine their dimension in order to give the complexity of the torus action on Y_π. Let us first consider the torus action on the matrix Schubert variety $\overline{X_\pi}$. Let m_{ij} be the ith row and jth column element of $M \in M_n$ and $a_1, ..., a_n$ and $b_1, ..., b_n$ be the diagonal elements of M_- and M_+. Without loss of generality, we pick the general point $(1, ..., 1)$ in $\overline{X_\pi}$ and we obtain that $(M_- M M_+^{-1}) = a_i b_j^{-1} m_{ij}$. The weights of the action are then $e_i - f_j$ where e_i denotes the canonical basis for $\mathbb{R}^n \times 0$ and f_j denotes the canonical basis for $0 \times \mathbb{R}^n$. One can project this cone to Y_π and obtain $\Phi(Y_\pi)$ as $\mathrm{Cone}(e_i - f_j \mid (i,j) \in L(\pi))$. Note that this cone is GL-equivalent to the edge cone associated to a bipartite graph. We now explain this relation.

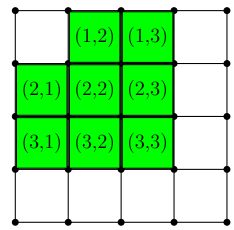
Recall from Lemma 3.2.1, one can calculate the dimension of an edge cone. To wit, for a bipartite graph $G \subseteq K_{m,n}$, $\dim(\sigma_G^\vee) = m + n - k$ where k is the number of connected components of G. Let $G_\pi \in K_{m,n}$ denote the bipartite graph associated to permutation π. We translate the information from Rothe diagram to G_π with the following trivial bijection:

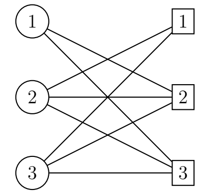
$$\begin{array}{c} L(\pi) \longrightarrow E(G^\pi) \\ (a,b) \mapsto (a,b) \end{array}$$

where for $(a,b) \in E(G^\pi)$, $a \in U_1$ and $b \in U_2$. Hence we obtain also the vertex set $V(G^\pi)$. We denote the associated edge cone by σ_π. Finally, we say that Y_π is a T-variety of complexity d with respect to the torus action T if and only if $\dim(\sigma_\pi^\vee) = L'(\pi) - d$.

Example 26. Let us consider again the matrix Schubert variety $\overline{X_{(12)(34)}} \cong Y_{(12)(34)} \times \mathbb{C}^7$. The second figure represents $L(\pi)$ and the third figure represents the bipartite graph σ_π. For each green cell (a,b) from the second figure, we construct an edge $(a,b) \in E(\sigma_\pi)$ with vertices $a \in U_1$ and $b \in U_2$. The dimension of the associated dual edge cone $\sigma_\pi^\vee$ is 5 and $|L'(\pi)| = 7$. Hence $Y_{(12)(34)}$ is a T-variety of complexity 2 with respect to the effective torus action of $T \cong (\mathbb{C}^*)^5$ with a weight cone linearly equivalent to $\sigma_\pi^\vee$.





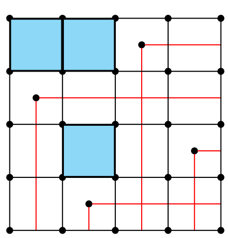


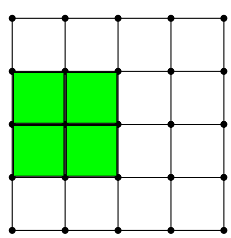
We utilize Graph Theory techniques for the next two theorems.

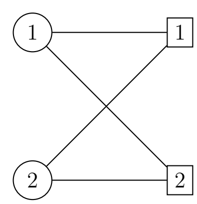
Theorem 6.2.2. *[[EM16], Theorem 3.4] Y_π is a toric variety if and only if $L'(\pi)$ consists of disjoint hooks not sharing a row or a column.*

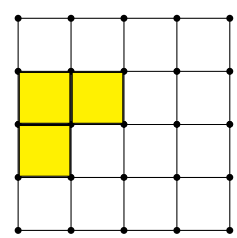
Proof. We want to characterize the case when $\dim(\sigma_\pi^\vee) = L'(\pi)$. Assume that $L(\pi)$ consists of k connected components with m_i rows and n_i columns for each $i \in [k]$. This means that we investigate the bipartite graph $G^\pi \subseteq K_{m,n}$ with k connected bipartite graph components $G_i^\pi \subseteq K_{m_i,n_i}$. Therefore, the dimension of the cone $\dim(\sigma_\pi^\vee)$ is $\sum_{i\in[k]}(m_i+n_i-1) = m+n-k$. Since $L(\pi)$ has k connected components, the components of $L'(\pi)$ for each $i \in [k]$ do not share a row or a column. Therefore, we are left with proving the statement for a connected component $L_i(\pi)$ of $L(\pi)$. The dimension of the dual edge cone of G_i^π is equal to $|L_i'(\pi)|$ if and only if $L_i'(\pi)$ has a hook shape. $\square$

Example 27. Let $\pi = (1243) \in S_4$. The first figure illustrates the Rothe diagram $D(\pi)$. The green colored boxes are $L(\pi)$ and the yellow colored boxes are $L'(\pi)$. The dimension of the associated bipartite graph and $|L'(\pi)|$ is three. Also, as seen in the last figure, $L'(\pi)$ has a hook shape. Thus, $Y_{(1243)}$ is a toric variety with respect to the effective torus action of $T = (\mathbb{C}^*)^3$.









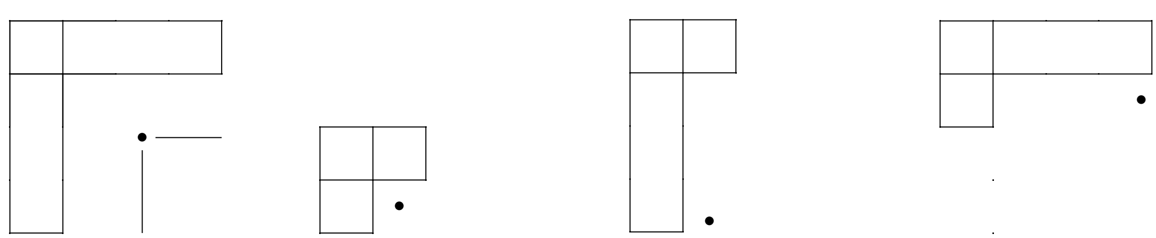
Our initial question was to study the deformations of complexity-one T-varieties Y_π with respect to the effective torus action of T. The question of the characterization of complexity-one T-varieties Y_π is originally due to Klaus Altmann.

A lot is known about T-varieties and the combinatorial techniques for deformations of complexity-one T-varieties have been developed in [IV09]. Therefore the next statement is considered unfortunate.

Theorem 6.2.3 (with Donten-Bury, Escobar). *There exists no complexity-one T-variety Y_π.*

Proof. Suppose that Y_π is a complexity-one T-Variety. With the same assumptions from the previous proof, this implies that $|L'(\pi)| = m + n - k + 1$. Suppose that there are two components that share a row (or a column). This means that there exist two boxes $(x_1, y), (x_2, y) \in L'(\pi)$ and $(x, y) \in D(\pi)$. The entry of (x', y) cannot be 1 for $x' < x$, otherwise $(x, y) \notin D(\pi)$. Therefore, there exists a $y' < y$ such that the entry of (x_1, y') is 1. Since $(x_2, y) \in L(\pi)$, the boxes (z, y') for $x_1 \leq z \leq x_2$ and (x_1, z') for $y' \leq z' \leq y$ are all in $L'(\pi)$. This implies that two components are either contained in the same component (a contradiction) or the component of (x_2, y) shares both a row and a column with the component of (x_1, y). In the latter case, the total number of rows and columns containing $L'(\pi)$ is not sufficient to get a complexity-1 torus action.

Now assume that, there exists a connected component of $L(\pi)$ where the related connected component of $L'(\pi)$ has $m_i + n_i$ entries, i.e. one more entry than the toric case. This component of $L'(\pi)$ can only be in a shape of a hook plus one box. The reason is the following: The shape of $L(\pi)$ is a skew diagram, i.e. the set theoretic difference of the shape of two Young tableaux. If there exists more than one north west corner in this connected component of $L(\pi)$, it results in two connected components of $L'(\pi)$ sharing a row or a column. In the first figure, we observe that if the marked entry belongs to $L'(\pi)$, then its row and column entries also belong to it. Since we want only one more entry, the marked entry must be as in the following three figures. However, it is also not possible, since these entries must be then contained in $D(\pi)$.



$\square$

Theorem 6.2.4 (with Donten-Bury, Escobar). *There exist complexity-d T-Varieties Y_π for $d \geq 2$.*

Proof. Let Y_α be a complexity-i T-variety and Y_β be a complexity-j T-variety, for $\beta \in S_m$ and $\alpha \in S_n$. Consider the Rothe diagram of some $\pi \in S_{m+n}$ constructed as follows:

$$\begin{bmatrix} (0)_{n\times m} & D(\alpha) \\ D(\beta) & (0)_{m\times n} \end{bmatrix}$$

We observe that $\pi = [\pi(1), \ldots, \pi(m+n)] = [\beta_1 + n, \ldots, \beta_m + n, \alpha_1, \ldots, \alpha_n] \in S_{m+n}$. We conclude that Y_π is a complexity-$(i+j)$ T-variety, because $\sigma_\pi = \sigma_\alpha + \sigma_\beta$. Finally, since $Y_{[2,1,4,3]}$ is a complexity-2 and $Y_{[1,2,4,3]}$ is a complexity-3 T-variety, the statement follows. $\square$

The deformation theory of T-varieties with complexity higher than 2 has been not yet studied combinatorially. Therefore for now, we steer our study in the direction of Kazhdan-Lusztig varieties as in Section 6.4. Before that, we put the toric case under our microscope.

6.3 Rigidity of Toric Matrix Schubert Varieties

This section is devoted to the study of the detailed structure of σ_π for matrix Schubert varieties $\overline{X_\pi}$ where Y_π is toric. First, we investigate the first independent sets of G^π and then by studying the three-dimensional faces of σ_π, we present the conditions for rigidity of toric matrix Schubert varieties. Without loss of generality, we can assume that $L(\pi)$ is connected. Throughout this section, $\overline{X_\pi}$ stands for the toric matrix Schubert variety with the permutation $\pi \in S_N$. Also, the connected bipartite graph $G^\pi \subseteq K_{m,n}$ denotes the associated bipartite graph of $L(\pi)$ which was constructed in the previous section.

Lemma 6.3.1. *For any permutation $\pi \in S_N$,*

(1) *The one-sided first independent sets of G^π are $U_i \backslash \{u_i\}$ for all $u_i \in U_1$ and for $i = 1, 2$.*

(2) *The two-sided first independent sets are all maximal two-sided independent sets of G_π.*

Proof. By Theorem 6.2.2, $L'(\pi)$ is a hook. The entries of $L(\pi)$ form a shape of a Ferrer diagram, i.e. we have $\lambda_1 \geq \ldots \geq \lambda_t$ where λ_i denotes the number of boxes at ith row of $L(\pi)$. Consider the smallest rectangle containing $L(\pi)$ of a length m and of a width n. The removed edges of the bipartite graph $G_\pi \subseteq K_{m,n}$ are linked with the free entries in the rectangle. Let $(x_i, y_i) \in \operatorname{Ess}(\pi)$, equivalently let $(x_i, y_i) \in E(G_\pi)$. Then one obtains naturally that there exists a two-sided maximal independent set $C = C_1 \sqcup C_2 = \{x_i + 1, \ldots, m\} \sqcup \{y_{i-1} + 1, \ldots, n\}$ where $(x_{i-1}, y_{i-1}) \in \operatorname{Ess}(\pi)$ with $x_{i-1} > x_i$ and $y_{i-1} < y_i$. Then the neighbor sets are $N(C_1) = U_2 \backslash C_2 = \{1, \ldots, y_{i-1}\}$ and $N(C_2) = U_1 \backslash C_1 = \{1, \ldots, x_i\}$. Therefore, the entries for the induced subgraphs $\mathrm{G}[C_1 \sqcup N(C_1)]$ and $\mathrm{G}[C_2 \sqcup N(C_2)]$ also form a shape of a Ferrer diagram and $\mathrm{G}\{C\}$ is indecomposable. In particular, $U_i \backslash \{u_i\}$ cannot be contained in a two-sided independent set. Suppose that $\mathrm{G}\{U_i \backslash \{u_i\}\}$ has more than three components. Then as in Theorem 3.1.10, there exist two-sided first independent sets $C^i \in \mathcal{I}_G^{(1)}$ such that $\bigsqcup C_1^i = U_i \backslash \{u_i\}$ which is not possible. $\square$

Lemma 6.3.2. *There exist k two-sided first independent sets of G_π where $|\operatorname{Ess}(\pi)| = k + 1$. Moreover, if $k \geq 2$ and, C and C' are two-sided first independent sets of G_π, then the pair (C, C') is of Type* (i)*, i.e. $C_1 \subsetneq C_1'$ and $C_2' \subsetneq C_2$.*

Proof. Consider again the smallest rectangle containing $L(\pi)$ of a length m and of a width n. If there exists only one essential set of π, then $G_\pi = K_{m,n}$. Assume that there are more than one essential entry. Let (x_j, y_j) and (x_i, y_i) be two essential entries with $x_j > x_i$ and $y_j < y_i$. By Lemma 6.3.1, we obtain two first independent sets $C = \{x_i+1, \dots, m\} \sqcup \{y_{i-1}+1, \dots, n\}$ and $C' = \{x_j+1, \dots, m\} \sqcup \{y_{j-1}+1, \dots, n\}$ of G_π. We infer that $C_1 \subsetneq C_1'$ and $C_2' \subsetneq C_2$. □

Example 28. We observe in Figure 6.3 the entries of $L(\pi)$ for some toric variety Y_π. The blue entries are removed edges between some vertex sets C_1 and C_2. We observe that $C := C_1 \sqcup C_2$ is maximal. In particular, the green color represents the edges of the induced subgraph $\mathrm{G}[C_1 \sqcup N(C_1)]$ and the yellow color represents the edges of the induced subgraph $\mathrm{G}[C_2 \sqcup N(C_2)]$. The crossed entries are the entries of the essential set $\mathrm{Ess}(\pi)$. The entries with a dot are the entries of $L'(\pi)$.

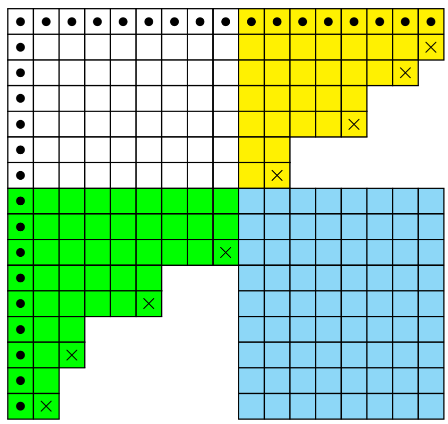


Figure 6.3: A representative figure of a first independent set of G_π associated to a toric matrix Schubert variety.

We covered the cases where there is one or there are two essential entries in Chapter 4. We state them in the following lemma.

Lemma 6.3.3. *Let $G_\pi \subseteq K_{m,n}$ be the associated connected bipartite graph to the toric variety Y_π.*

(1) *If $|\mathrm{Ess}(\pi)| = 1$, then the toric variety Y_π is isomorphic to $\mathrm{TV}(K_{m.n})$. In particular, Y_π is rigid if $m \neq 2$ and $n \neq 2$.*

(2) *If $|\mathrm{Ess}(\pi)| = 2$, then the toric variety Y_π is rigid if and only if $|C_1| \neq 1$ and $|C_2| \neq n-2$ or $|C_1| \neq m-2$ and $|C_2| \neq 1$.*

Proof. It follows from Theorem 4.1.4 and Theorem 4.3.3. □

From now on, we assume that $|\operatorname{Ess}(\pi)| \geq 3$. This means that we consider the associated connected bipartite graph $G_\pi \subsetneq K_{m,n}$ with $m, n \geq 4$. The following proposition is a result of Section 5.1. Nevertheless, we present a detailed proof in order to treat these results on a Rothe Diagram.

Proposition 6.3.4. *Let $A = U_1 \backslash \{i\}$, $B = U_2 \backslash \{j\}$, $C = C_1 \sqcup C_2$ be three types of first independent sets of the bipartite graph G_π.*

(1) *For any $A, B \in \mathcal{I}^{(1)}_{G_\pi}$, $(A, B) \in \mathcal{I}^{(2)}_{G_\pi}$.*

(2) *For any $C, C' \in \mathcal{I}^{(1)}_{G_\pi}$, $(C, C') \in \mathcal{I}^{(2)}_{G_\pi}$.*

(3) *$(A, A') \notin \mathcal{I}^{(2)}_{G_\pi}$ if and only if there exists a first independent set $U_1 \backslash \{i, i'\} \sqcup C_2$ where $C_2 \subsetneq U_2$ is some vertex set with $|C_2| \leq n - 2$.*

(4) *$(A, C) \notin \mathcal{I}^{(2)}_{G_\pi}$ if and only $C_1 = \{i\}$ or there exists $C' \in \mathcal{I}^{(1)}_{G_\pi}$ with $C_1 \backslash C'_1 = \{i\}$.*

Proof. **1.** Suppose that there exist a pair $(A, B) \notin \mathcal{I}^{(2)}_{G_\pi}$. Consider the intersection subgraph $\mathrm{G}\{A\} \cap \mathrm{G}\{B\}$ and assume that it has isolated vertices other than $\{i, j\}$. Consider the isolated vertices in U_1 other than $\{i\}$. This means that there exists a two-sided independent set consisting of these isolated vertices and the vertex set B which is impossible, because $B \in \mathcal{I}^{(1)}_{G_\pi}$. Now assume that $\mathrm{G}\{A\} \cap \mathrm{G}\{B\}$ consists of the isolated vertices $\{i, j\}$ and $k \geq 2$ connected bipartite graphs G_i. Let the vertex set of G_i consist of $V_i \subsetneq U_1$ and $W_i \subsetneq U_2$. Since $B \in \mathcal{I}^{(1)}_{G_\pi}$, there exist an edge $(i, w_i) \in E(G_\pi)$ for each $i \in [k]$ where $w_i \in W_i$. Symmetrically, since $A \in \mathcal{I}^{(1)}_{G_\pi}$, there exist an edge $(j, v_i) \in E(G_\pi)$ for each $i \in [k]$ where $v_i \in V_i$. However, then for $I \subsetneq [k]$, we obtain the two-sided maximal independent sets of form $\bigsqcup_{i \in I} V_i \sqcup (B \backslash (\bigsqcup_{i \in} W_i)$ which contradicts the construction of G_π.

2. Let (x_i, y_i) and (x_j, y_j) be two essential entries with $x_j > x_i$ and $y_j < y_i$, associated to two first independent sets C and C' in $\mathcal{I}^{(1)}_{G_\pi}$. We label the essential entries from the bottom of the diagram starting with (x_1, y_1) to the top ending with (x_k, y_k). It is enough to check if $\mathrm{G}[C'_1 \sqcup N(C'_1)] \cap \mathrm{G}[C_2 \cap N(C_2)]$ is connected. We observe that the edges of this graph are represented by the square with vertices $(x_i + 1, y_{j-1} + 1)$, $(x_i + 1, y_{i-1})$, $(x_j, y_{j-1} + 1)$, and (x_j, y_{i-1}), intersected with the diagram $D(\pi)$. This intersection is also a Ferrer diagram and connected.

3. Consider the intersection subgraph $\mathrm{G}\{A\} \cap \mathrm{G}\{A'\}$. Assume that it has only $\{i, i'\} \subsetneq U_1$ as isolated vertices and k connected bipartite graphs. Then, as in case **1**, there exist first independent sets C, C' with $C_1 \cap C'_1 = \emptyset$, which is impossible. Assume that it has the isolated vertices $\{i, i'\} \subsetneq U_1$ and $C_2 \subsetneq U_2$ with $|C_2| \leq n - 2$. Then $C := U_1 \backslash \{i, i'\} \sqcup C_2$ is maximal and thus a first independent set.

4. Suppose that $i \in C_1$ and $(A, C) \notin \mathcal{I}^{(2)}_{G_\pi}$. Consider the intersection subgraph $\mathrm{G}\{A\} \cap \mathrm{G}\{C\}$.

Similarly to last investigations, we conclude $\mathrm{G}[C_1 \sqcup N(C_1)]$ cannot admit $\{i\}$ as its only isolated vertex. If $C_1 = \{i\}$, then the intersection subgraph admits of $|N(C_1)| + 1$ isolated vertices and $\mathrm{G}[C_2 \sqcup N(C_2)]$. Assume that the intersection subgraph consists of the isolated vertex $\{i\} \subsetneq C_1$ and some vertex set $C_2' \subsetneq N(C_1)$. This means that $C' := C_1 \backslash \{i\} \sqcup C_2' \sqcup C_2$ is a maximal two-sided independent set. Hence $C' \in \mathcal{I}^{(1)}_{G_\pi}$. $\square$

Let us now eliminate the non-rigid cases of Y_π with non-simplicial three-dimensional faces of σ_π.

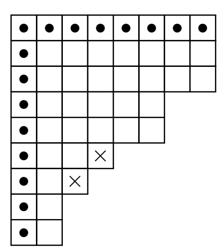
Lemma 6.3.5. *Assume that* $|\operatorname{Ess}(\pi)| \geq 3$.

(1) *Let* $C, C' \in \mathcal{I}^{(1)}_{G_\pi}$ *with* $C_1' \subsetneq C_1$ *and* $C_2 \subsetneq C_2'$. *If* $|C_1| - |C_1'| = 1$ *and* $|C_2'| - |C_2| = 1$, *then* Y_π *is not rigid.*

(2) *If there exists a first independent set* $C \in \mathcal{I}^{(1)}_{G_\pi}$ *with* $|C_1| = 1$ *and* $|C_2| = n - 2$ *or* $|C_1| = m - 2$ *and* $|C_2| = 1$, *then* Y_π *is not rigid.*

Proof. These are the cases from Proposition 5.2.2 (2) and Proposition 5.2.1 (2)(i). By Theorem 5.3.2, we conclude that Y_π is not rigid in these cases. $\square$

Example 29. Let $\pi = (21038569)(47) \in S_{10}$ and let us consider the diagram $L(\pi)$. We observe that the dotted entries form a hook and therefore Y_π is toric. Consider the first independent sets $C = \{8, 9\} \sqcup \{4, 5, 6, 7, 8\}$ and $C' = \{7, 8, 9\} \sqcup \{5, 6, 7, 8\}$ of the associated connected bipartite graph $G_\pi \subsetneq K_{9,8}$. By Lemma 6.3.5, $\langle \mathfrak{c}, \mathfrak{c}', e_7, f_4 \rangle$ spans a three-dimensional face of σ_π and hence Y_π is not rigid.



The cases in Lemma 6.3.5 are the only cases where σ_π has non-simplicial three-dimensional faces. We conclude this by examining the non 2-face pairs from Proposition 6.3.4 (3) and (4).

From now on, we assume that all three-dimensional faces of G_π are simplicial. In the next proposition, we examine the triples which do not form a three-dimensional face of σ_π.

Proposition 6.3.6. *Let* I *be a triple of first independent sets of* G_π *not forming a three-dimensional face. Assume that any pair of first independent sets of* I *forms a two-dimensional face. Then the triple* I *is*

(1) (A, A', A'') *if and only if there exists* $C \in \mathcal{I}^{(1)}_{G_\pi}$ *with* $C_1 = U_2\backslash\{i, i', i''\}$.

(2) (A, A', C) *if and only if* $C_1 = \{i, i'\}$ *or there exists* $C' \in \mathcal{I}^{(1)}_{G_\pi}$ *with* $C_1\backslash C'_1 = \{i, i'\}$.

Proof. The first case follows analogously as in the proof of Proposition 6.3.4 (3). Consider a triple of form (C, C', C'') with $C_1 \subsetneq C'_1 \subsetneq C''_1$ and $C''_2 \subsetneq C'_2 \subsetneq C_2$. Any such triple forms a 3-face, since the intersection graph $\mathrm{G}\{C\} \cap \mathrm{G}\{C'\} \cap \mathrm{G}\{C''\}$ is equal to

$$\mathrm{G}[C_1 \sqcup N(C_1)] \sqcup \mathrm{G}[(C'_1\backslash C_1) \sqcup (C_2\backslash C'_2)] \sqcup \mathrm{G}[(C''_1\backslash C') \sqcup (C'_2\backslash C''_2)] \sqcup \mathrm{G}[C''_2 \sqcup N(C''_2)].$$

For such triples containing both A and B, similar to the arguments in the proof of Proposition 6.3.4 (1), we conclude that they form 3-faces. Finally, consider the triple (A, A', C). Since $(A, A') \in \mathcal{I}^{(2)}_{G_\pi}$, i and i' cannot be both in $N(C_2)$. Assume that $i \in C_1$ and $i' \in N(C_2)$. Since (A, C) and (A', C) form 2-faces, the triple (A, A', C) forms a 3-face. Hence we have that $\{i, i'\} \subseteq C_1$. The statement follows by the analysis similar to that in the proof of Proposition 6.3.4 (4). □

Remark 11. In addition to the triple in Proposition 6.3.6, the triples of first independent sets of G_π, containing the pairs in Proposition 6.3.4 (3) and (4) do not form a three-dimensional face of σ_π.

The following theorem classifies the rigid toric matrix Schubert varieties.

Theorem 6.3.7. *The toric variety* $Y_\pi = \mathrm{TV}(\sigma_\pi)$ *is rigid if and only if the three-dimensional faces of* σ_π *are all simplicial.*

Proof. We have proven the statement for $|\operatorname{Ess}(\pi)| = 1, 2$. We prove it now for $|\operatorname{Ess}(\pi)| \geq 3$. We examine the non 2-faces pairs from Proposition 6.3.4 and non 3-face triples from Proposition 6.3.6.

1. Suppose that (e_1, e_2, e_3) does not span a 3-face and (e_1, e_2), (e_1, e_3) and (e_2, e_3) do span 2-faces. By Proposition 6.3.6, there exists a first independent set $C \in \mathcal{I}^{(1)}_{G_\pi}$ with $C_1 = U_1\backslash\{1, 2, 3\}$ and $|C_2| \leq n - 2$. Assume that $\overline{e_1}$, $\overline{e_2}$, and $\overline{e_3}$ are vertices in $Q(R)$ for some deformation degree $R \in M \cong \mathbb{Z}^{m+n}/\overline{(1, -1)}$. Let $a \in \sigma^{(1)}_\pi$ be an extremal ray. Since $(\mathfrak{a}, e_i, e_j)$ spans a 3-face of σ_π for every $i, j \in [3]$ and $i \neq j$, we are left with showing that there exists no such $\overline{\mathfrak{a}} \in Q(R)$. However, even though we have that $R_i \leq 0$, for every $i \in [m + n]\backslash\{1, 2, 3\}$, $\overline{\mathfrak{c}} \in Q(R)$.

2. Suppose that $(e_1, e_2, \mathfrak{c})$ does not span a 3-face and (e_1, e_2), $(e_1, \mathfrak{c})$ and $(e_2, \mathfrak{c})$ do span 2-faces. By Proposition 6.3.6, $|C_1| = \{1, 2\}$ or there exists $C' \in \mathcal{I}^{(1)}_{G_\pi}$ such that $C_1\backslash C'_1 = \{1, 2\}$. Assume that $\overline{e_1}$, $\overline{e_2}$, and $\overline{\mathfrak{c}}$ are vertices in $Q(R)$ for some deformation degree $R \in M$. If $|C_1| = \{1, 2\}$, then there exists $b \in N(C_1)$ such that $\overline{\mathfrak{b}} \in Q(R)$ is not a lattice vertex or there exist at least three vertices $b_i \in N(C_1)$ such that $\overline{\mathfrak{b}_\mathfrak{i}}$ is a lattice vertex in $Q(R)$. If

$C_1 \backslash C_1' = \{1,2\}$, then either $\overline{\mathfrak{c}'} \in Q(R)$ or $\overline{\mathfrak{b}} \in Q(R)$ for $b \in C_2' \backslash C_2$.

3. Suppose that (e_1, e_2) does not span a 2-face and $\overline{e_1}$ and $\overline{e_2}$ are in $Q(R)$ for some deformation degree $R \in M$. Then there exists a first independent set $C = C_1 \sqcup C_2 \in \mathcal{I}^{(1)}_{G_\pi}$ with $C_1 = U_1 \backslash \{1,2\}$ and $2 \le |C_2| \le n-2$. Remark that for any other two-sided first independent set $C' = C_1' \sqcup C_2' \in \mathcal{I}^{(1)}_{G_\pi}$, the pair $(C, C') \in \mathcal{I}^{(2)}_{G_\pi}$ is of Type (ii), i.e. $C_1' \subsetneq C_1$ and $C_2 \subsetneq C_2'$. Assume that there exist k vertices $\overline{f_j}$ in $Q(R)$ where $j \in [k] \subseteq [n]$. If $k = 0$, then $\mathfrak{c}$ is a non-lattice vertex in $Q(R)$. If $k = 1$, then $\overline{f_1}$ is a non-lattice vertex in $Q(R)$. If $k \ge 3$, there can be at most one non 2-face pair say (f_1, f_2). However, the other triples of type (A, B, B') not containing both $U_2 \backslash \{1\}$ and $U_2 \backslash \{2\}$ form 3-faces.

Suppose now that $\overline{\mathfrak{c}^i} \in Q(R)$ is a lattice vertex. We can assume that there exists only one such extremal ray $\mathfrak{c}^i$, since any triple of type $(C, C', U_1 \backslash \{1\})$ and $(C, C', U_1 \backslash \{2\})$ form 3-faces. Moreover there exists at most one $f_{j'}$ such that $(f_{j'}, \mathfrak{c}^i)$ do not span a two-dimensional face. Hence we obtain that $V(R)/\mathbb{C}(1,1) = 0$ for this deformation degree $R \in M$. It leaves us to check the case where $k = 2$. In this case, if the pair $\{f_1, f_2\}$ do not span a 2-face σ_π, then there exists a first independent set $C'' = C_1'' \sqcup C_2'' \in \mathcal{I}_{G_\pi}$ with $C_2'' = U_2 \backslash \{1,2\}$ and $|C_1''| \le m-3$. Then the only other vertex in $Q(R)$ is $\overline{\mathfrak{c}}$ and it is not a lattice vertex. Furthermore, $(e_i, f_j, \mathfrak{c})$ spans three-dimensional faces of σ_π for $i \in [2]$ and $j \in [2]$. Last, assume that (f_{j_1}, f_{j_2}) spans a 2-face of σ_{G_π}. As in the case where $k \ge 3$, it is enough to check the cases for only one vertex $\overline{\mathfrak{c}^i}$ in $Q(R)$. There exists at most one non 2-face pair containing $\mathfrak{c}^i$, say $(f_{j_1}, \mathfrak{c})$. But then $(\mathfrak{c}_j, f_{j_2}, e_1)$ is a 3-face of σ_{G_π}.

4. Lastly, suppose that $\{\mathfrak{c}, e_i\}$ does not span a 2-face and $\overline{\mathfrak{c}}$ and $\overline{e_i}$ are in $Q(R)$ for some deformation degree $R \in M$. Remark here that we excluded the cases where there exist non-simplicial three-dimensional faces. This means $\mathfrak{c}$ and e_i forms 2-faces with each extremal ray of σ_π. Assume that there exist more than three vertices in $Q(R)$ other than $\overline{\mathfrak{c}}$ and $\overline{e_i}$. We examined the cases where non 3-face (e_1, e_2, e_3) appears and where non 2-face (e_1, e_2) appears in $Q(R)$. Therefore we assume that there exists another non 2-face pair, say $(\mathfrak{c}^*, e_j)$. But, since $\mathfrak{c}^*$ and e_j also forms 2-faces with each extremal ray of σ_π, it is enough to check the cases where there exist less than five vertices in $Q(R)$.
Let us first consider the case where there exist exactly two more vertices in $Q(R)$ other than $\overline{\mathfrak{c}}$ and $\overline{e_i}$. We first start with the non 2-face pair (A, C) where $C_1 = \{m\}$ and $A = U_1 \backslash \{m\}$. Then there exists a non-lattice vertex $\overline{j} \in Q(R)$ where $j \in U_2 \backslash C_2$. We observe that there exists no other first independent set $C' \in \mathcal{I}^{(1)}_{G_\pi}$ such that $C_2 \subsetneq C_2'$. Therefore it is impossible that there exists another non 2-face pair containing $\overline{\mathfrak{c}'}$.

In the other case where $(\mathfrak{c}, e_i)$ does not span a 2-face, there exists an extremal ray, say $\mathfrak{c}'$ such that $\mathfrak{c}' = e_i + \mathfrak{c} - \sum_{j \in C_2' \backslash C_2} f_j$. The vertex $\overline{\mathfrak{c}'}$ is in $Q(R)$, unless there exists $\overline{f_j} \in Q(R)$ where $j \in C_2' \backslash C_2$. This vertex cannot be $\overline{f_j}$ with $\{j\} = C_2' \backslash C_2$, because then $(\mathfrak{c}, \mathfrak{c}', e_i, f_j)$ spans a

3-face. Hence $\overline{\mathfrak{c}'}$ is one of these two vertices. It remains to check the case where other vertex is $\overline{e_{i-1}}$. Then, there exists a first independent set $C'' \in \mathcal{I}^{(1)}_{G_\pi}$. We have that $\overline{\mathfrak{c}''} \notin Q(R)$ if and only if there exists $\overline{f'_j}$ with $j' \in C''_2 \backslash C'_2$, by the same reasoning as before. Lastly, assume that there exists only one lattice vertex in $Q(R)$ other than $\overline{\mathfrak{c}}$ and $\overline{e_i}$. We observe that $\overline{\mathfrak{c}'}$ is a lattice vertex of $Q(R)$ if there exist some $\overline{f_j} \in Q(R)$ where $j \in C'_2 \backslash C_2$. Therefore we assume that this lattice vertex is $\overline{f_j}$ for some $j \in [n]$. In order to obtain $\langle R, \mathfrak{c}' \rangle = 0$, we must have $\{j\} = C'_2 \backslash C_2$, but this implies that $(\mathfrak{c}, \mathfrak{c}', e_i, f_j)$ is a 3-face of σ_π.

□

We interpret the rigidity of Y_π by giving certain conditions on the Rothe diagram.

Corollary 6.3.8. *Let* $\mathrm{Ess}(\pi) = \{(x_i, y_i) \mid x_1 < \ldots < x_{k+1}$ *and* $y_{k+1} < \ldots < y_1\}$ *with* $k \geq 3$. *Then the toric variety* Y_π *is rigid if and only if*

- $(x_1, y_1) \neq (2, n)$ *and* $(x_{k+1}, y_{k+1}) \neq (m, 2)$
- *for any* $i \in [k]$,$(x_i, y_i) \neq (x_{i+1} - 1, y_{i+1} + 1)$.

Proof. It follows by Lemma 6.3.5 which characterizes the non-simplicial three-dimensional faces. □

Example 30. In the figure of Example 29, consider the essential entries (x_3, y_3) and (x_4, y_4) which are associated to the first independent sets C' and C. We obtain that $(x_3, y_3) = (6, 4) = (x_4 - 1, y_4 + 1)$. Therefore Y_π is not rigid.

6.4 Future Work

This section is an announcement for the going-on joint-work with Maria Donten-Bury and Laura Escobar.

Calculating the p-divisor of the T-Variety Y_π

As we studied shortly in Section 6.2, there exist matrix Schubert T-varieties of complexity-d, where $d \geq 2$. T-varieties naturally appear when one investigates torus invariant deformations of a toric variety. In this case, the total space has an effective torus action with positive complexity. They have also a nice analogous combinatorial construction to affine toric varieties. One can find a dense survey about T-varieties by Altmann, Ilten, Petersen, Suess and Vollmert in [AIPSV12]. We touch only a few aspects of the theory in this section in order to present our future work.

An affine T-variety X is in one to one correspondence with a so-called p-divisor $\mathcal{D}$. It is a formal sum $\mathcal{D} = \sum_i \triangle_i D_i$ where D_i is an effective Cartier divisor on the Chow quotient of X by the torus T and $\triangle_i$ is a polyhedron in the one-parameter subgroups lattice of T with the same tail cone. Also, one gets the complexity of the T-variety as the dimension of the Chow quotient. Our aim is to study matrix Schubert varieties of higher complexity. In particular, we plan to start classifying the complexity 2 cases of Y_π and write a script in `Singular` or in `Macaulay` to calculate the associated p-divisor for Y_π.

Complexity-1 Kazhdan-Lustig Varieties as subsets of matrix Schubert Varieties

Although one does not have a complexity one Schubert matrix variety, one can study the Kazhdan Lusztig varieties of complexity one which are subsets of matrix Schubert Varieties. We follow the definition of [WY12].

Definition 6.4.1. Let $\omega, \pi \in S_N$ be two permutations. Assume that $\omega \leq \pi$ where $\leq$ is the Bruhat order. Denote $\Omega_0^\omega \subseteq \mathbb{C}^{|D(\omega)|}$ as the $N \times N$ matrices such that

$$z_{i,j} = \begin{cases} 1, & \text{if } \omega(j) = i \\ 0, & \text{if } (i,j) \notin D(\omega) \end{cases}$$

We define the *KL-variety (Kazhdan-Lusztig variety)* as

$$\overline{X_\pi}^\omega := \Omega_0^\omega \cap \overline{X_\pi}.$$

Note that the assumption $\omega \leq \pi$ is required to obtain a non-empty KL-variety. Explicitly, the definition means that one imposes the conditions of Fulton's essential set as in Theorem 6.1.6 to the matrices in Ω_0^ω. The reason for prefering the name Kazhdan-Lusztig is because they are isomorphic to the Kazhdan-Lusztig variety $\overline{B\pi B/B} \cap B_{-}\omega B/B$ in the flag variety FL_n, This implies that

$$\dim(\overline{X_\pi}^\omega) = |D(\omega)| - |D(\pi)|$$

We observe in the next example the existence of complexity one KL-varieties. We can utilize the graphs again to determine the complexity of KL-varieties.

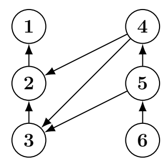
Example 31. Let $\pi = (15)(23)(46) \in S_6$ and $\omega = (16)(25)(34) \in S_6$. The dimension of the KL-variety $\overline{X_\pi^\omega}$ is $l(\omega) - l(\pi) = 15 - 9 = 6$. We impose the inequalities from the essential set of $D(\pi)$ and obtain the following matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & z_{1,5} & 1 \\ 0 & 0 & z_{2,3} & z_{2,4} & 1 & 0 \\ 0 & z_{3,2} & z_{3,3} & 1 & 0 & 0 \\ 0 & z_{4,2} & 1 & 0 & 0 & 0 \\ z_{5,1} & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $z_{3,2}z_{2,4} + z_{4,2}z_{2,3} - z_{2,4}z_{3,3}z_{4,2} = 0$. We consider the restriction of the torus action from $\overline{X_\pi}$ to $\overline{X_\pi^\omega}$.

$$\begin{array}{rcl} (\mathbb{C}^*)^6 \times (\mathbb{C}^*)^6 \times \overline{X_\pi^\omega} & \longrightarrow & \overline{X_\pi^\omega} \\ (t_i, u_j).z_{i,j} & \mapsto & t_i z_{i,j} u_j^{-1} \end{array}$$

Since, for $i + j = 7$, $z_{i,j} = 1$, we obtain that $t_i = u_{6-i+1}^{-1}$. Furthermore, since we want an effective torus action, we quotient out the case where $t_1 = \ldots = t_6$ from the torus $(\mathbb{C}^*)^6$. Therefore, the five dimensional torus acts effectively on KL-variety $\overline{X_\pi^\omega}$, i.e. $\overline{X_\pi^\omega}$ is a complexity-one T-variety with respect to the action of the torus $(\mathbb{C}^*)^5$ as described. In particular, this answer can be achieved by looking at the following simple directed graph and the rank of its incidence matrix. We denote this graph by G_π^ω.

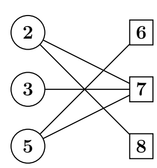


The rank of the incidence matrix of an simple directed graph is equal to the number of its vertices minus its connected component number. Therefore, we obtain the dimension of the effective torus action by calculating the rank of the incidence matrix of G_π^ω, which is equal to five. We observe that in the next example there are also toric KL-varieties.

Example 32. Let $\pi = (14785) \in S_8$ and $\omega = (16385274) \in S_8$. After imposing the conditions from essential set of $\overline{X_\pi}$ in $\Omega_0^{[6,7,8,1,2,3,4,5]}$, one gets the matrix as

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & z_{2,2} & z_{2,3} \\ 0 & z_{3,2} & 0 \\ 0 & 0 & 0 \\ z_{5,1} & z_{5,2} & 0 \end{bmatrix}$$

The KL-variety $\overline{X_\pi}^{(k,n)}$ is isomorphic to $\mathbb{C}^5$. After restricting the torus action from $\overline{X_\pi}$ as in Example 31, we obtain the bipartite directed graph G_π^ω. Since it is bipartite, the cone generated by the columns of the incidence matrix of G_π^ω is GL-equivalent to the dual edge cone of G_π^ω with directions omitted.



The dimension of the effective torus action on the KL-variety $\overline{X_\pi^\omega}$, equivalently the dimension of the dual edge cone of the bipartite graph above is five. Therefore, $\overline{X_\pi^\omega}$ is a toric variety.

First of all, we aim to classify all toric KL-varieties. For this, as we have done in the examples, we plan to use the graph G_π^ω. Note that $\overline{X_\pi^\omega}$ is a toric KL-variety with respect to the torus action coming from the matrix Schubert variety $\overline{X_\pi}$ if and only if

$$\dim(\overline{X_\pi^\omega}) = |D(\omega)| - |D(\pi)| = N - \#(\text{connected components of } G_\pi^\omega)$$

where $|N| = V(G_\pi^\omega)$. We observe that if the graph G_π^ω is a forest, then $\overline{X_\pi^\omega}$ is toric. The reason is that the dimension of the variety is the number of the edges of G_π^ω, since the graph admits no closed walks. However there are toric KL-varieties arising from graphs which are not forests. Next, we plan to classify the complexity-one cases and calculate their p-divisors. Once we obtain the classification, we will study the first order deformations of toric and complexity-one KL-varieties.

Chapter 7

Programming

Since our work on deformations is purely combinatorial, we chose `Polymake` [GJ00] to compute the face structure of an edge cone. We use the Fulton application within `Polymake` in order to calculate the dimension of the vector space T^1 of $\mathrm{TV}(G)$. Most functions we need are part of the `Polymake` and `Singular` core. The additional functionality deals with the face structure of the edge cone. If it detects a non-simplicial three-dimensional face of the given edge cone, the function returns early on the terms of Theorem 5.3.2. Although this is only for edge cone inputs, the subsequent code works for any toric variety smooth in codimension two. Here, the function asks for a deformation degree $R \in M$ and gives the skeleton of the crosscut picture $Q(R)$. Using `Singular`, we calculate algebraically the rigidity of $\mathrm{TV}(G)$.

7.1 Executing examples

A non-rigid bipartite graph

Let us consider the bipartite graph $G \subsetneq K_{4,4}$ with exactly one two-sided first independent set $A \in \mathcal{I}_G^{(1)}$ with $|A_1| = 1$ and $|A_2| = 2$.

```
polytope > $c = new Cone(INPUT_RAYS
    =>[[1,0,0,0,0,0,1,0],[1,0,0,0,0,0,0,1],[0,1,0,0,1,0,0,0],
[0,1,0,0,0,1,0,0],[0,1,0,0,0,0,1,0],[0,1,0,0,0,0,0,1],[0,0,1,0,1,0,0,0],
[0,0,1,0,0,1,0,0],[0,0,1,0,0,0,1,0],[0,0,1,0,0,0,0,1],[0,0,0,1,1,0,0,0],
[0,0,0,1,0,1,0,0],[0,0,0,1,0,0,1,0],[0,0,0,1,0,0,0,1]]);

polytope > is_tv_graph_rigid($c);

a_0 = 1 0 0 0 0 0 0 0

a_1 = 0 0 0 0 0 0 1 0

a_2 = 0 1 1 1 -1 -1 0 0
```

```
a_3 = 1 1 1 1 −1 −1 −1 0

a_4 = 0 0 0 0 1 0 0 0

a_5 = 0 1 0 0 0 0 0 0

a_6 = 0 0 0 0 0 1 0 0

a_7 = 0 0 1 0 0 0 0 0

a_8 = 0 0 0 1 0 0 0 0

There exist non−simplicial 3−faces:
{0 1 2 3}
TV(G) is not rigid.
```

The numbers in the set $\{0,1,2,3\}$ present the extremal rays $\{a_0,a_1,a_2,a_3\}$ printed in the beginning. We have shown in Theorem 5.3.2 that if the edge cone σ_G admits a non-simplicial three-dimensional face, then $\mathrm{TV}(G)$ is rigid. The function `is_tv_graph_rigid()` utilizes this fact. In the case where σ_G does not admit any non-simplicial face, the function asks for a deformation degree $R \in M$ and determines if the homogeneous piece $T^1(-R)$ is equal to zero. This part of the function works for any toric variety in codimension two, not just for the ones associated to bipartite graphs.

Complete bipartite graph with one edge removal

In this example, we will consider the bipartite graph $G \subsetneq K_{4,4}$ with one edge removal. Since the edge cone σ_G does not have any non-simplicial three-dimensional face, the function asks for a deformation degree. One has to keep in mind the chosen lattice M while inputting the deformation degree, e.g. in our case $M \cong \mathbb{Z}^8/\overline{(1,-1)}$.

```
polytope > $c = new Cone(INPUT_RAYS
    =>[[1,0,0,0,0,1,0,0],[1,0,0,0,0,0,1,0],[1,0,0,0,0,0,0,1],
[0,1,0,0,1,0,0,0],[0,1,0,0,0,1,0,0],[0,1,0,0,0,0,1,0],
[0,1,0,0,0,0,0,1],[0,0,1,0,1,0,0,0],[0,0,1,0,0,1,0,0],
[0,0,1,0,0,0,1,0],[0,0,1,0,0,0,0,1],[0,0,0,1,1,0,0,0],
[0,0,0,1,0,1,0,0],[0,0,0,1,0,0,1,0],[0,0,0,1,0,0,0,1]]);

polytope > is_tv_graph_rigid($c);
a_0 = 0 0 0 0 0 1 0 0

a_1 = 1 0 0 0 0 0 0 0

a_2 = 0 1 1 1 −1 0 0 0
```

```
a_3 = 0 0 0 0 0 0 1 0

a_4 = 1 1 1 1 -1 -1 -1 0

a_5 = 0 0 0 0 1 0 0 0

a_6 = 0 1 0 0 0 0 0 0

a_7 = 0 0 1 0 0 0 0 0

a_8 = 0 0 0 1 0 0 0 0

Enter 8 coordinates of a deformation degree R
1
0
0
0
-1
1
1
0
T1(-[1 0 0 0 -1 1 1 0]) is equal to zero
```

The idea here is to create a new graph, say $G(R)$, for the given deformation degree $R \in M$. For this, we first eliminate the extremal rays in $[R < 0]$. The vertices of $G(R)$ are identified with two-dimensional faces of $Q(R)$ and compact edges in $Q(R)$ which are not contained in any of these two-dimensional faces. These elements are collected with the function `crosscut_skeleton()`. We add an edge to $G(R)$, if two 2-faces are connected to each other by a common compact edge. In the other cases, we look at the vertex in $Q(R)$ which is connecting two faces. We add an edge to $G(R)$, if this vertex is a non-lattice vertex in $Q(R)$. The new graph $G(R)$ is produced by the function `crosscut_graph()`. In the end, if the new graph $G(R)$ is connected, the function `is_tv_graph_rigid()` returns that $T^1(-R) = 0$. An interactive and representative picture of $Q(R)$ is produced by the function `crosscut_picture()` by using the application "topaz" within `Polymake`.

```
polytope > $cdual = new Cone(INPUT_RAYS=>$c->FACETS);

polytope > $hasse = $cdual->HASSE_DIAGRAM;

polytope > $def_degree = new Vector<Rational>(1,0,1,0,-2,1,0,3);

polytope > print crosscut_skeleton($cdual,$hasse,$def_degree);
{0 1 4}{0 1 7}{0 2 4}{0 2 7}{0 4 7}{1 4 7}{2 4 7}

polytope > print crosscut_graph($cdual,$hasse,$def_degree)->EDGES;
{0 1}
```

```
{0 2}
{1 3}
{2 3}
{0 4}
{1 4}
{2 4}
{3 4}
{0 5}
{1 5}
{2 5}
{4 5}
{0 6}
{2 6}
{3 6}
{4 6}
{5 6}

polytope > application "topaz";

topaz > crosscut_picture($cdual,$hasse,$def_degree);
```

Cone over a Segre embedding

Let us consider the first rigid example, i.e. the cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^5$. Equivalently, this is the toric variety $\mathrm{TV}(K_{2,3})$. We calculate the rigidity by using `Singular`.

```
polytope > application "fulton";

fulton > $c = new Cone(INPUT_RAYS=>[[1,0,1,0,0],[1,0,0,1,0],
[1,0,0,0,1],[0,1,1,0,0],[0,1,0,1,0],[0,1,0,0,1]]);

fulton > T1_module($c);

Edge ideal generators related to G

toric_ideal[1]=-x_0*x_2+x_1*x_5
toric_ideal[2]=-x_0*x_3+x_1*x_4
toric_ideal[3]=-x_2*x_4+x_3*x_5

Generators of the module of infinitisimal deformations of TV(G)

M[1]=gen(6)
M[2]=gen(5)
M[3]=gen(4)
M[4]=gen(3)
M[5]=gen(2)
M[6]=gen(1)
```

```
The dimension of M as a module

-1

The dimension of M as a vector space

0
```

7.2 Code

7.2.1 Polymake

```
use application 'polytope';
use Array::Utils qw(intersect);

sub is_tv_graph_rigid {
    my($cone) = @_;
    my $conedual = new Cone(INPUT_RAYS=>$cone->FACETS);
    my $rays = $conedual->RAYS;
    for (my $i=0; $i < scalar(@{$rays}); $i++) {
        print "a_$i" . " = " . "$rays->[$i]\n\n";
    }
    my $hasse = $conedual->HASSE_DIAGRAM;
    my @bad_threefaces = get_threefaces($hasse, 1);
    if (@bad_threefaces) {
        print "There exist non-simplicial 3-faces: \n";
        print get_threefaces($hasse,1);
        print "\nTV(G) is not rigid.";
        return;
    }
    my $n_v = $cone->AMBIENT_DIM;
    print "Enter " . ($n_v) . " coordinates of a deformation degree R \n";
    my @input = ();
    for (my $i=0; $i < $n_v; $i++) {
    my $in = <STDIN>;
    push @input, $in;
    }
    my $def_degree = new Vector(@input);
    my $crosscut_graph = crosscut_graph($conedual, $hasse, $def_degree);
    my $cnncted = $crosscut_graph->CONNECTED;
    if ($cnncted == 1) {
        print "T1(-[" . $def_degree . "]) is equal to zero";
    }
    else {
        print "T1(-[" . $def_degree . "]) is not equal to zero. Therefore TV(G
            ) is not rigid."
```

```
    }
}

sub crosscut_graph {
    my($conedual, $hasse, $def_degree) = @_;
    my @edges = ();
    my @skeleton = crosscut_skeleton($conedual, $hasse, $def_degree);
    for (my $i=0; $i < scalar(@skeleton); $i++) {
        for (my $j=$i + 1; $j < scalar(@skeleton); $j++) {
            my $v = $skeleton[$i];
            my $w = $skeleton[$j];
            my @intersection = intersect(@{$v}, @{$w});
            my $interray = new Vector<Rational>($conedual->RAYS->[
                $intersection[0]]);
            if (scalar(@intersection) == 2){
                push @edges, [$i, $j];
            }
            elsif (scalar(@intersection) == 1 && $interray * $def_degree > 1){
                push @edges, [$i, $j];
            }
        }
    }
    my $newgraph = graph_from_edges([@edges]);
    return $newgraph;
}

sub crosscut_picture {
    my($conedual,$hasse,$def_degree) = @_;
    my @skeleton = crosscut_skeleton($conedual,$hasse,$def_degree);
    my $simplical_complex = new SimplicialComplex(INPUT_FACES=>[@skeleton]);
    $simplical_complex->VISUAL;
    graphviz($simplical_complex->VISUAL_FACE_LATTICE);
}

sub crosscut_skeleton {
    my($conedual, $hasse, $def_degree) = @_;
    my @skeleton = ();
    my @good_threefaces = get_threefaces($hasse, 0);
    foreach my $gtf (@good_threefaces) {
        my $gen1vec = new Vector<Rational>($conedual->RAYS->[$gtf->[0]]);
        my $gen2vec = new Vector<Rational>($conedual->RAYS->[$gtf->[1]]);
        my $gen3vec = new Vector<Rational>($conedual->RAYS->[$gtf->[2]]);
        my $scproduct1 = $gen1vec * $def_degree;
        my $scproduct2 = $gen2vec * $def_degree;
        my $scproduct3 = $gen3vec * $def_degree;
        if ($scproduct1 >= 1 && $scproduct2 >= 1 && $scproduct3 >= 1) {
            push @skeleton, $gtf;
        }
    }
```

```
    my @twofaces = get_twofaces($hasse);
    foreach my $twf (@twofaces) {
        my $gen1vec = new Vector<Rational>($conedual->RAYS->[$twf->[0]]);
        my $gen2vec = new Vector<Rational>($conedual->RAYS->[$twf->[1]]);
        my $scproduct1 = $gen1vec * $def_degree;
        my $scproduct2 = $gen2vec * $def_degree;
        if ($scproduct1 >= 1 && $scproduct2 >= 1) {
            my @subset = grep(/$twf->[0]/ && /$twf->[1]/, @skeleton);
            if(!@subset) {
                push @skeleton, $twf;
            }
        }
    }
    return @skeleton;
}

sub get_threefaces {
    my($hasse, $bad_flag) = @_;
    my @threefaceeics = (@{$hasse->nodes_of_dim(3)});
    my @threefaces = ();
    foreach my $tfi (@threefaceeics) {
        my $tf = $hasse->FACES->[$tfi];
        if (!$bad_flag && scalar(@{$tf})==3) {
            push @threefaces, $tf;
        } elsif ($bad_flag && scalar(@{$tf})>3) {
            push @threefaces, $tf;
        }
    }
    return @threefaces;
}

sub get_twofaces {
    my($hasse) = @_;
    my @twofaceeics = (@{$hasse->nodes_of_dim(2)});
    my @twofaces = ();
    foreach my $tf (@twofaceeics) {
        push @twofaces, $hasse->FACES->[$tf];
    }
    return @twofaces;
}
```

7.2.2 Interfacing Singular

This is the script which investigates the infinitesimal deformation of an affine toric variety algebraically. It interfaces `Singular` via application "Fulton".

```
use application 'polytope';
use application 'fulton';
load_singular_library("sing.lib");
```

```
sub T1_module {
    my($cone) = @_;
    # Calculate the toric ideal and its generators
    my $toric_ideal = $cone->TORIC_IDEAL;
    my $ti_gens = $toric_ideal->GENERATORS;
    my $cmd = "ideal toric_ideal =".join(",",@$ti_gens).";";

    # Use groebner basis to set up the polynomial ring in Singular
    my $G = $toric_ideal->add("GROEBNER", ORDER_NAME=>"dp");
    my $Gbasis = $G->BASIS;

    print "Edge ideal generators related to G\n\n";
    singular_eval($cmd);
    singular_eval("toric_ideal;");

    print "\nGenerators of the module of infinitisimal deformations of TV(G)\n
        \n";
    singular_eval("module M = T_1(toric_ideal);");
    singular_eval("M;");

    print "\nThe dimension of M as a module\n\n";
    singular_eval("dim(M);");

    print "\nThe dimension of M as a vector space\n\n";
    singular_eval("vdim(M);");
}
```

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Index of notation

General Terms

Specific for Bipartite Graphs

I_G	The edge ideal of a graph G	1
$\mathrm{TV}(G)$	The associated affine toric variety to graph G	1
$K_{m,n}$	The complete bipartite graph	1
$\sigma_G^\vee$	The dual edge cone of a graph G	2
σ_G	The edge cone of a graph G	2
U_i	Disjoint sets of a bipartite graph $G \subseteq K_{m,n}$	2
$\overline{(1,-1)}$	$\sum_{i=1}^m e_i - \sum_{j=1}^n f_j$, the minimal face of $\sigma_G \subsetneq \mathbb{Q}^{m+n}$	11
$N = \mathbb{Z}^{m+n}/\overline{(1,-1)}$	Lattice for the bipartite graph case	11
$M = \mathbb{Z}^{m+n} \cap \overline{(1,-1)}^\perp$	Dual lattice for the bipartite graph case	11
$N(A)$	The neighbor set of $A \subseteq V(G)$ is the set of vertices adjacent to some vertex in A	14
H_A	The supporting hyperplane in $M_\mathbb{Q}$ of an edge cone σ_G associated to an independent set A	14
$\mathrm{G}[S]$	The induced subgraph of $S \subseteq V(G)$ formed from the vertices of S and all of the edges connecting pairs of these vertices	14
$\mathrm{G}\{A\}$	The associated bipartite subgraph to the independent set $A = A_1 \sqcup A_2$, i.e. $\mathrm{G}[A_i \sqcup N(A_i)] \sqcup \mathrm{G}[(U_i \backslash A_i) \sqcup (U_j \backslash N(A_j))]$	16
$\mathcal{I}_G^{(*)}$	The set consisting of two-sided maximal independent sets and one-sided independent sets $U_i \backslash \{\bullet\}$ not contained in any two-sided maximal independent set	18
$\mathcal{I}_G^{(1)}$	The set of indecomposable elements of $\mathcal{I}_G^{(*)}$	20
$\mathcal{V}\mathfrak{al}_A$	The degree sequence of the bipartite graph $\mathrm{G}\{A\}$, i.e. the tuple of the degrees of its vertices	21
$\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$	The associated extremal rays of the first independent sets A, B, C	21
$\mathcal{I}_G^{(d)}$	The set of tuples of first independent sets forming a d-dimensional face of σ_G	42
$\overline{X_\pi}$	The matrix Schubert variety of permutation π	57
$D(\pi)$	The Rothe diagram of permutation π	58
$L(\pi)$	$NW(\pi) - \mathrm{dom}(\pi)$	58
$L'(\pi)$	$L(\pi) - D(\pi)$	58
σ_π	The edge cone of G^π	63
G^π	The associated bipartite graph to $L(\pi)$	63

List of Figures

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Last but not least, I would like to thank my lovely family and my beloved ones for always believing in me. Even though they did not have a clue what I was working on, they never had a doubt.

Zusammenfassung

Das Thema dieser Dissertation ist die Starrheit torischer Varietäten, die mit bipartiten Graphen assoziiert sind. Jeder bipartite Graph $G \subseteq K_{m,n}$ ist assoziiert zu einer affinen normalen torischen Varietät $\mathrm{TV}(G) := \mathrm{TV}(\sigma_G)$, deren sogenannter Kantenkegel $\sigma_G^\vee$ an seinen Rändern konstruiert wird. Die Deformationstheorie von affinen normalen torischen Varietäten wurde von K. Altmann untersucht. Wir wenden seine kombinatorische Formel für Deformationen erster Ordnung auf die torische Varietät $\mathrm{TV}(G)$ an. Eine affine Varietät bezeichnet man als starr, wenn sie keine nichttrivialen infinitesimalen Deformationen aufweist. Unser Ziel ist es, Kriterien für die Starrheit von $\mathrm{TV}(G)$ mit Hilfe bipartiter Graphen darzustellen. Es folgt ein Überblick über den Aufbau dieser Dissertation:

Die kombinatorische Formel für die Verformungen erster Ordnung torischer Varietäten erfordert die Untersuchung der zwei- und dreidimensionalen Flächen des Kantenkegels. Zu diesem Zweck charakterisieren wir zuerst die Flächen des Kantenkegels σ_G, indem wir Werkzeuge aus der Graphentheorie verwenden. Wir zeigen, dass die Extremalstrahlengeneratoren des Kantenkegels eins-zu-eins sogenannten ersten unabhängigen Knotenmengen entsprechen. Darüber hinaus entspricht jede Fläche des Kantenkegels einem Tupel erster unabhängiger Knotenmengen, die bestimmte Bedingungen erfüllen. Nach dieser Konstruktion nähern wir uns den Deformationen der torischen Varietäten an, die sich aus den zweiteiligen Graphen ergeben.

Mit einem klaren Verständnis der Flächenstruktur des Kantenkegels erhalten wir ein klassisches Ergebnis von Thom, Grauert-Kerner und Schlessinger über die Starrheit isolierter torischer Singularitäten in der Sprache von Graphen – nämlich vollständiger bipartiter Graphen. Als nächstes untersuchen wir solche Graphen, denen Kanten entfernt wurden. Wir leiten eine Bedingung für ihre Starrheit in Bezug auf die Anzahl der fehlenden Kanten ab.

Für den Fall eines allgemeinen bipartiten Graphen präsentieren wir eine vollständige Charakterisierung von zwei- und dreidimensionalen Flächen von σ_G. Wir beweisen, dass die entsprechenden torischen Varietäten glatt sind in Codimension zwei. Zusätzlich bestimmen wir die nicht-simpliziellen dreidimensionalen Flächen des Kantenkegels und schließen daraus, dass diese Flächen von genau vier Extremalstrahlen erzeugt werden. Für diese Fälle beweisen wir, dass die torische Varietät nicht starr ist.

Der letzte Teil handelt von Matrix-Schubert-Varietäten. Wir verwenden bipartite Graphen, um die Dimension der effektiven Toruswirkung auf sie zu bestimmen. Im torischen Fall klassifizieren wir mit Hilfe unserer Werkzeuge die starren torischen Matrix-Schubert-Varietäten. Als nächstes lenken wir unsere Aufmerksamkeit auf Kazhdan-Lusztig-Varietäten und präsentieren mögliche Anknüpfungspunkte.