Constrained Curve Flows

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Friederike Dittberner

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Erstgutachter: Prof. Dr. Klaus Ecker
Zweitgutachter: Prof. Dr. Jan Metzger

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Berlin, den 24. August 2017 Friederike Dittberner

Abstract

In this thesis we consider closed, embedded, smooth curves in the plane whose local total curvature does not lie below $-\pi$ and study their behaviour under the area preserving curve shortening flow (APCSF) and the length preserving curve flow (LPCF). For the APCSF, we show that under the above initial condition, the flow does not develop any singularities in finite time but exists for all times and converges smoothly and exponentially to a round circle after becoming convex in finite time. For the LPCF, we prove that under the above initial condition, the flow does not develop collapsed singularities and if it exists for all positive times, it converges smoothly and exponentially to a round circle after becoming convex in finite time. For these results, the above initial condition on the local total curvature is sharp. To exclude singularities, we introduce a distance comparison principle and a monotonicity formula and use methods from the theory of curve shortening flow.

Zusammenfassung

In dieser Arbeit betrachten wir geschlossene, eingebettete, glatte Kurven in der Ebene, deren lokale totale Krümmung nicht unter $-\pi$ liegt, und studieren ihr Verhalten unter dem flächenerhaltenenden, sowie dem längenerhaltenden Kurvenfluss. Für den flächenerhaltenden Kurvenfluss zeigen wir, dass unter der obigen Anfangsbedingung, der Fluss keine Singularitäten in endlicher Zeit entwickelt, aber stattdessen für alle Zeiten existiert. Nach einer bestimmten endlichen Zeit werden die Kurven konvex und konvergieren danach exponentiell schnell zu einem runden Kreis. Für den längenerhaltenden Kurvenfluss schließen wir unter der obigen Anfangsbedingung kollabierte Singularitäten aus und beweisen, dass Lösungen, die für alle positiven Zeiten existieren, glatt und exponentiell schnell zu einem runden Kreis konvergieren, nachdem sie in endlicher Zeit konvex wurden. Für diese Ergebnisse ist die obige Anfangsbedingung an die lokale totale Kümmung scharf. Um die Singularitäten auszuschließen, beweisen wir ein Abstandsvergleichsprinzip und eine Monotonieformel und benutzen außerdem Methoden aus der Theorie des Kurvenkürzungsflusses.

Contents

Chapter 1

Introduction

In this thesis we investigate constrained curve flows for closed curves in the plane. The classical curve shortening flow (CSF) is the gradient flow of the length functional of a given initial smooth curve Σ_0. That is, the flow decreases the length in the fastest possible way. In fact, one seeks a one-parameter family of embeddings $F : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ with $F(\mathbb{S}^1, 0) = \Sigma_0$ and

$$\frac{\partial F}{\partial t}(p,t) = -\kappa(p,t)\boldsymbol{\nu}(p,t) \tag{1.1}$$

for all $(p,t) \in \mathbb{S}^1 \times (0,T)$. Setting $\Sigma_t := F(\mathbb{S}^1, t)$, $\boldsymbol{\nu}$ is the outward pointing unit normal to Σ_t and κ is its curvature function.

Equation (1.1) can be seen as describing the motion of a superelastic rubber band, with small mass in a viscous medium. One of the earliest sources in the literature for the problem is the article by Mullins [Mul56], where it was used to model the behaviour of grain boundaries. After that, Brakke [Bra78] studied the motion for varifolds in the setting of geometric measure theory. In the parametric setting, the higher dimensional generalisation mean curvature flow (MCF) was first studied for smooth, compact, convex, n-dimensional hypersurfaces in $\mathbb{R}^{n+1}$ for $n \geq 2$ by Huisken [Hui84]. For an arbitrary smooth, embedded, compact, n-dimensional hypersurface Σ_0 in $\mathbb{R}^{n+1}$ the problem is given as follows. Let the embedding $F_0 : \Sigma^n \to \mathbb{R}^{n+1}$ be a parametrisation of Σ_0, where Σ^n is an abstract n-dimensional manifold. We seek a one parameter family of maps $F : \Sigma^n \times [0,T) \to \mathbb{R}^{n+1}$ with $F(\,\cdot\,, 0) = F_0$ satisfying

$$\frac{\partial F}{\partial t}(p,t) = -H(p,t)\boldsymbol{\nu}(p,t)\,, \tag{1.2}$$

for all $(p,t) \in \Sigma^n \times (0,T)$, where H is the mean curvature of the evolving smooth hypersurface $\Sigma_t := F(\Sigma^n, t)$. Huisken proved that closed, convex hypersurfaces evolving under (1.2) stay convex and shrink smoothly and exponentially in finite time to a round point, that is, they shrink to a point and when suitably rescaled converge to a round unit n-sphere. For curves in the plane, Gage and Hamilton [GH86] showed that embedded, closed, convex initial curves evolving under (1.1) stay convex and embedded until they smoothly and exponentially shrink to a round point. Grayson [Gra87] expanded the techniques from [GH86] and proved that embedded, closed, potentially non-convex curves stay embedded and become convex in finite before they shrink to round point. So, for CSF, arbitrary embedded curves in the plane stay embedded and only develop one singularity, that is, when they shrink to a round point in finite time. In [Hui95], Huisken gave a different proof for Grayson's result, by bounding the ratio of the exterior and a suitable function of the interior distance for the evolving curves. We give more details below.

In this thesis, we want to study the CSF, but ask that either the area enclosed by the evolving curve or its length is maintained during the flow. The resulting two flows have different motivations. The (enclosed) area preserving curve shortening flow (APCSF) can be seen as the motion of a super elastic rubber band that surrounds an incompressible fluid. It keeps the enclosed area of the curve fixed and decreases the length of the curve the fastest way possible. The flow has applications for shape recovery in image processing. The length preserving curve flow (LPCF) is the planar version of the so-called thread flow. Imagine a wire Γ in space which is either closed or has two endpoints. Let Σ be a space curve which is closed or attached to the ends of Γ. Consider a soap-film (minimal/least area surface) spanning this wire–thread boundary. An optimal theorem would be that the thread flow starts with an arbitrary smooth surface spanning the wire–thread combination and moves this towards a minimal surface by keeping the length of the thread fixed and decreasing the area of the spanning surface in the fastest possible way. In the limit the thread will consist of arcs of constant curvature. The latter is know to hold for the thread boundary of a spanning minimal surface. For the LPCF, we remove the wire and assume that the thread is closed and that the enclosed area increases the fastest way possible until the thread forms a circle. If we surrounded the region in which the thread is flowing by a large wire circle then the area between the thread and the wire circle would be decreased by this flow. Hence, for both the APCSF and the LPCF, the limit curves solve the isoperimetric problem. We call the resulting two flows constrained curve flows (CCF). Concerning the evolution equation for the embedding, either one of the constrains can be achieved by adding a global forcing term to the speed in (1.1).

We state the problem as follows. Let Σ_0 be an embedded, closed, smooth curve in $\mathbb{R}^2$, parametrised by the embedding $F_0 : \mathbb{S}^1 \to \mathbb{R}^2$. We seek a one-parameter family of maps $F : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ with $F(\,\cdot\,,0) = F_0$ satisfying the evolution equation

$$\frac{\partial F}{\partial t}(p,t) = \big(h(t) - \kappa(p,t)\big)\boldsymbol{\nu}(p,t) \tag{1.3}$$

for all $(p,t) \in \mathbb{S}^1 \times (0,T)$, where the global term $h : [0,T) \to \mathbb{R}$ is given by either

$$h_{\mathrm{ap}}(t) = \frac{2\pi}{L(\Sigma_t)} \qquad \text{or} \qquad h_{\mathrm{lp}}(t) = \frac{1}{2\pi}\int_{\Sigma_t} \kappa^2 \, d\mathcal{H}^1$$

for the APCSF respectively the LPCF and $L(\Sigma_t) := \int_{\Sigma_t} d\mathcal{H}^1$ is the length of the curve.

The APCSF was first studied by Gage [Gag86]. Using the techniques from [GH86], he proved that initially embedded, closed, convex curves stay embedded, smooth and convex, and converge smoothly to a circle of radius $\sqrt{A(\Sigma_0)/\pi}$, where $A(\Sigma_0)$ is the enclosed area of the initial curve. An analogous result for the LPCF was obtained by Pihan [Pih98], also using the techniques from [GH86] and [Gag86] and showing smooth, exponential convergence to a circle of radius $L(\Sigma_0)/(2\pi)$.

For dimensions $n \geq 2$, a generalisation of (1.3) are the (enclosed) volume preserving mean curvature flow (VPMCF) and the (surface) area preserving mean curvature flow (APMCF). Let Σ_0 be a smooth, embedded, compact, n-dimensional hypersurface in $\mathbb{R}^{n+1}$ and let the embedding $F_0 : \Sigma^n \to \mathbb{R}^{n+1}$ be a parametrisation of Σ_0, where Σ^n is an abstract n-dimensional manifold. We seek a one parameter family of maps $F : \Sigma^n \times [0,T) \to \mathbb{R}^{n+1}$ with $F(\,\cdot\,, 0) = F_0$ satisfying

$$\frac{\partial F}{\partial t}(p,t) = \big(h(t) - H(p,t)\big)\boldsymbol{\nu}(p,t) \tag{1.4}$$

for all $(p,t) \in \Sigma^n \times (0,T)$, where the global term suffices $h(0) > 0$ and is given by either

$$h_{\mathrm{vp}}(t) = \frac{\int_{\Sigma_t} H \, d\mathcal{H}^n}{\mathcal{H}^n(\Sigma_t)} \qquad \text{or} \qquad h_{\mathrm{ap}}(t) = \frac{\int_{\Sigma_t} H^2 \, d\mathcal{H}^n}{\int_{\Sigma_t} H \, d\mathcal{H}^n}$$

for the VPMCF respectively the APMCF.

The VPMCF has been first studied by Huisken in [Hui87] for embedded, closed, uniformly convex hypersurfaces. He proved that the solution exists globally, stays uniformly convex and converges smoothly and exponentially to a sphere. Escher and Simonett [ES98] weakened the assumption on the initial surface and showed that embedded, compact, closed, connected hypersurfaces in $\mathbb{R}^{n+1}$ that are in a certain sense $C^{1+\beta}$-Hölder close to a sphere converge smoothly to a sphere. In [Li09], Li proved smooth convergence of immersed, orientable, closed hypersurfaces in $\mathbb{R}^{n+1}$ to a sphere by only requiring that the traceless second fundamental form is sufficiently small. In [MSS16], Mugnai, Seis and Spadaro constructed global distributional solutions.

Similar results exist for the APMCF. McCoy [McC02, McC03] showed that every embedded, closed, compact, strictly convex hypersurface converges smoothly and exponentially to a sphere. In [HL15], Huang and Lin weakened the initial conditions, only requiring that the L^2-norm of the traceless second fundamental form of Σ_0 is small.

In [CRM16], Cabezas-Rivas and Miquel showed that mean convexity (that is, positivity of the mean curvature) and positivity of the scalar curvature are non-preserved curvature conditions for hypersurfaces of the Euclidean space evolving under either the VPMCF or the APMCF.

The VPMCF has also been studied in the following non-Euclidean settings. In [EH91], Ecker and Huisken used VPMCF to construct spacelike hypersurfaces of constant mean curvature in cosmological spacetimes. In [HY96], Huisken and Yau proved with VPMCF that every 3-dimensional, asymptotically flat manifold with positive mass is uniquely foliated at infinity by stable spheres of constant mean curvature. They defined the centre of mass using this constant mean curvature foliation. In particular, they showed that, for sufficiently large initial spheres, the solution of VPMCF exists for all times and converges smoothly to a constant mean curvature sphere. For arbitrary ambient compact Riemannian manifolds, Alikakos and Freire [AF03] showed that if the initial hypersurface is sufficiently close to a small geodesic sphere, the evolution is defined for all times and (under certain conditions) will converge smoothly to a leaf of a local foliation. In [Rig04], Rigger constructed a foliation of asymptotically hyperbolic 3-manifolds by 2-surfaces with constant mean curvature which are homeomorphic to spheres. Cabezas-Rivas and Miquel showed in [CRM07] that for compact hypersurface of the hyperbolic space which are convex in a certain sense, convexity is preserved for all times and the flow converges smoothly to a geodesic sphere.

Furthermore, there are the following results for VPMCF with Neumann free boundary conditions. Athanassenas [Ath97] investigated axially symmetric surfaces between two parallel hyperplanes and showed, for large volumes, smooth convergence to constant mean curvature surfaces. In [Ath03], she proved that singularities form a finite, discrete set along the axis of rotation and that type-I singularities are asymptotically cylindrical. In [AK12], Athanassenas and Kandanaarachchi studied the convergence of axially symmetric hypersurfaces. Assuming that the surface does not develop singularities along the axis of rotation at any time, they showed smooth convergence to a hemisphere, when the initial hypersurface has a free boundary and satisfies Neumann boundary data, and to a sphere when it is compact without boundary. Cabezas-Rivas and Miquel generalised these results for revolution hypersurfaces in a rotationally symmetric ambient space, see [CRM09] and [CRM12]. In [Har13], Hartley investigated the VPMCF with Neumann boundary condition for hypersurfaces that are graphs over a cylinder. If the initial hypersurfaces are sufficiently close to a cylinder of large enough radius, smooth convergence to a cylinder follows. Furthermore, he showed that there exist global solutions to the flow that converge to a cylinder, which are initially non-axially symmetric. In [MB14, MB15], Maeder-Baumdicker studied APCSF for convex curves in the plane with Neumann boundary on a convex support curve and showed smooth convergence to an arc for sufficiently short, convex, embedded initial curves.

In this thesis, we study the constrained curve flows (1.3) for embedded, compact, smooth initial curves Σ_0 which satisfy

$$\int_{F_0([p,q])} \kappa \, d\mathcal{H}^1 \geq -\pi \tag{1.5}$$

for all $p, q \in \mathbb{S}^1$ (see Figure 1.1 for an illustration). We show for both floes that an initially embedded curve satisfying (1.5) stays embedded as long as it is smooth. Moreover, condition (1.5) is sharp, that is, one can construct initial curves which violate (1.5) arbitrarily mildly and for which the resulting flow self-intersects in finite time. An example is the initial curve in Figure 1.2 with length sufficiently large compared to the $C^{3,\alpha}$-norm of its

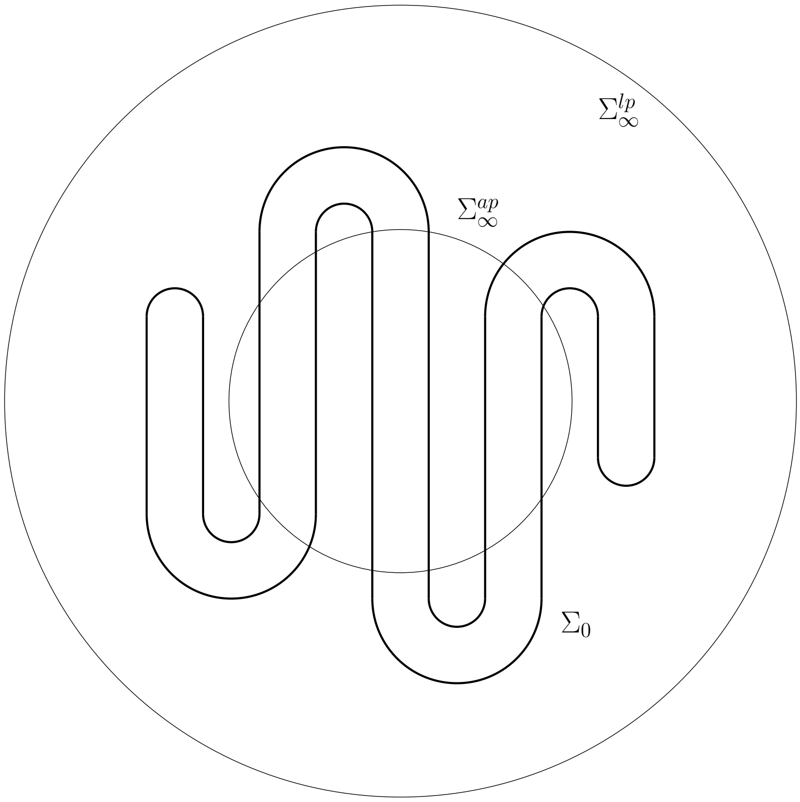

Figure 1.1: An initial curve and an approximate sketch of the limit circles for both flows.

embedding and for which

$$\int_{F_0([p,q])} \kappa \, d\mathcal{H}^1 < -\pi \, . \tag{1.6}$$

Note that for convex curves $\int_{F([p,q])} \kappa \, d\mathcal{H}^1 \geq 0$ for all $p, q \in \mathbb{S}^1$. For initial curves satisfying (1.5), we show that under the APCSF these curves do not develop any singularities but exist for all times and converge smoothly and exponentially to a round circle, like the convex initial curves above. For the LPCF, we can exclude a certain class of singularities and show that solutions which exist for all positive times, converge smoothly and exponentially to a round circle, like the convex initial curves above.

To explain our approach to proving the claim, let us concentrate on the theory of CSF. For CSF, the maximal time of existence T is finite. It follows that the curvature has to blow up for $t \to T$ with the lower bound

$$\max_{p \in \mathbb{S}^1} |\kappa(p,t)| \geq \frac{1}{\sqrt{2(T-t)}}$$

on its growth rate for all $t \in [0, T)$. We call a curvature blow-up a singularity and distinguish between two different kinds, as introduced in [Mul56]: For type-I singularities, there also exists an upper bound on the growth rate of the form

$$\max_{p \in \mathbb{S}^1} \kappa(p,t) \leq \frac{C_0}{\sqrt{2(T-t)}}$$

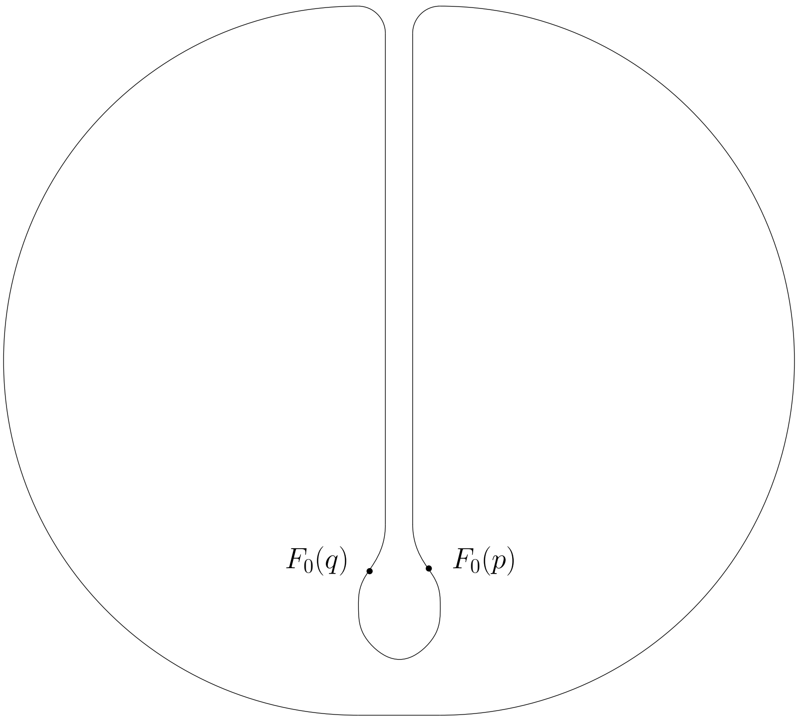

Figure 1.2: An initial curve satisfying (1.6).

for all $t \in [0, T)$ and some time-independent constant $C_0 < \infty$. For type-II singularities, there is no such bound, that is,

$$\limsup_{t \to T} \max_{p \in \mathbb{S}^1} \sqrt{2(T-t)}\, \kappa(p,t) = \infty\,.$$

To analyse singularities, one can follow Hamilton [Ham95a] (for Ricci flow) and Huisken–Sinestrari [HS99] (for MCF) and introduce a sequence of smooth parabolic rescalings near the point of highest curvature, that is,

$$F_k : \mathbb{S}^1 \times [-\alpha_k, T_k] \to \mathbb{R}^2$$

for $k \in \mathbb{N}$. One can show, that $\alpha_k \to \infty$ and $T_k \to T_\infty$, where $T_\infty = 0$ for a type-I singularity, and $T_\infty = \infty$ for a type-II singularity. Moreover, the embeddings F_k satisfy again (1.1) for all $k \in \mathbb{N}$. By curvature gradient estimates and the Arzelà–Ascoli theorem, smooth convergence of $F_k \to F_\infty$ follows, where for $S \in \{\mathbb{S}^1, \mathbb{R}\}$,

$$F_\infty : S \times (-\infty, T_\infty) \to \mathbb{R}^2$$

also satisfies (1.1). Using his famous monotonicity formula in [Hui90], Huisken showed that if an embedded curve develops a type-I singularity, the limit curves $\Sigma^\infty_\tau := F_\infty(S, \tau)$ of the rescaled solution have to satisfy the equation

$$\kappa_\infty(p,\tau) = \langle F_\infty(p,\tau), \boldsymbol{\nu}_\infty(p,\tau)\rangle$$

for all $(p, \tau) \in S \times (-\infty, T_\infty)$. (In fact, he showed this result for all $n \geq 1$ and immersed hypersurfaces.) Thus, it follows that the original curves Σ_t have to be asymptotic to a homothetically shrinking solution around the singular point for $t \to T$. Abresch and Langer [AL86] had previously classified all such solutions as circles and lines. One concludes, in case of a type-I singularity, that the curve shrinks to a round point. For the type-II singularities, Hamilton [Ham89] for convex initial curves and Altschuler [Alt91] for non-convex initial curves showed that each of the above rescaling sequences $(F_k)_{k\in\mathbb{N}}$ converges to a translating solution $F_\infty : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ and satisfies

$$\kappa_\infty(p,\tau) = \langle \mathbf{v}, \boldsymbol{\nu}_\infty(p,\tau)\rangle$$

for a fixed vector $\mathbf{v} \in \mathbb{R}^2$ and for all $(p,\tau) \in \mathbb{R} \times \mathbb{R}$. For curves in the plane, the only solution of this kind is the so-called grim reaper which is, for all $\tau \in \mathbb{R}$, given by the graph of the function

$$u(\sigma,\tau) = \tau - \log\cos(\sigma)\,,$$

where $\sigma \in (-\pi/2, \pi/2)$. To exclude type-II singularities Huisken [Hui95] considered the extrinsic distance

$$d(p,q,t) := \|F(p,t) - F(q,t)\|_{\mathbb{R}^2}$$

and the function

$$\psi(p,q,t) := \frac{L(\Sigma_t)}{\pi} \sin\left(\frac{\pi\, l(p,q,t)}{L(\Sigma_t)}\right)\,,$$

where $(p,q,t) \in \mathbb{S}^1 \times \mathbb{S}^1 \times [0,T)$ and

$$l(p,q,t) := \int_{F([p,q],t)} d\mathcal{H}^1$$

is the length of the segment $F([p,q],t)$. Then under (1.1), the infimum of the ratio d/ψ is strictly increasing in time unless Σ_t is a circle. On the grim reaper $\inf_{\mathbb{R}\times\mathbb{R}}(d/l) = 0$, so that type-II singularities can be excluded. Since $T < \infty$ and a singularity has to form, it has to be of type I.

Outline of this thesis

In Chapter 2, we derive the equation (1.4) via the gradient flow approach for the volume and the area functionals and define the VPMCF and the APMCF as well as the APCSF and the LPCF. In Chapter 3, we prove evolution equations for the geometric quantities under (1.3) and draw first conclusions. In Chapter 4, we estimate the derivatives of the curvature and show that if the flow exists only on a finite time interval then the curvature has to blow up.

In Chapter 5, we deduce a strong maximum principle for the local total curvature. In Chapter 6, we show that the curve Σ_t stays embedded for all $t \in [0,T)$ provided the initial embedding Σ_0 satisfies (1.5) and prove that this condition is sharp. In Chapter 7, we modify the distance comparison principle of Huisken [Hui95] and prove that, if the initial embedding Σ_0 satisfies (1.5), the ratio d/ψ is bounded from below away from zero uniformly in time.

In Chapter 8, we derive an analogue of Huisken's monotonicity formula for both APMCF and VPMCF, see also [MB14, MB15] for the APMCF.

In Chapter 9, we assume that the maximal time of existence is finite and study curvature blow ups via parabolic rescaling. We rule out type-I singularities for the APCSF using the monotonicity formula in a similar way as in [MB14, MB15]. Furthermore, we use the distance comparison principle from Chapter 6 in the same fashion as for CSF in [Hui95] to exclude singularities that, after rescaling, satisfy $\inf_{\mathbb{R}\times\mathbb{R}}(d/l) = 0$. For the APCSF, these are the type-II singularities.

In Chapter 10, we assume that a solution of (1.3) exists for all positive times and show that it becomes convex in finite time. In Chapter 11, we rework the arguments for convex theory from [GH86], [Gag86] and [Pih98] to prove smooth, exponential convergence to a circle. We summarise our results in Theorem 11.24 and Corollary 11.26.

In Appendix A, we give a short introduction to curves in the plane and to manifolds in $\mathbb{R}^{n+m}$. In Appendix B, we state useful theorems and equations.

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Chapter 2

Constrained flows

In this chapter, we derive the constrained gradient flows for the volume and the area functionals and define the (enclosed) volume and (surface) area preserving mean curvature flow (VPMCF and APMCF) as well as the (enclosed) area preserving curve shortening and length preserving curve flow (APCSF and LPCF).

2.1 Constrained gradient flows

Let $n, m \geq 1$ and $F : M^{n+1} \to \mathbb{R}^{n+m}$ be a smooth embedding of an orientable, compact, $(n+1)$-dimensional hypersurface $M := F(M^{n+1})$ with boundary, so that $\Sigma := F(\partial M^{n+1})$ is a smooth, compact, embedded, n-dimensional hypersurface without boundary in $\mathbb{R}^{n+m}$. (Refer to Appendix A.2 for an introduction to differentiable submanifolds of $\mathbb{R}^{n+m}$.)

We want to find two kinds of steepest descent L^2-gradient flows. The first one shall decrease the n-dimensional surface area of the boundary Σ and at the same time keep the $(n+1)$-dimensional enclosed volume of M fixed. The second one shall increase the enclosed volume and at the same time keep the surface area of the boundary fixed. The latter one was derived by Pihan [Pih98, Section B.2] in the setting of the thread flow, where a part of the boundary Σ stays fixed in time. We will adapt his method to both flows in the following. Let $\mathcal{I} := \mathcal{I}_1 \times \mathcal{I}_2$ be a vector space of functions with

$$\mathcal{I}_1 := \left\{ f_1 : M \to \mathbb{R}^{n+m} \mid f_1 \in C^\infty \right\}$$

and

$$\mathcal{I}_2 := \left\{ f_2 : \Sigma \to \mathbb{R}^{n+m} \mid f_2 \in C^\infty \right\}.$$

For $f = (f_1, f_2)$, $g = (g_1, g_2) \in \mathcal{I}$, we define the inner product on $\mathcal{I}$

$$\langle f, g \rangle_{\mathcal{I}} := \int_M \langle f_1, g_1 \rangle \, d\mathcal{H}^{n+1} + \int_\Sigma \langle f_2, g_2 \rangle \, d\mathcal{H}^n \,, \tag{2.1}$$

the volume functional

$$V(f) := \int_{f_1(M)} d\mathcal{H}^{n+1} = \mathcal{H}^{n+1}(f_1(M)) \,, \tag{2.2}$$

and the area functional

$$A(f) := \int_{f_2(\Sigma)} d\mathcal{H}^n = \mathcal{H}^n(f_2(\Sigma))\,. \tag{2.3}$$

Consider a path $\phi = (\phi_1, \phi_2) : (-1,1) \to \mathcal{I}$ in C^2, along which both V and A are continuously Fréchet differentiable, and which satisfies

$$\phi_1(0) \equiv \mathrm{id}_M \qquad \text{and} \qquad \phi_2(0) \equiv \mathrm{id}_\Sigma\ .$$

The existence of such a path for $m = 1$ follows from the short time existence, Theorem 2.3. Set

$$M_t := \phi_1(t)(M) \qquad \text{and} \qquad \Sigma_t := \phi_2(t)(\Sigma)$$

for $t \in (-1,1)$ and define the vector field $\mathbf{v}(t) = (\mathbf{v}_1(t), \mathbf{v}_2(t))$ by

$$\mathbf{v}_i(t) := \frac{d}{d\tau}_{|\tau=t} \phi_i(\tau)$$

for $i = 1,2$ and $t \in (-1,1)$. The Fréchet derivative of the volume functional V is then given by

$$\frac{d}{d\tau}_{|\tau=t} V(\phi(\tau)) = DV(\phi(t))(\mathbf{v}) = \langle \nabla V(\phi(t)), \mathbf{v}\rangle_{\mathcal{I}}\ .$$

On the other hand, the first variation of the area formula, Theorem B.6, and the divergence theorem, Theorem B.7, imply

$$\begin{aligned}\frac{d}{d\tau}_{|\tau=t} V(\phi(\tau)) &\overset{(2.2)}{=} \frac{d}{d\tau}_{|\tau=t} \mathcal{H}^{n+1}(\phi_1(\tau)(M)) \overset{\text{Thm. B.6}}{=} \int_{M_t} \mathrm{div}_{M_t}\, \mathbf{v}_1\, d\mathcal{H}^{n+1}\\ &\overset{\text{Thm. B.7}}{=} -\int_{M_t} \langle \mathbf{v}_1, \mathbf{H}_{M_t}\rangle\, d\mathcal{H}^{n+1} + \int_{\Sigma_t} \langle \mathbf{v}_2, \boldsymbol{\nu}_{\Sigma_t}\rangle\, d\mathcal{H}^n\,,\end{aligned}$$

where $\mathbf{H}_{M_t}$ is the mean curvature vector to M_t and $\boldsymbol{\nu}_{\Sigma_t}$ is the outward pointing unit co-normal to Σ_t. The above calculations hold for all differentiable vector fields $\mathbf{v}$. It therefore follows that the gradient of V exists with respect to $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{I}}$ and satisfies

$$\nabla V(\phi(t)) = \begin{cases} -\mathbf{H}_{M_t} & \text{on } M_t\\ \boldsymbol{\nu}_{\Sigma_t} & \text{on } \Sigma_t\end{cases} \tag{2.4}$$

for all $t \in (-1,1)$. The same observation can be applied to the area functional A, so that we obtain

$$\frac{d}{d\tau}_{|\tau=t} A(\phi(\tau)) = DA(\phi(t))(\mathbf{v}) = \langle \nabla A(\phi(t)), \mathbf{v}\rangle_{\mathcal{I}}$$

as well as

$$\begin{aligned}\frac{d}{d\tau}_{|\tau=t} A(\phi(\tau)) &\overset{(2.3)}{=} \frac{d}{d\tau}_{|\tau=t} \mathcal{H}^n(\phi_2(\tau)(\Sigma)) \overset{\text{Thm. B.6}}{=} \int_{\Sigma_t} \mathrm{div}_{\Sigma_t}\, \mathbf{v}_2\, d\mathcal{H}^n\\ &\overset{\text{Thm. B.7}}{=} -\int_{\Sigma_t} \langle \mathbf{v}_2, \mathbf{H}_{\Sigma_t}\rangle\, d\mathcal{H}^n\end{aligned}$$

since $\partial\Sigma_t = \emptyset$, and where $\mathbf{H}_{\Sigma_t}$ is the mean curvature vector to Σ_t. Hence, the gradient of A exists with respect to $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{I}}$ and can be written as

$$\nabla A(\phi(t)) = \begin{cases} 0 & \text{on } M_t\\ -\mathbf{H}_{\Sigma_t} & \text{on } \Sigma_t\end{cases} \tag{2.5}$$

for all $t \in (-1,1)$. We now derive both gradient flows separately.

Volume preserving gradient flow. We are looking for the steepest descent flow for the surface area A which at the same time keeps the enclosed volume V fixed. For given $\sigma > 0$, consider the level-set

$$\mathcal{S}_\sigma(V) := \{F \in \mathcal{I} \mid V(F) = \sigma\}\,.$$

To maintain the initial volume during the flow, we need to move tangentially to the set $\mathcal{S}_\sigma(V)$ with $\sigma = V(\phi(0))$. Since the surface area shall be decreasing, we have to move in opposite direction to the tangential gradient with respect to $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{I}}$ which is given by

$$-\nabla^\top = \nabla^\perp - \nabla\,,$$

where $\nabla^\perp$ is the normal part of the gradient. We calculate

$$\langle\nabla V(F), V(F)\rangle_{\mathcal{I}} = \frac{1}{2}\nabla\langle V(F), V(F)\rangle_{\mathcal{I}} = \frac{1}{2}\nabla\sigma^2 = 0$$

to see that the gradient of the level-set function is perpendicular to the level-set. It follows that $\nabla V/\|\nabla V\|$ defines a unit normal to $\mathcal{S}_\sigma(V)$ and $\langle\nabla A, \nabla V/\|\nabla V\|\rangle_{\mathcal{I}}$ is the length of the normal part of ∇A. We thus obtain

$$\begin{aligned}
-\nabla^\top A(\phi(t)) &= \left\langle \nabla A(\phi(t)), \frac{\nabla V(\phi(t))}{\|\nabla V(\phi(t))\|}\right\rangle_{\mathcal{I}} \frac{\nabla V(\phi(t))}{\|\nabla V(\phi(t))\|} - \nabla A(\phi(t)) \\
&= \frac{\langle\nabla A(\phi(t)), \nabla V(\phi(t))\rangle_{\mathcal{I}}}{\langle\nabla V(\phi(t)), \nabla V(\phi(t))\rangle_{\mathcal{I}}}\nabla V(\phi(t)) - \nabla A(\phi(t))\,.
\end{aligned} \tag{2.6}$$

The identity (A.16) for the mean curvature yields

$$\begin{aligned}
\langle\nabla A(\phi(t)), \nabla V(\phi(t))\rangle_{\mathcal{I}} &\overset{(2.1)}{=} \int_{M_t} \langle(\nabla A)_1(\phi(t)), (\nabla V)_1(\phi(t))\rangle\, d\mathcal{H}^{n+1} \\
&\quad + \int_{\Sigma_t} \langle(\nabla A)_2(\phi(t)), (\nabla V)_2(\phi(t))\rangle\, d\mathcal{H}^{n} \\
&\overset{(2.4),(2.5)}{=} \int_{M_t} \langle 0, -\mathbf{H}_{M_t}\rangle\, d\mathcal{H}^{n+1} + \int_{\Sigma_t} \langle -\mathbf{H}_{\Sigma_t}, \boldsymbol{\nu}_{\Sigma_t}\rangle\, d\mathcal{H}^{n} \\
&\overset{(A.16)}{=} \int_{\Sigma_t} \operatorname{div}_{\Sigma_t} \boldsymbol{\nu}_{\Sigma_t}\, d\mathcal{H}^{n}
\end{aligned} \tag{2.7}$$

and we calculate

$$\begin{aligned}
\langle\nabla V(\phi(t)), \nabla V(\phi(t))\rangle_{\mathcal{I}} &\overset{(2.1)}{=} \int_{M_t} \|(\nabla V)_1(\phi(t))\|^2\, d\mathcal{H}^{n+1} + \int_{\Sigma_t} \|(\nabla V)_2(\phi(t))\|^2\, d\mathcal{H}^{n} \\
&\overset{(2.4)}{=} \int_{M_t} \|\mathbf{H}_{M_t}\|^2\, d\mathcal{H}^{n+1} + \int_{\Sigma_t} \|\boldsymbol{\nu}_{\Sigma_t}\|^2\, d\mathcal{H}^{n} \\
&= \int_{M_t} \|\mathbf{H}_{M_t}\|^2\, d\mathcal{H}^{n+1} + \int_{\Sigma_t} d\mathcal{H}^{n} > 0\,.
\end{aligned} \tag{2.8}$$

The identities (2.4), (2.5), (2.7), and (2.8) applied to (2.6) then imply the gradient flow for the area functional with respect to $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{I}}$, up to tangential diffeomorphisms, namely,

$$-\nabla^\top A(t) = \begin{cases} -h(t)\mathbf{H}_{M_t} & \text{on } M_t \\ h(t)\boldsymbol{\nu}_{\Sigma_t} + \mathbf{H}_{\Sigma_t} & \text{on } \Sigma_t \end{cases}$$

for $t \in (-1, 1)$, where

$$h(t) := \frac{\int_{\Sigma_t} \operatorname{div}_{\Sigma_t} \boldsymbol{\nu}_{\Sigma_t} \, d\mathcal{H}^n}{\int_{M_t} \|\mathbf{H}_{M_t}\|^2 \, d\mathcal{H}^{n+1} + \int_{\Sigma_t} d\mathcal{H}^n} \, .$$

If we choose $m = 1$ so that Σ_t are closed n-dimensional hypersurfaces in $\mathbb{R}^{n+1}$ and $\mathbf{H}_{M_t} = \mathbf{H}_{\mathbb{R}^{n+1}} = 0$ for all $t \in (-1, 1)$, we obtain the (enclosed) volume preserving and (surface) area decreasing gradient flow

$$-\nabla^\top A(t) = h(t)\boldsymbol{\nu}_{\Sigma_t} + \mathbf{H}_{\Sigma_t} = (h(t) - H_{\Sigma_t})\, \boldsymbol{\nu}_{\Sigma_t} \, , \tag{2.9}$$

where $\boldsymbol{\nu}_{\Sigma_t}$ is the outward unit normal to Σ_t, H_{Σ_t} is the mean curvature and

$$h(t) = \frac{\int_{\Sigma_t} H_{\Sigma_t} \, d\mathcal{H}^n}{\int_{\Sigma_t} d\mathcal{H}^n} \, .$$

Area preserving gradient flow. Next, we are seeking the steepest descent flow for the enclosed volume V which at the same time keeps the surface area A fixed. To maintain the initial surface area, we have to move tangentially to the level-set

$$\mathcal{S}_\sigma(A) := \{F \in \mathcal{I} \mid A(F) = \sigma\}$$

for $\sigma = A(\phi(0))$. Since the enclosed volume shall be increasing, we move in the direction of the tangential gradient with respect to $\langle \cdot \, , \cdot \rangle_\mathcal{I}$ which is given by

$$\nabla^\top = \nabla - \nabla^\perp \, .$$

Here, $\nabla A / \|\nabla A\|$ is a normal to $\mathcal{S}_\sigma(A)$ and $\langle \nabla V, \nabla A / \|\nabla A\| \rangle_\mathcal{I}$ is the length of the perpendicular part of ∇V. This implies

$$\begin{aligned} \nabla^\top V(\phi(t)) &= \nabla V(\phi(t)) - \left\langle \nabla V(\phi(t)), \frac{\nabla A(\phi(t))}{\|\nabla A(\phi(t))\|} \right\rangle_\mathcal{I} \frac{\nabla A(\phi(t))}{\|\nabla A(\phi(t))\|} \\ &= \nabla V(\phi(t)) - \frac{\langle \nabla V(\phi(t)), \nabla A(\phi(t)) \rangle_\mathcal{I}}{\langle \nabla A(\phi(t)), \nabla A(\phi(t)) \rangle_\mathcal{I}} \nabla A(\phi(t)) \end{aligned} \tag{2.10}$$

and we calculate

$$\begin{aligned} \langle \nabla A(\phi(t)), \nabla A(\phi(t)) \rangle_\mathcal{I} &\overset{(2.1)}{=} \int_{M_t} \|(\nabla A)_1(\phi(t))\|^2 \, d\mathcal{H}^{n+1} + \int_{\Sigma_t} \|(\nabla A)_2(\phi(t))\|^2 \, d\mathcal{H}^n \\ &\overset{(2.5)}{=} \int_{M_t} 0^2 \, d\mathcal{H}^{n+1} + \int_{\Sigma_t} \|\mathbf{H}_{\Sigma_t}\|^2 \, d\mathcal{H}^n \\ &= \int_{\Sigma_t} \|\mathbf{H}_{\Sigma_t}\|^2 \, d\mathcal{H}^n > 0 \, . \end{aligned} \tag{2.11}$$

This time, we apply the identities (2.4), (2.5), (2.7), and (2.11) to (2.10) in order to deduce the gradient flow on M_t with respect to $\langle \cdot \, , \cdot \rangle_\mathcal{I}$ up to tangential diffeomorphisms, namely,

$$\nabla^\top V(t) = \begin{cases} -\mathbf{H}_{M_t} & \text{on } M_t \\ \boldsymbol{\nu}_{\Sigma_t} + h(t)\mathbf{H}_{\Sigma_t} & \text{on } \Sigma_t \end{cases}$$

for $t \in (-1, 1)$, where

$$h(t) := \frac{\int_{\Sigma_t} \operatorname{div}_{\Sigma_t} \boldsymbol{\nu}_{\Sigma_t} \, d\mathcal{H}^n}{\int_{\Sigma_t} \|\mathbf{H}_{\Sigma_t}\|^2 \, d\mathcal{H}^n} .$$

If we choose $m = 1$ so that Σ_t are closed n-dimensional hypersurfaces in $\mathbb{R}^{n+1}$ for all $t \in (-1, 1)$, then we find the (surface) area preserving and (enclosed) volume increasing gradient flow

$$\nabla^\top V(t) = h(t)\mathbf{H}_{\Sigma_t} + \boldsymbol{\nu}_{\Sigma_t} = (1 - h(t)H_{\Sigma_t})\, \boldsymbol{\nu}_{\Sigma_t} , \tag{2.12}$$

where

$$h(t) = \frac{\int_{\Sigma_t} H_{\Sigma_t} \, d\mathcal{H}^n}{\int_{\Sigma_t} H^2_{\Sigma_t} \, d\mathcal{H}^n} .$$

2.2 Constrained mean curvature flows

Let Σ_0 be a smooth, embedded, n-dimensional hypersurface in $\mathbb{R}^{n+1}$ without boundary. Let the embedding $F_0 : \Sigma^n \to \mathbb{R}^{n+1}$ be a smooth parametrisation of Σ_0. The initial value problem for the (enclosed) volume preserving mean curvature flow (VPMCF) and the (surface) area preserving mean curvature flow (APMCF) for hypersurfaces in $\mathbb{R}^{n+1}$ can be defined as follows.

Definition 2.1 (The volume and the area preserving mean curvature flow in $\mathbb{R}^{n+1}$)**.** Let $\Sigma_0 \subset \mathbb{R}^{n+1}$ be given as above. We seek a one parameter family of maps $F : \Sigma^n \times [0, T) \to \mathbb{R}^{n+1}$ satisfying $F(p, 0) = F_0(p)$ for all $p \in \Sigma^n$ and the evolution equation

$$\frac{\partial F}{\partial t}(p, t) = \big(h(t) - H(p, t)\big)\boldsymbol{\nu}(p, t) \tag{2.13}$$

for all $p \in \Sigma^n$ and $t \in (0, T)$, where H is the mean curvature of $\Sigma_t := F(\Sigma^n, t)$, $\boldsymbol{\nu}$ the outer unit normal, and $h : [0, T) \to \mathbb{R}$ is given by either

$$h_{\text{vp}}(t) := \frac{\int_{\Sigma_t} H \, d\mathcal{H}^n}{\int_{\Sigma_t} d\mathcal{H}^n}$$

for the VPMCF or by

$$h_{\text{ap}}(t) := \frac{\int_{\Sigma_t} H^2 \, d\mathcal{H}^n}{\int_{\Sigma_t} H \, d\mathcal{H}^n}$$

for the APMCF. Additionally, we assume that $h(0) > 0$.

In this thesis we will be mainly concerned with planar curves that is the case $n = 1$.

Remark 2.2. (i) For the VPMCF, the evolution equation (2.13) follows directly from the gradient flow (2.9). For the APMCF, the evolution equation that we deduce from the gradient flow (2.12) is

$$\frac{\partial F}{\partial t}(p, t) = \left(1 - \frac{1}{h_{\text{ap}}(t)} H(p, t)\right) \boldsymbol{\nu}(p, t) . \tag{2.14}$$

Note that Pihan works with this speed in his thesis [Pih98]. To find the speed (2.13) for the APMCF, we introduce a new time parameter

$$\tau := \int_0^t \frac{1}{h_{\mathrm{ap}}(\sigma)}\, d\sigma$$

and define the embedding $\bar{F} : \Sigma^n \times [0, T/h_{\mathrm{ap}}(T)) \to \mathbb{R}^2$ by

$$\bar{F}(\,\cdot\,, \tau) := F(\,\cdot\,, t)\,.$$

Moreover,

$$\frac{\partial}{\partial \tau} = h_{\mathrm{ap}}(t) \frac{\partial}{\partial t}$$

and, since the spatial derivatives are not affected by the transformation, $\bar{H}(\,\cdot\,, \tau) = H(\,\cdot\,, t)$, $\bar{\boldsymbol{\nu}}(\,\cdot\,, \tau) = \boldsymbol{\nu}(\,\cdot\,, t)$, and $\bar{h}_{\mathrm{ap}}(\tau) = h_{\mathrm{ap}}(t)$ which implies

$$\begin{aligned} \frac{\partial \bar{F}}{\partial \tau}(p, \tau) &= h_{\mathrm{ap}}(t) \frac{\partial F}{\partial t}(p, t) \overset{(2.14)}{=} h_{\mathrm{ap}}(t) \left(1 - \frac{1}{h_{\mathrm{ap}}(t)} H(p, t)\right) \boldsymbol{\nu}(p, t) \\ &= \left(\bar{h}_{\mathrm{ap}}(\tau) - \bar{H}(p, \tau)\right) \bar{\boldsymbol{\nu}}(p, \tau)\,. \end{aligned}$$

(ii) Note that (2.13) is a quasilinear parabolic equation since

$$-H(p, t)\boldsymbol{\nu}(p, t) = \mathbf{H}(p, t) \overset{(A.17)}{=} \Delta_{\Sigma_t} F(p, t)\,.$$

(iii) Static solutions to (2.13) are spheres. On $\mathbb{S}^n_R$, $H \equiv 1/R$ so that $h_{\mathrm{vp}} = h_{\mathrm{ap}} = 1/R$.

Theorem 2.3 (Short time existence, Huisken [Hui87, p. 36] for the VPMCF, Pihan [Pih98, Theorems 4.3 and Corollary 4.4] for the APMCF)**.** *For $\alpha \in (0, 1)$ and $k \geq 3$, let Σ_0 be an embedded, closed, n-dimensional $C^{k,\alpha}$-hypersurface in $\mathbb{R}^{n+1}$, parametrised by a $C^{k,\alpha}$-embedding $F_0 : \Sigma^n \to \mathbb{R}^2$. Then there exists a time $T = T(\|F_0\|_{C^{3,\alpha}}) > 0$ such that the initial value problem* (2.13) *has a unique solution $F \in C^{k,\alpha;\lfloor k/2 \rfloor,\alpha/2}(\Sigma^n \times (0, T))$. In particular, if Σ_0 is smooth so is Σ_t for all $t \in (0, T)$.*

Remark 2.4. The proof in [Pih98] establishes short time existence for the speed (2.14). As shown in Remark 2.2(i), the flows (2.13) and (2.14) can be transformed into each other by varying the speed. Hence, short time existence for one flow implies short time existence for the other.

2.3 Constrained curve flows

Constrained curve flows arise in the special case $n = 1$ of the constrained mean curvature flow from Section 2.2. Let Σ_0 be a smooth, embedded, closed curve in $\mathbb{R}^2$, parametrised by the smooth embedding $F_0 : \mathbb{S}^1 \to \mathbb{R}^2$. (Refer to Appendix A.1 for an introduction on curves.) The initial value problem for the (enclosed) area preserving curve shortening flow (APCSF) and the length preserving curve flow (LPCF) for curves in $\mathbb{R}^2$ can be defined as follows.

Definition 2.5 (The area preserving curve shortening and the length preserving curve flow for curves in $\mathbb{R}^2$)**.** Let $\Sigma_0 \subset \mathbb{R}^2$ be given as above. We seek a one parameter family of maps $F : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ satisfying the initial condition $F(p, 0) = F_0(p)$ for all $p \in \mathbb{S}^1$ and the evolution equation

$$\frac{\partial F}{\partial t}(p,t) = \big(h(t) - \kappa(p,t)\big)\boldsymbol{\nu}(p,t) \tag{2.15}$$

for all $p \in \mathbb{S}^1$ and $t \in (0, T)$. Here, κ is the curvature, $\boldsymbol{\nu}$ is the outer unit normal to $\Sigma_t := F(\mathbb{S}^1, t)$, and the global term $h : [0, T) \to \mathbb{R}$ is given by either

$$h_{\mathrm{ap}}(t) := \frac{\int_{\Sigma_t} \kappa \, d\mathcal{H}^1}{\int_{\Sigma_t} d\mathcal{H}^1}$$

for the APCSF or by

$$h_{\mathrm{lp}}(t) := \frac{\int_{\Sigma_t} \kappa^2 \, d\mathcal{H}^1}{\int_{\Sigma_t} \kappa \, d\mathcal{H}^1}$$

for the LPCF.

Remark 2.6. (i) The total curvature of an embedded closed curve is

$$\int_{\Sigma_t} \kappa \, d\mathcal{H}^1 = 2\pi$$

(see Theorem A.2). Thus, the global terms are given by

$$h_{\mathrm{ap}}(t) = \frac{2\pi}{L_t} \qquad \text{and} \qquad h_{\mathrm{lp}}(t) = \frac{1}{2\pi} \int_{\Sigma_t} \kappa^2 \, d\mathcal{H}^1 \,,$$

where $L_t := L(\Sigma_t)$ is the length of Σ_t.

(ii) Static solutions to (2.13) are circles. On $\mathbb{S}^1_R$, $\kappa \equiv 1/R$ so that $h_{\mathrm{ap}} = h_{\mathrm{lp}} = 1/R$.

Chapter 3

Evolution equations and first consequences

For fixed $t \in [0,T)$, we can parametrise Σ_t by arc length via the arc length parameter $s(\,\cdot\,,t)$ (see Section A.1 for details). Set $R_t := L_t/(2\pi)$. Then $s(\mathbb{S}^1,t) = \mathbb{S}^1_{R_t}$. The arc length parametrisation

$$\tilde{F}(\,\cdot\,,t) : \mathbb{S}^1_{R_t} \to \mathbb{R}^2$$

is given by

$$\tilde{F}(s,t) = F(p,t)$$

for $s = s(p,t) \in \mathbb{S}^1_{R_t}$, $p \in \mathbb{S}^1$ and $t \in [0,T)$. The evolution equation (2.15) applied to the arc length parametrisation reads

$$\frac{\partial \tilde{F}}{\partial t}(s,t) = \frac{\partial^2 \tilde{F}}{\partial s^2}(s,t) + h(t)\tilde{\boldsymbol{\nu}}(s,t) \tag{3.1}$$

for all $s \in \mathbb{S}^1_{R_t}$, where $\tilde{\boldsymbol{\nu}}(s,t) = \tilde{\boldsymbol{\nu}}(s(p,t),t) = \boldsymbol{\nu}(p,t)$ and we used the identity (A.9) for the curvature vector. The global term h is still given by either

$$h_{\mathrm{ap}}(t) = \frac{2\pi}{L_t} \qquad \text{or} \qquad h_{\mathrm{lp}}(t) = \frac{1}{2\pi}\int_{\mathbb{S}^1_{R_t}} \tilde{\kappa}^2 \, ds_t \,,$$

where $\tilde{\kappa}(s,t) = \tilde{\kappa}(s(p,t),t) = \kappa(p,t)$, and $ds_t := ds(p,t) = v(p,t)dp$. Whenever we will calculate via the arc length parametrisation, we will do so at a fixed time.

Remark 3.1. The total time derivative of the arc length parametrisation is given by

$$\frac{d\tilde{F}}{dt}(s(p,t),t) = \frac{\partial \tilde{F}}{\partial s}(s,t)\frac{\partial s}{\partial t}(p,t) + \frac{\partial \tilde{F}}{\partial t}(s,t)$$

for $s = s(p,t) \in \mathbb{S}^1_{R_t}$, $p \in \mathbb{S}^1$ and $t \in (0,T)$. Since $\frac{\partial}{\partial s}\tilde{F}$ is tangential to Σ_t,

$$\left(\frac{d\tilde{F}}{dt}\right)^{\perp} := \left\langle \frac{d\tilde{F}}{dt}, \tilde{\boldsymbol{\nu}} \right\rangle \tilde{\boldsymbol{\nu}} = \frac{\partial \tilde{F}}{\partial t} \,.$$

We only use the partial time derivative in (3.1) and omit the tangential movement since the image $\Sigma_t \subset \mathbb{R}^2$, $t \in (0,T)$, is the same for both motions. Therefore, we will below omit the "$\sim$" above geometric quantities related to $\tilde{F}$ if these depend on s rather than p.

The following evolution equations can also be found in [Pih98, Proposition 5.7 and Lemma 6.12] for the LPCF.

Lemma 3.2 (Evolution equations for geometric quantities)**.** *Let* $F : \mathbb{S}^1 \times (0,T) \to \mathbb{R}^2$ *be a solution of* (2.15)*. Then*

$$\frac{\partial v}{\partial t} = \kappa(h-\kappa)v \tag{3.2}$$

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s} = \frac{\partial}{\partial s}\frac{\partial}{\partial t} - \kappa(h-\kappa)\frac{\partial}{\partial s} \tag{3.3}$$

$$\frac{\partial \boldsymbol{\tau}}{\partial t} = -\frac{\partial \kappa}{\partial s}\boldsymbol{\nu} \tag{3.4}$$

$$\frac{\partial \boldsymbol{\nu}}{\partial t} = \frac{\partial \kappa}{\partial s}\boldsymbol{\tau}\,, \tag{3.5}$$

where v *is the length element and* $\boldsymbol{\tau}$ *is the unit tangent vector to* Σ_t *in direction of the arc length parametrisation.*

Proof of Lemma 3.2. We use the evolution equation (2.15) for the embedding, the definitions (A.3) and (A.4) for $\frac{\partial}{\partial s}$ and $\boldsymbol{\tau}$, and the identity (A.7) for κ to calculate at $s = s(p,t)$

$$\begin{aligned}
\frac{\partial v}{\partial t} &\overset{(A.1)}{=} \frac{\partial}{\partial t}\left\|\frac{\partial F}{\partial p}\right\| = \frac{\partial}{\partial t}\left\langle \frac{\partial F}{\partial p}, \frac{\partial F}{\partial p}\right\rangle^{1/2} = \frac{1}{v}\left\langle \frac{\partial}{\partial t}\frac{\partial F}{\partial p}, \frac{\partial F}{\partial p}\right\rangle \\
&= \frac{1}{v}\left\langle \frac{\partial}{\partial p}\frac{\partial F}{\partial t}, \frac{\partial F}{\partial p}\right\rangle \overset{(2.15)}{=} \frac{1}{v}\left\langle \frac{\partial}{\partial p}\big((h-\kappa)\boldsymbol{\nu}\big), \frac{\partial F}{\partial p}\right\rangle \\
&= \frac{1}{v}(h-\kappa)\left\langle \frac{\partial \boldsymbol{\nu}}{\partial p}, \frac{\partial F}{\partial p}\right\rangle = \frac{1}{v}v^2(h-\kappa)\left\langle \frac{1}{v}\frac{\partial \boldsymbol{\nu}}{\partial p}, \frac{1}{v}\frac{\partial F}{\partial p}\right\rangle \\
&\overset{(A.3),(A.4)}{=} v(h-\kappa)\left\langle \frac{\partial \boldsymbol{\nu}}{\partial s}, \boldsymbol{\tau}\right\rangle \overset{(A.7)}{=} v(h-\kappa)\kappa\,.
\end{aligned}$$

Next, we again use the definition (A.3) of $\frac{\partial}{\partial s}$ and the evolution equation (3.2) of v,

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial}{\partial s} &\overset{(A.3)}{=} \frac{\partial}{\partial t}\left(\frac{1}{v}\frac{\partial}{\partial p}\right) = \frac{1}{v}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial p}\right) + \frac{\partial}{\partial t}\left(\frac{1}{v}\right)\frac{\partial}{\partial p} \\
&\overset{(3.2)}{=} \frac{1}{v}\frac{\partial}{\partial p}\frac{\partial}{\partial t} - \frac{1}{v^2}\kappa(h-\kappa)v\frac{\partial}{\partial p} \\
&\overset{(A.3)}{=} \frac{\partial}{\partial s}\frac{\partial}{\partial t} - \kappa(h-\kappa)\frac{\partial}{\partial s}\,.
\end{aligned}$$

To show (3.4), we apply the definition (A.4) of the vector $\boldsymbol{\tau}$, the evolution equations (3.3) and (2.15) for $\frac{\partial}{\partial s}$ and F respectively, and the Frenet–Serret equation, Lemma A.1, to obtain

$$\begin{aligned}
\frac{\partial \boldsymbol{\tau}}{\partial t} &\overset{(A.4)}{=} \frac{\partial}{\partial t}\frac{\partial \tilde{F}}{\partial s} \overset{(3.3)}{=} \frac{\partial}{\partial s}\frac{\partial \tilde{F}}{\partial t} - \kappa(h-\kappa)\frac{\partial \tilde{F}}{\partial s} \\
&\overset{(2.15)}{=} \frac{\partial}{\partial s}\big((h-\kappa)\boldsymbol{\nu}\big) - \kappa(h-\kappa)\boldsymbol{\tau} \\
&\overset{\text{Lem. A.1}}{=} -\boldsymbol{\nu}\frac{\partial \kappa}{\partial s} + (h-\kappa)\frac{\partial \boldsymbol{\nu}}{\partial s} - (h-\kappa)\frac{\partial \boldsymbol{\nu}}{\partial s} = -\frac{\partial \kappa}{\partial s}\boldsymbol{\nu}\,.
\end{aligned}$$

For (3.5), we observe that

$$0 = \frac{\partial 0}{\partial t} = \frac{\partial}{\partial t}\langle \boldsymbol{\nu}, \boldsymbol{\tau}\rangle = \left\langle \frac{\partial \boldsymbol{\nu}}{\partial t}, \boldsymbol{\tau}\right\rangle + \left\langle \boldsymbol{\nu}, \frac{\partial \boldsymbol{\tau}}{\partial t}\right\rangle,$$

so that (3.4) implies

$$\left\langle \frac{\partial \boldsymbol{\nu}}{\partial t}, \boldsymbol{\tau}\right\rangle = -\left\langle \boldsymbol{\nu}, \frac{\partial \boldsymbol{\tau}}{\partial t}\right\rangle \overset{(3.4)}{=} \frac{\partial \kappa}{\partial s}$$

which together with $\left\langle \frac{\partial}{\partial t}\boldsymbol{\nu}, \boldsymbol{\nu}\right\rangle = 0$ yields

$$\frac{\partial \boldsymbol{\nu}}{\partial t} = \left\langle \frac{\partial \boldsymbol{\nu}}{\partial t}, \boldsymbol{\tau}\right\rangle \boldsymbol{\tau} = \frac{\partial \kappa}{\partial s}\boldsymbol{\tau}\,.$$ □

For $t \in [0, T)$, let

$$L(t) := \int_{\Sigma_t} d\mathcal{H}^1 = \int_{\mathbb{S}^1_{R_t}} ds_t = \int_{\mathbb{S}^1} v\, dp \tag{3.6}$$

be the length of the curve Σ_t, and

$$A(t) := \int_{\Omega_t} d\mathcal{H}^2 \tag{3.7}$$

be the enclosed area of Σ_t where $\Omega_t \subset \mathbb{R}^2$ with $\partial\Omega_t = \Sigma_t$.

Lemma 3.3 (Evolution equations for L and A)**.** *Let $F : \mathbb{S}^1 \times (0,T) \to \mathbb{R}^2$ be a solution of* (2.15)*. Then*

$$\frac{dL}{dt} = 2\pi h - \int_{\mathbb{S}^1_{R_t}} \kappa^2\, ds_t \tag{3.8}$$

and

$$\frac{dA}{dt} = hL - 2\pi\,. \tag{3.9}$$

for all $t \in (0, T)$.

Proof. The evolution equation (3.2) of the length element and Theorem A.2 for the total curvature yield

$$\begin{aligned}\frac{dL}{dt} &\overset{(3.6)}{=} \frac{d}{dt}\left(\int_{\mathbb{S}^1} v\, dp\right) = \int_{\mathbb{S}^1} \frac{\partial v}{\partial t}\, dp \overset{(3.2)}{=} \int_{\mathbb{S}^1} \kappa(h-\kappa) v\, dp \\ &= h\int_{\mathbb{S}^1_{R_t}} \kappa\, ds_t - \int_{\mathbb{S}^1_{R_t}} \kappa^2\, ds_t \overset{\text{Thm. A.2}}{=} 2\pi h - \int_{\mathbb{S}^1_{R_t}} \kappa^2\, ds_t\,.\end{aligned}$$

The first variation of the area formula, Theorem B.6, the divergence theorem, Theorem B.7, the evolution equation (2.15) of the embedding, and again Theorem A.2 imply

$$\begin{aligned}\frac{dA}{dt} &\overset{(3.7)}{=} \frac{d}{dt}\int_{\Omega_t} d\mathcal{H}^2 \overset{\text{Thm. B.6}}{=} \int_{\Omega_t} \operatorname{div}_{\Omega_t}\left(\frac{\partial F}{\partial t}\right) d\mathcal{H}^2 \\ &\overset{\text{Thm. B.7}}{=} \int_{\Sigma_t} \left\langle \frac{\partial F}{\partial t}, \boldsymbol{\nu}\right\rangle d\mathcal{H}^1 \overset{(2.15)}{=} \int_{\Sigma_t} (h-\kappa)\langle \boldsymbol{\nu}, \boldsymbol{\nu}\rangle\, d\mathcal{H}^1 \\ &= \int_{\Sigma_t} (h-\kappa)\, d\mathcal{H}^1 = h\int_{\Sigma_t} d\mathcal{H}^1 - \int_{\Sigma_t} \kappa\, d\mathcal{H}^1 \overset{\text{Thm. A.2}}{=} hL - 2\pi\,.\end{aligned}$$ □

Lemma 3.4 (Isoperimetric inequality)**.** *For an embedded closed curve* Σ *in the plane,*

$$L^2 \geq 4\pi A$$

with equality if and only if Σ *is a circle.*

Corollary 3.5 (L and A under the APCSF)**.** *Under the APCSF, for* $t \in [0, T)$*, the enclosed area* $A_t \equiv A_0$ *is constant and the length of the curve* L_t *is strictly decreasing unless* Σ_t *is a circle of radius* $\sqrt{A_0/\pi}$*. Consequently,*

$$L_0 \geq L_t \geq 2\sqrt{\pi A_0}$$

for all $t \in (0, T)$ *with equalities if and only if* Σ_t *is a circle.*

Proof. Recall that $h_{\mathrm{ap}} = 2\pi/L$. We use the evolution equation (3.9) for the enclosed volume of the curve and conclude that $\frac{d}{dt}A = 0$ which implies $A_t = A_0$ for all $t \in [0, T)$. The evolution equation (3.8) for the length of the curve, Cauchy–Schwarz (B.3), and Theorem A.2 for the total curvature yield

$$\frac{dL}{dt} \overset{(3.8),(B.3)}{\leq} \frac{4\pi^2}{L} - \frac{1}{L}\left(\int_{\Sigma_t} \kappa \, d\mathcal{H}^1\right)^2 \overset{\text{Thm. A.2}}{=} \frac{4\pi^2}{L} - \frac{(2\pi)^2}{L} = 0$$

with equality if and only if the curvature is constant on Σ_t. The isoperimetric inequality, Lemma 3.4, leads to the lower bound $L_t^2 \geq 4\pi A_t = 4\pi A_0$, with equality only on the circle. If Σ_t is a circle, the radius is thus given by $L_t/(2\pi) = \sqrt{A_0/\pi}$. □

Corollary 3.6 (L and A under the LPCF)**.** *Under the LPCF, for* $t \in [0, T)$*, the length of the curve* $L_t \equiv L_0$ *is constant and the enclosed area* A_t *is strictly increasing unless* Σ_t *is a circle of radius* $L_0/(2\pi)$*. Consequently,*

$$A_0 \leq A_t \leq L_0^2/(4\pi)$$

for all $t \in (0, T)$ *with equalities if and only if* Σ_t *is a circle.*

Proof. Recall that $h_{\mathrm{lp}} = \int_{\mathbb{S}^1_{R_t}} \kappa^2 \, ds_t / 2\pi$. The evolution equation (3.8) for the length of the curve then immediately implies that $\frac{d}{dt}L = 0$ and $L_t = L_0$ for all $t \in [0, T)$. By the evolution equation (3.9) for the enclosed volume of the curve and by Cauchy–Schwarz (B.3), we obtain

$$\frac{dA}{dt} \geq \frac{2\pi^2}{2\pi L} L - 2\pi = 0$$

with equality if and only if the curvature is constant on Σ_t. This holds for embedded curves if and only if Σ_t is a circle of radius $L_t/2\pi = L_0/2\pi$. Furthermore, the isoperimetric inequality, Lemma 3.4, leads to the upper bound $4\pi A_t \leq L_t^2 = L_0^2$ with equality only on the circle. □

Remark 3.7. Since, by Corollary 3.6, for the LPCF the length of the curve is indeed not changing, we can parametrise by arc length via the arc length parameter $s : \mathbb{S}^1 \times [0, T) \to \mathbb{R}$ with $s(\mathbb{S}^1, t) = \mathbb{S}^1_{R_t} \equiv \mathbb{S}^1_{R_0}$ for all $t \in [0, T)$. The arc length parametrisation $\tilde{F} : \mathbb{S}^1_{R_0} \times (0, T) \to \mathbb{R}^2$ evolves according to (3.1) and

$$h_{\mathrm{lp}}(t) = \frac{1}{2\pi} \int_{\mathbb{S}^1_{R_0}} \kappa^2 \, ds_t \, .$$

Lemma 3.8. *Let $F : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ be a solution of* (2.15) *with initial curve Σ_0. Then,*

$$h_{\mathrm{ap}}(t) \le |\kappa|_{\max}(t) \qquad \text{and} \qquad h_{\mathrm{lp}}(t) \le \min\left\{ \frac{|\kappa|_{\max}(t)}{2\pi} \int_{\mathbb{S}^1_{R_0}} |\kappa| \, ds_t, \, \frac{L(\Sigma_0)}{2\pi} \kappa^2_{\max}(t) \right\}$$

for all $t \in [0,T)$.

Proof. For all $t \in [0,T)$, we can estimate the global terms by

$$h_{\mathrm{ap}} = \frac{1}{L} \int_{\mathbb{S}^1_{R_t}} \kappa \, ds_t \le |\kappa|_{\max} \frac{1}{L} \int_{\mathbb{S}^1_{R_t}} ds_t = |\kappa|_{\max} \, .$$

and

$$h_{\mathrm{lp}} = \frac{1}{2\pi} \int_{\mathbb{S}^1_{R_0}} \kappa^2 \, ds_t \le \frac{|\kappa|_{\max}}{2\pi} \int_{\mathbb{S}^1_{R_0}} |\kappa| \, ds_t \, .$$

□

Lemma 3.9 (Evolution and bounds for h_{ap})**.** *Let $F : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ be a solution of* (2.15) *with initial curve Σ_0. Then h_{ap} is strictly increasing on $[0,T)$ unless Σ_t is a circle of radius $\sqrt{A_0/\pi}$. Consequently,*

$$0 < \frac{2\pi}{L_0} = h_{\mathrm{ap}}(0) \le h_{\mathrm{ap}}(t) \le \sqrt{\frac{\pi}{A_0}} \tag{3.10}$$

for all $t \in (0,T)$ with equalities if and only if Σ_t is a circle.

Proof. Recall that $h_{\mathrm{ap}} = 2\pi/L$ and apply Corollary 3.5. □

Lemma 3.10 (Lower bound for h_{lp})**.** *Let $F : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ be a solution of* (2.15) *with initial curve Σ_0. Then*

$$0 < \frac{2\pi}{L_0} \le h_{\mathrm{lp}}(t) \tag{3.11}$$

for all $t \in (0,T)$ with equality in the middle inequality if and only if Σ_t is a circle.

Proof for Lemma 3.10. Cauchy–Schwarz (B.3) and Theorem A.2 for the total curvature yield

$$h_{\mathrm{lp}} = \frac{1}{2\pi} \int_{\mathbb{S}^1_{R_0}} \kappa^2 \, ds_t \overset{\text{(B.3)}}{\ge} \frac{1}{2\pi L_0} \left(\int_{\mathbb{S}^1_{R_0}} \kappa \, ds_t \right)^2 \overset{\text{Thm. A.2}}{=} \frac{(2\pi)^2}{2\pi L_0} = \frac{2\pi}{L_0}$$

with equality if and only if Σ_t is a circle. □

Lemma 3.11. *Let $F : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ be a solution of* (2.15) *with initial curve Σ_0. Then*

$$v(p,t) = \exp\left(\int_0^t \kappa(h - \kappa)(p,\tau) \, d\tau \right) v(p,0)$$

for every $(p,t) \in \mathbb{S}^1 \times (0,T)$. Hence, the curve Σ_t is regular for all $t \in [0,T)$ (see Section A.1 for a definition).

Proof. The claim follows directly from the evolution equation (3.2) of the length element and Lemma B.1. □

Corollary 3.12. *Let $F : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ be a solution of (2.15) with initial curve Σ_0 and let $t_0 \in [0,T)$. Then the operator $\frac{\partial^2}{\partial s^2}$ is uniformly elliptic on Σ_t for all $t \in [0,t_0]$.*

Proof. For $(p,t) \in \mathbb{S}^1 \times [0,T)$ and $s = s(p,t) \in \mathbb{S}^1_{R_t}$,

$$\frac{\partial^2}{\partial s^2} = \frac{1}{v}\frac{\partial}{\partial p}\left(\frac{1}{v}\frac{\partial}{\partial p}\right) = \frac{1}{v^2}\frac{\partial^2}{\partial p^2} - \frac{1}{v^4}\left\langle \frac{\partial F}{\partial p}, \frac{\partial^2 F}{\partial p^2}\right\rangle \frac{\partial}{\partial p}\,.$$

Since $F \in C^\infty(\mathbb{S}^1 \times [0,T))$, $\frac{\partial}{\partial p}F$, $\frac{\partial^2}{\partial p^2}F$ and $|\kappa|_{\max}$ are bounded on $[0,t_0]$. Lemma 3.11 yields that $1/v$ is bounded on $[0,t_0]$ so that the operator $\frac{\partial^2}{\partial s^2}$ is elliptic on $[0,t_0]$. □

Next, we show that the solution stays in a bounded region during the time of existence if $T < \infty$. We thank Theodora Bourni for the idea to the proof of the following Lemma.

Lemma 3.13 (Boundedness on finite time intervals)**.** *Let $F : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ be a solution of (2.15) with initial curve Σ_0. Let $T < \infty$ and suppose that $\sup_{[0,T)} h < \infty$. Then there exists a constant $c = c\big(|\kappa|_{\max}(0), \sup_{[0,T)} h\big) < \infty$ such that*

$$\|F(p,t) - F(p,0)\| \le 2ct$$

for all $p \in \mathbb{S}^1$ and $t \in [0,T)$.

Proof. Set

$$c := \max\left\{|\kappa|_{\max}(0), \sup_{[0,T)} h\right\}$$

and let $\mathbf{v} \in \mathbb{R}^2$, $\|\mathbf{v}\| = 1$, be arbitrary. We use the evolution equation (3.1) and the identity (A.9) of the curvature vector to calculate

$$\begin{aligned}&\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial s^2}\right)\big(\langle F(p,t) - F(p,0), \mathbf{v}\rangle + 2ct\big)\\ &\overset{(3.1),(A.9)}{=} h(t)\langle \boldsymbol{\nu}(p,t), \mathbf{v}\rangle - \kappa(p,0)\,\langle \boldsymbol{\nu}(p,0), \mathbf{v}\rangle + 2c\\ &\ge -\sup_{[0,T)} h - |\kappa|_{\max}(0) + 2c \ge 0\end{aligned}$$

and

$$\begin{aligned}&\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial s^2}\right)\big(\langle F(p,t) - F(p,0), \mathbf{v}\rangle - 2ct\big)\\ &= h(t)\langle \boldsymbol{\nu}(p,t), \mathbf{v}\rangle - \kappa(p,0)\,\langle \boldsymbol{\nu}(p,0), \mathbf{v}\rangle - 2c\\ &\le \sup_{[0,T)} h + |\kappa|_{\max}(0) - 2c \le 0\end{aligned}$$

for $s \in \mathbb{S}^1_{R_t}$ and $t \in (0,T)$. The weak maximum principle, Theorem B.16, yields

$$-2ct \le \langle F(p,t) - F(p,0), \mathbf{v}\rangle \le 2ct$$

for all $p \in \mathbb{S}^1$ and $t \in [0,T)$. For fixed $p \in \mathbb{S}^1$ and $t \in (0,T)$, choose $\mathbf{v} = F(p,t) - F(p,0)/\|F(p,t) - F(p,0)\|$ to obtain

$$\|F(p,t) - F(p,0)\| \le 2ct\,.$$

□

Chapter 4

Estimates on curvature derivatives

In this chapter we bound the derivatives of the curvature for smooth solutions of (2.15) like in [Pih98, Section 6.3]. His methods work for both flows. In [Gag86] very little detail is given. We also show that if the maximal existence time T is finite, the maximum of the curvature has to blow up when time approaches T.

Lemma 4.1 (Evolution equations for the curvature)**.** *Let $F : \mathbb{S}^1 \times (0, T) \to \mathbb{R}^2$ be a solution of* (2.15)*. Then*

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial s^2}\right)\kappa = (\kappa - h)\kappa^2 \tag{4.1}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial s^2}\right)\frac{\partial^n \kappa}{\partial s^n} = \big((n+3)\kappa - (n+2)h\big)\kappa \frac{\partial^n \kappa}{\partial s^n} + P_{n-1} \tag{4.2}$$

for all $t \in (0, T)$ and $n \in \mathbb{N}$, where

$$P_n = P_n\left(h, \kappa, \frac{\partial \kappa}{\partial s}, \ldots, \frac{\partial^n \kappa}{\partial s^n}\right) \tag{4.3}$$

is a polynomial in all its entries and $P_0 \equiv 0$.

Remark 4.2. The evolution equation (4.1) of the curvature consists of a diffusion term $\frac{\partial^2}{\partial s^2}\kappa$ and a reaction term $(\kappa - h)\kappa^2$. For the classical CSF (1.1), the reaction term is κ^3 which causes the curvature to blow up in finite time. For the constrained curve flow, this behaviour is weakened by the global term for points with positive curvature and amplified for points with negative curvature.

Proof. We follow the lines of [Pih98, Proposition 5.7 and Lemma 6.13]. For (4.1), we use the definition (A.6) of κ, the evolution equations (3.3), (3.4) and (3.5) for $\frac{\partial}{\partial s}$, $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ respectively, the fact that $\left\langle \frac{\partial}{\partial s}\boldsymbol{\tau}, \boldsymbol{\tau} \right\rangle = 0$, as well as the Frenet–Serret equation, Lemma A.1, to obtain

$$\begin{aligned}
\frac{\partial \kappa}{\partial t} &\overset{(A.6)}{=} -\frac{\partial}{\partial t}\left\langle \boldsymbol{\nu}, \frac{\partial \boldsymbol{\tau}}{\partial s} \right\rangle = -\left\langle \frac{\partial \boldsymbol{\nu}}{\partial t}, \frac{\partial \boldsymbol{\tau}}{\partial s} \right\rangle - \left\langle \boldsymbol{\nu}, \frac{\partial}{\partial t}\frac{\partial \boldsymbol{\tau}}{\partial s} \right\rangle \\
&\overset{(3.5),(3.3)}{=} -\left\langle \frac{\partial \kappa}{\partial s}\boldsymbol{\tau}, \frac{\partial \boldsymbol{\tau}}{\partial s} \right\rangle - \left\langle \boldsymbol{\nu}, \frac{\partial}{\partial s}\frac{\partial \boldsymbol{\tau}}{\partial t} - \kappa(h-\kappa)\frac{\partial \boldsymbol{\tau}}{\partial s} \right\rangle \\
&\underset{(3.4)}{\overset{\text{Lem. A.1}}{=}} -\left\langle \boldsymbol{\nu}, \frac{\partial}{\partial s}\left(-\frac{\partial \kappa}{\partial s}\boldsymbol{\nu}\right) \right\rangle + \kappa(h-\kappa)\left\langle \boldsymbol{\nu}, \frac{\partial \boldsymbol{\tau}}{\partial s} \right\rangle \overset{(A.6)}{=} \frac{\partial^2 \kappa}{\partial s^2} - \kappa^2(h-\kappa)\,.
\end{aligned}$$

We prove (4.2) by induction over n. The rule (3.3) for interchanging the spatial and time derivative and the evolution equations (4.1) for the curvature yield

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial\kappa}{\partial s} &\overset{(3.3)}{=} \frac{\partial}{\partial s}\frac{\partial\kappa}{\partial t} - (h-\kappa)\kappa\frac{\partial\kappa}{\partial s}\\ &\overset{(4.1)}{=} \frac{\partial}{\partial s}\left(\frac{\partial^2\kappa}{\partial s^2} - (h-\kappa)\kappa^2\right) - (h-\kappa)\kappa\frac{\partial\kappa}{\partial s}\\ &= \frac{\partial^3\kappa}{\partial s^3} - 2h\kappa\frac{\partial\kappa}{\partial s} + 3\kappa^2\frac{\partial\kappa}{\partial s} - (h-\kappa)\kappa\frac{\partial\kappa}{\partial s}\\ &= \frac{\partial^3\kappa}{\partial s^3} + (4\kappa-3h)\kappa\frac{\partial\kappa}{\partial s}\,.\end{aligned}$$

For the induction step, we assume (4.2) to hold for all $i \in \{1,\dots,n\}$ for a fixed but arbitrary $n \in \mathbb{N}$. We abbreviate

$$\kappa' := \frac{\partial\kappa}{\partial s} \qquad \text{and} \qquad \kappa^{(n)} := \frac{\partial^n\kappa}{\partial s^n}$$

and again use (3.3) to calculate

$$\begin{aligned}\frac{\partial}{\partial t}\kappa^{(n+1)} &\overset{(3.3)}{=} \frac{\partial}{\partial s}\frac{\partial}{\partial t}\kappa^{(n)} - (h-\kappa)\kappa\frac{\partial}{\partial s}\kappa^{(n)}\\ &\overset{(4.2)}{=} \frac{\partial}{\partial s}\left(\frac{\partial^2\kappa^{(n)}}{\partial s^2} + \big((n+3)\kappa-(n+2)h\big)\kappa\kappa^{(n)} + P_{n-1}\right) + (\kappa-h)\kappa\kappa^{(n+1)}\\ &= \frac{\partial^3\kappa^{(n)}}{\partial s^3} + (n+3)2\kappa\kappa'\kappa^{(n)} - (n+2)h\kappa'\kappa^{(n)}\\ &\quad + \big((n+3)\kappa-(n+2)h\big)\kappa\kappa^{(n+1)} + \frac{\partial P_{n-1}}{\partial s} + (\kappa-h)\kappa\kappa^{(n+1)}\\ &= \frac{\partial^2\kappa^{(n+1)}}{\partial s^2} + \big((n+4)\kappa-(n+3)h\big)\kappa\kappa^{(n+1)} + P_n\,.\end{aligned}$$

□

Corollary 4.3 (Pihan [Pih98, Proposition 5.8], see also [Hui87, Theorem 1.3])**.** *Let $F : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ be a solution of* (2.15) *and let $\kappa \geq 0$ on Σ_0. Then $\kappa > 0$ on Σ_t for all $t \in (0,T)$.*

Proof. Assume that there exists a point $(p_0,t_0) \in \mathbb{S}^1 \times (0,T)$ with $\kappa(p_0,t_0) = 0$. We estimate

$$|(h-\kappa)\kappa| \overset{\text{Lem. 3.8}}{\leq} c(\Sigma_0)|\kappa|^3_{\max} < \infty$$

on $\mathbb{S}^1 \times [0,T)$. Hence, we can apply the strong maximum principle, Theorem B.17(iii), with respect to the evolution equation (4.1) of κ and obtain that $\kappa \equiv 0$ on $\mathbb{S}^1 \times [0,t_0]$. Since Σ_t is closed for $t \in [0,t_0]$, this is a contradiction. □

Corollary 4.4. *Let $F : \mathbb{S}^1 \times (0,T) \to \mathbb{R}^2$ be a solution of* (2.15)*. Then*

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial s^2}\right)\kappa^2 = -2\left(\frac{\partial\kappa}{\partial s}\right)^2 + 2(\kappa-h)\kappa^3 \tag{4.4}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial s^2}\right)\left(\frac{\partial^n\kappa}{\partial s^n}\right)^2 &= -2\left(\frac{\partial^{n+1}\kappa}{\partial s^{n+1}}\right)^2 + 2\big((n+3)\kappa-(n+2)h\big)\kappa\left(\frac{\partial^n\kappa}{\partial s^n}\right)^2\\ &\quad + 2\frac{\partial^n\kappa}{\partial s^n}P_{n-1}\end{aligned} \tag{4.5}$$

for all $t \in (0,T)$ and $n \in \mathbb{N}$, where P_{n-1} is defined in (4.3)*.*

Proof. We follow the lines of [Pih98, Corollary 6.14]. The evolution equation (4.1) of the curvature implies

$$\frac{\partial \kappa^2}{\partial t} = 2\kappa \frac{\partial^2 \kappa}{\partial s^2} + 2(\kappa - h)\kappa^3 .$$

Furthermore,

$$\frac{\partial^2 \kappa^2}{\partial s^2} = \frac{\partial}{\partial s}\left(2\kappa \frac{\partial \kappa}{\partial s}\right) = 2\kappa \frac{\partial^2 \kappa}{\partial s^2} + 2\left(\frac{\partial \kappa}{\partial s}\right)^2 .$$

Subtracting the above two equalities yields (4.4). To prove (4.5), we again abbreviate $\kappa^{(n)} := \frac{\partial^n}{\partial s^n}\kappa$. By the evolution equation (4.2),

$$\frac{\partial}{\partial t}\left(\kappa^{(n)}\right)^2 = 2\kappa^{(n)}\frac{\partial^2 \kappa^{(n)}}{\partial s^2} + 2\big((n+3)\kappa - (n+2)h\big)\kappa\left(\kappa^{(n)}\right)^2 + 2\kappa^{(n)}P_{n-1}$$

and again

$$\frac{\partial^2}{\partial s^2}\left(\kappa^{(n)}\right)^2 = \frac{\partial}{\partial s}\left(2\kappa^{(n)}\frac{\partial \kappa^{(n)}}{\partial s}\right) = 2\kappa^{(n)}\frac{\partial^2 \kappa^{(n)}}{\partial s^2} + 2\left(\frac{\partial \kappa^{(n)}}{\partial s}\right)^2 .$$

Subtracting the above two equalities yields the claim. □

Proposition 4.5 (Bounds on curvature derivatives)**.** *Let $F : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ be a solution of (2.15) with initial curve Σ_0. Let $n \in \mathbb{N}$ and, for $l \in \{0,\dots,n-1\}$, C_l be a constant such that*

$$\sup_{t\in[0,T)} \left|\frac{\partial^l \kappa}{\partial s^l}\right|_{\max}(t) \le C_l .$$

Then there exists a constant $C_n = C_n\big(n, C_0, \dots, C_{n-1}, \left|\frac{\partial^n}{\partial s^n}\kappa\right|_{\max}(0)\big)$ such that

$$\sup_{t\in[0,T)} \left|\frac{\partial^n \kappa}{\partial s^n}\right|_{\max} \le C_n .$$

Proof. We prove the claim by induction over n in a similar fashion to [Pih98, Proposition 6.15]. Assume that there exists a constant C_0 so that $|\kappa|_{\max} \le C_0$ on $[0,T)$. We abbreviate $\kappa' := \frac{\partial}{\partial s}\kappa$ and $\kappa^{(n)} := \frac{\partial^n}{\partial s^n}\kappa$. Let $\Lambda = \Lambda(C_0)$ be a positive constant to be chosen later. The evolution equations (4.4) and (4.5), and the estimate $h \le c(C_0)$ from Lemma 3.8 imply

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial s^2}\right)\big((\kappa')^2 + \Lambda\kappa^2\big) &\overset{(4.4),(4.5)}{=} -2\left(\frac{\partial \kappa'}{\partial s}\right)^2 + 2(4\kappa - 3h)\kappa\big(\kappa'\big)^2 \\ &\qquad - 2\Lambda\left(\frac{\partial \kappa}{\partial s}\right)^2 + 2\Lambda(\kappa - h)\kappa^3 \\ &\overset{\text{Lem. 3.8}}{\le} 2\big((4\kappa - 3h)\kappa - \Lambda\big)\big(\kappa'\big)^2 + c_1(C_0) . \end{aligned} \tag{4.6}$$

We estimate

$$(4\kappa - 3h)\kappa \le \lambda(C_0)$$

and set

$$\Lambda(C_0) := \lambda + \frac{1}{2} .$$

Then, $(4\kappa - 3h)\kappa - \Lambda \leq -1/2$ and

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial s^2}\right)\big((\kappa')^2 + \Lambda\kappa^2\big) \overset{(4.6)}{\leq} -(\kappa')^2 + c_1 = -(\kappa')^2 - \Lambda\kappa^2 + \Lambda\kappa^2 + c_1$$
$$\leq -\big((\kappa')^2 + \Lambda\kappa^2\big) + c_2(C_0)\,.$$

Assume, that $\big((\kappa')^2 + \Lambda\kappa^2\big)$ reaches a value

$$K > \max\big\{\big((\kappa')^2 + \Lambda\kappa^2\big)_{\max}(0), c_2\big\}$$

for the first time. Then, at that time

$$0 \leq \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial s^2}\right)\big((\kappa')^2 + \Lambda\kappa^2\big) \leq -K + c_2 < 0$$

which is a contradiction. Hence,

$$\big((\kappa')^2 + \Lambda\kappa^2\big)_{\max}(t) \leq \max\big\{\big((\kappa')^2 + \Lambda\kappa^2\big)_{\max}(0), c_2\big\}$$

and thus

$$|\kappa'|_{\max}(t) \leq C_1(C_0, |\kappa'|_{\max}(0))$$

for all $t \in [0, T)$. The induction step is done in the same manner. We assume for arbitrary but fixed $n \in \mathbb{N}$ that there exist constants C_l for $l \in \{0, \ldots, n-1\}$ so that

$$\sup_{t \in [0,T)} \left|\frac{\partial^l \kappa}{\partial s^l}\right|_{\max}(t) \leq C_l\,.$$

Let Λ be again a positive constant to be chosen later. We use Young's inequality (B.1) to estimate $2\kappa^{(n+1)}P_n \leq 2(\kappa^{(n+1)})^2 + 2(P_n)^2$ and the evolution equation (4.5) to obtain

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial s^2}\right)\left(\left(\kappa^{(n+1)}\right)^2 + \Lambda\left(\kappa^{(n)}\right)^2\right) \\
&= -2\left(\frac{\partial \kappa^{(n+1)}}{\partial s}\right)^2 + 2\big((n+4)\kappa - (n+3)h\big)\kappa\left(\kappa^{(n+1)}\right)^2 + 2\kappa^{(n+1)}P_n \\
&\quad - 2\Lambda\left(\frac{\partial \kappa^{(n)}}{\partial s}\right)^2 + 2\Lambda\big((n+2)\kappa - (n+1)h\big)\kappa\left(\kappa^{(n)}\right)^2 + 2\Lambda\kappa^{(n)}P_{n-1} \\
&\leq 2\Big(\big((n+4)\kappa - (n+3)h\big)\kappa + 1 - \Lambda\Big)\left(\kappa^{(n+1)}\right)^2 + c_3(\Lambda)P_n\,. \qquad (4.7)
\end{aligned}$$

We estimate

$$\big((n+4)\kappa - (n+3)h\big)\kappa \leq \lambda(n, C_0)$$

and choose

$$\Lambda(n, C_0) := \lambda + \frac{3}{2}\,.$$

Then

$$\big((n+4)\kappa - (n+3)h\big)\kappa + 1 - \Lambda \leq -\frac{1}{2}$$

and from (4.7) it follows that

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial s^2}\right)\left(\left(\kappa^{(n+1)}\right)^2+\Lambda\left(\kappa^{(n)}\right)^2\right) \leq -\left(\kappa^{(n+1)}\right)^2+c_3(C_0)P_n$$
$$\leq -\left(\left(\kappa^{(n+1)}\right)^2+\Lambda\left(\kappa^{(n)}\right)^2\right)+c_4\,,$$

where $c_4 = c_4(n, C_0, \ldots, C_n)$. Like before, we assume that $\left(\kappa^{(n+1)}\right)^2+\Lambda\left(\kappa^{(n)}\right)^2$, reaches a value

$$K > \max\left\{\left(\left(\kappa^{(n+1)}\right)^2+\Lambda\left(\kappa^{(n)}\right)^2\right)_{\max}(0), c_4\right\}$$

for the first time. Then, at that time,

$$0 \leq \left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial s^2}\right)\left(\left(\kappa^{(n+1)}\right)^2+\Lambda\left(\kappa^{(n)}\right)^2\right) \leq -K + c_4 < 0$$

which is a contradiction. Hence,

$$\left(\left(\kappa^{(n+1)}\right)^2+\Lambda\left(\kappa^{(n)}\right)^2\right)_{\max}(t) \leq \max\left\{\left(\left(\kappa^{(n+1)}\right)^2+\Lambda\left(\kappa^{(n)}\right)^2\right)_{\max}(0), c_4\right\}$$

and thus

$$\left|\kappa^{(n+1)}\right|_{\max}(t) \leq c\Big(C_0, \ldots, C_n, \left|\kappa^{(n+1)}\right|_{\max}(0)\Big) =: C_{n+1}$$

for all $t \in [0, T)$. □

Lemma 4.6. *Let $F : \mathbb{S}^1 \times (0, T) \to \mathbb{R}^2$ be a solution of* (2.15). *Then,*

$$\left(\frac{\partial^m}{\partial t^m}-\frac{\partial^{2m}}{\partial s^{2m}}\right)\frac{\partial^n \kappa}{\partial s^n} = P_1 \frac{\partial^{n+2m-2}\kappa}{\partial s^{n+2m-2}} + P_{n+2m-3} \tag{4.8}$$

for all $t \in (0, T)$ and $n, m \in \mathbb{N}$, where P_n is defined in (4.3).

Proof. We prove the claim by induction over m. The evolution equation (4.2) yields the claim for $m = 1$. We assume that (4.8) holds for some $m \geq 2$ and use again (4.2) to obtain

$$\begin{aligned}
\frac{\partial^{m+1}}{\partial t^{m+1}}\kappa^{(n)} &= \frac{\partial}{\partial t}\Big(\kappa^{(n+2)} + P_1\kappa^{(n+2m-2)} + P_{n+2m-3}\Big)\\
&\overset{(4.2)}{=} \kappa^{(n+2m+2)} + \big((n+2m+3)\kappa - (n+2m+2)h\big)\kappa\kappa^{(n+2m)} + P_{n+2m-1}\\
&\quad + P_1\kappa^{(n+2m-1)} + P_{n+2m-2}\\
&= \kappa^{(n+2(m+1))} + P_1\kappa^{(n+2(m+1)-2)} + P_{n+2(m+1)-3}\,.
\end{aligned}$$

□

Corollary 4.7. *Let $F : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ be a solution of* (2.15) *with initial curve Σ_0. Let $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ and*

$$\sup_{t\in[0,T)} |\kappa|_{\max}(t) \leq C_0\,.$$

Then there exist constants $C_{n,m} = C_{n,m}\big(n, m, \Sigma_0, C_0\big)$ such that

$$\sup_{t\in[0,T)} \left|\frac{\partial^m}{\partial t^m}\frac{\partial^n \kappa}{\partial s^n}\right|_{\max}(t) \leq C_{n,m}\,.$$

Proof. The case $m = 0$ and $n = 1$ is Proposition 4.5, where

$$C_1 = C_1\left(C_0, \left|\frac{\partial \kappa}{\partial s}\right|_{\max}(0)\right) = C_1(\Sigma_0, C_0)\,.$$

For $m = 0$ and $n = 2$, Proposition 4.5 yields

$$\sup_{t\in[0,T)} \left|\frac{\partial^2 \kappa}{\partial s^2}\right|_{\max}(t) \le C_2$$

with

$$C_2 = C_2\left(C_0, C_1, \left|\frac{\partial^2 \kappa}{\partial s^2}\right|_{\max}(0)\right) = C_2(\Sigma_0, C_0)\,.$$

For arbitrary $m, n \in \mathbb{N}$, Lemma 4.6 and again Proposition 4.5 imply

$$\left(\frac{\partial^m}{\partial t^m}\frac{\partial^n \kappa}{\partial s^n}\right)^2 \overset{\text{Lem. 4.6}}{=} \left(\frac{\partial^{n+2m}\kappa}{\partial s^{n+2m}} + P_1 \frac{\partial^{n+2m-2}\kappa}{\partial s^{n+2m-2}} + P_{n+2m-3}\right)^2 \overset{\text{Lem. 4.5}}{\le} C_{n+2m} =: C_{n,m}\,,$$

where

$$C_{n,m} = C_{n,m}\left(n, m, C_0, \ldots, C_{n+2m-1}, \left|\frac{\partial^{n+2m}\kappa}{\partial s^{n+2m}}\right|_{\max}(0)\right) = C_{n,m}(n, m, \Sigma_0, C_0)\,. \qquad \square$$

Corollary 4.8. *Let $F : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ be a solution of* (2.15) *with initial curve Σ_0. Let $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ and*

$$\sup_{t\in[0,T)} |\kappa|_{\max}(t) \le C_0\,.$$

Then there exist constants $\bar{C}_{n,m} = \bar{C}_{n,m}(n, m, T, \Sigma_0, C_0)$ such that

$$\sup_{t\in[0,T)} \left|\frac{\partial^m}{\partial t^m}\frac{\partial^n \kappa}{\partial p^n}\right|_{\max}(t) \le \bar{C}_{n,m}\,.$$

Proof. Lemmata 3.8 and 3.11 implies

$$0 < c_1(T, \Sigma_0, C_0) \le v(p,t) \le c_2(T, \Sigma_0, C_0) \tag{4.9}$$

for all $(p,t) \in \mathbb{S}^1 \times [0,T)$. By identity (A.3) for the arc length differentiation and Proposition 4.5,

$$\left|\frac{\partial \kappa}{\partial p}(p,t)\right| \overset{\text{(A.3)}}{=} \left|v(p,t)\frac{\partial \kappa}{\partial s}(s(p,t),t)\right| \underset{\text{(4.9)}}{\overset{\text{Prop. 4.5}}{\le}} \bar{C}_1(T, \Sigma_0, C_0)$$

for all $(p,t) \in \mathbb{S}^1 \times [0,T)$. Furthermore, we estimate

$$\left|v\frac{\partial v}{\partial s}(p,t)\right| = \left|\frac{\partial v}{\partial p}(p,t)\right| = v^{-1}\left|\left\langle \frac{\partial F}{\partial p}, \frac{\partial^2 F}{\partial p^2}\right\rangle\right| \overset{\text{(A.8)}}{\le} v^2|\kappa| \overset{\text{(4.9)}}{\le} c(T, \Sigma_0, C_0)$$

so that, for $n \in \mathbb{N}$, by Proposition 4.5 and an induction argument over n,

$$\left|\frac{\partial^n \kappa}{\partial p^n}\right| = \left|\left(v\frac{\partial}{\partial s}\right)^n \kappa\right| \overset{\text{Prop. 4.5}}{\le} \bar{C}_n(n, T, \Sigma_0, C_0, C_1, \ldots, C_n) = \bar{C}_n(n, T, \Sigma_0, C_0)\,. \tag{4.10}$$

For $n, m \in \mathbb{N} \cup \{0\}$, we use Lemma 4.6 and (4.10) to obtain

$$\left|\frac{\partial^m}{\partial t^m}\frac{\partial^n \kappa}{\partial p^n}\right| \overset{\text{Lem. 4.6}}{=} |P_{n+2m}| \overset{\text{(4.10)}}{\le} \bar{C}_{n+2m} =: \bar{C}_{n,m}(n, m, T, \Sigma_0, C_0)\,. \qquad \square$$

Proposition 4.9. *Let $F : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ be a solution of* (2.15) *with initial curve Σ_0, where T is the maximal time of existence. If $T < \infty$, then $|\kappa|_{\max} \to \infty$ for $t \to T$.*

Proof. We follow the lines of [Hui84, Theorem 8.1] and show the claim by contradiction. Assume that there exists a constant $C_0 < \infty$ with $\sup_{[0,T)} |\kappa|_{\max} \le C_0$. By Lemma 3.8,

$$\sup_{[0,T)} h \le c(\Sigma_0, C_0)\,. \tag{4.11}$$

Let $(t_k)_{k\in\mathbb{N}}$ be a sequence in $[0,T)$ with $t_k \nearrow T$ for $k \to \infty$. Then, for all $p \in \mathbb{S}^1$ and $k < l$, by the evolution equation (2.15) and (4.11),

$$\|F(p,t_l) - F(p,t_k)\| \le \int_{t_k}^{t_l} \left\| \frac{\partial F}{\partial t} \right\| dt \overset{(2.15)}{=} \int_{t_k}^{t_l} |h-\kappa|\, dt \overset{(4.11)}{\le} 2C_0(t_l - t_k)\,.$$

Hence, $(F(\,\cdot\,,t_k))_{k\in\mathbb{N}}$ is a uniform Cauchy sequence of smooth functions on $\mathbb{S}^1$ which, by Theorem B.13, has a continuous limit $F_T : \mathbb{S}^1 \to \mathbb{R}^2$. By standard real analysis F_T is independent of the chosen sequence. By Corollary 4.8, for all $n, m \in \mathbb{N}\cup\{0\}$, $i \in \{0,\ldots,n\}$ and $j \in \{0,\ldots,m\}$, there exist constants $C_{i,j} = C_{i,j}(i,j,T,\Sigma_0,C_0)$, so that, for all $p \in \mathbb{S}^1$ and $k < l$,

$$\begin{aligned}\left\| \frac{\partial^m}{\partial t^m}\frac{\partial^n F}{\partial p^n}(p,t_l) - \frac{\partial^m}{\partial t^m}\frac{\partial^n F}{\partial p^n}(p,t_k) \right\| &\le \int_{t_k}^{t_l} \left\| \frac{\partial^{m+1}}{\partial t^{m+1}}\frac{\partial^n F}{\partial p^n} \right\| dt \\ &\overset{(2.15)}{=} \int_{t_k}^{t_l} \left\| \frac{\partial^m}{\partial t^m}\frac{\partial^n}{\partial p^n}((h-\kappa)\boldsymbol{\nu}) \right\| dt \overset{\text{Cor. 4.8}}{\le} c_{n,m}(t_l - t_k)\,,\end{aligned}$$

where $c_{n,m} = c_{n,m}(n,m,C_0,C_1,\ldots,C_{i,j},\ldots,C_{n,m}) = c_{n,m}(n,m,T,\Sigma_0,C_0)$. Thus, for all $m,n \in \mathbb{N}\cup\{0\}$, $\left(\frac{\partial^m}{\partial t^m}\frac{\partial^n}{\partial p^n}F(\,\cdot\,,t_k)\right)_{k\in\mathbb{N}}$ is a uniform Cauchy sequence of smooth functions on $\mathbb{S}^1$ which has a continuous limit $F_T^{n,m} : \mathbb{S}^1 \to \mathbb{R}^2$. Since the convergences are uniform, Theorem B.14 yields that $F_T \in C^\infty(\mathbb{S}^1)$ with

$$\frac{\partial^m}{\partial t^m}\frac{\partial^n F_T}{\partial p^n} = F_T^{n,m}$$

for every $n,m \in \mathbb{N}\cup\{0\}$. Furthermore, $F(\,\cdot\,,t) \to F_T(\,\cdot\,)$ smoothly for $t \nearrow T$. The short time existence, Theorem 2.3, applied to the initial smooth curve Σ_T yields that there exists a time

$$\bar{T} = \bar{T}(\|F_T\|_{C^{3,\alpha}}) > T$$

and a smooth solution $\bar{F} : \mathbb{S}^1 \times (T,\bar{T}) \to \mathbb{R}^2$ of (2.15) with $\bar{F}(\,\cdot\,,t) \to F_T(\,\cdot\,)$ smoothly for $t \searrow T$. Hence, we have found a smooth extension of F to $[0,\bar{T})$. This is a contradiction. □

Chapter 5

Angles and local total curvature

In this chapter, we exploit the relationship between angles of tangent vectors and local total curvatures and prove a strong maximum principle for the latter.

Define $\vartheta : \mathbb{S}^1 \times [0,T) \to [0,2\pi)$ to be the angle between the x_1-axis and the tangent vector, such that

$$\vartheta(p,t) = \begin{cases} \arccos(\langle \mathbf{e}_1, \boldsymbol{\tau}_p \rangle) & \text{if } \langle \mathbf{e}_2, \boldsymbol{\tau}_p \rangle \geq 0 \\ 2\pi - \arccos(\langle \mathbf{e}_1, \boldsymbol{\tau}_p \rangle) & \text{if } \langle \mathbf{e}_2, \boldsymbol{\tau}_p \rangle \leq 0 \,. \end{cases}$$

Since $\boldsymbol{\nu} = (\boldsymbol{\tau}_2, -\boldsymbol{\tau}_1)$,

$$\cos(\vartheta) = \langle \mathbf{e}_1, \boldsymbol{\tau} \rangle = -\langle \mathbf{e}_2, \boldsymbol{\nu} \rangle \qquad \text{and} \qquad \sin(\vartheta) = \langle \mathbf{e}_2, \boldsymbol{\tau} \rangle = \langle \mathbf{e}_1, \boldsymbol{\nu} \rangle \,. \tag{5.1}$$

For a fixed time $t \in [0,T)$, we can define the angle $\tilde{\vartheta}$ via the arc length parameter by $\tilde{\vartheta} : \mathbb{S}^1_{R_t} \to \mathbb{R}$. As explained in Remark 3.1, we can omit the "$\sim$" for simplicity.

Lemma 5.1 (Derivatives of ϑ, [GH86, Lemma 3.1.5])**.** *Let $F : \mathbb{S}^1 \times (0,T) \to \mathbb{R}^2$ be a solution of* (2.15)*. Then*

$$\frac{\partial \vartheta}{\partial s} = \kappa \qquad \textit{and} \qquad \frac{\partial \vartheta}{\partial t} = \frac{\partial \kappa}{\partial s} \,. \tag{5.2}$$

Proof. By (5.1), we can write $\boldsymbol{\tau} = (\cos(\vartheta), \sin(\vartheta))$ and $\boldsymbol{\nu} = (\sin(\vartheta), -\cos(\vartheta))$. We differentiate the tangent vector

$$\frac{\partial \boldsymbol{\tau}}{\partial s} = \frac{\partial \boldsymbol{\tau}}{\partial \vartheta}\frac{\partial \vartheta}{\partial s} = -\boldsymbol{\nu}\frac{\partial \vartheta}{\partial s}$$

and combine the above calculation with the Frenet–Serret equation, Lemma A.1. Differentiating in time yields

$$\frac{\partial \boldsymbol{\tau}}{\partial t} = \frac{\partial \boldsymbol{\tau}}{\partial \vartheta}\frac{\partial \vartheta}{\partial t} = -\boldsymbol{\nu}\frac{\partial \vartheta}{\partial t} \,.$$

The claim follows from the evolution equation (3.4) of $\boldsymbol{\tau}$. ☐

According to the definition of the total curvature, we define the *local total curvature* $\theta : \mathbb{S}^1 \times \mathbb{S}^1 \times [0,T) \to \mathbb{R}$ by

$$\theta(p,q.t) := \int_p^q \kappa(r,t)\, v(r,t) dr \,, \tag{5.3}$$

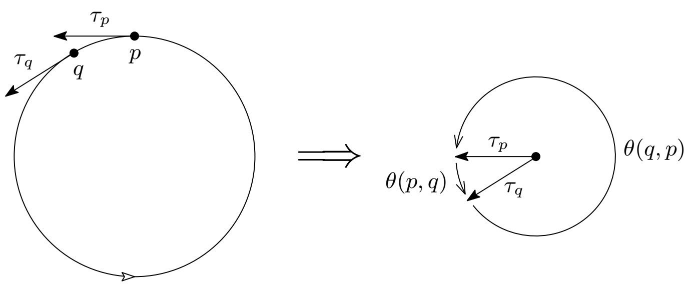

Figure 5.1: $\theta(p,q) = 2\pi - \theta(p,q)$.

where we integrate in direction of the parametrisation with outward unit normal (A.5), that is, for $p, q \in [0, 2\pi)$,

$$\theta(p,q.t) = \begin{cases} \int_p^q \kappa\, v dr & \text{if } p \leq q \\ \int_p^{2\pi} \kappa\, v dr + \int_0^q \kappa\, v dr & \text{if } q < p\,. \end{cases} \tag{5.4}$$

By Theorem A.2, for $p, q \in [0, 2\pi)$, $p < q$,

$$2\pi \overset{\text{Thm. A.2}}{=} \int_{\mathbb{S}^1} \kappa\, v dr = \int_0^p \kappa\, v dr + \int_p^q \kappa\, v dr + \int_q^{2\pi} \kappa\, v dr \overset{(5.4)}{=} \theta(p,q,t) + \theta(q,p,t) \tag{5.5}$$

(see Figure 5.1). Furthermore,

$$\theta(p,q,t) = \int_p^q \kappa(r,t) v(r,t)\, dr \overset{\text{Lem. 5.1}}{=} \int_p^q \frac{1}{v_r} \frac{\partial \vartheta}{\partial r}\, v dr = \vartheta(q,t) - \vartheta(p,t) + 2\pi\omega\,, \tag{5.6}$$

where $\omega(p,q,t) \in \mathbb{Z}$ is the local winding number. Hence, θ is the angle between the tangent vectors at two points $F(p,t)$ and $F(q,t)$ modulo the local winding number (see Figure 5.2). If Σ_t is embedded and convex, then

$$0 \leq \theta(p,q,t) = \int_p^q \kappa\, v dr \leq \int_{\mathbb{S}^1} \kappa\, v dr \overset{\text{Thm. A.2}}{=} 2\pi \tag{5.7}$$

for all $p, q \in \mathbb{S}^1$.

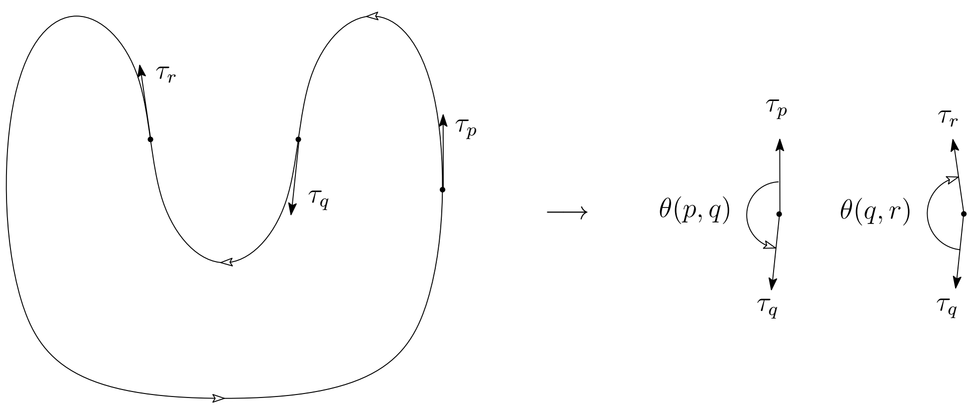

Figure 5.2: $\theta(p,q) > 0$ and $\theta(q,r) < 0$.

Lemma 5.2. *Let $\Sigma = F(\mathbb{S}^1)$ be an embedded, closed curve in the plane. Then*

$$\sup_{\mathbb{S}^1\times\mathbb{S}^1} \theta = 2\pi - \min_{\mathbb{S}^1\times\mathbb{S}^1} \theta \,.$$

Proof. If Σ is convex, the claim follows directly from (5.7). If Σ is non-convex, then $\min_{\mathbb{S}^1\times\mathbb{S}^1} \theta < 0$ and $\max_{\mathbb{S}^1\times\mathbb{S}^1} \theta > 2\pi$. Let the maximum of θ be attained at $p_0, q_0 \in \mathbb{S}^1$, that is,

$$\max_{\mathbb{S}^1\times\mathbb{S}^1} \theta = \theta(p_0, q_0) \overset{(5.5)}{=} 2\pi - \theta(q_0, p_0) \,. \tag{5.8}$$

Then, for all $p, q \in \mathbb{S}^1$, $p \neq q$,

$$2\pi - \theta(q_0, p_0) \overset{(5.8)}{=} \theta(p_0, q_0) \geq \theta(q, p) \overset{(5.5)}{=} 2\pi - \theta(p, q) \,.$$

Consequently,

$$\theta(q_0, p_0) \leq \theta(p, q)$$

for all $p, q \in \mathbb{S}^1$, $p \neq q$, which implies

$$\min_{\mathbb{S}^1\times\mathbb{S}^1} \theta = \theta(q_0, p_0) \overset{(5.8)}{=} 2\pi - \max_{\mathbb{S}^1\times\mathbb{S}^1} \theta \,.$$

□

Theorem 5.3. *Let $F : \mathbb{S}^1 \times (0, T) \to \mathbb{R}^2$ be a solution of (2.15). Then*

$$\left(\frac{\partial}{\partial t} - \Delta_{\Sigma_t}\right) \theta(p, q, t) = 0$$

for all $p, q \in \mathbb{S}^1$ and $t \in (0, T)$. Moreover, all spatial and time derivatives of θ are smooth in $\mathbb{S}^1 \times \mathbb{S}^1 \times [0, T)$.

Proof. We differentiate at $p, q \in \mathbb{S}^1$,

$$\boldsymbol{\tau}_p(\theta) \overset{(5.3)}{=} \frac{1}{v_p}\frac{\partial}{\partial p}\int_p^q \kappa v \, dr = -\kappa_p \quad \text{and} \quad \boldsymbol{\tau}_q(\theta) \overset{(5.3)}{=} \frac{1}{v_q}\frac{\partial}{\partial q}\int_p^q \kappa v \, dr = \kappa_q \,. \tag{5.9}$$

According to the two-point differentiation (A.10), for $a, b \in \mathbb{R}$,

$$(a\boldsymbol{\tau}_p \oplus b\boldsymbol{\tau}_q)^2(\theta) = a^2\boldsymbol{\tau}_p^2(\theta) + b^2\boldsymbol{\tau}_q^2(\theta)$$

and

$$\Delta_{\Sigma_t}\theta = \boldsymbol{\tau}_p^2(\theta) + \boldsymbol{\tau}_q^2(\theta) \overset{(5.9)}{=} \boldsymbol{\tau}_q(\kappa_q) - \boldsymbol{\tau}_p(\kappa_p) \,. \tag{5.10}$$

The evolution equations (3.2) and (4.1) of the length element and the curvature and integration by parts imply

$$\begin{aligned}
\frac{\partial\theta}{\partial t} &\overset{(5.3)}{=} \frac{\partial}{\partial t}\int_p^q \kappa \, v dr \overset{(3.2)}{=} \int_{s(p,t)}^{s(q,t)} \frac{\partial\kappa}{\partial t} \, ds_t + \int_p^q \kappa\frac{\partial v}{\partial t} \, dr \\
&\overset{(4.1)}{=} \int_{s(p,t)}^{s(q,t)} \left(\frac{\partial^2\kappa}{\partial s^2} - (h-\kappa)\kappa^2\right) ds_t + \int_p^q \kappa(h-\kappa)\kappa \, v dr \\
&= \boldsymbol{\tau}_q(\kappa_q) - \boldsymbol{\tau}_p(\kappa_p) \,.
\end{aligned} \tag{5.11}$$

Subtracting (5.10) from (5.11) yields the claim. Moreover, since

$$\boldsymbol{\tau}_p^n(\kappa_p) = \frac{\partial^n \kappa}{\partial s^n}(s(p,t),t)\,,$$

we obtain for $n, m \in \mathbb{N}$,

$$(a\boldsymbol{\tau}_p \oplus b\boldsymbol{\tau}_q)^n \theta(p,q,t) \overset{(5.9)}{=} b^n \frac{\partial^{n-1}\kappa}{\partial s^{n-1}}(s(q,t),t) - a^n \frac{\partial^{n-1}\kappa}{\partial s^{n-1}}(s(p,t),t)$$

as well as

$$\frac{\partial^m \theta}{\partial t^m}(p,q,t) \overset{(5.11)}{=} \frac{\partial^{m-1}}{\partial t^{m-1}}\frac{\partial \kappa}{\partial s}(s(q,t),t) - \frac{\partial^{m-1}}{\partial t^{m-1}}\frac{\partial \kappa}{\partial s}(s(p,t),t)$$

and for $n \geq 2$

$$\frac{\partial^m}{\partial t^m}(a\boldsymbol{\tau}_p \oplus b\,\boldsymbol{\tau}_q)^n \theta(p,q,t) = b^n \frac{\partial^m}{\partial t^m}\frac{\partial^{n-1}\kappa}{\partial s^{n-1}}(s(q,t),t) - a^n \frac{\partial^m}{\partial t^m}\frac{\partial^{n-1}\kappa}{\partial s^{n-1}}(s(p,t),t)\,.$$

Since F is closed and smooth, all spatial and time derivatives of θ are smooth in $\mathbb{S}^1 \times \mathbb{S}^1 \times [0,T)$. □

For $t \in [0,T)$, define

$$\theta_{\min}(t) := \min_{(p,q)\in\mathbb{S}^1\times\mathbb{S}^1} \theta(p,q,t) \qquad \text{and} \qquad \theta_{\max}(t) := \max_{(p,q)\in\mathbb{S}^1\times\mathbb{S}^1} \theta(p,q,t)\,.$$

Corollary 5.4 (Maximum principles for θ)**.** *Let $F : \mathbb{S}^1 \times (0,T) \to \mathbb{R}^2$ be an embedded solution of* (2.15)*.*

(i) (Weak maximum principle). For all $t \in [0,T)$ and for all $p, q \in \mathbb{S}^1$ there holds

$$\theta_{\min}(0) \leq \theta(p,q,t) \leq \theta_{\sup}(0)\,.$$

(ii) (Strong maximum principle). Let $t_0 \in (0,T)$ such that for some $p_0, q_0 \in \mathbb{S}^1$, $p_0 \neq q_0$,

(a) $\theta(p_0,q_0,t_0) = \sup_{\mathbb{S}^1\times\mathbb{S}^1\times[0,T)} \theta$, or

(b) $\theta(p_0,q_0,t_0) = \inf_{\mathbb{S}^1\times\mathbb{S}^1\times[0,T)} \theta$.

Then,

(a) $\theta \equiv \max_{\mathbb{S}^1\times\mathbb{S}^1\times[0,t_0]} \theta$ in $\mathbb{S}^1 \times \mathbb{S}^1 \times [0,t_0]$, or

(b) $\theta \equiv \min_{\mathbb{S}^1\times\mathbb{S}^1\times[0,t_0]} \theta$ in $\mathbb{S}^1 \times \mathbb{S}^1 \times [0,t_0]$.

(iii) Suppose $\theta_{\min}(0) < 0$, then $\theta_{\min}(0) < \theta(p,q,t)$ for all $p, q \in \mathbb{S}^1$ and $t \in (0,T)$.

Remark 5.5. A similar result for the angle ϑ was attained in [Gra87, Lemma 1.9].

Proof of Corollary 5.4. By the definition (5.4) of θ, for fixed $p \in \mathbb{S}^1$ and $t \in [0,T)$,

$$\lim_{q\searrow p} \theta(p,q,t) \overset{(5.4)}{=} \lim_{q\searrow p} \int_p^q \kappa\, v dr = 0$$

and

$$\lim_{q\nearrow p} \theta(p,q,t) \overset{(5.4)}{=} \lim_{q\nearrow p} \int_0^q \kappa\, v dr + \int_p^{2\pi} \kappa\, v dr = \int_{\mathbb{S}^1} \kappa\, v dr = 2\pi\,.$$

Hence, θ is discontinuous along the diagonal $\{p = q\} \subset \mathbb{S}^1 \times \mathbb{S}^1$. The set

$$S := \mathbb{S}^1 \times \mathbb{S}^1 \setminus \{p = q\}$$

is an oriented cylinder. The closure $\bar{S}$ has two boundaries

$$(\partial S)_- = \left\{(p,p) \,\middle|\, p \in \mathbb{S}^1\right\} \qquad \text{and} \qquad (\partial S)_+ = \left\{\left(\lim_{r \nearrow p} r, p\right) \,\middle|\, p \in \mathbb{S}^1\right\},$$

where

$$\theta \equiv 0 \ \text{ on } \ (\partial S)_- \times [0,T) \qquad \text{and} \qquad \theta \equiv 2\pi \ \text{ on } \ (\partial S)_+ \times [0,T).$$

However, by Theorem 5.3 and the continuity of the integral, θ is smooth in $S \times [0,T)$. Claims (i) and (ii) are immediate consequences of the weak and strong maximum principle, Theorems B.16 and B.17, and the evolution equation of θ, Theorem 5.3, applied to $S \times [0,T)$. For claim (iii), we first state that by (i),

$$\theta_{\min}(0) \le \theta(p,q,t) \le \theta_{\sup}(0)$$

for all $t \in [0,T)$ and for all $p, q \in \mathbb{S}^1$. Suppose that there exists a time $t_0 \in (0,T)$ and $(p_0, q_0) \in S$ with

$$\theta(p_0, q_0, t_0) = \theta_{\min}(0) < 0.$$

Then (ii) yields

$$\theta \equiv \min_{\mathbb{S}^1 \times \mathbb{S}^1 \times [0,t_0]} \theta = \theta_{\min}(0) < 0$$

in $\mathbb{S}^1 \times \mathbb{S}^1 \times [0, t_0]$. But $\theta(p,p,t) = 0$ for all $p \in \mathbb{S}^1$ and for all $t \in [0,T)$ which is a contradiction. □

We need the following result from Angenent.

Proposition 5.6 (Zero sets of solutions of parabolic equations, Angenent [Ang88, Theorems A–D]). *Let $u : S \times (0,t_0) \to \mathbb{R}$ be a classical solution of*

$$\frac{\partial u}{\partial t} = a(p,t)\frac{\partial^2 u}{\partial p^2} + b(p,t)\frac{\partial u}{\partial p} + c(p,t)u$$

with either

(i) $S = [0,1]$ and u is bounded with either Dirichlet, Neumann or periodic boundary conditions, or

(ii) $S = \mathbb{R}$ and $|u(p,t)| \le A\exp(Bp^2)$ for some $A, B < \infty$.

Assume that the coefficients satisfy

$$a\,,\ a^{-1}\,,\ \frac{\partial}{\partial t}a\,,\ \frac{\partial}{\partial p}a\,,\ \frac{\partial^2}{\partial p^2}a\,,\ b\,,\ \frac{\partial}{\partial t}b\,,\ \frac{\partial}{\partial p}b \ \text{ and } \ c \in L^\infty$$

on $S \times (0,t_0)$. Then, for any $t \in (0,t_0)$, the set of zeros of $u(\,\cdot\,,t)$

$$z(t) := \{p \in S \,|\, u(p,t) = 0\}$$

is finite for $S = [0,1]$ and discrete for $S = \mathbb{R}$. In addition, if for some $p_1 \in S$ and $t_1 \in (0,t_0)$, $u(p_1,t_1)$ has a multiple zero (i.e., if u and $\frac{\partial}{\partial p}u$ vanish simultaneously), then, *(i) for $S = [0,1]$, $\#z(t)$ strictly decreases for $t \in (t_1, t_0)$, and*

(ii) for $S = \mathbb{R}$, there exits a neighbourhood $U = [p_1 - \varepsilon, p_1 + \varepsilon] \times [t_1 - \delta, t_1 + \delta]$ such that

- *$u(p_1 \pm \varepsilon, t) \neq 0$ for $|t - t_1| \leq \delta$*
- *$u(\,\cdot\,, t + \delta)$ has at most one zero on the interval $[p_1 - \varepsilon, p_1 + \varepsilon]$*
- *$u(\,\cdot\,, t - \delta)$ has at least two zeros on the interval $[p_1 - \varepsilon, p_1 + \varepsilon]$.*

Corollary 5.7 (Zero set of the curvature)**.** *Let $S \in \{\mathbb{S}^1, \mathbb{R}\}$. Let $F : S \times (0,T) \to \mathbb{R}^2$ be an embedded solution of* (2.15) *for $S = \mathbb{S}^1$ or of CSF for $S = \mathbb{R}$ with $\kappa \not\equiv 0$. Let $t_0 \in (0,T)$. Then, for any $t \in (0, t_0)$, the set*

$$z(t) = \{p \in S \mid \kappa(p,t) = 0\}$$

is finite for $S = \mathbb{S}^1$ and discrete for $S = \mathbb{R}$. In addition, if at some point $(p_1, t_1) \in S \times (0, t_0)$ we have $\kappa(p_1, t_1) = 0$ and $\frac{\partial}{\partial s}\kappa(p_1, t_1) = 0$, then

(i) for $S = \mathbb{S}^1$, $\#z(t)$ strictly decreases for $t \in (t_1, t_0)$, and

(ii) for $S = \mathbb{R}$, there exits a neighbourhood $U = [p_1 - \varepsilon, p_1 + \varepsilon] \times [t_1 - \delta, t_1 + \delta]$ such that

- *$\kappa(p_1 \pm \varepsilon, t) \neq 0$ for $|t - t_1| \leq \delta$*
- *$\kappa(\,\cdot\,, t + \delta)$ has at most one zero on the interval $[p_1 - \varepsilon, p_1 + \varepsilon]$*
- *$\kappa(\,\cdot\,, t - \delta)$ has at least two zeros on the interval $[p_1 - \varepsilon, p_1 + \varepsilon]$.*

Remark 5.8. A similar version of this statement for CSF can be found in [Gra87, Lemma 1.9], see also [Man11, Proposition 4.3.1].

Proof of Corollary 5.7. The curvature suffices the evolution equation

$$\begin{aligned}\frac{\partial \kappa}{\partial t} &\overset{(4.1)}{=} \frac{\partial^2 \kappa}{\partial s^2} + (\kappa - h)\kappa^2 \overset{(A.3)}{=} \frac{1}{v}\frac{\partial}{\partial p}\left(\frac{1}{v}\frac{\partial \kappa}{\partial p}\right) + (\kappa - h)\kappa^2 \\ &= \frac{1}{v^2}\frac{\partial^2 \kappa}{\partial p^2} - \frac{1}{v^4}\left\langle \frac{\partial F}{\partial p}, \frac{\partial^2 F}{\partial p^2}\right\rangle \frac{\partial \kappa}{\partial p} + (\kappa - h)\kappa^2\end{aligned}$$

with $h = 0$ for CSF. Define the coefficients

$$a := \frac{1}{v^2}\,, \quad b := -\frac{1}{v^4}\left\langle \frac{\partial F}{\partial p}, \frac{\partial^2 F}{\partial p^2}\right\rangle \quad \text{and} \quad c := (\kappa - h)\kappa$$

and let $t_0 \in (0,T)$. Lemma 3.11 about the regularity of the curve can also be applied to $S = \mathbb{R}$, that is, the length element v is bounded from above and from below away from zero on $[0, t_0]$. Furthermore, also v^{-1} is bounded on $[0, t_0]$. By Corollary 4.7, all derivatives of κ are bounded in $(0, t_0)$. Thus, the derivatives

$$\frac{\partial}{\partial t}a\,, \ \frac{\partial}{\partial p}a\,, \ \frac{\partial^2}{\partial p^2}a\,, \ \frac{\partial}{\partial t}b \ \text{ and } \ \frac{\partial}{\partial p}b$$

are bounded on $[0, t_0]$ and we can apply Proposition 5.6 to $\kappa : S \times (0, t_0) \to \mathbb{R}$. □

Corollary 5.9. *Let $F : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ be a solution of (2.15) with initial curve Σ_0. Then,*

$$\frac{d}{dt}\int_{\Sigma_t} |\kappa|\, d\mathcal{H}^1 = -2 \sum_{\{s\in s(\mathbb{S}^1,\,t)\,|\,\kappa(s,t)=0\}} \left|\frac{\partial \kappa}{\partial s}(s,t)\right|$$

for $t \in (0,T)$. Consequently, there exists a scaling invariant constant $C_h = C_h(\Sigma_0) \geq 1$ such that

$$h(t) \leq C_h |\kappa|_{\max}(t)$$

for all $t \in [0,T)$, where $C_h = 1$ for the APCSF for all embedded curves and for the LPCF in case the curve is convex. Moreover,

$$\exp\left(-c(C_h)\int_0^t \kappa_{\max}^2(\tau)\, d\tau\right) v(p,0) \leq v(p,t) \leq \exp\left(c(C_h)\int_0^t \kappa_{\max}^2(\tau)\, d\tau\right) v(p,0)$$

for every $(p,t) \in \mathbb{S}^1 \times (0,T)$.

Proof. We follow the lines of [Alt91, Theorem 5.14]. A similar proof can be found in [Man11, Proposition 4.3.2]. For $t \in (0,T)$, the integral $\int_{\mathbb{S}^1_{R_t}} |\kappa|\, ds_t$ is positive and finite. Let

$$\mathcal{S}_t := \left\{S \subset \mathbb{S}^1_{R_t} \,\middle|\, \kappa(s,t) > 0 \text{ for all } s \in S \text{ or } \kappa(s,t) < 0 \text{ for all } s \in S\right\}$$

be the family of open intervals S in $\mathbb{S}^1_{R_t}$ where $\kappa \neq 0$. By Corollary 5.7, $\overline{\mathcal{S}_t} = \mathbb{S}^1_{R_t}$. For $S \in \mathcal{S}_t$, we define $[s_1^S, s_2^S] := \overline{S}$ and $\operatorname{sign}(S) := \operatorname{sign}(\kappa(s,t))$ for $s \in S$. Then $\kappa(s_i^S,t) = 0$ for $i = 1,2$ and, for consecutive segments S_1, S_2 and S_3,

$$s_2^{S_1} = s_1^{S_2} \qquad \text{and} \qquad s_2^{S_2} = s_1^{S_3}\,. \tag{5.12}$$

Furthermore, we observe that either

$$\operatorname{sign}(S) > 0\,, \qquad \frac{\partial \kappa}{\partial s}(s_1^S,t) \geq 0 \qquad \text{and} \qquad \frac{\partial \kappa}{\partial s}(s_2^S,t) \leq 0$$

or

$$\operatorname{sign}(S) < 0\,, \qquad \frac{\partial \kappa}{\partial s}(s_1^S,t) \leq 0 \qquad \text{and} \qquad \frac{\partial \kappa}{\partial s}(s_2^S,t) \geq 0$$

so that in both cases

$$\operatorname{sign}(S)\left(\frac{\partial \kappa}{\partial s}(s_2^S,t) - \frac{\partial \kappa}{\partial s}(s_1^S,t)\right) = -\left|\frac{\partial \kappa}{\partial s}(s_1^S,t)\right| - \left|\frac{\partial \kappa}{\partial s}(s_2^S,t)\right|\,. \tag{5.13}$$

Hence,

$$
\begin{aligned}
\frac{d}{dt}\int_{\Sigma_t}|\kappa|\,d\mathcal{H}^1 &= \frac{d}{dt}\left(\sum_{S\in\mathcal{S}_t}\int_{\tilde{F}(S,t)}\operatorname{sign}(\kappa)\kappa\,d\mathcal{H}^1\right)\\
&\overset{(3.2),(4.1)}{=} \sum_{S\in\mathcal{S}_t}\int_{s_1^S}^{s_2^S}\operatorname{sign}(S)\left[\left(\frac{\partial^2\kappa}{\partial s^2}-(h-\kappa)\kappa^2\right)+\kappa(h-\kappa)\kappa\right]ds_t\\
&= \sum_{S\in\mathcal{S}_t}\operatorname{sign}(S)\int_{s_1^S}^{s_2^S}\frac{\partial^2\kappa}{\partial s^2}\,ds_t\\
&= \sum_{S\in\mathcal{S}_t}\operatorname{sign}(S)\left(\frac{\partial\kappa}{\partial s}(s_2^S,t)-\frac{\partial\kappa}{\partial s}(s_1^S,t)\right)\\
&\overset{(5.12),(5.13)}{=} -2\sum_{\{s\in s(\mathbb{S}^1,t)\,|\,\kappa(s,t)=0\}}\left|\frac{\partial\kappa}{\partial s}(s,t)\right|\leq 0
\end{aligned}
$$

for $t\in(0,T)$. Thus, $\int_{\Sigma_t}|\kappa|\,d\mathcal{H}^1$ is decreasing in time on $[0,T)$. By (5.6), the integral is the sum over the absolute value of the angles between inflection points, it is scaling invariant.

The conclusions about the local terms and the length element follow directly from Lemmata 3.8 and 3.11. □

Chapter 6

Preservation of embeddedness

In this chapter, we show that under the initial condition $\theta_{\min}(0) \geq -\pi$ the curves Σ_t stay embedded for $t \in [0, T)$.

The extrinsic distance function $d : \mathbb{S}^1 \times \mathbb{S}^1 \times [0, T) \to \mathbb{R}$ is given by

$$d(p, q, t) := \|F(q, t) - F(p, t)\| .$$

We define the vector $\mathbf{w} : \big(\mathbb{S}^1 \times \mathbb{S}^1 \times [0, T)\big) \setminus \{d = 0\} \to \mathbb{R}^2$ by

$$\mathbf{w}(p, q, t) := \frac{F(q, t) - F(p, t)}{d(p, q, t)}$$

(see Figure 6.1 for an illustration).

Lemma 6.1. *Let $\Sigma = F(\mathbb{S}^1)$ be a curve in the plane and $p, q \in \mathbb{S}^1$ with $d(p, q) \neq 0$. If $\langle \mathbf{w}, \boldsymbol{\tau}_q - \boldsymbol{\tau}_p \rangle = 0$, then $\langle \mathbf{w}, \boldsymbol{\tau}_q + \boldsymbol{\tau}_p \rangle^2 = \|\boldsymbol{\tau}_q + \boldsymbol{\tau}_p\|^2$.*

Proof. For unit tangent vectors, we have

$$\langle \boldsymbol{\tau}_p + \boldsymbol{\tau}_q, \boldsymbol{\tau}_q - \boldsymbol{\tau}_p \rangle = \|\boldsymbol{\tau}_p\|^2 - \|\boldsymbol{\tau}_q\|^2 = 0 .$$

Thus, $\mathbf{w}$ and $\boldsymbol{\tau}_p + \boldsymbol{\tau}_q$ are both perpendicular to $\boldsymbol{\tau}_q - \boldsymbol{\tau}_p$ and are therefore parallel, that is, $\measuredangle(\mathbf{w}, \boldsymbol{\tau}_q + \boldsymbol{\tau}_p) = 0$. Using $\|\mathbf{w}\| = 1$, we calculate

$$\langle \mathbf{w}, \boldsymbol{\tau}_q + \boldsymbol{\tau}_p \rangle^2 = \|\mathbf{w}\|^2 \|\boldsymbol{\tau}_q + \boldsymbol{\tau}_p\|^2 \arccos^2(\measuredangle(\mathbf{w}, \boldsymbol{\tau}_q + \boldsymbol{\tau}_p)) = \|\boldsymbol{\tau}_q + \boldsymbol{\tau}_p\|^2 . \qquad \square$$

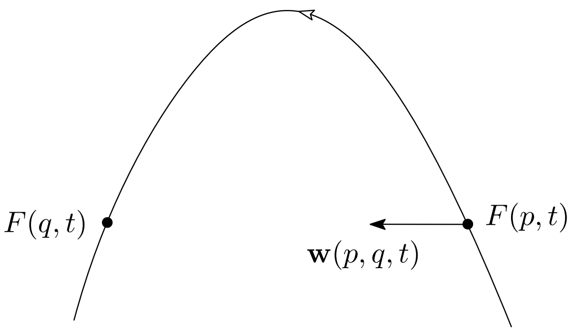

Figure 6.1: The vector $\mathbf{w}$.

Lemma 6.2 (Spatial derivatives of d)**.** *Let* $p, q \in \mathbb{S}^1$, $d(p,q) \neq 0$, $a, b \in \mathbb{R}$ *and* $\xi(p,q) := a\boldsymbol{\tau}_p \oplus b\boldsymbol{\tau}_q$. *Then*

$$\xi(d) = \langle \mathbf{w}, b\boldsymbol{\tau}_q - a\boldsymbol{\tau}_p \rangle \tag{6.1}$$

and

$$\xi^2(d) = -\frac{1}{d}\langle \mathbf{w}, b\boldsymbol{\tau}_q - a\boldsymbol{\tau}_p \rangle^2 + \frac{1}{d}\|b\boldsymbol{\tau}_q - a\boldsymbol{\tau}_p\|^2 + \left\langle \mathbf{w}, b^2\boldsymbol{\kappa}_q - a^2\boldsymbol{\kappa}_p \right\rangle , \tag{6.2}$$

where $\boldsymbol{\kappa} = -\kappa\boldsymbol{\nu}$ *is the curvature vector.*

Proof. The definition (A.10) of the two-point differentiation implies

$$\begin{aligned}
\xi(d) &= \xi(\|F(q,t) - F(p,t)\|) \\
&= \frac{1}{d}\langle F(q,t) - F(p,t), (a\boldsymbol{\tau}_p \oplus b\boldsymbol{\tau}_q)(F(q,t) - F(p,t)) \rangle \\
&\overset{\text{(A.10)}}{=} \frac{1}{d}\langle F(q,t) - F(p,t), b\boldsymbol{\tau}_q(F(q,t)) - a\boldsymbol{\tau}_p(F(p,t)) \rangle \\
&= \langle \mathbf{w}, b\boldsymbol{\tau}_q - a\boldsymbol{\tau}_p \rangle \ .
\end{aligned}$$

Next, we calculate

$$\nabla_{\boldsymbol{\tau}_q}\boldsymbol{\tau}_p = \frac{1}{v(q,t)}\frac{\partial}{\partial q}\left(\frac{1}{v(p,t)}\frac{\partial F}{\partial p}(p,t)\right) = 0 \tag{6.3}$$

and use (6.1) and the Frenet–Serret equation, Lemma A.1, to differentiate twice:

$$\begin{aligned}
\xi^2(d) = \xi(\xi(d)) &\overset{(6.1)}{=} \xi\left(\frac{1}{d}\langle F(q,t) - F(p,t), b\boldsymbol{\tau}_q - a\boldsymbol{\tau}_p \rangle\right) \\
&\overset{(6.1),(6.3)}{=} -\frac{1}{d^3}\langle F(q,t) - F(p,t), b\boldsymbol{\tau}_q - a\boldsymbol{\tau}_p \rangle^2 + \frac{1}{d}\|b\boldsymbol{\tau}_q - a\boldsymbol{\tau}_p\|^2 \\
&\quad + \frac{1}{d}\langle F(q,t) - F(p,t), b^2\nabla_{\boldsymbol{\tau}_q}\boldsymbol{\tau}_q - a^2\nabla_{\boldsymbol{\tau}_p}\boldsymbol{\tau}_p \rangle \\
&\overset{\text{Lem. A.1}}{=} -\frac{1}{d}\langle \mathbf{w}, b\boldsymbol{\tau}_q - a\boldsymbol{\tau}_p \rangle^2 + \frac{1}{d}\|b\boldsymbol{\tau}_q - a\boldsymbol{\tau}_p\|^2 + \left\langle \mathbf{w}, b^2\boldsymbol{\kappa}_q - a^2\boldsymbol{\kappa}_p \right\rangle \ . \qquad \square
\end{aligned}$$

Corollary 6.3. *For* $p, q \in \mathbb{S}^1$ *with* $d(p,q) \neq 0$,

$$(\boldsymbol{\tau}_p \oplus 0)(d) = -\langle \mathbf{w}, \boldsymbol{\tau}_p \rangle \qquad \textit{and} \qquad (0 \oplus \boldsymbol{\tau}_q)(d) = \langle \mathbf{w}, \boldsymbol{\tau}_q \rangle$$

as well as

$$(\boldsymbol{\tau}_p \ominus \boldsymbol{\tau}_q)^2(d) = -\frac{1}{d}\langle \mathbf{w}, \boldsymbol{\tau}_q + \boldsymbol{\tau}_p \rangle^2 + \frac{1}{d}\|\boldsymbol{\tau}_q + \boldsymbol{\tau}_p\|^2 + \langle \mathbf{w}, \boldsymbol{\kappa}_q - \boldsymbol{\kappa}_p \rangle \ .$$

Lemma 6.4 (Evolution equation for d)**.** *Let* $F : \mathbb{S}^1 \times (0,T) \to \mathbb{R}^2$ *be a solution of* (2.15). *Then*

$$\frac{\partial d}{\partial t} = \langle \mathbf{w}, \boldsymbol{\kappa}_q - \boldsymbol{\kappa}_p \rangle + h\,\langle \mathbf{w}, \boldsymbol{\nu}_q - \boldsymbol{\nu}_p \rangle \tag{6.4}$$

for $p, q \in \mathbb{S}^1$ *and* $t \in (0,T)$ *with* $d(p,q,t) \neq 0$.

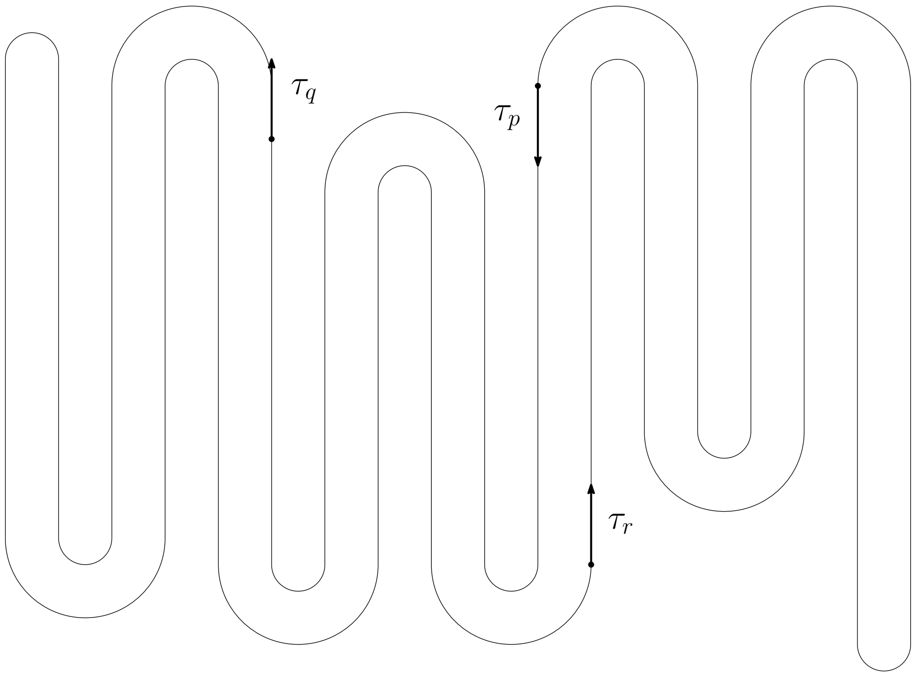

Figure 6.2: See Remark 6.6(iii).

Proof. The evolution equation (2.15) for the embedding yields

$$\begin{aligned}\frac{\partial d}{\partial t} &= \frac{\partial}{\partial t}\|F(q,t) - F(p,t)\| \\ &= \frac{1}{d}\left\langle F(q,t) - F(p,t), \frac{\partial F}{\partial t}(q,t) - \frac{\partial F}{\partial t}(p,t)\right\rangle \\ &\overset{(2.15)}{=} \langle \mathbf{w}, (h-\kappa_q)\boldsymbol{\nu}_q + (h-\kappa_p)\boldsymbol{\nu}_p\rangle \\ &= h\,\langle \mathbf{w}, \boldsymbol{\nu}_q - \boldsymbol{\nu}_p\rangle + \langle \mathbf{w}, \boldsymbol{\kappa}_q - \boldsymbol{\kappa}_p\rangle\ . \end{aligned}$$ ☐

Theorem 6.5 (Preservation of embeddedness)**.** *Let Σ_0 be a smooth, embedded curve satisfying $\theta_{\min} \geq -\pi$, and let $F : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ be a solution of* (2.15) *with initial curve Σ_0. Then $\Sigma_t = F(\mathbb{S}^1, t)$ is embedded for all $t \in (0,T)$.*

Remark 6.6. (i) Lemma 5.2 yields that $\min_{\mathbb{S}^1\times\mathbb{S}^1}\theta \geq -\pi$ implies $\max_{\mathbb{S}^1\times\mathbb{S}^1}\theta \leq 3\pi$.

(ii) Counterexample 6.7 shows that in order for embeddedness to be preserved it is crucial to assume that the initial local total curvature lies in the interval $[-\pi, 3\pi]$.

(iii) In Figure 6.2 the angles are all between $-\pi$ and 3π, for example,

$$\theta(p,q) = -\pi\,,\ \ \theta(q,p) = 3\pi\,,\ \ \theta(q,r) = 2\pi\,,\ \ \theta(r,q) = 0\,,\ \ \theta(r,p) = \pi\,.$$

Proof of Theorem 6.5. Let

$$\varepsilon \in \left(0, \frac{4\pi}{L_0}\right) \tag{6.5}$$

and define $d_\varepsilon : \mathbb{S}^1 \times \mathbb{S}^1 \times [0,T) \to \mathbb{R}$ by

$$d_\varepsilon(p,q,t) := d(p,q,t) - \varepsilon t\,.$$

Since Σ_0 is embedded, $d_\varepsilon(p,q,0) = d(p,q,0) > 0$ for all $p,q \in \mathbb{S}^1$, $p \neq q$. Assume that the curve Σ_t touches itself for the first time at time $t_0 \in (0,T)$ and points $p_0, q_0 \in \mathbb{S}^1$, $p_0 \neq q_0$. Since

$$d_\varepsilon(p_0,q_0,t_0) = -\varepsilon t_0 < 0$$

and d_ε is in $C^0(\mathbb{S}^1 \times \mathbb{S}^1 \times [0,t_0))$, there exists a time $t_1 \in (0,t_0)$ and points $p,q \in \mathbb{S}^1$, $p \neq q$, with

$$d_\varepsilon(p,q,t_1) = 0$$

for the first time. Then

$$\frac{\partial d_\varepsilon}{\partial t} \leq 0\,, \qquad \xi(d_\varepsilon) = 0 \qquad \text{and} \qquad \xi^2(d_\varepsilon) \geq 0 \tag{6.6}$$

at (p,q,t_1) and for all $\xi \in T_{F(p,t_1)}\Sigma_t \oplus T_{F(q,t_1)}\Sigma_t$. Furthermore,

$$d(p,q,t_1) = \varepsilon t_1 > 0 \tag{6.7}$$

and

$$\frac{\partial d}{\partial t}_{|t=t_1}(p,q,t) = \frac{\partial d_\varepsilon}{\partial t}_{|t=t_1}(p,q,t) + \varepsilon \overset{(6.6),(6.5)}{<} \frac{4\pi}{L_0}\,. \tag{6.8}$$

By (6.6), (6.7) and $\xi(d_\varepsilon) = \xi(d)$ for all $\xi \in T_{F(p,t_1)}\Sigma_t \oplus T_{F(q,t_1)}\Sigma_t$, Corollary 6.3 yields

$$0 = -\langle \mathbf{w}, \boldsymbol{\tau}_p \rangle \qquad \text{and} \qquad 0 = \langle \mathbf{w}, \boldsymbol{\tau}_q \rangle \tag{6.9}$$

at (p,q,t_1) so that $\boldsymbol{\tau}_p = \pm\boldsymbol{\tau}_q$. Assume that $\boldsymbol{\tau}_p = \boldsymbol{\tau}_q$. Since Σ_{t_1} is embedded, the curve has to cross the connecting line between $F(p,t_1)$ and $F(q,t_1)$, that is, there exists a point $p < r < q$ with

$$d_\varepsilon(p,r,t_1) = d(p,r,t_1) - \varepsilon t_1 < d(p,q,t_1) - \varepsilon t_1 = d_\varepsilon(p,q,t_1) = 0$$

This is a contradiction. Hence, $\boldsymbol{\tau}_p = -\boldsymbol{\tau}_q$. By (5.1),

$$\cos(\vartheta_p) = -\cos(\vartheta_q) \qquad \text{and} \qquad \sin(\vartheta_p) = -\sin(\vartheta_q)$$

so that $\vartheta_q - \vartheta_p = \pm\pi$ and

$$\theta(p,q,t_1) \overset{(5.6)}{=} 2\pi k \pm \pi \tag{6.10}$$

for $k \in \mathbb{Z}$. By Remark 6.6, $\theta \in [-\pi, 3\pi]$ initially and, by Corollary 5.4(iii),

$$\theta_{\min}(t) > \theta_{\min}(0) \geq -\pi$$

for all $t \in (0,T)$. Lemma 5.2 yields

$$\theta_{\max}(t) < 3\pi$$

for all $t \in (0, T)$. Thus, (6.10) implies that $\theta(p, q, t_1) = \pi$ and since Σ_t is embedded for all $t \in [0, t_1]$,

$$-\boldsymbol{\nu}_p = \boldsymbol{\nu}_q = \mathbf{w}\,. \tag{6.11}$$

By (6.7), (6.9) and Lemma 6.1,

$$\langle \mathbf{w}, \boldsymbol{\tau}_p + \boldsymbol{\tau}_q \rangle^2 = \|\boldsymbol{\tau}_p + \boldsymbol{\tau}_q\|^2 \tag{6.12}$$

and, by Corollary 6.3, (6.6) and $\xi^2(d_\varepsilon) = \xi^2(d)$,

$$\begin{aligned} 0 &\overset{\text{Cor. 6.3,(6.6)}}{\leq} -\frac{1}{d}\langle \mathbf{w}, \boldsymbol{\tau}_p + \boldsymbol{\tau}_q \rangle^2 + \frac{1}{d}\|\boldsymbol{\tau}_p + \boldsymbol{\tau}_q\|^2 + \langle \mathbf{w}, \boldsymbol{\kappa}_q - \boldsymbol{\kappa}_p \rangle \\ &\overset{(6.12)}{=} \langle \mathbf{w}, \boldsymbol{\kappa}_q - \boldsymbol{\kappa}_p \rangle \end{aligned} \tag{6.13}$$

at (p, q, t_1). However, the evolution equation (6.4) of the distance and the bounds (3.10) and (3.11) for the global terms yield

$$\frac{\partial d}{\partial t}_{|t=t_1}(p, q, t) \overset{(6.4)}{=} h\,\langle \mathbf{w}, \boldsymbol{\nu}_q - \boldsymbol{\nu}_p \rangle + \langle \mathbf{w}, \boldsymbol{\kappa}_q - \boldsymbol{\kappa}_p \rangle \overset{(6.11),(6.13)}{\geq} 2h \overset{(3.10),(3.11)}{\geq} \frac{4\pi}{L_0}\,,$$

which contradicts (6.8). □

The next example shows, why the condition $\theta_{\min}(0) \geq -\pi$ is sharp.

Counterexample 6.7. Gage [Gag86, p. 53] suggested the following counterexample. Pihan [Pih98, Section 5.4] gave an incomplete proof for its validity which we will fix here. If we allow local total curvature smaller than $-\pi$, then there exist counterexamples for any given minimum

$$\theta_{\min}(0) < -\pi\,.$$

For the curve in Figure 6.3, $\theta_{\min} = \theta(p_1, p_2) \in (-2\pi, -\pi)$. We will construct a solution of (2.15) with embedded initial curve Σ_0 that intersects itself in finite time. Fix $K_0 > 0$. Let $\mathcal{S}$ be the set of all smooth, embedded, closed curves in $\mathbb{R}^2$ that satisfy $\theta_{\min} < -\pi$,

$$\|F_0\|_{C^{3,\alpha}(\mathbb{S}^1)} \leq K_0 \tag{6.14}$$

and

$$L(\Sigma) = L_0 \geq 8\pi K_0\,, \tag{6.15}$$

where L_0 is chosen big enough so that curves like in Figure 6.3 are in $\mathcal{S}$. By the short time existence, Theorem 2.3, there exists a time $T = T(K_0)$ so that

$$\|F\|_{C^{3,\alpha;1\lfloor\alpha/2\rfloor}(\mathbb{S}^1 \times [0, T/2])} \leq K_1(K_0)\,.$$

In particular,

$$\left|\frac{\partial F^1}{\partial t}(p, t) - \frac{\partial F^1}{\partial t}(p, 0)\right| \leq K_1 t^{\alpha/2}\,, \tag{6.16}$$

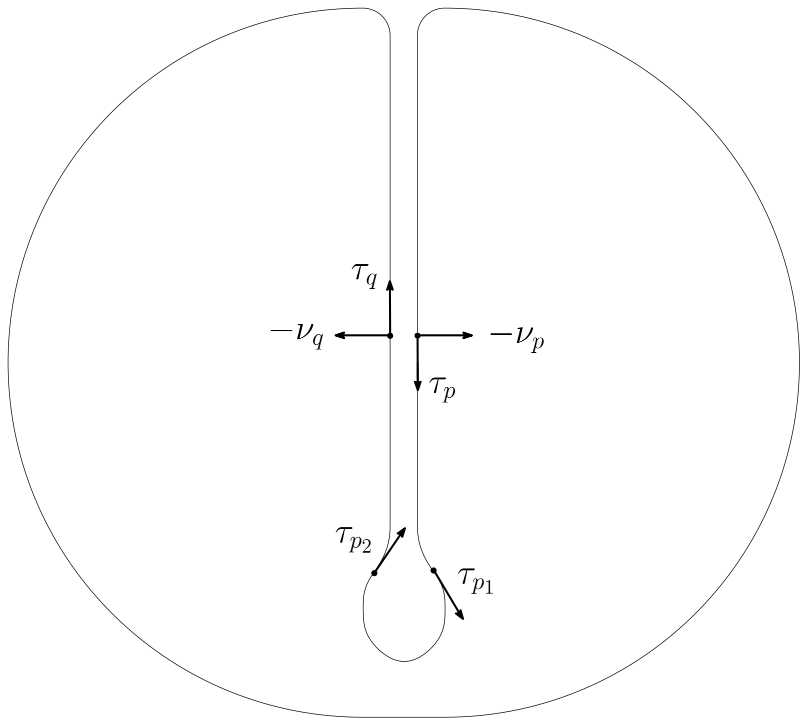

Figure 6.3: See Counterexamples 6.7

where $F^1 := \langle F, \mathbf{e}_1 \rangle$, and, by (2.15) and (6.16),

$$-K_1 t^{\alpha/2} \leq (h(t) - \kappa(p,t))\boldsymbol{\nu}^1(p,t) - (h(0) - \kappa(p,0))\boldsymbol{\nu}^1(p,0) \leq K_1 t^{\alpha/2} \tag{6.17}$$

for all $p \in \mathbb{S}^1$ and for all $t \in [0, T/2]$, where $\boldsymbol{\nu}^1 := \langle \boldsymbol{\nu}, \mathbf{e}_1 \rangle$. Set

$$t_1 = t_1(K_0) := \min \left\{ \frac{T}{2}, \left(\frac{\pi}{L_0 K_1} \right)^{-\alpha/2} \right\}. \tag{6.18}$$

Then (6.17) holds for $t \in [0, t_1]$. Let $\Sigma \in \mathcal{S}$ be a curve like in Figure 6.3, which is symmetric about the x_2-axis. Let $p, q \in \mathbb{S}^1$ be located as in the picture so that

$$\boldsymbol{\nu}(p,0) = -\boldsymbol{\nu}(q,0) = -\mathbf{e}_1 \qquad \text{and} \qquad \kappa(p,0) = \kappa(q,0) = 0\,. \tag{6.19}$$

We use the lower bounds (3.10) and (3.11) for the global term to estimate

$$\begin{aligned} \frac{\partial F^1}{\partial t}(p,t) &\overset{(2.15)}{=} (h(t) - \kappa(p,t))\boldsymbol{\nu}^1(p,t) \overset{(6.17)}{\leq} (h(0) - \kappa(p,0))\boldsymbol{\nu}^1(p,0) + K_1 t_1^{\alpha/2} \\ &\underset{(6.18),(6.19)}{\overset{(3.10),(3.11)}{\leq}} -\frac{2\pi}{L(\Sigma_0)} + \frac{\pi}{L_0} \overset{(6.15)}{=} -\frac{\pi}{L_0} \end{aligned} \tag{6.20}$$

and likewise

$$\begin{aligned} \frac{\partial F^1}{\partial t}(q,t) &\overset{(2.15)}{=} (h(t) - \kappa(q,t))\boldsymbol{\nu}^1(q,t) \overset{(6.17)}{\geq} (h(0) - \kappa(q,0))\boldsymbol{\nu}^1(q,0) - K_1 t_1^{\alpha/2} \\ &\underset{(6.18),(6.19)}{\overset{(3.10),(3.11)}{\geq}} \frac{2\pi}{L(\Sigma_0)} - \frac{\pi}{L_0} \overset{(6.15)}{=} \frac{\pi}{L_0} \end{aligned} \tag{6.21}$$

for $t \in [0, t_1]$. We can smoothly deform a curve like in Figure 6.3 to achieve arbitrarily small distance between $F(p,0)$ and $F(q,0)$ without exceeding the upper bound (6.14) or changing the length (6.15). Hence, we can choose an embedded initial curve Σ_0 with

$$F^1(p,0) = -F^1(q,0) = \frac{\pi t_1}{2L_0}\,. \tag{6.22}$$

Then,

$$F^1(p,t_1) = F^1(p,0) + \int_0^{t_1} \frac{\partial F^1}{\partial t}(p,t)\,dt \overset{(6.20),(6.22)}{\leq} \frac{\pi t_1}{2L_0} - \frac{\pi t_1}{L_0} < 0$$

and

$$F^1(q,t_1) = F^1(q,0) + \int_0^{t_1} \frac{\partial F^1}{\partial t}(q,t)\,dt \overset{(6.21),(6.22)}{\geq} -\frac{\pi t_1}{2L_0} + \frac{\pi t_1}{L_0} > 0$$

so that the curve has crossed itself by the time t_1.

Chapter 7

A non-collapsing estimate

In this chapter we will adapt the methods from Huisken [Hui95] to obtain estimates that imply a certain non-collapsing behaviour of the evolving curves. The main tool used in this chapter is a comparison between the behaviour of the extrinsic and intrinsic distance function. First, we continue to calculate various spatial and time derivatives of the distance functions. The expressions for the spatial derivatives can all be found in [Hui95].

7.1 Interior distance functions

The intrinsic distance function $l : \mathbb{S}^1 \times \mathbb{S}^1 \times [0, T) \to \mathbb{R}$ is given by

$$l(p, q, t) := \int_p^q v(r, t)\, dr\,.$$

Remark 7.1. Notice that for an embedded closed curve $F(\mathbb{S}^1)$, $l(p, q) = L - l(q, p)$ and $d/l > 0$, where we set $(d/l)(p, p) \equiv 1$ for $p \in \mathbb{S}^1$. The curve segment $F([p, q])$ is a straight line if and only if $d \equiv l$ on $[p, q] \times [p, q]$. Otherwise, there exist $p_0, q_0 \in \mathbb{S}^1$ with $d(p_0, q_0) < l(p_0, q_0)$, implying $\min_{[p,q]\times[p,q]}(d/l) < 1$. The infimum of the ratio d/l thus measures how close a curve is to being a line.

Let $F(\mathbb{S}^1)$ be a circle of radius R. Then, for all $p, q \in \mathbb{S}^1$, there exists an angle $\beta(p, q) \in (0, \pi]$ with

$$l(p, q) = \beta(p, q)R\,,$$

where $R := L/(2\pi)$. By the geometric definition of the sine function,

$$\sin\left(\frac{\beta}{2}\right) = \frac{d/2}{R} \qquad \text{and} \qquad \beta = \frac{l}{R}$$

(see Figure 7.1 for an illustration) so that

$$d = 2R\sin\left(\frac{l}{2R}\right) = \frac{L}{\pi}\sin\left(\frac{\pi l}{L}\right)\,.$$

This motivates the definition of the function $\psi : \mathbb{S}^1 \times \mathbb{S}^1 \times [0, T) \to \mathbb{R}$ with

$$\psi(p, q, t) := \frac{L_t}{\pi}\sin\left(\frac{\pi l(p, q, t)}{L_t}\right)\,, \tag{7.1}$$

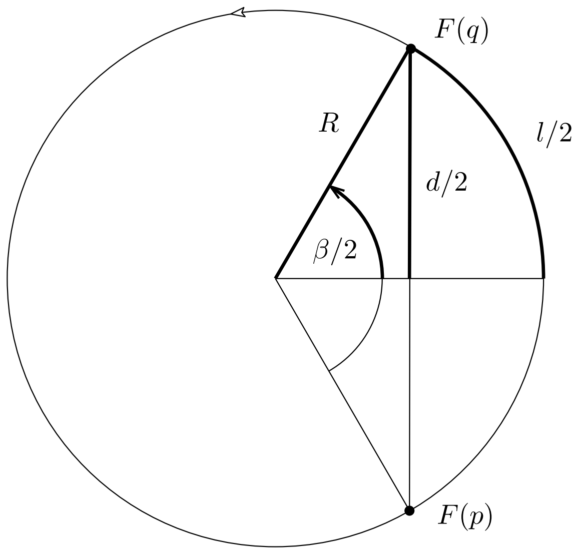

Figure 7.1: Motivation for the function ψ

where $L_t < \infty$. We set $(d/\psi)(p,p,t) \equiv 1$ for $p \in \mathbb{S}^1$ and $t \in [0,T)$, then $(d/\psi)(\,\cdot\,,\,\cdot\,,t) \in C^0(\mathbb{S}^1 \times \mathbb{S}^1)$.

Lemma 7.2 (Sine and Cosine)**.** *For $\alpha, \beta \in \mathbb{R}$,*

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \tag{7.2}$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)\,. \tag{7.3}$$

Remark 7.3. Since $\sin(\pi - \alpha) = \sin(\alpha)$,

$$\sin\left(\frac{\pi l(p,q,t)}{L_t}\right) = \sin\left(\frac{\pi(L_t - l(q,p,t))}{L_t}\right) = \sin\left(\frac{\pi l(q,p,t)}{L_t}\right)$$

so that

$$\psi(p,q,t) = \psi(q,p,t)\,.$$

Hence, we will later assume that $l \leq L_t/2$. For embedded closed curves, we have $d/\psi > 0$. On a circle $d \equiv \psi$ and thus $d/\psi \equiv 1$. If a closed curve Σ_t is not a circle, then there exist $p, q \in \mathbb{S}^1$ so that $d(p,q,t) < \psi(p,q,t)$ and thus $\min_{\mathbb{S}^1\times\mathbb{S}^1}(d/\psi) < 1$. The minimum of the ratio d/ψ is thus a measurement of how close the curve is to being a circle.

Lemma 7.4 (Spatial derivatives of l)**.** *Let $p, q \in \mathbb{S}^1$, $a, b \in \mathbb{R}$ and $\xi := a\boldsymbol{\tau}_p \oplus b\boldsymbol{\tau}_q$. Then*

$$\xi(l) = b - a \qquad \textit{and} \qquad \xi^2(l) = 0\,. \tag{7.4}$$

Proof. By definitions (A.4) and (A.10) of the tangent vector and the two-point differentiation,

$$\xi(l) = (a\boldsymbol{\tau}_p \oplus b\boldsymbol{\tau}_q)\left(\int_p^q v\,dr\right) = \left(\frac{a}{v}\frac{\partial}{\partial p} + \frac{b}{v}\frac{\partial}{\partial q}\right)\left(\int_p^q v\,dr\right) = b - a$$

and it follows that $\xi^2(l) = 0$. □

Lemma 7.5 (Evolution equation for l)**.** *Let* $F : \mathbb{S}^1 \times (0,T) \to \mathbb{R}^2$ *be a solution of* (2.15). *Then*

$$\frac{\partial l}{\partial t} = h \int_p^q \kappa \, v dr - \int_p^q \kappa^2 \, v dr \,. \tag{7.5}$$

Proof. The evolution equation (3.2) for the length element yields

$$\frac{\partial l}{\partial t} = \frac{\partial}{\partial t} \int_p^q v dr = \int_p^q \frac{\partial v}{\partial t} dr = \int_p^q \kappa (h - \kappa) \, v dr \,. \qquad \square$$

Lemma 7.6 (Spatial derivatives of ψ)**.** *Let* $p, q \in \mathbb{S}^1$, $a, b \in \mathbb{R}$ *and* $\xi := a \boldsymbol{\tau}_p \oplus b \boldsymbol{\tau}_q$. *Then*

$$\xi(\psi) = \cos\left(\frac{\pi l}{L}\right) (b - a) \qquad \text{and} \qquad \xi^2(\psi) = -\frac{\pi}{L} \sin\left(\frac{\pi l}{L}\right) (b - a)^2 \,. \tag{7.6}$$

Proof. The spatial derivatives (7.4) of l imply

$$\xi(\psi) = \cos\left(\frac{\pi l}{L}\right) \xi(l) = \cos\left(\frac{\pi l}{L}\right) (b - a)$$

and

$$\begin{aligned} \xi^2(\psi) = \xi\left(\cos\left(\frac{\pi l}{L}\right) (b - a)\right) &= -\frac{\pi}{L} \sin\left(\frac{\pi l}{L}\right) (b - a) \, \xi(l) \\ &= -\frac{\pi}{L} \sin\left(\frac{\pi l}{L}\right) (b - a)^2 \,. \end{aligned} \qquad \square$$

Lemma 7.7 (Evolution equation for ψ)**.** *Let* $F : \mathbb{S}^1 \times (0,T) \to \mathbb{R}^2$ *be a solution of* (2.15). *Then*

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= \cos\left(\frac{\pi l}{L}\right) \left(h \int_p^q \kappa \, v dr - \int_p^q \kappa^2 \, v dr\right) \\ &\quad + \frac{1}{\pi} \left(2\pi h - \int_{\mathbb{S}^1} \kappa^2 \, v dr\right) \left\{\sin\left(\frac{\pi l}{L}\right) - \frac{\pi l}{L} \cos\left(\frac{\pi l}{L}\right)\right\} \,. \end{aligned} \tag{7.7}$$

Proof. The evolution equations (3.8) and (7.5) for L and l imply

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= \frac{\partial}{\partial t} \left(\frac{L}{\pi} \sin\left(\frac{\pi l}{L}\right)\right) \\ &= \frac{1}{\pi} \frac{dL}{dt} \sin\left(\frac{\pi l}{L}\right) + \frac{L}{\pi} \cos\left(\frac{\pi l}{L}\right) \left(\frac{\pi}{L} \frac{\partial l}{\partial t} - \frac{\pi l}{L^2} \frac{dL}{dt}\right) \\ &\overset{(3.8),(7.5)}{=} \frac{1}{\pi} \left(2\pi h - \int_{\mathbb{S}^1} \kappa^2 \, v dr\right) \sin\left(\frac{\pi l}{L}\right) \\ &\qquad + \cos\left(\frac{\pi l}{L}\right) \left\{ \left(h \int_p^q \kappa \, v dr - \int_p^q \kappa^2 \, v dr\right) \right. \\ &\qquad\qquad \left. - \frac{l}{L} \left(2\pi h - \int_{\mathbb{S}^1} \kappa^2 \, v dr\right) \right\} \\ &= \cos\left(\frac{\pi l}{L}\right) \left(h \int_p^q \kappa \, v dr - \int_p^q \kappa^2 \, v dr\right) \\ &\quad + \frac{1}{\pi} \left(2\pi h - \int_{\mathbb{S}^1} \kappa^2 \, v dr\right) \left\{\sin\left(\frac{\pi l}{L}\right) - \frac{\pi l}{L} \cos\left(\frac{\pi l}{L}\right)\right\} , \end{aligned}$$

where we only rearranged terms in the last line. $\square$

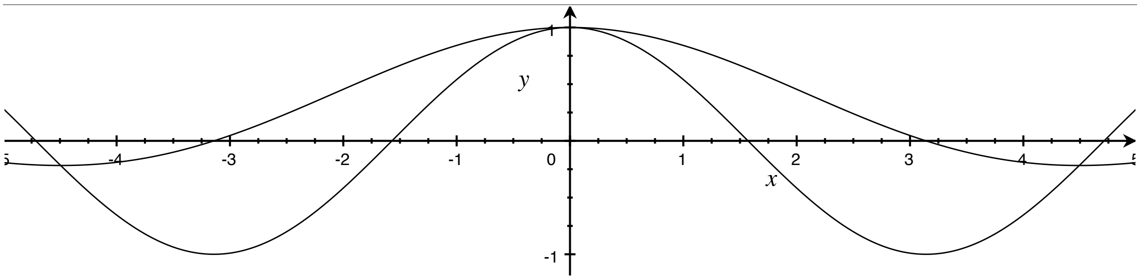

Figure 7.2: The upper graph is $\sin(x)/x$, the lower one is $\cos(x)$.

Lemma 7.8. *For $x \in [-\pi, \pi]$,*

$$\sin(x) \geq x\cos(x)$$

with equality if and only if $x = 0$ (see Figure 7.2 for an illustration).

Proof. For $x \in (-\pi/2, 0) \cup (0, \pi/2)$, we calculate,

$$\frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) = 1 + \frac{\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} > 1\,.$$

Hence, $|\sin(x)/\cos(x)| \geq |x|$ for $x \in (-\pi/2, 0) \cup (0, \pi/2)$. On $(-\pi, -\pi/2) \cup (\pi/2, \pi)$, $\sin(x)/x > 0$ and $\cos(x) \leq 0$. For $x = 0$, we have that $\sin(x)/x = 1 = \cos(x)$. For $x = \pm\pi$, $\sin(x)/x = 0$ and $\cos(x) < 0$. □

Corollary 7.9. *Under the APCSF the function ψ evolves according to*

$$\begin{aligned}\frac{\partial \psi}{\partial t} &= \cos\left(\frac{\pi l}{L}\right)\left(\frac{2\pi}{L}\int_p^q \kappa\, v dr - \int_p^q \kappa^2\, v dr\right)\\ &\quad + \frac{1}{\pi}\left(\frac{(2\pi)^2}{L} - \int_{\mathbb{S}^1} \kappa^2\, v dr\right)\left\{\sin\left(\frac{\pi l}{L}\right) - \frac{\pi l}{L}\cos\left(\frac{\pi l}{L}\right)\right\} \qquad (7.8)\\ &\leq \cos\left(\frac{\pi l}{L}\right)\left(\frac{2\pi}{L}\int_p^q \kappa\, v dr - \int_p^q \kappa^2\, v dr\right), \qquad (7.9)\end{aligned}$$

with equality in (7.9) *if and only if either Σ_t is a circle, or $l = 0$.*

Proof. Since $l \in [0, L]$, we have $\pi l/L \in [0, \pi]$, and Lemma 7.8 implies

$$\sin\left(\frac{\pi l}{L}\right) - \frac{\pi l}{L}\cos\left(\frac{\pi l}{L}\right) \geq 0$$

with equality if and only if $l = 0$. By Cauchy–Schwarz (B.3),

$$\frac{(2\pi)^2}{L} - \int_{\mathbb{S}^1} \kappa^2\, v dr \leq \frac{(2\pi)^2}{L} - \frac{(2\pi)^2}{L} = 0$$

with equality if and only if Σ_t is a circle. Hence, (7.9) equals (7.8) if and only if $l = 0$ or Σ_t is a circle. □

Corollary 7.10. *Under the LPCF the function ψ evolves according to*

$$\frac{\partial \psi}{\partial t} = \cos\left(\frac{\pi l}{L}\right)\left(h\int_p^q \kappa\, v dr - \int_p^q \kappa^2\, v dr\right). \qquad (7.10)$$

Proof. For the LPCF, the global term is given by $h = \int_{\mathbb{S}^1} \kappa^2 \, v dr/(2\pi)$, so that the second term in the evolution equation (7.7) of ψ is zero and we are left with (7.10). □

Next, we calculate the derivatives of the ratio d/ψ.

Lemma 7.11 (Spatial derivatives of d/ψ)**.** *Let $p, q \in \mathbb{S}^1$, $d(p,q) \neq 0$, $a, b \in \mathbb{R}$ and $\xi := a\boldsymbol{\tau}_p \oplus b\boldsymbol{\tau}_q$. Then*

$$\xi\left(\frac{d}{\psi}\right) = \frac{1}{\psi}\left\langle \mathbf{w}, b\boldsymbol{\tau}_q - a\boldsymbol{\tau}_p \right\rangle - \frac{d}{\psi^2}\cos\left(\frac{\pi l}{L}\right)(b-a)$$

and

$$\begin{aligned}\xi^2\left(\frac{d}{\psi}\right) &= -\frac{1}{d\psi}\left\langle \mathbf{w}, b\boldsymbol{\tau}_q - a\boldsymbol{\tau}_p \right\rangle^2 + \frac{1}{d\psi}\|b\boldsymbol{\tau}_q - a\boldsymbol{\tau}_p\|^2 \\ &\quad - \frac{1}{\psi}\left\langle \mathbf{w}, b^2\kappa_q\boldsymbol{\nu}_q - a^2\kappa_p\boldsymbol{\nu}_p \right\rangle - \frac{2(b-a)}{\psi^2}\left\langle \mathbf{w}, b\boldsymbol{\tau}_q - a\boldsymbol{\tau}_p \right\rangle \cos\left(\frac{\pi l}{L}\right) \\ &\quad + \frac{d}{\psi^2}\frac{\pi}{L}\sin\left(\frac{\pi l}{L}\right)(b-a)^2 + 2\frac{d}{\psi^3}\cos^2\left(\frac{\pi l}{L}\right)(b-a)^2 .\end{aligned}$$

Proof. The spatial derivatives (6.1) and (7.6) of d and ψ imply

$$\xi\left(\frac{d}{\psi}\right) = \frac{1}{\psi}\xi(d) - \frac{d}{\psi^2}\xi(\psi) = \frac{1}{\psi}\left\langle \mathbf{w}, b\boldsymbol{\tau}_q - a\boldsymbol{\tau}_p \right\rangle - \frac{d}{\psi^2}\cos\left(\frac{\pi l}{L}\right)(b-a) .$$

By also applying the second spatial derivative (6.2) and (7.6) of d and ψ, we obtain

$$\begin{aligned}\xi^2\left(\frac{d}{\psi}\right) &= \xi\left(\frac{1}{\psi}\xi(d) - \frac{d}{\psi^2}\xi(\psi)\right) \\ &= \frac{1}{\psi}\xi^2(d) - \frac{1}{\psi^2}\left\langle \xi(d), \xi(\psi) \right\rangle - \frac{d}{\psi^2}\xi^2(\psi) - \frac{1}{\psi^2}\left\langle \xi(\psi), \xi(d) \right\rangle + 2\frac{d}{\psi^3}\|\xi(\psi)\|^2 \\ &= \frac{1}{\psi}\left(-\frac{1}{d}\left\langle \mathbf{w}, b\boldsymbol{\tau}_q - a\boldsymbol{\tau}_p \right\rangle^2 + \frac{1}{d}\|b\boldsymbol{\tau}_q - a\boldsymbol{\tau}_p\|^2 - \left\langle \mathbf{w}, b^2\kappa_q\boldsymbol{\nu}_q - a^2\kappa_p\boldsymbol{\nu}_p \right\rangle\right) \\ &\quad - \frac{2}{\psi^2}\left\langle \mathbf{w}, b\boldsymbol{\tau}_q - a\boldsymbol{\tau}_p \right\rangle \cos\left(\frac{\pi l}{L}\right)(b-a) + \frac{d}{\psi^2}\frac{\pi}{L}\sin\left(\frac{\pi l}{L}\right)(b-a)^2 \\ &\quad + 2\frac{d}{\psi^3}\cos^2\left(\frac{\pi l}{L}\right)(b-a)^2 .\end{aligned}$$

□

Corollary 7.12. *For $p, q \in \mathbb{S}^1$ with $d(p,q) \neq 0$,*

$$(\boldsymbol{\tau}_p \oplus 0)\left(\frac{d}{\psi}\right) = -\frac{1}{\psi}\left\langle \mathbf{w}, \boldsymbol{\tau}_p \right\rangle + \frac{d}{\psi^2}\cos\left(\frac{\pi l}{L}\right) \tag{7.11}$$

$$(0 \oplus \boldsymbol{\tau}_q)\left(\frac{d}{\psi}\right) = \frac{1}{\psi}\left\langle \mathbf{w}, \boldsymbol{\tau}_q \right\rangle - \frac{d}{\psi^2}\cos\left(\frac{\pi l}{L}\right) \tag{7.12}$$

as well as

$$\begin{aligned}(\boldsymbol{\tau}_p \oplus \boldsymbol{\tau}_q)^2\left(\frac{d}{\psi}\right) &= -\frac{1}{d\psi}\left\langle \mathbf{w}, \boldsymbol{\tau}_q - \boldsymbol{\tau}_p \right\rangle^2 + \frac{1}{d\psi}\|\boldsymbol{\tau}_q - \boldsymbol{\tau}_p\|^2 \\ &\quad + \frac{1}{\psi}\left\langle \mathbf{w}, \boldsymbol{\kappa}_q - \boldsymbol{\kappa}_p \right\rangle\end{aligned} \tag{7.13}$$

and

$$\begin{aligned}(\boldsymbol{\tau}_p \ominus \boldsymbol{\tau}_q)^2\left(\frac{d}{\psi}\right) &= -\frac{1}{d\psi}\left\langle \mathbf{w}, \boldsymbol{\tau}_q + \boldsymbol{\tau}_p\right\rangle^2 + \frac{1}{d\psi}\|\boldsymbol{\tau}_q + \boldsymbol{\tau}_p\|^2 \\ &\quad + \frac{1}{\psi}\left\langle \mathbf{w}, \boldsymbol{\kappa}_q - \boldsymbol{\kappa}_p\right\rangle - \frac{4}{\psi^2}\left\langle \mathbf{w}, \boldsymbol{\tau}_q + \boldsymbol{\tau}_p\right\rangle \cos\left(\frac{\pi l}{L}\right) \\ &\quad + 4\frac{d}{\psi^2}\frac{\pi}{L}\sin\left(\frac{\pi l}{L}\right) + 8\frac{d}{\psi^3}\cos^2\left(\frac{\pi l}{L}\right). \end{aligned} \tag{7.14}$$

Lemma 7.13 (Evolution equation for d/ψ)**.** *Let $F : \mathbb{S}^1 \times (0,T) \to \mathbb{R}^2$ be a solution of* (2.15)*. Then*

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{d}{\psi}\right) &= \frac{1}{\psi}\Big(\left\langle \mathbf{w}, \boldsymbol{\kappa}_q - \boldsymbol{\kappa}_p\right\rangle + h\left\langle \mathbf{w}, \boldsymbol{\nu}_q - \boldsymbol{\nu}_p\right\rangle\Big) \\ &\quad - \frac{d}{\psi^2}\cos\left(\frac{\pi l}{L}\right)\left(h\int_p^q \kappa\, v dr - \int_p^q \kappa^2\, v dr\right) \\ &\quad + \frac{d}{\pi\psi^2}\left(\int_{\mathbb{S}^1}\kappa^2\, v dr - 2\pi h\right)\left\{\sin\left(\frac{\pi l}{L}\right) - \frac{\pi l}{L}\cos\left(\frac{\pi l}{L}\right)\right\}. \end{aligned}$$

for $p, q \in \mathbb{S}^1$ and $t \in (0,T)$ with $d(p,q,t) \neq 0$.

Proof. The identity follows directly from the evolution equations (6.4) and (7.7) of the distance functions d and ψ. □

Corollary 7.14. *Under the APCSF the ratio d/ψ evolves according to*

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{d}{\psi}\right) &= \frac{1}{\psi}\left(\left\langle \mathbf{w}, \boldsymbol{\kappa}_q - \boldsymbol{\kappa}_p\right\rangle + \frac{2\pi}{L}\left\langle \mathbf{w}, \boldsymbol{\nu}_q - \boldsymbol{\nu}_p\right\rangle\right) \\ &\quad - \frac{d}{\psi^2}\cos\left(\frac{\pi l}{L}\right)\left(\frac{2\pi}{L}\int_p^q \kappa\, v dr - \int_p^q \kappa^2\, v dr\right) \\ &\quad + \frac{d}{\pi\psi^2}\left(\int_{\mathbb{S}^1}\kappa^2\, v dr - \frac{(2\pi)^2}{L}\right)\left\{\sin\left(\frac{\pi l}{L}\right) - \frac{\pi l}{L}\cos\left(\frac{\pi l}{L}\right)\right\} \qquad (7.15) \\ &\geq \frac{1}{\psi}\left(\left\langle \mathbf{w}, \boldsymbol{\kappa}_q - \boldsymbol{\kappa}_p\right\rangle + \frac{2\pi}{L}\left\langle \mathbf{w}, \boldsymbol{\nu}_q - \boldsymbol{\nu}_p\right\rangle\right) \\ &\quad - \frac{d}{\psi^2}\cos\left(\frac{\pi l}{L}\right)\left(\frac{2\pi}{L}\int_p^q \kappa\, v dr - \int_p^q \kappa^2\, v dr\right), \qquad (7.16)\end{aligned}$$

for $p, q \in \mathbb{S}^1$ and $t \in (0,T)$ with $d(p,q,t) \neq 0$, where the third term in (7.15) *vanishes if and only if either Σ_t is a circle or $l = 0$, such that in particular equality holds in* (7.16)*.*

Proof. For the APCSF, we have that $h = 2\pi/L$. We use the estimate (7.9) for the evolution of ψ to estimate (7.15) from below. □

Corollary 7.15. *Under the LPCF the ratio d/ψ evolves according to*

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{d}{\psi}\right) &= \frac{1}{\psi}\Big(\left\langle \mathbf{w}, \boldsymbol{\kappa}_q - \boldsymbol{\kappa}_p\right\rangle + h\left\langle \mathbf{w}, \boldsymbol{\nu}_q - \boldsymbol{\nu}_p\right\rangle\Big) \\ &\quad - \frac{d}{\psi^2}\cos\left(\frac{\pi l}{L}\right)\left(h\int_p^q \kappa\, v dr - \int_p^q \kappa^2\, v dr\right). \end{aligned} \tag{7.17}$$

for $p, q \in \mathbb{S}^1$ and $t \in (0,T)$ with $d(p,q,t) \neq 0$.

Proof. The claim follows from Lemma 7.13 since for the LPCF

$$\int_{\mathbb{S}^1} \kappa^2 \, v dr - 2\pi h = 0 \, . \qquad \square$$

7.2 Behaviour at a minimum of the ratio of the distance functions

In this section we prove that, under the initial condition $\theta_{\min}(0) \geq -\pi$, the ratio d/ψ is bounded from below uniformly in time. We start with a lemma which hold for general closed, embedded, planar curves.

Lemma 7.16. *For $\alpha \in \mathbb{R}$, $1 + \cos(\alpha) = 2\cos^2(\alpha/2)$.*

Proof. By (7.3),

$$1 + \cos(\alpha) \overset{(7.3)}{=} 1 + \cos^2(\alpha/2) - \sin^2(\alpha/2) = \cos^2(\alpha/2) + \cos^2(\alpha/2) = 2\cos^2(\alpha/2) \, . \qquad \square$$

Lemma 7.17. *For $p, q \in \mathbb{S}^1$,*

$$\|\boldsymbol{\tau}_p + \boldsymbol{\tau}_q\|^2 = 4\cos^2\left(\frac{\theta(p,q)}{2}\right) \, .$$

Proof. We abbreviate $\theta := \theta(p,q)$ and use the identity (5.6), that is, $\theta = \vartheta_q - \vartheta_p + 2\pi\omega$. The representation (5.1) of $\boldsymbol{\tau}$ and the subtraction rule (7.3) for the cosine function imply

$$\begin{aligned} \langle \boldsymbol{\tau}_p, \boldsymbol{\tau}_q \rangle &\overset{(5.1)}{=} \big\langle \big(\cos(\vartheta_p), \sin(\vartheta_p)\big), \big(\cos(\vartheta_q), \sin(\vartheta_q)\big) \big\rangle \\ &= \cos(\vartheta_p)\cos(\vartheta_q) + \sin(\vartheta_p)\sin(\vartheta_q) \overset{(7.3)}{=} \cos(\vartheta_q - \vartheta_p) \overset{(5.6)}{=} \cos(\theta) \, . \end{aligned}$$

Lemma 7.16 yields

$$\|\boldsymbol{\tau}_p + \boldsymbol{\tau}_q\|^2 = \langle \boldsymbol{\tau}_p, \boldsymbol{\tau}_p \rangle + \langle \boldsymbol{\tau}_q, \boldsymbol{\tau}_q \rangle + 2\langle \boldsymbol{\tau}_p, \boldsymbol{\tau}_q \rangle = 2 + 2\cos(\theta) \overset{\text{Lem. 7.16}}{=} 4\cos^2\left(\frac{\theta}{2}\right) \, . \qquad \square$$

Lemma 7.18. *Let $\theta(p,q) \in (0,\pi]$ and $\langle \mathbf{w}, \boldsymbol{\tau}_p \rangle = \langle \mathbf{w}, \boldsymbol{\tau}_q \rangle = \cos(\theta(p,q)/2)$ for a vector $\mathbf{w} \in \mathbb{R}^2$. Then either*

(i) $\langle \mathbf{w}, \boldsymbol{\nu}_p \rangle = -\langle \mathbf{w}, \boldsymbol{\nu}_q \rangle = -\sin(\theta(p,q)/2)$, *or*

(ii) $\langle \mathbf{w}, \boldsymbol{\nu}_p \rangle = -\langle \mathbf{w}, \boldsymbol{\nu}_q \rangle = 1$.

Proof. The angle $\theta := \theta(p,q)$ is invariant under rotations in the plane, thus we may assume $\mathbf{w} = \mathbf{e}_1$. Since $\theta/2 \in (0, \pi/2]$, the definition (5.1) of ϑ and the assumptions yield

$$\cos(\vartheta_p) \overset{(5.1)}{=} \langle \mathbf{e}_1, \boldsymbol{\tau}_p \rangle = \cos\left(\frac{\theta}{2}\right) \geq 0$$

and

$$\cos(\vartheta_q) \overset{(5.1)}{=} \langle \mathbf{e}_1, \boldsymbol{\tau}_q \rangle = \cos\left(\frac{\theta}{2}\right) \geq 0 \, .$$

Hence,

$$\vartheta_p, \vartheta_q \in \left\{ \frac{\theta}{2}, 2\pi - \frac{\theta}{2} \right\} \qquad \text{and} \qquad \vartheta_q - \vartheta_p + 2\pi\omega \overset{(5.6)}{=} \theta \in (0, \pi] \,, \tag{7.18}$$

where $\omega \in \mathbb{Z}$.

Case (a): Assume that

$$\vartheta_p = \vartheta_q = \frac{\theta}{2} \qquad \text{or} \qquad \vartheta_p = \vartheta_q = 2\pi - \frac{\theta}{2} \,.$$

From (7.18) follows that $2\pi\omega \in (0, \pi]$ which is impossible for $\omega \in \mathbb{Z}$.

Case (b): Assume that

$$\vartheta_p = 2\pi - \frac{\theta}{2} \qquad \text{and} \qquad \vartheta_q = \frac{\theta}{2} \,.$$

Then

$$\langle \mathbf{e}_1, \boldsymbol{\nu}_p \rangle \overset{(5.1)}{=} \sin(\vartheta_p) = \sin\left(-\frac{\theta}{2} + 2\pi k \right) = -\sin\left(\frac{\theta}{2} \right)$$

and

$$\langle \mathbf{e}_1, \boldsymbol{\nu}_q \rangle \overset{(5.1)}{=} \sin(\vartheta_q) = \sin\left(\frac{\theta}{2} + 2\pi l \right) = \sin\left(\frac{\theta}{2} \right)$$

as claimed.

Case (c): Assume that

$$\vartheta_p = \frac{\theta}{2} \qquad \text{and} \qquad \vartheta_q = 2\pi - \frac{\theta}{2} \,.$$

From (7.18) follows that

$$-\theta + 2\pi\omega = \theta \in (0, \pi] \,,$$

so that

$$\pi\omega = \theta \in (0, \pi]$$

which yields $\theta = \pi$ and $\omega = 1$. Hence,

$$\vartheta_p = \frac{\pi}{2} \qquad \text{and} \qquad \vartheta_q = -\frac{3\pi}{2}$$

so that

$$\langle \mathbf{e}_1, \boldsymbol{\nu}_p \rangle \overset{(5.1)}{=} \sin(\vartheta_p) = \sin\left(\frac{\pi}{2} \right) = 1$$

and

$$\langle \mathbf{e}_1, \boldsymbol{\nu}_q \rangle \overset{(5.1)}{=} \sin(\vartheta_q) = \sin\left(\frac{3\pi}{2} \right) = -1$$

as claimed. □

We thank Theodora Bourni for ideas used in the proof of the next Lemma.

Lemma 7.19 (Global spatial minima of d/ψ)**.** *Let $\Sigma = F(\mathbb{S}^1)$ be an embedded, closed curve in the plane. Let $p, q \in \mathbb{S}^1$, $p \neq q$, such that Σ crosses the connecting line between $F(p)$ and $F(q)$. Then $(d/\psi)(p,q)$ cannot be a global spatial minimum for the function d/ψ.*

Proof. Let $\Sigma = F(\mathbb{S}^1)$ be an embedded, closed curve in $\mathbb{R}^2$ so that it crosses the connecting line between $F(p)$ and $F(q)$ (see Figure 7.3 for illustrations). That is, there exists an $r \in \mathbb{S}^1$, $r \neq p, q$, with

$$F(r) = F(p) + w(p,q)\|F(r) - F(p)\| \,.$$

Set

$$d := d(p,q)\,, \qquad d_1 := d(p,r) \qquad \text{and} \qquad d_2 := d(r,q)\,.$$

Then

$$d = d_1 + d_2\,. \tag{7.19}$$

Furthermore, set

$$l := l(p,q)\,, \qquad l_1 := l(p,r) \qquad \text{and} \qquad l_2 := l(r,q)\,.$$

Since $l_1, l_2 \in (0, L)$,

$$\sin\left(\frac{\pi l_1}{L}\right) > 0 \qquad \text{and} \qquad \sin\left(\frac{\pi l_2}{L}\right) > 0 \tag{7.20}$$

as well as

$$\cos\left(\frac{\pi l_1}{L}\right) < 1 \qquad \text{and} \qquad \cos\left(\frac{\pi l_2}{L}\right) < 1\,. \tag{7.21}$$

This and the addition rule (7.2) for the sine function implies

$$\begin{aligned}\sin\left(\frac{\pi(l_1+l_2)}{L}\right) &= \sin\left(\frac{\pi l_1}{L}\right)\cos\left(\frac{\pi l_2}{L}\right) + \sin\left(\frac{\pi l_2}{L}\right)\cos\left(\frac{\pi l_1}{L}\right) \\ &\overset{(7.20),(7.21)}{<} \sin\left(\frac{\pi l_1}{L}\right) + \sin\left(\frac{\pi l_2}{L}\right)\,. \end{aligned} \tag{7.22}$$

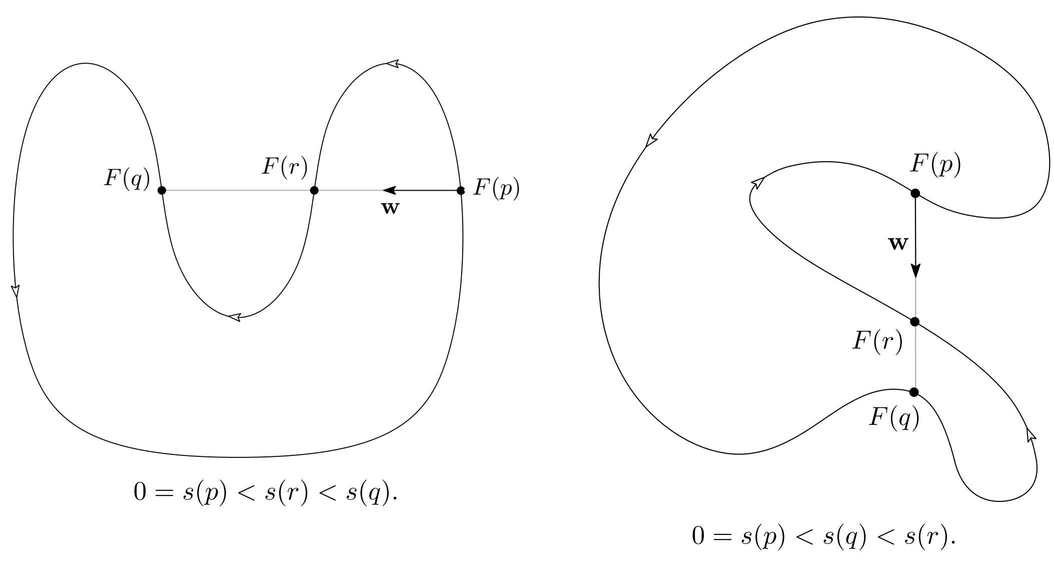

Figure 7.3: $F(r)$ lies on the connecting line between $F(p)$ and $F(q)$.

Set

$$\psi := \psi(p,q)\,, \qquad \psi_1 := \psi(p,r) \qquad \text{and} \qquad \psi_2 := \psi(r,q)$$

and assume that d/ψ attains its global minimum at (p,q). Then

$$\frac{d}{\psi} \leq \frac{d_1}{\psi_1} \qquad \text{and} \qquad \frac{d}{\psi} \leq \frac{d_2}{\psi_2}\,. \tag{7.23}$$

We parametrise Σ by arc length, so that $s(p) = 0$. Then either $0 = s(p) < s(r) < s(q)$ or $0 = s(p) < s(q) < s(r)$ (see Figure 7.3).

(a) Assume that $0 = s(p) < s(r) < s(q)$. Then $l = l_1 + l_2$ and the definition (7.1) of ψ and (7.22) imply

$$\frac{\pi}{L}\psi \overset{(7.1)}{=} \sin\left(\frac{\pi l}{L}\right) \overset{(7.22)}{<} \sin\left(\frac{\pi l_1}{L}\right) + \sin\left(\frac{\pi l_2}{L}\right) \overset{(7.1)}{=} \frac{\pi}{L}\psi_1 + \frac{\pi}{L}\psi_2\,. \tag{7.24}$$

(b) Assume $0 = s(p) < s(q) < s(r)$. Then $l = L - (l_1 + l_2)$ so that the subtraction rule (7.2) for the sine function and (7.22) yield

$$\begin{aligned}
\frac{\pi}{L}\psi &\overset{(7.1)}{=} \sin\left(\frac{\pi l}{L}\right) = \sin\left(\frac{\pi L}{L} - \frac{\pi(l_1+l_2)}{L}\right) \\
&\overset{(7.2)}{=} \sin(\pi)\cos\left(\frac{\pi(l_1+l_2)}{L}\right) - \cos(\pi)\sin\left(\frac{\pi(l_1+l_2)}{L}\right) \\
&= \sin\left(\frac{\pi(l_1+l_2)}{L}\right) \overset{(7.22)}{<} \sin\left(\frac{\pi l_1}{L}\right) + \sin\left(\frac{\pi l_2}{L}\right) \\
&\overset{(7.1)}{=} \frac{\pi}{L}\psi_1 + \frac{\pi}{L}\psi_2\,.
\end{aligned} \tag{7.25}$$

Hence, in both cases,

$$\psi \overset{(7.24),(7.25)}{<} \psi_1 + \psi_2 \tag{7.26}$$

which together with (7.19) and (7.23) yields

$$\frac{d_1}{\psi_1} \overset{(7.23)}{\geq} \frac{d}{\psi} \overset{(7.19),(7.26)}{>} \frac{d_1+d_2}{\psi_1+\psi_2} \qquad \text{and} \qquad \frac{d_2}{\psi_2} \overset{(7.23)}{\geq} \frac{d}{\psi} \overset{(7.19),(7.26)}{>} \frac{d_1+d_2}{\psi_1+\psi_2}$$

so that

$$d_1(\psi_1+\psi_2) > (d_1+d_2)\psi_1 \qquad \text{and} \qquad d_2(\psi_1+\psi_2) > (d_1+d_2)\psi_2\,.$$

Adding both inequalities implies

$$(d_1+d_2)(\psi_1+\psi_2) > (d_1+d_2)(\psi_1+\psi_2)$$

which is a contradiction. Thus, d/ψ cannot have a global minimum at (p,q). □

Now we can prove a similar result to [Hui95, Theorem 2.3].

Proposition 7.20 (Behaviour at a minimum of d/ψ)**.** *Let Σ_0 be a smooth, embedded curve satisfying $\theta_{\min} \geq -\pi$. Let $F : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ be a solution of* (2.15) *with initial curve Σ_0 and let $t_0 \in (0,T)$. Suppose that d/ψ attains a local spatial minimum at (p,q) at time t_0. Then the following holds:*

(i) if $\theta(p,q,t_0) \in (0,\pi]$ and $\langle \mathbf{w}, \boldsymbol{\nu}_p \rangle = -\langle \mathbf{w}, \boldsymbol{\nu}_q \rangle = -\sin(\theta/2)$, then

$$\frac{\partial}{\partial t}_{|t=t_0} \left(\frac{d}{\psi}\right)(p,q,t) > 0\,,$$

or

(ii) if $\theta \in (\pi, 2\pi) \cup (2\pi, 3\pi)$ or $\theta = \pi$ and $\langle \mathbf{w}, \boldsymbol{\nu}_p \rangle = -\langle \mathbf{w}, \boldsymbol{\nu}_q \rangle = 1$, then $(d/\psi)(p,q,t_0)$ cannot be a global spatial minimum, that is,

$$\frac{d}{\psi}(p,q,t_0) > \min_{\mathbb{S}^1\times\mathbb{S}^1} \frac{d}{\psi}(\,\cdot\,,\,\cdot\,,t_0)\,;$$

(iii) if $\theta(p,q,t_0) \in (-\pi, 0)$, then $(d/\psi)(p,q,t_0)$ is bounded below by a positive constant

$$C^* = C^*(\Sigma_0) \leq \min_{\mathbb{S}^1\times\mathbb{S}^1} \frac{d}{\psi}(\,\cdot\,,\,\cdot\,,0)\,;$$

(iv) if $\theta(p,q,t_0) \in \{0, 2\pi\}$, then

(a) for the APCFS,

$$\frac{\partial}{\partial t}_{|t=t_0} \left(\frac{d}{\psi}\right)(p,q,t) > 0\,,$$

(b) for the LPCF, $(d/\psi)(p,q,t_0)$ cannot be a global spatial minimum, that is,

$$\frac{d}{\psi}(p,q,t_0) > \min_{\mathbb{S}^1\times\mathbb{S}^1} \frac{d}{\psi}(\,\cdot\,,\,\cdot\,,t_0)\,.$$

Theorem 7.21 (Lower bound on d/ψ)**.** *Let Σ_0 be a smooth, embedded curve satisfying $\theta_{\min} \geq -\pi$. Let $F : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ be a solution of* (2.15) *with initial curve Σ_0. Then*

$$\frac{d}{\psi}(p,q,t) \geq C^* > 0$$

for all $p,q \in \mathbb{S}^1$ and all $t \in [0,T)$ and where $C^* = C^*(\Sigma_0) > 0$ is the constant from Proposition 7.20.

Proof. Assume that d/ψ falls below C^* and attains $\gamma \in (0, C^*)$ for the first time at time $t_1 \in (0,T)$ and points $p,q \in \mathbb{S}^1$, $p \neq q$, so that

$$C^* > \gamma = \frac{d}{\psi}(p,q,t_1) = \min_{\mathbb{S}^1\times\mathbb{S}^1} \frac{d}{\psi}(\,\cdot\,,\,\cdot\,,t_1) \tag{7.27}$$

is a global minimum and

$$\frac{\partial}{\partial t}_{|t=t_1} \left(\frac{d}{\psi}\right)(p,q,t) \leq 0\,. \tag{7.28}$$

Recall that in Proposition 7.20(ii) and (iv)(b) the minimum is not a global minimum. Hence, for the **APCSF**, Proposition 7.20(i) and (iv) contradict (7.28), and Proposition 7.20(ii) and (iii) contradict (7.27). On the other hand, for the **LPCF**, Proposition 7.20(i) contradicts (7.28), and Proposition 7.20(ii)–(iv) contradict (7.27). □

Proof of Proposition 7.20. Let Σ_0 be an embedded closed curve in $\mathbb{R}^2$ satisfying the initial condition $\theta(p, q, 0) \geq -\pi$ for all $p, q \in \mathbb{S}^1$. By Theorem 6.5, Σ_t is embedded for all $t \in [0, T)$. Remark 6.6 implies that $\theta(p, q, 0) \in [-\pi, 3\pi]$ for all $p, q \in \mathbb{S}^1$. From the maximum principle for θ, Corollary 5.4, it follows that

$$\theta(p, q, t) \in (-\pi, 3\pi) \tag{7.29}$$

for all $p, q \in \mathbb{S}^1$ and $t \in (0, T)$.

Fix $t_0 \in [0, T)$. If Σ_{t_0} is a circle, then Remark 2.6(ii) implies that Σ_t is a circle for all $t \geq t_0$. Furthermore, Remark 7.3 yields that $d/\psi \equiv 1$ on $\mathbb{S}^1 \times \mathbb{S}^1$ for all $t \geq t_0$.

From now on assume that $t_0 \in (0, T)$ and that Σ_{t_0} is not a circle. As stated in Remark 7.3, $\min_{\mathbb{S}^1 \times \mathbb{S}^1}(d/\psi) < 1$ at t_0. Let $p, q \in \mathbb{S}^1$, $p \neq q$, be points where the spatial minimum of d/ψ at t_0 is attained and assume w.l.o.g. that $s(p, t_0) < s(q, t_0)$. Again by Remark 7.3, we can assume that $l(p, q, t_0) \leq L_{t_0}/2$. We have

$$0 < \frac{d}{\psi}(p, q, t_0) < 1\,,$$

and for all $\xi \in T_{F(p,t_0)}\Sigma_{t_0} \bigoplus T_{F(q,t_0)}\Sigma_{t_0}$,

$$\xi\left(\frac{d}{\psi}\right)(p, q, t_0) = 0 \tag{7.30}$$

as well as

$$\xi^2\left(\frac{d}{\psi}\right)(p, q, t_0) \geq 0\,. \tag{7.31}$$

In the following, we abbreviate the distance functions

$$d := d(p, q, t_0)\,, \quad l := l(p, q, t_0)\,, \quad \psi := \psi(p, q, t_0) \quad \text{and} \quad L_{t_0} := L\,,$$

the unit tangent and normal vectors

$$\boldsymbol{\tau}_p := \boldsymbol{\tau}(p, t_0)\,, \quad \boldsymbol{\tau}_q := \boldsymbol{\tau}(q, t_0)\,, \quad \boldsymbol{\nu}_p := \boldsymbol{\nu}(p, t_0) \quad \text{and} \quad \boldsymbol{\nu}_q := \boldsymbol{\nu}(q, t_0)\,,$$

the curvature and the curvature vectors

$$\kappa_p := \kappa(p, t_0)\,, \quad \kappa_q := \kappa(q, t_0)\,, \quad \boldsymbol{\kappa}_p := \boldsymbol{\kappa}(p, t_0) \quad \text{and} \quad \boldsymbol{\kappa}_q := \boldsymbol{\kappa}(q, t_0)\,,$$

and the length element, the local total curvature as well as the vector $\mathbf{w}$

$$v := v(r, t_0)\,, \quad \theta := \theta(p, q, t_0) \quad \text{and} \quad \mathbf{w} := \mathbf{w}(p, q, t_0) = \frac{F(q, t_0) - F(p, t_0)}{d(p, q, t_0)}\,.$$

The first spatial derivative of d/ψ in the direction of the vector $\xi = \boldsymbol{\tau}_p \oplus 0$ is given by (7.11). Combined with (7.30) this yields

$$0 \overset{(7.30)}{=} (\boldsymbol{\tau}_p \oplus 0)\left(\frac{d}{\psi}\right) \overset{(7.11)}{=} -\frac{1}{\psi}\langle \mathbf{w}, \boldsymbol{\tau}_p \rangle + \frac{d}{\psi^2}\cos\left(\frac{\pi l}{L}\right),$$

so that at (p, t_0),

$$\langle \mathbf{w}, \boldsymbol{\tau}_p \rangle = \frac{d}{\psi}\cos\left(\frac{\pi l}{L}\right) \geq 0\,. \tag{7.32}$$

The left hand side is non-negative since $d/\psi > 0$ and $l \leq L/2$. Equality holds if and only if $l = L/2$. For the vector $\xi = 0 \oplus \boldsymbol{\tau}_q$ we refer to (7.12) to obtain

$$0 \overset{(7.30)}{=} (0 \oplus \boldsymbol{\tau}_q)\left(\frac{d}{\psi}\right) \overset{(7.12)}{=} \frac{1}{\psi}\langle \mathbf{w}, \boldsymbol{\tau}_q \rangle - \frac{d}{\psi^2}\cos\left(\frac{\pi l}{L}\right),$$

so that at (q, t_0) we also have

$$\langle \mathbf{w}, \boldsymbol{\tau}_q \rangle = \frac{d}{\psi}\cos\left(\frac{\pi l}{L}\right) \geq 0\,. \tag{7.33}$$

We now consider the two cases $\boldsymbol{\tau}_p \neq \boldsymbol{\tau}_q$ and $\boldsymbol{\tau}_p = \boldsymbol{\tau}_q$.

Case 1: Assume that $\boldsymbol{\tau}_p \neq \boldsymbol{\tau}_q$.Subtracting (7.32) from (7.33) yields $\langle \mathbf{w}, \boldsymbol{\tau}_q - \boldsymbol{\tau}_p \rangle = 0$. By Lemmata 6.1 and 7.17,

$$\langle \mathbf{w}, \boldsymbol{\tau}_q + \boldsymbol{\tau}_p \rangle^2 \overset{\text{Lem. 6.1}}{=} \|\boldsymbol{\tau}_q + \boldsymbol{\tau}_p\|^2 \overset{\text{Lem. 7.17}}{=} 4\cos^2\left(\frac{\theta}{2}\right). \tag{7.34}$$

On the other hand, adding (7.32) and (7.33) yields

$$\langle \mathbf{w}, \boldsymbol{\tau}_q + \boldsymbol{\tau}_p \rangle = 2\frac{d}{\psi}\cos\left(\frac{\pi l}{L}\right), \tag{7.35}$$

so that, by (7.34), (7.35) and $d/\psi < 1$,

$$\left|\cos\left(\frac{\theta}{2}\right)\right| = \frac{d}{\psi}\cos\left(\frac{\pi l}{L}\right) < \cos\left(\frac{\pi l}{L}\right). \tag{7.36}$$

From (7.29) it follows that

$$\theta \in (-\pi, 0) \cup (0, 2\pi) \cup (2\pi, 3\pi)\,.$$

For $\theta = \pi$, Lemma 7.18 provides that either

$$\langle \mathbf{w}, \boldsymbol{\nu}_p \rangle = -\langle \mathbf{w}, \boldsymbol{\nu}_q \rangle = -\sin\left(\frac{\theta}{2}\right) \qquad \text{or} \qquad \langle \mathbf{w}, \boldsymbol{\nu}_p \rangle = -\langle \mathbf{w}, \boldsymbol{\nu}_q \rangle = 1\,.$$

We now look at the different intervals separately.

(a) Assume that $\theta \in (0, \pi]$ and

$$\langle \mathbf{w}, \boldsymbol{\nu}_p \rangle = -\langle \mathbf{w}, \boldsymbol{\nu}_q \rangle = -\sin\left(\frac{\theta}{2}\right) \tag{7.37}$$

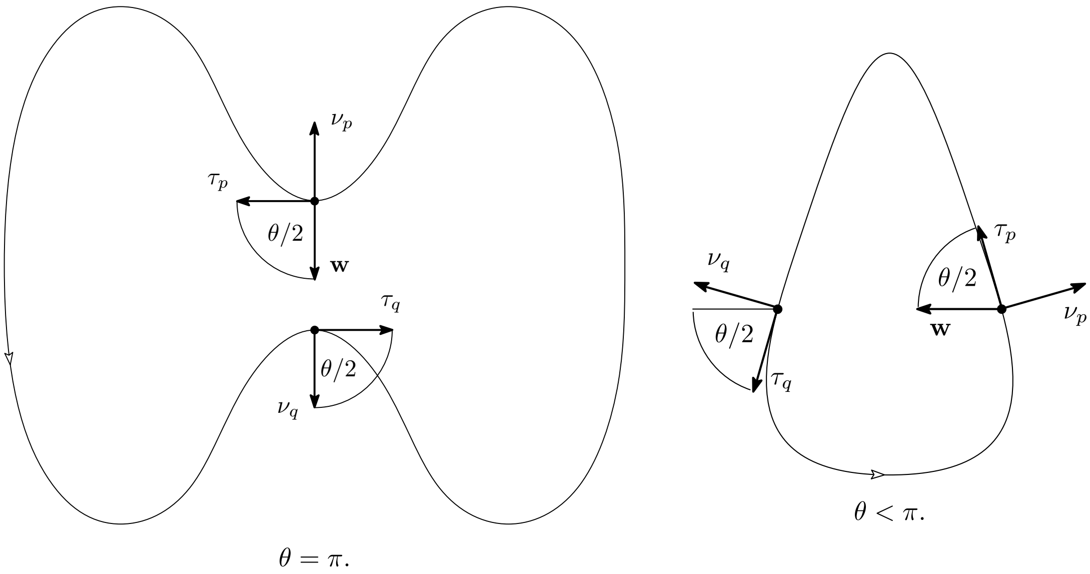

Figure 7.4: Case 1(a), $\theta \in (0, \pi]$ and $\langle \mathbf{w}, \boldsymbol{\nu}_p \rangle = -\langle \mathbf{w}, \boldsymbol{\nu}_q \rangle = -\sin(\theta/2)$.

(see Figure 7.4 for an illustration). Since $\pi l/L \in (0, \pi/2]$ and the cosine function is axially symmetric and monotonically decreasing on $(0, \pi/2]$, (7.36) implies

$$\theta > \frac{2\pi l}{L}\,. \tag{7.38}$$

Furthermore,

$$\langle \mathbf{w}, \boldsymbol{\kappa}_q - \boldsymbol{\kappa}_p \rangle = -\kappa_q \langle \mathbf{w}, \boldsymbol{\nu}_q \rangle + \kappa_p \langle \mathbf{w}, \boldsymbol{\nu}_p \rangle \overset{(7.37)}{=} -(\kappa_p + \kappa_q) \sin\left(\frac{\theta}{2}\right)\,. \tag{7.39}$$

We differentiate d/ψ at (p, q, t_0) twice with respect to the vector $\xi = \boldsymbol{\tau}_p \ominus \boldsymbol{\tau}_q$ (see (7.14)) and calculate using the definition (7.1) of ψ,

$$\begin{aligned}
0 &\overset{(7.31)}{\leq} (\boldsymbol{\tau}_p \ominus \boldsymbol{\tau}_q)^2 \left(\frac{d}{\psi}\right) \\
&\overset{(7.14)}{=} -\frac{1}{d\psi} \langle \mathbf{w}, \boldsymbol{\tau}_q + \boldsymbol{\tau}_p \rangle^2 + \frac{1}{d\psi} \|\boldsymbol{\tau}_q + \boldsymbol{\tau}_p\|^2 + \frac{1}{\psi} \langle \mathbf{w}, \boldsymbol{\kappa}_q - \boldsymbol{\kappa}_p \rangle \\
&\qquad - \frac{4}{\psi^2} \langle \mathbf{w}, \boldsymbol{\tau}_q + \boldsymbol{\tau}_p \rangle \cos\left(\frac{\pi l}{L}\right) + \frac{4d}{\psi^2} \frac{\pi}{L} \sin\left(\frac{\pi l}{L}\right) + 8\frac{d}{\psi^3} \cos^2\left(\frac{\pi l}{L}\right) \\
&\underset{(7.35)}{\overset{(7.1),(7.34)}{=}} -\frac{1}{d\psi} \|\boldsymbol{\tau}_q + \boldsymbol{\tau}_p\|^2 + \frac{1}{d\psi} \|\boldsymbol{\tau}_q + \boldsymbol{\tau}_p\|^2 + \frac{1}{\psi} \langle \mathbf{w}, \boldsymbol{\kappa}_q - \boldsymbol{\kappa}_p \rangle \\
&\qquad - 8\frac{d}{\psi^3} \cos^2\left(\frac{\pi l}{L}\right) + \frac{4\pi^2 d}{L^2 \psi^2} \psi + 8\frac{d}{\psi^3} \cos^2\left(\frac{\pi l}{L}\right) \\
&\overset{(7.39)}{=} -\frac{1}{\psi} (\kappa_p + \kappa_q) \sin\left(\frac{\theta}{2}\right) + \frac{4\pi^2 d}{L^2 \psi}\,.
\end{aligned}$$

We abbreviate $\kappa := (\kappa_p + \kappa_q)/2$ and obtain

$$2\kappa \sin\left(\frac{\theta}{2}\right) \leq \frac{4\pi^2}{L^2} d\,. \tag{7.40}$$

Since the sine function is positive and monotonically increasing on $(0, \pi/2]$, we conclude with $d/\psi < 1$ that

$$d < \psi \overset{(7.1)}{=} \frac{L}{\pi} \sin\left(\frac{\pi l}{L}\right) \overset{(7.38)}{<} \frac{L}{\pi} \sin\left(\frac{\theta}{2}\right) ,$$

so that (7.40) implies

$$2\kappa \sin\left(\frac{\theta}{2}\right) \overset{(7.40)}{\leq} \frac{4\pi^2}{L^2} d < \frac{4\pi^2}{L^2} \frac{L}{\pi} \sin\left(\frac{\theta}{2}\right) = \frac{4\pi}{L} \sin\left(\frac{\theta}{2}\right) .$$

Since $\sin(\theta/2) > 0$ for $\theta \in (0, \pi]$, we can divide by it to achieve

$$\kappa < \frac{2\pi}{L} .$$

Since $h_{\mathrm{ap}} = 2\pi/L$ (see (3.10)) and $h_{\mathrm{lp}} \geq 2\pi/L_0 = 2\pi/L$ (see (3.11)), we have $h \geq 2\pi/L > \kappa$ for both flows and

$$h - \kappa > h - \frac{2\pi}{L} \geq 0 . \tag{7.41}$$

Furthermore, the inequality

$$\sin\left(\frac{\theta}{2}\right) > \frac{\theta}{2} \cos\left(\frac{\theta}{2}\right) \tag{7.42}$$

holds for all $\theta \in (0, \pi]$ (see Lemma 7.8), and we obtain

$$(h - \kappa) \sin\left(\frac{\theta}{2}\right) \overset{(7.41),(7.42)}{>} \frac{\theta}{2} \cos\left(\frac{\theta}{2}\right) \left(h - \frac{2\pi}{L}\right) . \tag{7.43}$$

Cauchy–Schwarz (B.3) and the definition (5.3) of θ imply

$$\int_p^q \kappa^2 \, v dr \overset{(B.3)}{\geq} \frac{1}{l} \left(\int_p^q \kappa \, v dr\right)^2 \overset{(5.3)}{=} \frac{\theta^2}{l} . \tag{7.44}$$

We use the definition (5.3) of θ to estimate the evolution equations (7.16) and (7.17) of d/ψ for both flows

$$\begin{aligned} \frac{\partial}{\partial t}_{|t=t_0} \left(\frac{d}{\psi}\right) &\overset{(7.16),(7.17)}{\geq} \frac{1}{\psi}\Big(h \left\langle \mathbf{w}, \boldsymbol{\nu}_q - \boldsymbol{\nu}_p \right\rangle + \left\langle \mathbf{w}, \boldsymbol{\kappa}_q - \boldsymbol{\kappa}_p \right\rangle\Big) \\ &\qquad + \frac{d}{\psi^2} \cos\left(\frac{\pi l}{L}\right) \left(\int_p^q \kappa^2 \, v dr - h \int_p^q \kappa \, v dr\right) \\ &\underset{(7.39),(7.44)}{\overset{(5.3),(7.36),(7.37)}{\geq}} \frac{1}{\psi}\left((2h - 2\kappa) \sin\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right) \left(\frac{\theta^2}{l} - h\theta\right)\right) \end{aligned} \tag{7.45}$$

with equality in the first line for the LPCF and strict inequality for the APCSF (see Corollary 7.14), since Σ_{t_0} is not a circle. By (7.38),

$$\frac{\theta^2}{l} > \frac{\theta}{l} \frac{2\pi l}{L} = \frac{2\pi\theta}{L} ,$$

so that

$$\frac{\theta^2}{l} - h\theta > \theta \left(\frac{2\pi}{L} - h\right) .$$

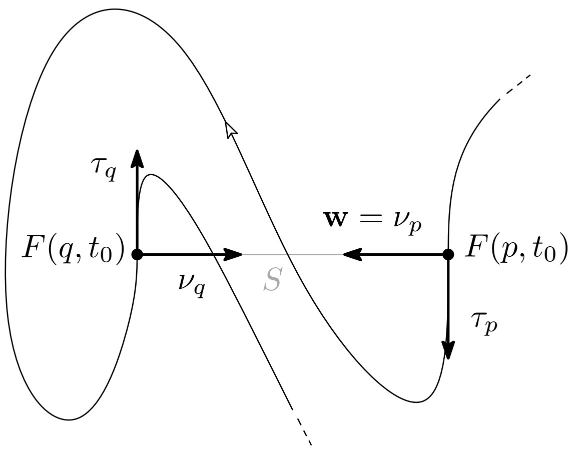

Figure 7.5: Case 1(b). Σ_{t_0} is crossing the connecting line between $F(p, t_0)$ and $F(q, t_0)$.

We can therefore estimate (7.45) further to deduce

$$\frac{\partial}{\partial t}_{|t=t_0}\left(\frac{d}{\psi}\right) \geq \frac{1}{\psi}\left(2\,(h-\kappa)\sin\left(\frac{\theta}{2}\right) - \theta\cos\left(\frac{\theta}{2}\right)\left(h - \frac{2\pi}{L}\right)\right). \tag{7.46}$$

By (7.43), the left-hand side of (7.46) is strictly positive, and

$$\frac{\partial}{\partial t}_{|t=t_0}\left(\frac{d}{\psi}\right) > 0$$

for $\theta \in (0, \pi]$ as claimed.

(b) Assume that $\theta \in (\pi, 2\pi) \cup (2\pi, 3\pi)$ or $\theta = \pi$ and $\langle \mathbf{w}, \boldsymbol{\nu}_p \rangle = -\langle \mathbf{w}, \boldsymbol{\nu}_q \rangle = 1$ (see Figure 7.5 for an illustration). Define the straight line segment

$$S := \{x \in \mathbb{R}^2 \,|\, x = (1-\lambda)F(p, t_0) + \lambda F(q, t_0) \text{ for } \lambda \in [0, 1]\}$$

through $F(p, t_0)$ and $F(q, t_0)$. By (7.32) and (7.33), $\langle \mathbf{w}, \boldsymbol{\tau}_q \rangle = \langle \mathbf{w}, \boldsymbol{\tau}_p \rangle \in [0, 1)$. Since Σ_{t_0} is closed and F_{t_0} is continuous, Σ_{t_0} has to cross S between $F(p, t_0)$ and $F(q, t_0)$ at least once. Lemma 7.19 implies that the ratio d/ψ cannot have a global minimum at (p, q, t_0) (it could still, however, attain a local minimum a this point). Hence,

$$\frac{d}{\psi} > \min_{\mathbb{S}^1\times\mathbb{S}^1} \frac{d}{\psi}(\,\cdot\,, \,\cdot\,, t_0)\,.$$

By the compactness of $\mathbb{S}^1 \times \mathbb{S}^1$, the global minimum of $(d/\psi)(\,\cdot\,, \,\cdot\,, t_0)$ exists at a point (p_0, q_0) which was treated in Case 1(a) and will be further treated in Case 1(c) and 2.

(c) Assume that $\theta \in (-\pi, 0)$ (see Figure 7.6 for an illustration). By the short time existence, Theorem 2.3, $F \in C^\infty\big(\mathbb{S}^1 \times [0, T)\big)$ with $T = T(\|F(\,\cdot\,, 0)\|_{C^{3,\alpha}}) > 0$. Theorem 6.5 and Lemma 7.13 imply that

$$\frac{d}{\psi} \in C^\infty\big((\mathbb{S}^1 \times \mathbb{S}^1 \setminus \{\bar{p} = \bar{q}\}) \times [0, T)\big) \cap C^0\big(\mathbb{S}^1 \times \mathbb{S}^1 \times [0, T)\big)\,. \tag{7.47}$$

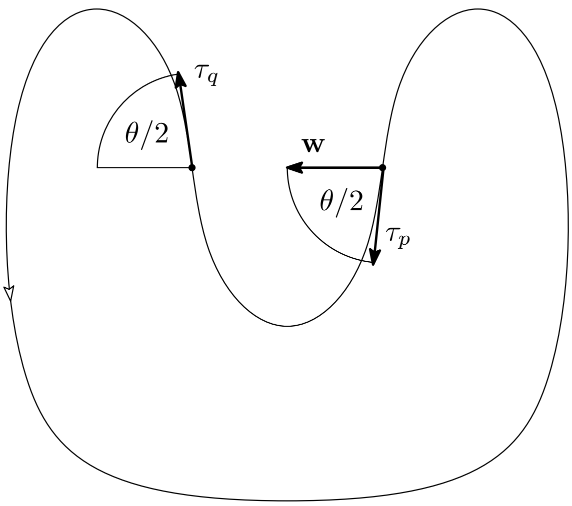

Figure 7.6: Case 1(c), $\theta \in (-\pi, 0)$.

Since $(d/\psi)(\bar{p}, \bar{p}, t) = 1$ for all $\bar{p} \in \mathbb{S}^1$ and $t \in [0, T)$ and $\min_{\mathbb{S}^1\times\mathbb{S}^1} d/\psi(\,\cdot\,, \,\cdot\,, 0) < 1$, there exists $\varepsilon > 0$ so that for $|\bar{p} - \bar{q}| < \varepsilon \mod (2\pi)$,

$$\frac{d}{\psi}(\bar{p}, \bar{q}, t) > \frac{3}{4} \min_{\mathbb{S}^1\times\mathbb{S}^1} \frac{d}{\psi}(\,\cdot\,, \,\cdot\,, 0)$$

for all $t \in [0, T/2]$. We define the set

$$A := \left\{ (\bar{p}, \bar{q}, t) \in \mathbb{S}^1 \times \mathbb{S}^1 \times \left[0, \frac{T}{2}\right] \;\middle|\; \frac{d}{\psi}(\bar{p}, \bar{q}, t) > \frac{3}{4} \min_{\mathbb{S}^1\times\mathbb{S}^1} \frac{d}{\psi}(\,\cdot\,, \,\cdot\,, 0) \right\} \tag{7.48}$$

with $\{p = q\} \times [0, T/2] \subset A$, and the closed complement

$$B := \left(\mathbb{S}^1 \times \mathbb{S}^1 \times \left[0, \frac{T}{2}\right] \right) \setminus A\,, \tag{7.49}$$

where $\mathbb{S}^1 \times \mathbb{S}^1 \times \{0\} \not\subseteq B$. By (7.47),

$$\Lambda(\|F(\,\cdot\,, 0)\|_{C^{3,\alpha}}) := \max_B \left| \frac{\partial}{\partial t}\left(\frac{d}{\psi}\right) \right| < \infty\,. \tag{7.50}$$

For $(\bar{p}, \bar{q}, t) \in B$, let

$$\tau_0 := \min\{\tau \in (0, t] \mid (\bar{p}, \bar{q}, \tau) \in B\}\,.$$

Then, by the definition (7.49) of B,

$$\frac{d}{\psi}(\bar{p}, \bar{q}, \tau_0) \overset{(7.49)}{=} \frac{3}{4} \min_{\mathbb{S}^1\times\mathbb{S}^1} \frac{d}{\psi}(\,\cdot\,, \,\cdot\,, 0)$$

so that

$$\begin{aligned} \left| \frac{d}{\psi}(\bar{p}, \bar{q}, t) - \frac{3}{4} \min_{\mathbb{S}^1\times\mathbb{S}^1} \frac{d}{\psi}(\,\cdot\,, \,\cdot\,, 0) \right| &= \left| \frac{d}{\psi}(\bar{p}, \bar{q}, t) - \frac{d}{\psi}(\bar{p}, \bar{q}, \tau_0) \right| \\ &\le \int_{\tau_0}^{t} \left| \frac{\partial}{\partial t}\left(\frac{d}{\psi}\right)(\bar{p}, \bar{q}, \tau) \right| d\tau \le \Lambda(t - \tau_0) \le \Lambda t\,. \end{aligned} \tag{7.51}$$

We choose

$$t_1 = t_1(\Sigma_0) \overset{(7.50)}{:=} \min\left\{\frac{1}{4\Lambda}\min_{\mathbb{S}^1\times\mathbb{S}^1}\frac{d}{\psi}(\,\cdot\,,\,\cdot\,,0), \frac{T}{2}\right\} \in \left(0, \frac{T}{2}\right] \tag{7.52}$$

and distinguish between three different cases.

(i) Assume that $t_0 \in [0, t_1]$ and $(p, q, t_0) \in A$. Then

$$\frac{d}{\psi} \equiv \frac{d}{\psi}(p, q, t_0) \overset{(7.48)}{\geq} \frac{3}{4}\min_{\mathbb{S}^1\times\mathbb{S}^1}\frac{d}{\psi}(\,\cdot\,,\,\cdot\,,0) \tag{7.53}$$

by definition of the set A.

(ii) Assume that $t_0 \in [0, t_1]$ and $(p, q, t_0) \in B$. Then

$$\begin{aligned}\frac{d}{\psi} \equiv \frac{d}{\psi}(p, q, t_0) &\overset{(7.51)}{\geq} -\Lambda t_0 + \frac{3}{4}\min_{\mathbb{S}^1\times\mathbb{S}^1}\frac{d}{\psi}(\,\cdot\,,\,\cdot\,,0)\\ &\geq -\Lambda t_1 + \frac{3}{4}\min_{\mathbb{S}^1\times\mathbb{S}^1}\frac{d}{\psi}(\,\cdot\,,\,\cdot\,,0) \overset{(7.52)}{\geq} \frac{1}{2}\min_{\mathbb{S}^1\times\mathbb{S}^1}\frac{d}{\psi}(\,\cdot\,,\,\cdot\,,0)\end{aligned} \tag{7.54}$$

is a lower bound.

(iii) Assume that $t_0 \in (t_1, T)$. Corollary 5.4(iii) applied with the initial time t_1 yields

$$-\pi \overset{(7.29)}{<} \theta_{\min}(t_1) < \theta_{\min}(t_0) < 0$$

so that the monotone behaviour of the cosine on $(-\pi, 0)$ implies

$$\cos\left(\frac{\theta_{\min}(t_0)}{2}\right) > \cos\left(\frac{\theta_{\min}(t_1)}{2}\right) > 0\,. \tag{7.55}$$

From $0 < l \leq L/2$ it follows that $1 > \cos(\pi l/L) \geq 0$ so that again the monotone behaviour of the cosine on $(-\pi, 0)$,

$$\frac{d}{\psi} > \frac{d}{\psi}\cos\left(\frac{\pi l}{L}\right) \overset{(7.36)}{=} \cos\left(\frac{\theta}{2}\right) \geq \cos\left(\frac{\theta_{\min}(t_0)}{2}\right) \overset{(7.55)}{\geq} \cos\left(\frac{\theta_{\min}(t_1)}{2}\right)\,. \tag{7.56}$$

We deduce that, for $\theta \in (-\pi, 0)$ at (p, q, t_0),

$$\frac{d}{\psi} \underset{(7.56)}{\overset{(7.53),(7.54)}{\geq}} \min\left\{\frac{1}{2}\min_{\mathbb{S}^1\times\mathbb{S}^1}\frac{d}{\psi}(\,\cdot\,,\,\cdot\,,0), \cos\left(\frac{\theta_{\min}(t_1)}{2}\right)\right\} =: C^*(\Sigma_0) > 0$$

as claimed.

Case 2: Assume that $\boldsymbol{\tau}_p = \boldsymbol{\tau}_q$. By the definition (5.1), $\vartheta_p = \vartheta_q + 2\pi k$ for some $k \in \mathbb{Z}$ so that $\theta = 2k\pi$ (see (5.6)). By (7.29), θ can only be zero or 2π (see Figure 7.7 for an illustration). We now look at both flows separately.

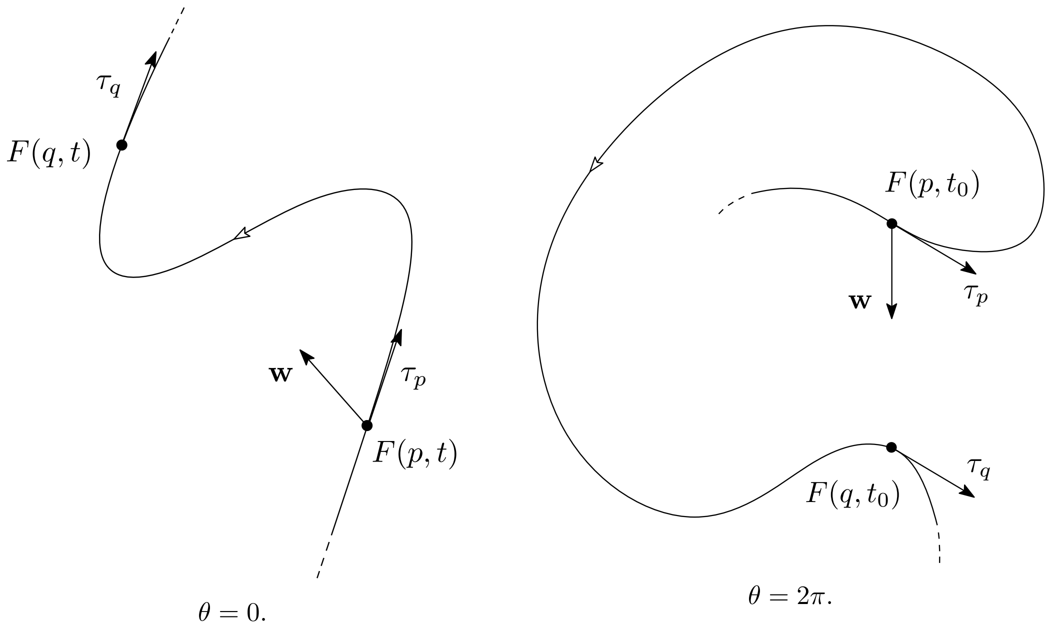

Figure 7.7: Case 2.

(a) For the **APCSF**, it follows from $\boldsymbol{\tau}_p = \boldsymbol{\tau}_q$ and embeddedness of Σ_{t_0} that $\boldsymbol{\nu}_p = \boldsymbol{\nu}_q$ which implies

$$\langle \mathbf{w}, \boldsymbol{\nu}_q - \boldsymbol{\nu}_p \rangle = 0\,. \tag{7.57}$$

The identity (7.13) for the second derivative of d/ψ along $\xi = \boldsymbol{\tau}_p \oplus \boldsymbol{\tau}_q$ and (7.31) yield

$$0 \overset{(7.31)}{\leq} (\boldsymbol{\tau}_p \oplus \boldsymbol{\tau}_q)^2 \left(\frac{d}{\psi}\right) \overset{(7.13)}{=} \frac{1}{\psi} \left\langle \mathbf{w}, \boldsymbol{\kappa}_q - \boldsymbol{\kappa}_p \right\rangle \,. \tag{7.58}$$

By (7.57), (7.58), the definition (5.3) of θ, and $h_{\mathrm{ap}} = 2\pi/L$, the evolution equation (7.15) for d/ψ reduces to

$$\begin{aligned} \frac{\partial}{\partial t}_{|t=t_0} \left(\frac{d}{\psi}\right) \geq \frac{d}{\psi^2} \cos\left(\frac{\pi l}{L}\right) \left(\int_p^q \kappa^2\, v dr - \frac{2\pi\theta}{L}\right) \\ + \frac{d}{\pi\psi^2} \left(\int_{\mathbb{S}^1} \kappa^2\, v dr - \frac{(2\pi)^2}{L}\right) \left\{\sin\left(\frac{\pi l}{L}\right) - \frac{\pi l}{L} \cos\left(\frac{\pi l}{L}\right)\right\}. \end{aligned} \tag{7.59}$$

We now examine different lengths separately.

(i) Assume that $l = L/2$. Then $\cos(\pi l/L) = 0$ and $\sin(\pi l/L) = 1$, so that (7.59) yields

$$\frac{\partial}{\partial t}_{|t=t_0} \left(\frac{d}{\psi}\right) \overset{(7.59)}{\geq} \frac{d}{\pi\psi^2} \left(\int_{\mathbb{S}^1} \kappa^2\, v dr - \frac{(2\pi)^2}{L}\right) > 0$$

since, by assumption, Σ_{t_0} is not a circle.

(ii) Assume that $l < L/2$. Since Σ_{t_0} is not a circle, Corollary 7.14 implies that the second term on the right-hand side of (7.59) is positive. Moreover, $\cos(\pi l/L) >$

$\cos(\pi/2) = 0$, as the cosine function is strictly decreasing on the interval $[0, \pi/2]$. This and (7.44) reduce (7.59) to

$$\frac{\partial}{\partial t}_{|t=t_0}\left(\frac{d}{\psi}\right) > \frac{d}{\psi^2}\cos\left(\frac{\pi l}{L}\right)\theta\left(\frac{\theta}{l} - \frac{2\pi}{L}\right).$$

For $\theta = 0$, the right-hand side is zero. For $\theta = 2\pi$, we conclude from $l < L$, that $\theta/l = 2\pi/l > 2\pi/L$, and thus

$$\frac{\theta}{l} - \frac{2\pi}{L} > 0$$

which also leads to $\frac{\partial}{\partial t}_{|t=t_0}(d/\psi) > 0$.

Hence, in both cases,

$$\frac{\partial}{\partial t}_{|t=t_0}\left(\frac{d}{\psi}\right) > 0$$

for the APCSF and $\theta \in \{0, 2\pi\}$ as claimed.

(b) For the **LPCF**, the evolution equation (7.17) of d/ψ cannot be estimated as in the case of the APCSF. Thus, we argument like in Case 1(a)(ii). Since $l \in (0, L/2]$ and $d/\psi \in (0, 1)$, and by (7.32) and (7.33), we obtain

$$\langle \mathbf{w}, \boldsymbol{\tau}_q \rangle, \langle \mathbf{w}, \boldsymbol{\tau}_p \rangle \overset{(7.32),(7.33)}{=} \frac{d}{\psi}\cos\left(\frac{\pi l}{L}\right) \in [0, 1).$$

Define the straight line segment

$$S := \{x \in \mathbb{R}^2 \,|\, x = (1-\lambda)F(p, t_0) + \lambda F(q, t_0) \text{ for } \lambda \in [0, 1]\}$$

through $F(p, t_0)$ and $F(q, t_0)$. Since $\boldsymbol{\tau}_p = \boldsymbol{\tau}_q$ and $\langle \mathbf{w}, \boldsymbol{\tau}_q \rangle, \langle \mathbf{w}, \boldsymbol{\tau}_p \rangle \in [0, 1)$, both vectors point to the same side of S. Since Σ_{t_0} is closed and F_{t_0} is continuous, Σ_{t_0} has to cross S between $F(p, t_0)$ and $F(q, t_0)$ at least once (see Figure 7.8 for an illustration). Lemma 7.19 implies that the ratio d/ψ cannot have a global minimum at (p, q, t_0) (it could still, however, attain a local minimum a this point). Hence,

$$\frac{d}{\psi} > \min_{\mathbb{S}^1\times\mathbb{S}^1}\frac{d}{\psi}(\,\cdot\,,\,\cdot\,, t_0).$$

By the compactness of $\mathbb{S}^1 \times \mathbb{S}^1$, the global minimum of $(d/\psi)(\,\cdot\,,\,\cdot\,, t_0)$ exists at a point (p_0, q_0) with $\boldsymbol{\tau}_{p_0} \neq \boldsymbol{\tau}_{q_0}$ which was treated in Case 1(a)(i) and (b). $\square$

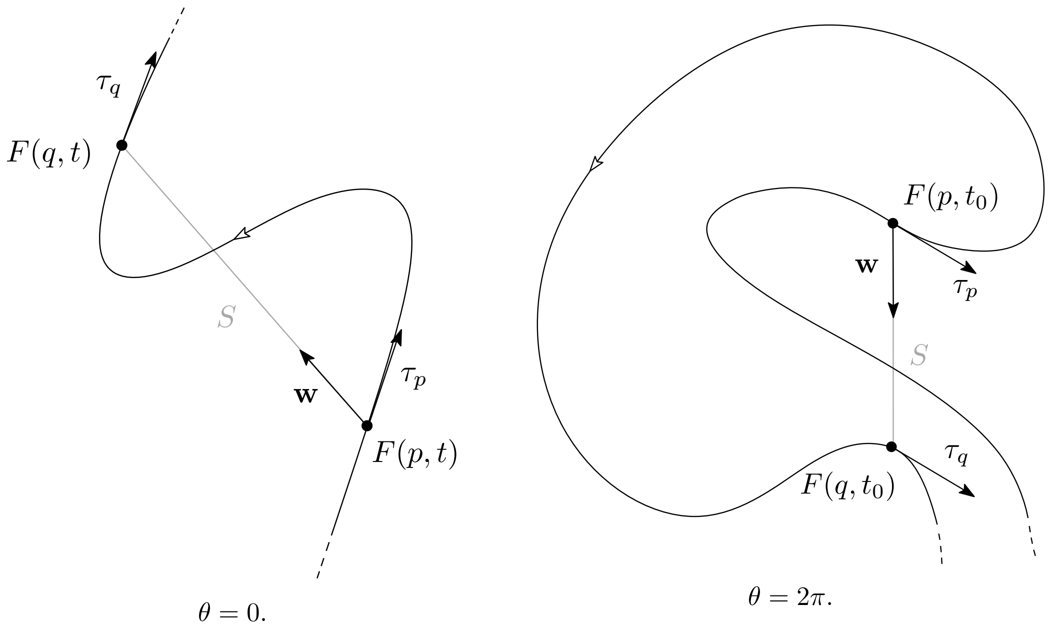

Figure 7.8: Case 2(b), Σ_{t_0} is crossing the connecting line between $F(p,t_0)$ and $F(q,t_0)$.

Chapter 8

A monotonicity formula

In this chapter, we consider smooth, embedded, n-dimensional hypersurfaces in $\mathbb{R}^{n+1}$ without boundary. For solutions $F : \Sigma^n \times (0,T) \to \mathbb{R}^{n+1}$ to the problem (2.13), we derive a monotonicity formula analogue to the one in [Hui90] for MCF. For the APMCF with boundary the formula has already been introduced in [MB14, MB15]. The same proof holds for the VPMCF. (Refer to Appendix A.2 for an introduction to differentiable hypersurfaces of $\mathbb{R}^{n+1}$.)

Definition 8.1. For $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}$, define the *backward heat kernel* $\Phi_{(x_0,t_0)} : \mathbb{R}^n \times (-\infty, t_0) \to \mathbb{R}$ by

$$\Phi_{(x_0,t_0)}(x,t) := \frac{1}{(4\pi(t_0-t))^{n/2}} \exp\left(-\frac{\|x-x_0\|^2}{4(t_0-t)}\right). \tag{8.1}$$

Lemma 8.2. *Let $x, x_0, y_0 \in \mathbb{R}^n$, $t_0, \tau_0 \in \mathbb{R}$, $t \in (-\infty, t_0)$, $\lambda > 0$ and $\tau_0 > \lambda^2(t-t_0)$. Then*

$$\Phi_{(y_0,\tau_0)}\big(\lambda(x-x_0), \lambda^2(t-t_0)\big) = \frac{1}{\lambda^n}\Phi_{(x_0+y_0/\lambda, t_0+\tau_0/\lambda^2)}(x,t).$$

Proof. For $y = \lambda(x-x_0)$, we calculate

$$\|y-y_0\|^2 = \|\lambda(x-x_0)-y_0\|^2 = \lambda^2 \left\|x - \left(x_0 + \frac{y_0}{\lambda}\right)\right\|^2$$

and for $\tau = \lambda^2(t-t_0)$,

$$\tau_0 - \tau = \tau_0 - \lambda^2(t-t_0) = \lambda^2\left(\left(\frac{\tau_0}{\lambda^2} + t_0\right) - t\right).$$

Then, by the definition (8.1) of the backward heat kernel,

$$\begin{aligned}
\Phi_{(y_0,\tau_0)}(y,\tau) &= \frac{1}{(4\pi(\tau_0-\tau))^{n/2}} \exp\left(-\frac{\|y-y_0\|^2}{4(\tau_0-\tau)}\right) \\
&= \frac{\lambda^{-n}}{(4\pi((\tau_0/\lambda^2+t_0)-t))^{n/2}} \exp\left(-\frac{\|x-(x_0+y_0/\lambda)\|^2}{4((\tau_0/\lambda^2+t_0)-t)}\right) \\
&= \frac{1}{\lambda^n}\Phi_{(x_0+y_0/\lambda, t_0+\tau_0/\lambda^2)}(x,t).
\end{aligned}$$

□

For a solution $F : \Sigma^n \times [0,T) \to \mathbb{R}^{n+1}$ to the problem (2.13) with initial surface Σ_0, define the auxiliary function $f : [0,T) \to \mathbb{R}$ by

$$f(t) := \exp\left(-\frac{1}{2}\int_0^t h^2(\sigma)d\sigma\right) \tag{8.2}$$

(see [Ath03, Proposition 3.2] or [MB15, Proposition 4.9]).

Lemma 8.3. *We have $f(t) \in (0,1]$ for all $t \in [0,T)$. Furthermore, if $\sup_{[0,T)} h < \infty$, then there exists a constant $c = c(\sup_{[0,T)} h) > 0$ such that $f \geq c$ on $[0,T)$.*

Proof. This follows directly from the definition (8.2) of f. □

Lemma 8.4. *Let $F : \Sigma^n \times (0,T) \to \mathbb{R}^{n+1}$ be a solution of (2.13). Then*

$$\frac{df}{dt} = -\frac{1}{2}h^2 f \qquad \text{and} \qquad \frac{d}{dt}\int_{\Sigma_t} d\mathcal{H}^n = \int_{\Sigma_t} (h-H)H\, d\mathcal{H}^n\,.$$

Proof. We differentiate

$$\frac{df}{dt} = \frac{d}{dt}\exp\left(-\frac{1}{2}\int_0^t h^2(\sigma)d\sigma\right) = -\frac{1}{2}h^2\exp\left(-\frac{1}{2}\int_0^t h^2(\sigma)d\sigma\right) = -\frac{1}{2}h^2 f$$

and

$$\begin{aligned}\frac{\partial\sqrt{g}}{\partial t} &= \frac{\partial}{\partial t}\sqrt{\det(g_{ij})} = \frac{g}{\sqrt{g}}g^{ij}\left\langle\frac{\partial F}{\partial p_i}, \frac{\partial}{\partial p_j}\frac{\partial F}{\partial t}\right\rangle = \sqrt{g}g^{ij}\left\langle\frac{\partial F}{\partial p_i}, \frac{\partial}{\partial p_j}((h-H)\boldsymbol{\nu})\right\rangle\\ &= \sqrt{g}(h-H)g^{ij}\left\langle\frac{\partial F}{\partial p_i}, \frac{\partial\boldsymbol{\nu}}{\partial p_j}\right\rangle = \sqrt{g}(h-H)H\end{aligned}$$

so that

$$\frac{d}{dt}\int_{\Sigma_t} d\mathcal{H}^n = \int_{\Sigma^n}\frac{\partial}{\partial t}\sqrt{g(p,t)}dp = \int_{\Sigma^n}(h-H)H\sqrt{g(p,t)}dp = \int_{\Sigma_t}(h-H)H\, d\mathcal{H}^n\,. \quad \square$$

In the following, we set $H(x,t) = H(p,t)$ and $\boldsymbol{\nu}(x,t) = \boldsymbol{\nu}(p,t)$ for $x = F(p,t)$.

Theorem 8.5 (Monotonicity formula, [Hui90, Theorem 3.1], see also [MB15, Proposition 4.9])**.** *Let $F : \Sigma^n \times (0,T) \to \mathbb{R}^{n+1}$ be a solution of (2.13). Then*

$$\begin{aligned}&\frac{d}{dt}\left(f\int_{\Sigma_t}\Phi_{(x_0,t_0)}\, d\mathcal{H}^n\right)\\ &\quad = -\frac{f}{2}\int_{\Sigma_t}\left(\left\|(H-h)\boldsymbol{\nu} - \frac{(x-x_0)^\perp}{2(t_0-t)}\right\|^2 + \left\|\mathbf{H} + \frac{(x-x_0)^\perp}{2(t_0-t)}\right\|^2\right)\Phi_{(x_0,t_0)}\, d\mathcal{H}^n\end{aligned}$$

for $t_0 \in (0,T]$ and $t \in (0,t_0)$, where $(x-x_0)^\perp := \langle x-x_0, \boldsymbol{\nu}\rangle\,\boldsymbol{\nu}$ is the normal part of the vector $x - x_0$.

Proof. Since $x = x(t)$ with $\frac{\partial}{\partial t} x(t) = (h - H)\boldsymbol{\nu}$, we derive

$$\begin{aligned}\frac{d}{dt}\Phi_{(x_0,t_0)} &= \frac{(n/2)4\pi}{(4\pi(t_0-t))^{n/2+1}} \exp\Big(-\frac{\|x-x_0\|^2}{4(t_0-t)}\Big) \\ &\quad + \frac{1}{(4\pi(t_0-t))^{n/2}} \exp\Big(-\frac{\|x-x_0\|^2}{4(t_0-t)}\Big) \\ &\qquad \times \left(-\frac{2\,\langle x-x_0,(h-H)\boldsymbol{\nu}\rangle}{4(t_0-t)} - \frac{\|x-x_0\|^2}{4(t_0-t)^2}\right) \\ &= \frac{1}{2}\left(\frac{n}{(t_0-t)} - (h-H)\frac{\langle x-x_0,\boldsymbol{\nu}\rangle}{(t_0-t)} - \frac{\|x-x_0\|^2}{2(t_0-t)^2}\right)\Phi_{(x_0,t_0)} \end{aligned} \tag{8.3}$$

so that

$$\begin{aligned}&\frac{d}{dt}\left(f\int_{\Sigma_t}\Phi_{(x_0,t_0)}\,d\mathcal{H}^n\right) \\ &\overset{\text{Lem. 8.4}}{\underset{(8.3)}{=}} -\frac{f}{2}\int_{\Sigma_t} h^2\Phi_{(x_0,t_0)}\,d\mathcal{H}^n + f\int_{\Sigma_t}(h-H)H\,\Phi_{(x_0,t_0)}\,d\mathcal{H}^n \\ &\qquad + \frac{f}{2}\int_{\Sigma_t}\left(\frac{n}{(t_0-t)} - (h-H)\frac{\langle x-x_0,\boldsymbol{\nu}\rangle}{(t_0-t)} - \frac{\|x-x_0\|^2}{2(t_0-t)^2}\right)\Phi_{(x_0,t_0)}\,d\mathcal{H}^n\,.\end{aligned}$$

Rearranging terms yields

$$\begin{aligned}&\frac{d}{dt}\left(f\int_{\Sigma_t}\Phi_{(x_0,t_0)}\,d\mathcal{H}^n\right) \\ &= f\int_{\Sigma_t}\left(-\frac{h^2}{2} + h\left(H - \frac{\langle x-x_0,\boldsymbol{\nu}\rangle}{2(t_0-t)}\right)\right)\Phi_{(x_0,t_0)}\,d\mathcal{H}^n - f\int_{\Sigma_t} H^2\,\Phi_{(x_0,t_0)}\,d\mathcal{H}^n \\ &\quad + f\int_{\Sigma_t}\left(\frac{n}{2(t_0-t)} + H\frac{\langle x-x_0,\boldsymbol{\nu}\rangle}{2(t_0-t)} - \frac{\|x-x_0\|^2}{4(t_0-t)^2}\right)\Phi_{(x_0,t_0)}\,d\mathcal{H}^n\,.\end{aligned} \tag{8.4}$$

We apply the divergence theorem, Theorem B.7, to the vector $\mathbf{v} = (x - x_0)\Phi_{(x_0,t_0)}$ and obtain

$$\int_{\Sigma_t}\langle x-x_0, H\boldsymbol{\nu}\rangle\,\Phi_{(x_0,t_0)}\,d\mathcal{H}^n = \int_{\Sigma_t}\operatorname{div}_{\Sigma_t}\big((x-x_0)\,\Phi_{(x_0,t_0)}\big)\,d\mathcal{H}^n\,, \tag{8.5}$$

where

$$\operatorname{div}_{\Sigma_t}\big((x-x_0)\,\Phi_{(x_0,t_0)}\big) = \operatorname{div}_{\Sigma_t}(x-x_0)\,\Phi_{(x_0,t_0)} + \big\langle (x-x_0), \nabla^{\Sigma_t}\Phi_{(x_0,t_0)}\big\rangle\,.$$

For $x \in \Sigma_t$, (A.13) yields

$$\operatorname{div}_{\Sigma_t} x = \operatorname{div}_{\Sigma^n} F(p,t) \overset{(A.13)}{=} n$$

and (A.12) implies for a unit tangent frame $\{\boldsymbol{\tau}_i\}_{i\in\mathbb{N}}$ of Σ_t,

$$\begin{aligned}\nabla^{\Sigma_t}\Phi_{(x_0,t_0)} &\overset{(A.12)}{=} -\frac{1}{(4\pi(t_0-t))^{n/2}}\exp\Big(-\frac{\|x-x_0\|^2}{4(t_0-t)}\Big)\sum_{i=1}^n \frac{2\,\langle x-x_0,\boldsymbol{\tau}_i(x)\rangle}{4(t_0-t)}\boldsymbol{\tau}_i \\ &= -\Phi_{(x_0,t_0)}\sum_{i=1}^n\frac{\langle x-x_0,\boldsymbol{\tau}_i\rangle}{2(t_0-t)}\boldsymbol{\tau}_i\,.\end{aligned} \tag{8.6}$$

By combining the last four identities, we conclude

$$\int_{\Sigma_t} \frac{\langle x - x_0, H\boldsymbol{\nu}\rangle}{2(t_0 - t)} \Phi_{(x_0,t_0)}\, d\mathcal{H}^n = \int_{\Sigma_t} \left(\frac{n}{2(t_0 - t)} - \sum_{i=1}^{n} \frac{\langle x - x_0, \boldsymbol{\tau}_i\rangle^2}{4(t_0 - t)^2} \right) \Phi_{(x_0,t_0)}\, d\mathcal{H}^n$$

so that the last integral at the right-hand side of (8.4) becomes

$$\int_{\Sigma_t} \left(\frac{n}{2(t_0 - t)} + H\frac{\langle x - x_0, \boldsymbol{\nu}\rangle}{2(t_0 - t)} - \frac{\|x - x_0\|^2}{4(t_0 - t)^2} \right) \Phi_{(x_0,t_0)}\, d\mathcal{H}^n = \int_{\Sigma_t} \left(2H\frac{\langle x - x_0, \boldsymbol{\nu}\rangle}{2(t_0 - t)} + \sum_{i=1}^{n} \frac{\langle x - x_0, \boldsymbol{\tau}_i\rangle^2}{4(t_0 - t)^2} - \frac{\|x - x_0\|^2}{4(t_0 - t)^2} \right) \Phi_{(x_0,t_0)}\, d\mathcal{H}^n . \tag{8.7}$$

Next, we observe that

$$-\|x - x_0\|^2 + \sum_{i=1}^{n} \langle x - x_0, \boldsymbol{\tau}_i\rangle^2 = -\langle x - x_0, \boldsymbol{\nu}\rangle^2 ,$$

and

$$-H^2 + H\frac{\langle x - x_0, \boldsymbol{\nu}\rangle}{(t_0 - t)} - \frac{\langle x - x_0, \boldsymbol{\nu}\rangle^2}{4(t_0 - t)^2} = -\left| H - \frac{\langle x - x_0, \boldsymbol{\nu}\rangle}{2(t_0 - t)} \right|^2 ,$$

so that (8.4) reduces to

$$\frac{d}{dt}\left(f \int_{\Sigma_t} \Phi_{(x_0,t_0)}\, d\mathcal{H}^n \right) = -\frac{f}{2}\int_{\Sigma_t} h^2 \Phi_{(x_0,t_0)}\, d\mathcal{H}^n + f\int_{\Sigma_t} h \left(H - \frac{\langle x - x_0, \boldsymbol{\nu}\rangle}{2(t_0 - t)} \right) \Phi_{(x_0,t_0)}\, d\mathcal{H}^n - f\int_{\Sigma_t} \left| H - \frac{\langle x - x_0, \boldsymbol{\nu}\rangle}{2(t_0 - t)} \right|^2 \Phi_{(x_0,t_0)}\, d\mathcal{H}^n .$$

Following [MB14, p. 11], we observe that

$$-\frac{h^2}{2} + ha - a^2 = -\frac{1}{2}a^2 - \frac{1}{2}(h - a)^2$$

so that, for $a = H - \langle x - x_0, \boldsymbol{\nu}\rangle / (2(t_0 - t))$,

$$\frac{d}{dt}\left(f \int_{\Sigma_t} \Phi_{(x_0,t_0)}\, d\mathcal{H}^n \right) = -\frac{f}{2}\int_{\Sigma_t} \left(\left| H - \frac{\langle x - x_0, \boldsymbol{\nu}\rangle}{2(t_0 - t)} \right|^2 + \left| (h - H) + \frac{\langle x - x_0, \boldsymbol{\nu}\rangle}{2(t_0 - t)} \right|^2 \right) \Phi_{(x_0,t_0)}\, d\mathcal{H}^n .$$

As a last step we calculate

$$\left| H - \frac{\langle x - x_0, \boldsymbol{\nu}\rangle}{2(t_0 - t)} \right|^2 = \left\| H\boldsymbol{\nu} - \frac{\langle x - x_0, \boldsymbol{\nu}\rangle \boldsymbol{\nu}}{2(t_0 - t)} \right\|^2 = \left\| \mathbf{H} + \frac{(x - x_0)^\perp}{2(t_0 - t)} \right\|^2 ,$$

where $(x - x_0)^\perp := \langle x - x_0, \boldsymbol{\nu}\rangle \boldsymbol{\nu}$ is the normal part of the vector $x - x_0$. $\square$

Theorem 8.6 (Weighted monotonicity formula, [Eck04, Theorem 4.13])**.** *Let* $F : \Sigma^n \times (0,T) \to \mathbb{R}^{n+1}$ *be a solution of* (2.13) *and* $\varphi : \mathbb{R}^{n+1} \times (0,T) \to \mathbb{R}$ *in* $C^{2;1}$*. Then*

$$\begin{aligned}
&\frac{d}{dt}\left(f \int_{\Sigma_t} \varphi\, \Phi_{(x_0,t_0)}\, d\mathcal{H}^n\right) \\
&\quad = -\frac{f}{2}\int_{\Sigma_t}\left(\left\|\mathbf{H} + \frac{(x-x_0)^\perp}{2(t_0-t)}\right\|^2 + \left\|(h-H)\boldsymbol{\nu} + \frac{(x-x_0)^\perp}{2(t_0-t)}\right\|^2\right)\varphi\, \Phi_{(x_0,t_0)}\, d\mathcal{H}^n \\
&\qquad + \frac{f}{2}\int_{\Sigma_t}\left(\frac{\partial}{\partial t} - \Delta_{\Sigma_t}\right)\varphi\, \Phi_{(x_0,t_0)}\, d\mathcal{H}^n
\end{aligned}$$

for $t_0 \in (0,T]$ *and* $t \in (0,t_0)$*.*

Proof. The proof is like the one for Theorem 8.5 with one additional step. When applying the divergence theorem, Theorem B.7, in (8.5), we now use the vector $\mathbf{v} = (x-x_0)\varphi\, \Phi_{(x_0,t_0)}$ instead and deduce

$$\int_{\Sigma_t} \langle x - x_0, H\boldsymbol{\nu}\rangle\, \varphi\, \Phi_{(x_0,t_0)}\, d\mathcal{H}^n = \int_{\Sigma_t} \operatorname{div}_{\Sigma_t}\big((x-x_0)\varphi\, \Phi_{(x_0,t_0)}\big)\, d\mathcal{H}^n\,,$$

where

$$\begin{aligned}
\operatorname{div}_{\Sigma_t}\big((x-x_0)\varphi\, \Phi_{(x_0,t_0)}\big) &= n\varphi\, \Phi_{(x_0,t_0)} + \varphi\left\langle (x-x_0), \nabla^{\Sigma_t}\Phi_{(x_0,t_0)}\right\rangle \\
&\quad + \left\langle (x-x_0), \nabla^{\Sigma_t}\varphi\right\rangle \Phi_{(x_0,t_0)}\,.
\end{aligned}$$

Since $\nabla^{\Sigma_t}\varphi = \boldsymbol{\tau}_i(\varphi)\boldsymbol{\tau}_i$ we can utilise the gradient of $\Phi_{(x_0,t_0)}$ (see (8.6)) again to find

$$\frac{\left\langle (x-x_0), \nabla^{\Sigma_t}\varphi\right\rangle}{2(t_0-t)}\Phi_{(x_0,t_0)} = -\left\langle \nabla^{\Sigma_t}\Phi_{(x_0,t_0)}, \nabla^{\Sigma_t}\varphi\right\rangle$$

so that integration by parts (B.4) yields the extra term

$$\int_{\Sigma_t} \frac{\left\langle (x-x_0), \nabla^{\Sigma_t}\varphi\right\rangle}{2(t_0-t)}\Phi_{(x_0,t_0)}\, d\mathcal{H}^n = \int_{\Sigma_t} \Delta_{\Sigma_t}\varphi\, \Phi_{(x_0,t_0)}\, d\mathcal{H}^n\,.$$

The minus sign comes from the operation in (8.7). □

Chapter 9

Singularity analysis

In Proposition 4.9 we have shown that the curvature blows up if $T < \infty$. In this chapter, we assume $T < \infty$ and investigate curvature blow-ups for embedded constrained curve flows with $\theta_{\min}(0) \geq -\pi$. We adapt the techniques from CSF to show that for the APCSF, the curvature does not blow up in finite time and conclude $T = \infty$. For the LPCF, we exclude collapsed curvature blow ups. Proposition 4.9 motivates the following definition.

Definition 9.1 (Singularity, blow-up sequence)**.** We say that a solution $F : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ of (2.15) develops a *singularity* at $T \leq \infty$ if

$$|\kappa|_{\max}(t) \to \infty$$

for $t \nearrow T$. A sequence $(p_k, t_k)_{k\in\mathbb{N}}$ in $\mathbb{S}^1 \times [0, T)$ with

$$|\kappa(p_k, t_k)| = |\kappa|_{\max}(t_k) \to \infty$$

is called *blow-up* sequence.

Lemma 9.2 (Singular point for the APCSF)**.** *Let $F : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ be a solution of the APCSF with $T < \infty$. Then there exists a point $x_0 \in \mathbb{R}^2$ and a blow-up sequence $(p_k, t_k)_{k\in\mathbb{N}}$ with*

$$F(p_k, t_k) \to x_0$$

for $k \to \infty$ so that the solution $(\Sigma_t)_{t\in[0,T)}$ has no smooth extension beyond time T in any neighbourhood of x_0. The point x_0 is called a singular *or* blow-up point *of the flow at time T.*

Proof. For the APCSF, Corollary 3.5 and Lemma 3.9 imply that the length of the curve L_t and the global term h_{ap} are bounded for all $t \in [0, T)$. Hence, Lemma 3.13 yields that there exists a radius $R = R(\Sigma_0, T) > 0$ so that $\Sigma_t \subset B_R(0)$ for all $t \in [0, T)$. By Proposition 4.9,

$$|\kappa|_{\max}(t) \to \infty$$

for $t \nearrow T$. Let $(p_k, t_k)_{k\in\mathbb{N}}$ be a blow-up sequence. Since $(F(p_k, t_k))_{k\in\mathbb{N}}$ is a bounded sequence in $B_R(0)$, there exists a point $x_0 \in \mathbb{R}^2$ and a subsequence with

$$F(p_k, t_k) \to x_0$$

for $k \to \infty$. ◻

Lemma 9.3 (Hamilton's trick [Ham86, Lemma 3.5])**.** *Let $f : [a,b] \times (0,T) \to \mathbb{R}$ be in C^1. Then $f_{\max}(t) := \max_{p\in[a,b]} f(p,t)$ is locally Lipschitz for $t \in (0,T)$ and at a differentiable time,*

$$\frac{df_{\max}}{dt}(t) \leq \sup\left\{ \frac{\partial f}{\partial t}(p,t) \,\middle|\, p \in [a,b] \text{ with } f(p,t) = f_{\max}(t) \right\}.$$

Lemma 9.4. *Let $f : [a,b] \to \mathbb{R}$ be Lipschitz. Then f' exists almost everywhere, is bounded and*

$$f(b) - f(a) = \int_a^b f'\, dt\,.$$

Lemma 9.5 (Lower blow-up rate for the curvature)**.** *Let $F : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ be a solution of* (2.15) *with initial curve Σ_0 and $T < \infty$. Then*

$$|\kappa|_{\max}(t) \geq \frac{1}{\sqrt{2(C_h+1)(T-t)}}\,.$$

for all $t \in [0,T)$, where the constant C_h is defined in Corollary 5.9.

Proof. We follow the lines of [Hui90, Lemma 1.2]. By Proposition 4.9, $|\kappa|_{\max}(t) \to \infty$ for $t \to T$. For $t \in (0,T)$, let $s \in \mathbb{S}^1_{R_t}$ so that $\kappa^2(s,t) = \kappa^2_{\max}(t)$. Then

$$\frac{\partial^2 \kappa^2}{\partial s^2}(s,t) \leq 0\,.$$

By the evolution equation (4.4) of curvature and Corollary 5.9,

$$\frac{\partial \kappa^2}{\partial t} \overset{(4.4)}{=} \frac{\partial^2 \kappa^2}{\partial s^2} + 2(\kappa - h)\kappa^3 \overset{\text{Cor. 5.9}}{\leq} 2(C_h+1)\kappa^4 \tag{9.1}$$

at (s,t). Since $\kappa^2_{\max}$ is Lipschitz, by Rademacher's theorem, Theorem B.5, $\frac{d}{dt}\kappa^2_{\max}$ exists for almost every $t \in (0,T)$. By Hamilton's trick, Lemma 9.3,

$$\begin{aligned}\frac{d\kappa^2_{\max}}{dt}(t) &\overset{\text{Lem. 9.3}}{\leq} \max\left\{ \frac{\partial \kappa^2}{\partial t}(p,t) \,\middle|\, p \in \mathbb{S}^1 \text{ with } \kappa^2(p,t) = \kappa^2_{\max} \right\} \\ &\overset{(9.1)}{\leq} \max\left\{ 2(C_h+1)\kappa^4(p,t) \,\middle|\, p \in \mathbb{S}^1 \text{ with } \kappa^2(p,t) = \kappa^2_{\max} \right\} \\ &= 2(C_h+1)\kappa^4_{\max}(t) \end{aligned} \tag{9.2}$$

for almost every $t \in (0,T)$. Assume that there exists a time $t_0 \in [0,T)$ where $\kappa^2_{\max} = 0$. Then Σ_{t_0} is a line segment in $\mathbb{R}^2$ which contradicts that Σ_t is closed for all $t \in [0,T)$. Hence, $\kappa^2_{\max}(t) > 0$ for all $t \in [0,T)$ and $\kappa^{-2}_{\max}$ is Lipschitz as well. Rademacher's theorem, Theorem B.5, implies that $\frac{d}{dt}\kappa^{-2}_{\max}(t)$ exists for almost every $t \in (0,T)$. Thus,

$$\frac{d\kappa^{-2}_{\max}}{dt} = -\kappa^{-4}_{\max}\frac{d\kappa^2_{\max}}{dt} \overset{(9.2)}{\geq} -2(C_h+1) \tag{9.3}$$

for almost every $t \in (0,T)$. Since $\kappa^{-2}_{\max}$ is Lipschitz, we can apply Lemma 9.4 and integrate (9.3) over an interval $[t,t_k] \subset [0,T)$ to obtain

$$\frac{1}{\kappa^2_{\max}(t_k)} - \frac{1}{\kappa^2_{\max}(t)} \geq -2(C_h+1)(t_k - t)\,. \tag{9.4}$$

Let $t \in [0,T)$ and $(t_k)_{k\in\mathbb{N}}$ be a sequence with $t_k \in (t,T)$ for all $k \in \mathbb{N}$, $t_k \nearrow T$ and $\kappa^2_{\max}(t_k) \to \infty$ for $k \to \infty$. Taking the limit $k \to \infty$ in (9.4) yields

$$\frac{1}{\kappa^2_{\max}(t)} \leq 2(C_h+1)(T-t)$$

for all $t \in [0,T)$. □

Like for MCF, we distinguish between singularities with controlled curvature growth and those without.

Definition 9.6 (Type-I and type-II singularities)**.** Let $F : \mathbb{S}^1 \times (0,T) \to \mathbb{R}^2$ be a solution of (2.15) with $T < \infty$. We say that a singularity is of *type I*, if there exists a constant $C_0 > 1$ so that

$$|\kappa|_{\max}(t) \leq \frac{C_0}{\sqrt{2(C_h+1)(T-t)}} \tag{9.5}$$

for all $t \in [0,T)$, where the constant C_h is defined in Corollary 5.9, and of *type II*, if such a constant does not exist, that is,

$$\limsup_{t\to T} |\kappa|_{\max}(t)\sqrt{T-t} = \infty\,. \tag{9.6}$$

9.1 Rescaling

We want to rescale the curves Σ_t near a singular point as $t \to T < \infty$. The following rescaling technique for type-I singularities was introduced in [HS99, Remark 4.6]. We will use for type-I singularities of the APCSF, since, by Lemma 9.2 the existence of a singular point $x_0 \in \mathbb{R}^2$ is guaranteed.

Definition 9.7 (Type-I rescaling for the APCSF)**.** Let $(p_k,t_k)_{k\in\mathbb{N}}$ be a blow-up sequence in $\mathbb{S}^1 \times [0,T)$ with $t_k \nearrow T$ for $k \to \infty$ and

$$\kappa^2(p_k,t_k) = \max_{p\in\mathbb{S}^1} \kappa^2(p,t_k) = \max_{\mathbb{S}^1\times[0,t_k]} \kappa^2(p,t)$$

for each $k \in \mathbb{N}$. Furthermore, we assume that there exists a singular point $x_0 \in \mathbb{R}^2$ so that

$$F(p_k,t_k) \to x_0$$

for $k \to \infty$. We set

$$\lambda_k^2 := \kappa^2(p_k,t_k) \qquad \text{and} \qquad \alpha_k := -\lambda_k^2 T$$

and define the rescaled embeddings $F_k : \mathbb{S}^1 \times [\alpha_k, 0) \to \mathbb{R}^2$ by

$$F_k(p,\tau) := \lambda_k \left[F\left(p, T + \frac{\tau}{\lambda_k^2}\right) - x_0 \right] . \tag{9.7}$$

The next parabolic rescaling we will use to cover type-I singularities of the LPCF.

Definition 9.8 (Type-I rescaling for the LPCF)**.** Let $(p_k, t_k)_{k\in\mathbb{N}}$ be a blow-up sequence in $\mathbb{S}^1 \times [0,T)$ with $t_k \nearrow T$ for $k \to \infty$ and

$$\kappa^2(p_k, t_k) = \max_{p\in\mathbb{S}^1} \kappa^2(p, t_k) = \max_{\mathbb{S}^1\times[0,t_k]} \kappa^2(p,t)$$

for each $k \in \mathbb{N}$. We set

$$\lambda_k^2 := \kappa^2(p_k, t_k) \qquad \text{and} \qquad \alpha_k := -\lambda_k^2 T$$

and define the rescaled embeddings $F_k : \mathbb{S}^1 \times [\alpha_k, 0) \to \mathbb{R}^2$ by

$$F_k(p,\tau) := \lambda_k \left[F\left(p, T + \frac{\tau}{\lambda_k^2}\right) - F(p_k, t_k) \right] . \tag{9.8}$$

The following rescaling technique for type-II singularities was introduced in [Ham95a, Proof of Theorem 16.4] for Ricci flow, and applied to type-II singularities of MCF in [HS99, p. 11]. We will use it for type-II singularities of both flows.

Definition 9.9 (Type-II rescaling)**.** Let $(p_k, t_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{S}^1 \times [0, T - 1/k]$ with

$$\kappa^2(p_k, t_k) \left(T - \frac{1}{k} - t_k \right) = \max_{(p,t)\in\mathbb{S}^1\times[0,T-1/k]} \left[\kappa^2(p,t) \left(T - \frac{1}{k} - t \right) \right]$$

for each $k \in \mathbb{N}$. We set

$$\lambda_k^2 := \kappa^2(p_k, t_k) , \qquad \alpha_k := -\lambda_k^2 t_k \qquad \text{and} \qquad T_k := \lambda_k^2 \left(T - \frac{1}{k} - t_k \right) .$$

and define the rescaled embeddings $F_k : \mathbb{S}^1 \times [\alpha_k, T_k] \to \mathbb{R}^2$ by

$$F_k(p,\tau) := \lambda_k \left[F\left(p, t_k + \frac{\tau}{\lambda_k^2}\right) - F(p_k, t_k) \right] . \tag{9.9}$$

To treat all rescaling techniques at once, we define

$$\bar{t}_k := \begin{cases} T \\ t_k \end{cases} , \qquad \mathcal{J}_k := \begin{cases} [\alpha_k, 0) \\ [\alpha_k, T_k] \end{cases} \qquad \text{and} \qquad x_k := \begin{cases} x_0 & \text{(type-I)} \\ F(p_k, t_k) & \text{(type-II).} \end{cases} \tag{9.10}$$

Then (9.7), (9.8) and (9.9) combine to the rescaled embedding $F_k : \mathbb{S}^1 \times \mathcal{J}_k \to \mathbb{R}^2$ with

$$F_k(p,\tau) := \lambda_k \left[F\left(p, \bar{t}_k + \frac{\tau}{\lambda_k^2}\right) - x_k \right] . \tag{9.11}$$

For every $k \in \mathbb{N}$ and $\tau \in \mathcal{J}_k$, we can parametrise the rescaled curve $\Sigma_\tau^k := F_k(\mathbb{S}^1, \tau)$ by arc length (see Section A.1 for further details). In the following, set

$$t = \bar{t}_k + \frac{\tau}{\lambda_k^2} . \tag{9.12}$$

For $p \in \mathbb{S}^1$, the rescaled length element is given by

$$v_k(p,\tau) := \left\| \frac{\partial F_k}{\partial p}(p,\tau) \right\| = \lambda_k \left\| \frac{\partial F}{\partial p}(p,t) \right\| = \lambda_k v(p,t) \tag{9.13}$$

and the rescaled arc length parameter by

$$s_k(p,\tau) := \sigma_k(p,\tau)\int_{p_k}^{p} v_k(q,\tau)\,dq = \sigma_k(p,\tau)\int_{p_k}^{p} \lambda_k v(q,\tau)\,dq$$
$$= \sigma_k(p,\tau)\lambda_k[s(p,t) - s(p_k,t)]\,,$$

where we define

$$\sigma_k(p,\tau) := \begin{cases} 1\,, & \text{if} \quad \int_{p_k}^{p} v_k(q,\tau)\,dq \le \frac{L(\Sigma_\tau^k)}{2} \\ -1\,, & \text{if} \quad \int_{p_k}^{p} v_k(q,\tau)\,dq > \frac{L(\Sigma_\tau^k)}{2}\,. \end{cases}$$

Then we have for all $k \in \mathbb{N}$ and for all $\tau \in \mathcal{J}_k$,

$$s_k(p_k,\tau) = 0 \tag{9.14}$$

and

$$s_k(\mathbb{S}^1,\tau) = \left[-\frac{L(\Sigma_\tau^k)}{2}, \frac{L(\Sigma_\tau^k)}{2}\right] = \mathbb{S}^1_{L(\Sigma_\tau^k)/(2\pi)}\,, \tag{9.15}$$

where we identified $-L(\Sigma_\tau^k)/2$ and $L(\Sigma_\tau^k)/2$ in the last equality. For $k \in \mathbb{N}$ and $\tau \in \mathcal{J}_k$, we can parametrise the curve Σ_τ^k by arc length via the parametrisation $\tilde{F}_k(\,\cdot\,,\tau) : s_k(\mathbb{S}^1,\tau) \to \mathbb{R}^2$ with

$$\tilde{F}_k(s,\tau) := F_k\big(s_k^{-1}(s,\tau),\tau\big) \qquad \text{and} \qquad \tilde{F}_k(s_k(p,\tau),\tau) = F_k(p,\tau)\,. \tag{9.16}$$

The unit tangent vector is invariant under rescaling

$$\boldsymbol{\tau}_k(p,\tau) \overset{\text{(A.4)}}{:=} \frac{1}{v_k(p,\tau)}\frac{\partial F_k}{\partial p}(p,\tau) \overset{\text{(9.13)}}{=} \frac{\lambda_k}{\lambda_k}\frac{1}{v(p,t)}\frac{\partial F}{\partial p}(p,t) \overset{\text{(A.4)}}{=} \boldsymbol{\tau}(p,t)$$

where still $t = \bar{t}_k + \tau/\lambda_k^2$, as well as the the unit outer normal

$$\boldsymbol{\nu}_k(p,\tau) := (\boldsymbol{\tau}_k^2(p,\tau), -\boldsymbol{\tau}_k^1(p,\tau))\,.$$

The rescaled curvature is given by

$$\kappa_k(p,\tau) \overset{\text{(A.6)}}{=} -\left\langle \frac{1}{v_k}\frac{\partial \boldsymbol{\tau}_k}{\partial p}, \boldsymbol{\nu}_k \right\rangle(p,\tau)$$
$$\overset{\text{(9.13)}}{=} -\frac{1}{\lambda_k}\left\langle \frac{1}{v}\frac{\partial \boldsymbol{\tau}}{\partial p}, \boldsymbol{\nu} \right\rangle(p,t) \overset{\text{(A.6)}}{=} \frac{1}{\lambda_k}\kappa(p,t)\,. \tag{9.17}$$

We use the scaling behaviours (9.13) and (9.17) of the length element and the curvature to calculate the rescaled global term

$$h_{\text{ap},k}(\tau) := \frac{\int_{\Sigma_\tau^k} \kappa_k\,d\mathcal{H}^1}{\int_{\Sigma_\tau^k} d\mathcal{H}^1} \overset{\text{(9.13),(9.17)}}{=} \frac{\lambda_k}{\lambda_k^2}\frac{\int_{\Sigma_t} \kappa\,d\mathcal{H}^1}{\int_{\Sigma_t} d\mathcal{H}^1} = \frac{1}{\lambda_k}h_{\text{ap}}(t) \tag{9.18}$$

for the APCSF and

$$h_{\mathrm{lp},k}(\tau) := \frac{\int_{\Sigma_\tau^k} \kappa_k^2 \, d\mathcal{H}^1}{\int_{\Sigma_\tau^k} \kappa_k \, d\mathcal{H}^1} \overset{(9.13),(9.17)}{=} \frac{\lambda_k^2}{\lambda_k^3} \frac{\int_{\Sigma_t} \kappa^2 \, d\mathcal{H}^1}{\int_{\Sigma_t} \kappa \, d\mathcal{H}^1} = \frac{1}{\lambda_k} h_{\mathrm{lp}}(t) \tag{9.19}$$

for the LPCF. From (9.12) follows

$$\frac{\partial}{\partial \tau} = \frac{1}{\lambda_k^2} \frac{\partial}{\partial t} \qquad \text{and} \qquad d\tau = \lambda_k^2 dt \,,$$

thus differentiation of the rescaled embedding (9.11) yields

$$\begin{aligned} \frac{\partial F_k}{\partial \tau}(p,\tau) &= \frac{\partial}{\partial \tau} \left(\lambda_k \left[F(p,t) - x_k\right]\right) = \frac{\lambda_k}{\lambda_k^2} \frac{\partial F}{\partial t}(p,t) \\ &= \frac{1}{\lambda_k} (h(t) - \kappa(p,t)) \boldsymbol{\nu}(p,t) = \left(h_k(\tau) - \kappa_k(p,\tau)\right) \boldsymbol{\nu}_k(p,\tau) \,. \end{aligned} \tag{9.20}$$

For each $k \in \mathbb{N}$, the rescaled evolution equation (9.20) is again a constrained curve flow. Like for the original embedding, the evolution equation (9.20) holds also for the arc length parametrisation (see (3.1)). To show convergence of the rescalings, we need to introduce yet another parametrisation. We follow the idea of [MB15, Proposition 7.1.10]. Define the intervals

$$I_k := \begin{cases} \left[-\sqrt{\pi A_{\alpha_k}^k}, \sqrt{\pi A_{\alpha_k}^k}\right] \overset{\text{Cor. 3.5}}{\subset} \left[-\frac{L_\tau^k}{2}, \frac{L_\tau^k}{2}\right] \overset{(9.15)}{=} s_k(\mathbb{S}^1, \tau) & \text{for APCSF} \\ \left[-\frac{L_{\alpha_k}^k}{2}, \frac{L_{\alpha_k}^k}{2}\right] \overset{\text{Cor. 3.6}}{=} \left[-\frac{L_\tau^k}{2}, \frac{L_\tau^k}{2}\right] \overset{(9.15)}{=} s_k(\mathbb{S}^1, \tau) & \text{for LPCF,} \end{cases} \tag{9.21}$$

for all $\tau \in \mathcal{J}_k$, where $L_\tau^k := L(\Sigma_\tau^k)$ and $A_\tau^k := A(\Sigma_\tau^k)$. For $k \in \mathbb{N}$, let $\tau_0 \in \mathcal{J}_k$. By (9.21), $I_k \subset s_k\big(\mathbb{S}^1, \tau_0\big)$ so that $s_k^{-1}(I_k, \tau_0) \subset \mathbb{S}^1$ is well-defined. Define the embeddings $F_{k,\tau_0} : I_k \times \mathcal{J}_k \to \mathbb{R}^2$ by

$$F_{k,\tau_0}(s,\tau) := F_k\big(s_k^{-1}(s,\tau_0), \tau\big) \,. \tag{9.22}$$

Then, for each $k \in \mathbb{N}$,

$$F_{k,\tau_0}(\,\cdot\,, \tau_0) = F_k\big(s_k^{-1}(\,\cdot\,,\tau_0), \tau_0\big) \overset{(9.16)}{=} \tilde{F}_k(\,\cdot\,, \tau_0) : I_k \to \mathbb{R}^2 \tag{9.23}$$

is an arc length parametrisation and

$$F_{k,\tau_0}(0,\tau) \overset{(9.22)}{=} F_k\big(s_k^{-1}(0,\tau_0), \tau\big) \overset{(9.14)}{=} F_k(p_k, \tau) \tag{9.24}$$

for all $\tau \in \mathcal{J}_k$. Furthermore, F_{k,τ_0} satisfies again the evolution equation (9.20) and

$$|\kappa_{k,\tau_0}|_{\max}(\tau) = \max_{p \in I_k} |\kappa_{k,\tau_0}(p,\tau)| \overset{(9.21)}{\leq} \max_{p \in \mathbb{S}^1} |\kappa_k(p,\tau)| = |\kappa_k|_{\max}(\tau) \tag{9.25}$$

for all $\tau \in \mathcal{J}_k$.

Lemma 9.10 (Properties of the type-I rescaling)**.** *Let $F : \mathbb{S}^1 \times (0,T) \to \mathbb{R}^2$ be a solution of* (2.15) *with $T < \infty$. For the type-I rescalings 9.7 and 9.8 in case of a type-I singularity,*

$$\lambda_k \to \infty \qquad \text{and} \qquad \alpha_k \to -\infty \tag{9.26}$$

for $k \to \infty$. Let $\tau_0 \in \mathbb{R}$ and $k_0 \in \mathbb{N}$ such that $\tau_0 \in \mathcal{J}_k$ for all $k \geq k_0$, then

$$F_{k,\tau_0}(0,\tau_k) \in B_{3C_0^2}(0) \qquad \text{and} \qquad \kappa^2_{k,\tau_0}(0,\tau_k) = 1\,,$$

where

$$\tau_k := -\lambda_k^2(T - t_k) \in \left[-\frac{C_0^2}{2(C_h+1)}, -\frac{1}{2(C_h+1)}\right]$$

for all $k \geq k_0$. Moreover, for $\delta > 0$,

$$\max_{I_k \times [\alpha_k, -\delta^2]} |\kappa_{k,\tau_0}| \leq \frac{C_0}{\delta} \qquad \text{and} \qquad \max_{[\alpha_k,-\delta^2]} h_k \leq \frac{C_h C_0}{\delta}$$

for all $k \geq k_0$.

Proof. First, we consider the type-I rescaling 9.7 for the APCSF. We follow in parts [MB14, Lemma 7.1.8 and Proposition 7.1.10]. By Lemma 9.2, there exists a singular point $x_0 \in \mathbb{R}^2$ and with corresponding blow-up sequence $(p_k, t_k)_{k\in\mathbb{N}}$ in $\mathbb{S}^1 \times [0,T)$. By the definition (9.5) of a type-I singularity and Corollary 5.9, we calculate for $p \in \mathbb{S}^1$ and $t_k, t_l \in [0,T)$,

$$\begin{aligned}
\|F(p,t_l) - F(p,t_k)\| &\leq \int_{t_k}^{t_l} \left\|\frac{\partial F}{\partial t}(p,t)\right\| dt \overset{(2.15)}{\leq} \int_{t_k}^{t_l} |h(t) - \kappa(p,t)|\, dt \\
&\overset{\text{Cor. 5.9}}{\leq} 2\int_{t_k}^{t_l} |\kappa|_{\max}(t)\, dt \overset{(9.5)}{\leq} 2\int_{t_k}^{t_l} \frac{C_0}{\sqrt{4(T-t)}}\, dt \\
&= C_0\left[-\sqrt{4(T-t_l)} + \sqrt{4(T-t_k)}\right] \leq C_0\sqrt{4(T-t_k)}\,.
\end{aligned} \tag{9.27}$$

Since the sequence $(p_k)_{k\in\mathbb{N}}$ is bounded, there exist a point $p_0 \in \mathbb{S}^1$ and a subsequence with

$$p_k \to p_0 \tag{9.28}$$

for $k \to \infty$. We employ (9.27) for $p = p_l$, and obtain

$$\|F(p_l,t_l) - F(p_l,t_k)\| \leq C_0\sqrt{4(T-t_k)} \tag{9.29}$$

for all $k, l \in \mathbb{N}$. By Definition 9.7, we can choose $l_0 = l_0(k)$ large enough so that, for fixed $k \in \mathbb{N}$,

$$\|F(p_l,t_l) - x_0\| \leq C_0\sqrt{4(T-t_k)} \tag{9.30}$$

for all $l \geq l_0$. Estimates (9.29) and (9.30) imply

$$\begin{aligned}
\|F(p_l,t_k) - x_0\| &\leq \|F(p_l,t_k) - F(p_l,t_l)\| + \|F(p_l,t_l) - x_0\| \\
&\overset{(9.29),(9.30)}{\leq} 3C_0\sqrt{4(T-t_k)}
\end{aligned} \tag{9.31}$$

for fixed $k \in \mathbb{N}$ and for all $l \geq l_0(k)$. For given $\varepsilon > 0$, choose $k_0 = k_0(\varepsilon)$ large enough, so that

$$3C_0\sqrt{4(T-t_k)} < \frac{\varepsilon}{2}\,.$$

for all $k \geq k_0$. Then (9.31) yields

$$\|F(p_l, t_k) - x_0\| < \frac{\varepsilon}{2} \tag{9.32}$$

for all $k \geq k_0(\varepsilon)$ and $l \geq l_0(k)$. By the convergence (9.28) and the continuity of the embedding F in its spatial argument, we can further choose l_0 large enough, so that also

$$\|F(p_0, t_k) - F(p_l, t_k)\| < \frac{\varepsilon}{2} \tag{9.33}$$

for $l \geq l_0$. Hence,

$$\|F(p_0, t_k) - x_0\| \leq \|F(p_0, t_k) - F(p_{l_0}, t_k)\| + \|F(p_{l_0}, t_k) - x_0\| \overset{(9.32),(9.33)}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $k \geq k_0(\varepsilon)$. Since $\varepsilon > 0$ was chosen arbitrarily, we obtain

$$F(p_0, t_k) \to x_0 \tag{9.34}$$

for $k \to \infty$. Definition 9.7 and the type-I condition (9.5) yield

$$\lambda_k \overset{\text{Def. 9.7}}{=} |\kappa(p_k, t_k)| \overset{(9.5)}{\leq} \frac{C_0}{\sqrt{4(T-t_k)}} \tag{9.35}$$

and the estimate (9.27) implies

$$\|F(p_0, t_l) - F(p_0, t_k)\| \overset{(9.27)}{\leq} 2C_0\sqrt{4(T-t_k)} \overset{(9.35)}{\leq} \frac{2C_0^2}{\lambda_k}\,.$$

We send $l \to \infty$ in the above inequality and obtain with (9.34),

$$\lambda_k\|x_0 - F(p_0, t_k)\| \leq 2C_0^2 \tag{9.36}$$

for all $k \in \mathbb{N}$. The definition (9.7) of the rescaled embedding provides, for $\tau_k := \lambda_k^2(t_k - T)$,

$$\|F_k(p_0, \tau_k)\| \overset{(9.7)}{=} \lambda_k \left\|F\left(p_0, T + \frac{\tau_k}{\lambda_k^2}\right) - x_0\right\| \overset{(9.36)}{\leq} 2C_0^2 \tag{9.37}$$

for all $k \in \mathbb{N}$. By the convergence (9.28), for given $\delta > 0$, there exists $k_1 \in \mathbb{N}$ so that $|p_k - p_0| < \delta$ for all $k \geq k_0$. By the continuity of the rescaled embedding, for given $\varepsilon > 0$, there exists $\delta > 0$ so that, for $|p_k - p_0| < \delta$, we have

$$\|F_k(p_k, \tau_k) - F_k(p_0, \tau_k)\| < \varepsilon\,. \tag{9.38}$$

Hence, for given $\varepsilon > 0$, there exists $k_1 \in \mathbb{N}$ so that

$$\begin{aligned}\|F_{k,\tau_0}(0, \tau_k)\| &\overset{(9.24)}{=} \|F_k(p_k, \tau_k)\| \leq \|F_k(p_k, \tau_k) - F_k(p_0, \tau_k)\| + \|F_k(p_0, \tau_k)\| \\ &\overset{(9.37),(9.38)}{<} \varepsilon + 2C_0^2\end{aligned}$$

for all $k \geq k_1$. Choosing $\varepsilon = C_0^2$ yields $F_{k,\tau_0}(0, \tau_k) \in B_{3C_0^2}(0)$ for all $k \geq k_1$.

For the type-I rescaling 9.8 for the LPCF, we immediately obtain, for $\tau_k := \lambda_k^2(t_k - T)$,

$$F_{k,\tau_0}(0,\tau_k) \overset{(9.24)}{=} F_k(p_k,\tau_k) \overset{(9.8)}{=} \lambda_k^2 \left[F\left(p_k, T + \frac{\tau_k}{\lambda_k^2}\right) - F(p_k,t_k)\right] = 0$$

for all $k \geq k_1$.

We follow the lines of [MB15, Corollary 4.8], to bound the sequence

$$\left(\tau_k = -\lambda_k^2(T - t_k)\right)_{k\in\mathbb{N}} . \tag{9.39}$$

We estimate

$$\alpha_k \overset{\text{Defs. 9.7,9.8}}{=} -\lambda_k^2 T < -\lambda_k^2 T + \lambda_k^2 t_k \overset{(9.39)}{=} \tau_k < 0$$

for all $k \in \mathbb{N}$. The rescaling behaviour (9.17) of the curvature yields

$$\kappa^2_{k,\tau_0}(0,\tau_k) \overset{(9.24)}{=} \kappa_k^2(p_k,\tau_k) \overset{(9.17)}{=} \frac{1}{\lambda_k^2}\kappa^2\left(p_k, T + \frac{\tau_k}{\lambda_k^2}\right) \overset{(9.39)}{=} \frac{1}{\lambda_k^2}\kappa^2(p_k,t_k) \overset{\text{Defs. 9.7,9.8}}{=} 1\,.$$

Using Definition 9.7 and 9.8 and the lower blow-up rate from Lemma 9.5, we estimate

$$\begin{aligned}\tau_k &\overset{(9.39)}{=} -\lambda_k^2(T-t_k) \overset{\text{Defs. 9.7,9.8}}{=} -\kappa^2(p_k,t_k)(T-t_k)\\ &\overset{\text{Lem. 9.5}}{\leq} -\frac{(T-t_k)}{2(C_h+1)(T-t_k)} = -\frac{1}{2(C_h+1)}\end{aligned}$$

and, by the type-I assumption (9.5),

$$\begin{aligned}\tau_k &\overset{(9.39)}{=} -\lambda_k^2(T-t_k) \overset{\text{Defs. 9.7,9.8}}{=} -\kappa^2(p_k,t_k)(T-t_k)\\ &\overset{(9.5)}{\geq} -\frac{C_0^2(T-t_k)}{2(C_h+1)(T-t_k)} = -\frac{C_0^2}{2(C_h+1)}\end{aligned}$$

for all $k \in \mathbb{N}$.

For the curvature estimate for both rescalings, let $\delta > 0$, $k \in \mathbb{N}$, $\tau \in [\alpha_k, -\delta^2]$ and $p \in \mathbb{S}^1$. Then, the type-I condition (9.5) rescales to

$$|\kappa_k(p,\tau)| \overset{(9.7),(9.8)}{=} \frac{1}{\lambda_k}\left|\kappa\left(p, T + \frac{\tau}{\lambda_k^2}\right)\right| \overset{(9.5)}{\leq} \frac{1}{\lambda_k}\frac{C_0}{\sqrt{-2(C_h+1)\tau/\lambda_k^2}} \leq \frac{C_0}{\sqrt{-\tau}}\,. \tag{9.40}$$

Hence,

$$\max_{I_k\times[\alpha_k,-\delta^2]} |\kappa_{k,\tau_0}| \overset{(9.25)}{\leq} \max_{\mathbb{S}^1\times[\alpha_k,-\delta^2]} |\kappa_k| \overset{(9.40)}{\leq} \frac{C_0}{\delta}$$

and

$$\max_{[\alpha_k,-\delta^2]} h_k \overset{\text{Cor. 5.9}}{\leq} C_h \max_{[\alpha_k,-\delta^2]} |\kappa_k|_{\max} \overset{(9.40)}{\leq} \frac{C_h C_0}{\delta}$$

for each $k \in \mathbb{N}$. □

Lemma 9.11 (Properties of the type-II rescaling, Huisken-Sinestrari [HS99, Lemma 4.3])**.** *Let $F : \mathbb{S}^1 \times (0,T) \to \mathbb{R}^2$ be a solution of (2.15) with $T < \infty$. For the type-II rescaling 9.9 in case of a type-II singularity,*

$$\lambda_k \to \infty\,, \qquad \alpha_k \to -\infty \qquad \text{and} \qquad T_k \to \infty \tag{9.41}$$

for $k \to \infty$. Let $\tau_0 \in \mathbb{R}$ and $k_0 \in \mathbb{N}$ such that $\tau_0 \in \mathcal{J}_k$ for all $k \geq k_0$, then

$$F_{k,\tau_0}(0,0) = 0 \qquad \text{and} \qquad \kappa^2_{k,\tau_0}(0,0) = 1$$

for every $k \geq k_0$ and for any $\varepsilon > 0$ and any $\bar{T} > 0$, there exists a $k_1 \geq k_0$ such that

$$\max_{I_k \times [\alpha_k, \bar{T}]} \kappa^2_{k,\tau_0} < 1 + \varepsilon \qquad \text{and} \qquad \max_{[\alpha_k, \bar{T}]} h_k < C_h \sqrt{1+\varepsilon}$$

for all $k \geq k_1$.

Proof. We follow the lines of [HS99, Lemma 4.3]. Let $\tau_0 \in \mathbb{R}$ and $k_0 \in \mathbb{N}$ so that $\tau_0 \in \mathcal{J}_k$ for all $k \geq k_0$. By definition,

$$F_{k,\tau_0}(0,0) \overset{(9.22)}{=} F_k(p_k, 0) \overset{(9.9)}{=} 0$$

and

$$\kappa^2_{k,\tau_0}(0,0) \overset{(9.24)}{=} \kappa^2_k(p_k, 0) \overset{(9.17)}{=} \frac{1}{\lambda_k^2} \kappa^2(p_k, t_k) \overset{\text{Def. 9.9}}{=} 1\,.$$

for every $k \geq k_0$. Let $M > 0$ be arbitrary. By the definition (9.6) of a type-II singularity, there exist $\bar{t} \in [0,T)$ and $\bar{p} \in \mathbb{S}^1$ so that

$$\kappa^2(\bar{p}, \bar{t})(T - \bar{t}) > 2M\,.$$

We fix $\bar{t}$ and choose $k_1 \geq k_0$, so that $\bar{t} < T - 1/k$ and $\kappa^2(\bar{p}, \bar{t})/k < M$ for all $k \geq k_1$. Then

$$\kappa^2(\bar{p}, \bar{t}) \left(T - \frac{1}{k} - \bar{t} \right) = \kappa^2(\bar{p}, \bar{t})(T - \bar{t}) - \frac{1}{k} \kappa^2(\bar{p}, \bar{t}) > M$$

and Definition 9.9 yields

$$T_k \overset{\text{Def. 9.9}}{=} \kappa^2(p_k, t_k) \left(T - \frac{1}{k} - t_k \right) \overset{\text{Def. 9.9}}{\geq} \kappa^2(\bar{p}, \bar{t}) \left(T - \frac{1}{k} - \bar{t} \right) > M\,.$$

Since M was chosen arbitrarily, it follows that $T_k \to \infty$ and thus also $\lambda_k = \kappa^2(p_k, t_k) \to \infty$ for $k \to \infty$. Since $t_k \nearrow T$, we conclude that $\alpha_k = -\lambda_k^2 t_k \to -\infty$ for $k \to \infty$. For the curvature estimate, it again follows from Definition 9.9 that

$$\kappa^2(p,t) \left(T - \frac{1}{k} - t \right) \overset{\text{Def. 9.9}}{\leq} \kappa^2(p_k, t_k) \left(T - \frac{1}{k} - t_k \right) \overset{\text{Def. 9.9}}{=} T_k \tag{9.42}$$

for all $p \in \mathbb{S}^1$, $t \in [0, T - 1/k]$ and $k \in \mathbb{N}$. Let $\varepsilon > 0$ and $\bar{T} > 0$ be given. Since $T_k \to \infty$, there exists again $k_1 \in \mathbb{N}$ so that, for all $k \geq k_1$, $\bar{T} < T_k$ and

$$0 < \frac{\bar{T}}{T_k - \bar{T}} < \varepsilon\,. \tag{9.43}$$

For $\tau \in [\alpha_k, \bar{T}]$, it is $t := t_k + \tau/\lambda_k^2 \in [0, T - 1/k)$, and we can use the scaling behaviour of the curvature and (9.42) to estimate

$$\begin{aligned}\kappa_k^2(p,\tau) &\overset{(9.17)}{=} \frac{1}{\lambda_k^2}\kappa^2\left(p, t_k + \frac{\tau}{\lambda_k^2}\right) \overset{(9.42)}{\leq} \frac{T - 1/k - t_k}{T - 1/k - (t_k + \tau/\lambda_k^2)} \\ &= \frac{T_k}{T_k - \tau} \leq \frac{T_k}{T_k - \bar{T}} = 1 + \frac{\bar{T}}{T_k - \bar{T}} \overset{(9.43)}{<} 1 + \varepsilon \end{aligned} \tag{9.44}$$

for all $p \in \mathbb{S}^1$ and $k \geq k_1$. Hence,

$$\max_{I_k \times [\alpha_k, \bar{T}]} \kappa_{k,\tau_0}^2 \overset{(9.25)}{\leq} \max_{\mathbb{S}^1 \times [\alpha_k, \bar{T}]} \kappa_k^2 \overset{(9.44)}{<} 1 + \varepsilon$$

and

$$\max_{[\alpha_k, \bar{T}]} h_k \overset{\text{Cor. 5.9}}{\leq} C_h \max_{[\alpha_k, \bar{T}]} |\kappa_k|_{\max} \overset{(9.44)}{<} C_h\sqrt{1+\varepsilon}$$

for all $k \geq \max\{k_0, k_1\}$. □

Define

$$T_\infty := \begin{cases} 0 & \text{for the type-I rescaling} \\ \infty & \text{for the type-II rescaling.} \end{cases}$$

Lemma 9.12. *Let $F : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ be an embedded solution of* (2.15) *with initial curve Σ_0 and $T < \infty$. Let $\tau \in (-\infty, T_\infty)$ and $M, \varepsilon > 0$. Then, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$,*

$$L(\Sigma_\tau^k) > M\,, \qquad A(\Sigma_\tau^k) > M$$

and

$$h_k(\tau) < \varepsilon$$

for the APCSF and for the LPCF if $\sup_{[0,T)} h_{\text{lp}} < \infty$. *Moreover, let $\tau_0 \in (-\infty, T_\infty)$ and let $I \subset \mathbb{R}$ be bounded and $J \subset (-\infty, T_\infty)$ be compact and $k_0 \in \mathbb{N}$ so that $I \subset I_k$, $\tau_0 \in \mathcal{J}_k$ and $J \subset \mathcal{J}_k$. Then there exist $c_1 = c_1(\Sigma_0, \tau_0, \tau)$ and $c_2 = c_2(\Sigma_0, \tau_0, \tau)$ such that*

$$c_1|I| \leq L(F_{k,\tau_0}(I, \tau)) \leq c_2|I|$$

for all $\tau \in \mathcal{J}_k$ and $k \geq k_0$, and there exists $R = R(\Sigma_0, C_0, \tau_0, |I|, |J|)$ such that

$$F_{k,\tau_0}(I, \tau) \subset B_R(0)$$

for all $\tau \in J$ and $k \geq k_0$.

Proof. Let $\tau \in (-\infty, T_\infty)$. Let $k_0 \in \mathbb{N}$ so that $\tau \in \mathcal{J}_k$ for all $k \geq k_0$. By the scaling behaviour (9.13) of the length element, the behaviour of the length of the curve (see Corollaries 3.5 and 3.6) and the behaviours (9.26) and (9.41) of λ_k,

$$\begin{aligned} L(\Sigma_\tau^k) &= \int_{\Sigma_\tau^k} d\mathcal{H}^1 \overset{(9.13)}{=} \lambda_k \int_{\Sigma_{\bar{t}_k + \tau/\lambda_k^2}} d\mathcal{H}^1 \\ &= \lambda_k L\left(\Sigma_{\bar{t}_k + \tau/\lambda_k^2}\right) \overset{\text{Cors. 3.5,3.6}}{\geq} \lambda_k L(\Sigma_0) \overset{(9.26),(9.41)}{\longrightarrow} \infty \end{aligned}$$

for $k \to \infty$. For $k \geq k_0$, let $\Omega_\tau^k \subset \mathbb{R}^2$ be the domain with $\Sigma_\tau^k = \partial\Omega_\tau^k$. Then

$$\begin{aligned} A(\Sigma_\tau^k) &= \int_{\Omega_\tau^k} d\mathcal{H}^2 \overset{(9.11)}{=} \lambda_k \int_{\Omega_{\bar{t}_k+\tau/\lambda_k^2}} d\mathcal{H}^2 \\ &= \lambda_k A(\Sigma_{\bar{t}_k+\tau/\lambda_k^2}) \overset{\text{Cors. 3.5,3.6}}{\geq} \lambda_k A(\Sigma_0) \overset{(9.26),(9.41)}{\longrightarrow} \infty \end{aligned}$$

for $k \to \infty$. The scaling behaviour (9.18) and the bound (3.10) for h_{ap} yield

$$h_{\text{ap},k}(\tau) \overset{(9.18)}{=} \frac{1}{\lambda_k} h_{\text{ap}}\left(\bar{t}_k + \frac{\tau}{\lambda_k^2}\right) \overset{(3.10)}{\leq} \frac{\sqrt{\pi}}{\lambda_k\sqrt{A_0}} \to 0$$

for $k \to \infty$. Likewise, if $\sup_{[0,T)} h_{\text{lp}} = c < \infty$, the scaling behaviour (9.19) for h_{lp} implies

$$h_{\text{lp},k}(\tau) \overset{(9.19)}{=} \frac{1}{\lambda_k} h_{\text{lp}}\left(\bar{t}_k + \frac{\tau}{\lambda_k^2}\right) \leq \frac{c}{\lambda_k} \to 0$$

for $k \to \infty$. Let $\tau_0 \in (-\infty, T_\infty)$ and $I \subset \mathbb{R}$ be a bounded interval. Then there exists $k_0 \in \mathbb{N}$ so that $I \subset I_k$ for all $k \geq k_0$. Corollary 5.9 and Lemmata 9.10 and 9.11 yield for $p \in I$ and for $\tau \in (\tau_0, T_\infty)$,

$$\begin{aligned} \frac{1}{c(\Sigma_0,\tau_0,\tau)} v_{k,\tau_0}(p,\tau_0) &\overset{\text{Lems. 9.10,9.11}}{\leq} \exp\left(-c(\Sigma_0) \int_{\tau_0}^{\tau} \left(\kappa_{k,\tau_0}^2\right)_{\max}(\sigma)\,d\sigma\right) v_{k,\tau_0}(p,\tau_0) \\ &\overset{\text{Cor. 5.9}}{\leq} v_{k,\tau_0}(p,\tau) \\ &\overset{\text{Cor. 5.9}}{\leq} \exp\left(c(\Sigma_0) \int_{\tau_0}^{\tau} \left(\kappa_{k,\tau_0}^2\right)_{\max}(\sigma)\,d\sigma\right) v_{k,\tau_0}(p,\tau_0) \\ &\overset{\text{Lems. 9.10,9.11}}{\leq} c(\Sigma_0,\tau_0,\tau) v_{k,\tau_0}(p,\tau_0) \end{aligned}$$

and for $\tau \in (-\infty, \tau_0)$,

$$\begin{aligned} c(\Sigma_0,\tau_0,\tau) v_{k,\tau_0}(p,\tau) &\overset{\text{Lems. 9.10,9.11}}{\leq} \exp\left(-c(\Sigma_0) \int_{\tau}^{\tau_0} \left(\kappa_{k,\tau_0}^2\right)_{\max}(\sigma)\,d\sigma\right) v_{k,\tau_0}(p,\tau) \\ &\overset{\text{Cor. 5.9}}{\leq} v_{k,\tau_0}(p,\tau_0) \\ &\overset{\text{Cor. 5.9}}{\leq} \exp\left(c(\Sigma_0) \int_{\tau}^{\tau_0} \left(\kappa_{k,\tau_0}^2\right)_{\max}(\sigma)\,d\sigma\right) v_{k,\tau_0}(p,\tau) \\ &\overset{\text{Lems. 9.10,9.11}}{\leq} \frac{1}{c(\Sigma_0,\tau_0,\tau)} v_{k,\tau_0}(p,\tau) \end{aligned}$$

for all $k \geq k_0$. By (9.23), $F_{k,\tau_0}(\,\cdot\,,\tau_0)$ is an arclength parametrisation so that $v_{k,\tau_0}(\,\cdot\,,\tau_0) \equiv 1$ and in both above cases,

$$0 < c_1(\Sigma_0,\tau_0,\tau) \leq v_{k,\tau_0}(p,\tau) \leq c_2(\Sigma_0,\tau_0,\tau) < \infty$$

for all $p \in I$, $\tau \in \mathcal{J}_k$ and $k \geq k_0$. (In fact, the constants only depend on τ in case of a type-II singularity and for $\tau \in (\tau_0, \infty)$.) Moreover,

$$c_1(\Sigma_0,\tau_0,\tau)|I| \leq L(F_{k,\tau_0}(I,\tau)) = \int_I v_{k,\tau_0}(p,\tau)\,dp \leq c_2(\Sigma_0,\tau_0,\tau)|I|$$

for all $\tau \in \mathcal{J}_k$ and $k \geq k_0$. Let $J \subset (-\infty, T_\infty)$ be compact and $k_1 \geq k_0$ so that $J \subset \mathcal{J}_k$ for all $k \geq k_1$. By the above estimate,

$$c_1(\Sigma_0, \tau_0, J)|I| \leq L(F_{k,\tau_0}(I, \tau)) \leq c_2(\Sigma_0, \tau_0, J)|I| \tag{9.45}$$

for all $\tau \in J$ and $k \geq k_1$. By Lemmata 9.10 and 9.11,

$$F_{k,\tau_0}(0, \tau_k) \overset{\text{Lem. 9.10}}{\in} B_{3C_0}(0) \qquad \text{or} \qquad F_{k,\tau_0}(0, 0) \overset{\text{Lem. 9.11}}{=} 0 \tag{9.46}$$

where $\tau_k \in [-C_0^2/4, 1/4]$, and by Lemma 3.13,

$$\|F_{k,\tau_0}(0, \tau_k) - F_{k,\tau_0}(0, \tau)\| \leq 2R|\tau_k - \tau| \qquad \text{or} \qquad \|F_{k,\tau_0}(0, 0) - F_{k,\tau_0}(0, \tau)\| \leq 2R|\tau|$$

with $R = R\big(\Sigma_0, \sup_{[\alpha_k,\tau_2]} h_k\big)$. In view of the upper bound on h_k from Lemmata 9.10 and 9.11, and utilising (9.45) and (9.46), we can choose $R = R(\Sigma_0, C_0, \tau_0, I, J) > 0$ to obtain

$$F_{k,\tau_0}(I, \tau) \subset B_R(0)$$

for all $\tau \in J$ and $k \geq k_1$. □

9.2 Convergence

In this section we show that the sequence (9.22) of rescaled embeddings converges locally in the domain of definition and the ambient space, smoothly along a subsequence to a maximal, embedded, convex or concave, smooth, ancient solutions.

For $k \in \mathbb{N}$, we define the intervals

$$J_k := [\alpha_k, T_k] \begin{cases} \subset \mathcal{J}_k \subset (-\infty, 0) & \text{for the type-I rescaling} \\ = \mathcal{J}_k \subset \mathbb{R} & \text{for the type-II rescaling,} \end{cases} \tag{9.47}$$

where, for the type-I rescaling, $(T_k)_{k\in\mathbb{N}}$ is a sequence with

$$\alpha_k < T_k \nearrow 0 \tag{9.48}$$

for $k \to \infty$. For the type-II rescaling, $(T_k)_{k\in\mathbb{N}}$ is defined as in Definition 9.9.

Theorem 9.13 (Convergence of the rescaling). *Let $F : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ be a smooth, embedded solution of* (2.15) *with initial curve Σ_0 and $T < \infty$. Let $\tau_0 \in (-\infty, T_\infty)$ and $k_0 \in \mathbb{N}$ such that $\tau_0 \in \mathcal{J}_k$ for all $k \geq k_0$. Then, for the rescaling* (9.11)*, the sequence of embeddings*

$$\big(F_{k,\tau_0} : I_k \times J_k \to \mathbb{R}^2\big)_{k\geq k_0}$$

(see (9.22)*) converges for $k \to \infty$ along a subsequence, uniformly and smoothly on compact subsets $I \times J \subset \mathbb{R} \times (-\infty, T_\infty)$ with $0 \in I$ to a maximal, smooth, ancient limit solution $F_{\infty,\tau_0} : \mathbb{R} \times (-\infty, T_\infty) \to \mathbb{R}^2$ which satisfies*

$$\frac{\partial F_{\infty,\tau_0}}{\partial \tau}(p, \tau) = \big(h_\infty(\tau) - \kappa_{\infty,\tau_0}(p, \tau)\big)\boldsymbol{\nu}_{\infty,\tau_0}(p, \tau)\,, \tag{9.49}$$

where

$$h_{\mathrm{ap},\infty}(\tau) = 0 \qquad \textit{and} \qquad h_{\mathrm{lp},\infty}(\tau) \geq 0$$

for $\tau \in (-\infty, T_\infty)$*. Moreover,* $F_{\infty,\tau_0}(\,\cdot\,,\tau_0)$ *is an arc length parametrisation and* $L(\Sigma_\tau^{\infty,\tau_0}) = \infty$ *for all* $\tau \in (-\infty, T_\infty)$*. For the type-I rescaling in case of a type-I singularity,* $T_\infty = 0$ *and there exists a time* $\tau_\infty \in [-C_0^2/4, -1/4]$ *such that*

$$F_{\infty,\tau_0}(0,\tau_\infty) \in B_{3C_0}(0) \qquad \textit{and} \qquad |\kappa_{\infty,\tau_0}(0,\tau_\infty)| = 1$$

as well as

$$\sup_{\mathbb{R}\times(-\infty,-\delta^2]} |\kappa_{\infty,\tau_0}| \leq \frac{C_0}{\delta} \qquad \textit{and} \qquad \sup_{(-\infty,-\delta^2]} h_{\mathrm{lp},\infty} \leq \frac{C_h C_0}{\delta}$$

for all $\delta < 0$*. For the type-II rescaling in case of a type-II singularity,* $T_\infty = \infty$ *as well as*

$$F_{\infty,\tau_0}(0,0) = 0\,, \qquad \sup_{\mathbb{R}\times\mathbb{R}} |\kappa_{\infty,\tau_0}| = |\kappa_{\infty,\tau_0}(0,0)| = 1 \qquad \textit{and} \qquad \sup_{\mathbb{R}} h_{\mathrm{lp},\infty} \leq C_h\,.$$

Remark 9.14. For the APCSF, the limit solution satisfies CSF, that is,

$$\frac{\partial F_{\infty,\tau_0}}{\partial \tau}(p,\tau) = -\kappa_{\infty,\tau_0}(p,\tau)\boldsymbol{\nu}_{\infty,\tau_0}(p,\tau)$$

for $(p,\tau) \in \mathbb{R} \times (-\infty, T_\infty)$.

Proof of Theorem 9.13. The proof follows similar lines to those of [Eck04, Remark 4.22 (2)] and [MB15, Proposition 4.7]. Let $(p_k, t_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{S}^1 \times [0,T)$ according to Definition 9.7, 9.8 or 9.9. By Lemma 9.12, $L(\Sigma_{\alpha_k}^k) \to \infty$ and $A(\Sigma_{\alpha_k}^k) \to \infty$ for $k \to \infty$, so that we can find a subsequence $(p_k, t_k)_{k\in\mathbb{N}}$ with

$$I_k \subset I_{k+1} \qquad \text{and} \qquad I_k \to (-\infty, \infty) \tag{9.50}$$

for all $k \in \mathbb{N}$ and for $k \to \infty$ (see the definition (9.21) of I_k). By the behaviours of α_k and T_k (see (9.26) and (9.48) for a type-I singularity and (9.41) for a type-II singularity), we can find a subsubsequence $(p_k, t_k)_{k\in\mathbb{N}}$ so that

$$J_k \subset J_{k+1} \qquad \text{and} \qquad J_k \to (-\infty, T_\infty) \tag{9.51}$$

for all $k \in \mathbb{N}$ and for $k \to \infty$ (see the definition (9.47) of J_k). Let $\tau_0 \in (-\infty, T_\infty)$, $0 \in I \subset \mathbb{R}$ and $J \subset (-\infty, T_\infty)$, I and J intervals, and $k_0 \in \mathbb{N}$ so that $\tau_0 \in \mathcal{J}_k$, $I \subset I_k$ and $J \subset J_k$ for all $k \geq k_0$. We have that the first spatial derivative

$$\left\| \frac{\partial \tilde{F}_{k,\tau_0}}{\partial s} \right\| = \|\boldsymbol{\tau}_{k,\tau_0}\| = 1$$

and, by Lemmata 9.10 and 9.11,

$$\max_{(p,\tau)\in I\times J} |\kappa_{k,\tau_0}(p,\tau)| \overset{\text{Lems. 9.10,9.11}}{\leq} c$$

for all $k \geq k_0$, where $c = c(C_0, T_{k_0})$ for the type-I rescalings. With the above uniform bound for the curvature, we can apply Corollary 4.7 and have bounded derivatives $\frac{\partial^m}{\partial t^m}\frac{\partial^n}{\partial s^n}\kappa_{k,\tau_0}$

of the curvature on $I \times J$ for all $n, m \in \mathbb{N} \cup \{0\}$ and $k \geq k_0$ as well, where the bounds only depend on n and Σ_0, and also C_0 and T_{k_0} for the type-I rescalings. Thus, all derivatives of the curvature are uniformly bounded on $I \times J$. By Lemma 9.12, there exists $R = R(\Sigma_0, C_0, \tau_0, I, J) > 0$ so that

$$F_{k,\tau_0}(I, \tau) \subset B_R(0)$$

for all $\tau \in J$ and $k \geq k_0$. For every $n, m \in \mathbb{N} \cup \{0\}$ and $k \geq k_0$, the functions

$$F^{(n,m)}_{k,\tau_0} := \frac{\partial^m}{\partial t^m} \frac{\partial^n}{\partial s^n} \tilde{F}_{k,\tau_0}$$

are bounded and equicontinuous in k by the fundamental theorem of calculus. For each $n, m \in \mathbb{N} \cup \{0\}$, the Arzelà–Ascoli theorem, Theorem B.12, yields that sequence

$$\left(F^{(n,m)}_{k,\tau_0} : I \times J \to \mathbb{R}^2\right)_{k \geq k_0}$$

has a uniformly converging subsequence for each $n, m \in \mathbb{N} \cup \{0\}$. Theorem B.14 employed to the functions $F^{(n,m)}_{k,\tau_0}$ for every $n, m \in \mathbb{N} \cup \{0\}$ implies that $\left(F_{k,\tau_0}\right)_{k \geq k_0}$ converges smoothly in space and time to the smooth limit

$$F_{\infty,\tau_0} : I \times J \to \mathbb{R}^2 \,.$$

We pick sequences $(I_l)_{l \in \mathbb{N}}$ and $(J_l)_{l \in \mathbb{N}}$, with

$$0 \in I_l \subset I_{l+1} \subset \mathbb{R} \qquad \text{and} \qquad J_l \subset J_{l+1} \subset (-\infty, T_\infty)$$

for all $l \in \mathbb{N}$ and

$$I_l \to \mathbb{R} \qquad \text{and} \qquad J_l \to (-\infty, T_\infty) \tag{9.52}$$

for $l \to \infty$. By (9.50) and (9.51), we can repeat the above argument for every $l \in \mathbb{N}$. The sequence

$$\left(F^{(n,m)}_{k,\tau_0} : I_l \times J_l \to \mathbb{R}^2\right)_{k \geq k_l}$$

coincides with $\left(F^{(n,m)}_{k,\tau_0}\right)_{k \geq k_{l-1}}$ on $I_{l-1} \times J_{l-1}$. By the same argument as above, the sequence $\left(F_{k,\tau_0}\right)_{k \geq k_l}$ has a subsequence that converges smoothly in space and time to the smooth limit

$$F_{\infty,\tau_0,l} : I_l \times J_l \to \mathbb{R}^2$$

which equals $F_{\infty,\tau_0,l-1}$ on $I_{l-1} \times J_{l-1}$. Hence, the diagonal subsequence

$$\left(F_{k,\tau_0} : I_l \times J_l \to \mathbb{R}^2\right)_{k \geq k_l}$$

converges for $k, l \to \infty$ smoothly in space and time to the smooth limit flow

$$F_{\infty,\tau_0} : \mathbb{R} \times (-\infty, T_\infty) \to \mathbb{R}^2$$

which equals $F_{\infty,\tau_0,l}$ on $I_l \times J_l$ for every $l \in \mathbb{N}$. In the above step we have applied the limit behaviours (9.52) of I_l and J_l, and where $T_\infty = 0$ for the type-I rescalings (see (9.48)) and $T_\infty = \infty$ for the type-II rescaling (see (9.41)). Since $F_{k,\tau_0}(\,\cdot\,, \tau_0)$ is an arc length parametrisation for every $k \in \mathbb{N}$, $F_{\infty,\tau_0}(\,\cdot\,,\tau_0)$ is as well by the smooth convergence. Lemma 9.12 yields for bounded intervals $I \subset \mathbb{R}$ that

$$c_1(\Sigma_0,\tau_0,\tau)|I| \leq L(F_{\infty,\tau_0}(I,\tau)) \leq c_2(\Sigma_0,\tau_0,\tau)|I|$$

for all $\tau \in (-\infty, T_\infty)$, which implies $L(\Sigma_\tau^{\infty,\tau_0}) = L(F_{\infty,\tau_0}(\mathbb{R},\tau)) = \infty$.

For a **type-I** singularity, by Lemma 9.10, the sequence $(\tau_k)_{k\in\mathbb{N}}$ is bounded and has a convergent subsequence with

$$\tau_k \to \tau_\infty \in \left[-\frac{C_0^2}{2(C_h+1)}, -\frac{1}{2(C_h+1)}\right]$$

so that

$$\kappa^2_{\infty,\tau_0}(0,\tau_\infty) = 1\,, \qquad F_{\infty,\tau_0}(0,\tau_\infty) \in B_{3C_0}(0) \qquad \text{and} \qquad \sup_{\mathbb{R}\times(-\infty,-\delta^2]} |\kappa_{\infty,\tau_0}| \leq \frac{C_0}{\delta}$$

for all $\delta < 0$. For a **type-II** singularity, Lemma 9.11 implies that

$$F_{\infty,\tau_0}(0,0) = 0 \qquad \text{and} \qquad \kappa^2_{\infty,\tau_0}(0,0) = 1 \tag{9.53}$$

and that for any $\varepsilon > 0$ and any $\bar{T} > 0$,

$$\sup_{\mathbb{R}\times(-\infty,\bar{T}]} \kappa^2_{\infty,\tau_0} \leq 1 + \varepsilon\,.$$

Sending $\bar{T} \to \infty$ and $\varepsilon \to 0$ yields

$$\sup_{\mathbb{R}\times\mathbb{R}} \kappa^2_{\infty,\tau_0} \leq 1 \overset{(9.53)}{=} \kappa^2_{\infty,\tau_0}(0,0)\,.$$

The evolution equation (9.49) for the **APCSF** follows from the behaviour of the global term $h_{\mathrm{ap},k}$ for $k \to \infty$ (see Lemma 9.12). For the **LPCF**, by Lemmata 3.10, 9.10, 9.11 and 9.12,

$$0 \overset{\text{Lems. 3.10,9.12}}{\leq} h_{\mathrm{lp},\infty}(\tau) \overset{\text{Lems. 9.10,9.11}}{\leq} \begin{cases} \dfrac{C_h C_0}{\delta} & \text{for } \tau \in (-\infty,-\delta^2] \subset (-\infty,0) \text{ for the type-I rescaling} \\ C_h & \text{for } \tau \in \mathbb{R} \text{ for the type-II rescaling.} \end{cases}$$

□

9.3 Limit flow

In this section, we study the limit flows obtained in Theorem 9.13 and show in particular, that each curve is strictly convex or concave.

Lemma 9.15 (Preservation of strict convexity/concavity of the limit flow)**.** *Let* $F : \mathbb{R} \times [0,T) \to \mathbb{R}^2$ *be a solution of* (9.49) *and let* $\kappa > (<)\ 0$ *on* Σ_0*. Then* $\kappa > (<)\ 0$ *on* Σ_t *for all* $t \in (0,T)$*.*

Proof. Assume that there exists a point $(p_0, t_0) \in \mathbb{R}\times(0,T)$ with $\kappa(p_0, t_0) = 0$. We proceed as in Corollary 4.3 on the interval $I := (p_0 - 1, p_0 + 1) \subset \mathbb{R}$. We estimate

$$|(h-\kappa)\kappa| \overset{\text{Cor. 5.9}}{\leq} (C_h + 1)\kappa_{\max}^2 < \infty$$

on $I \times [0, T)$ and employ the strong maximum principle, Theorem B.17, with respect to the evolution equation (4.1) of κ. It follows that $\kappa \equiv 0$ on $I \times [0, t_0]$, which is a contradiction. □

Proposition 9.16 (Convexity/Concavity of the limit flows)**.** *Let $F : \mathbb{S}^1\times(0,T) \to \mathbb{R}^2$ be a smooth, embedded solution of* (2.15) *with $T < \infty$. Each limit flow $F_{\infty,\tau_0} : \mathbb{R}\times(-\infty, T_\infty) \to \mathbb{R}^2$, as obtained in Theorem 9.13, is either strictly convex or strictly concave.*

Proof of Proposition 9.16. We follow the lines of [Alt91, Theorems 5.14 and 7.7]. A similar proof can be found in [Man11, Proposition 4.3.2]. Corollary 5.9 implies

$$\frac{d}{dt}\int_{\Sigma_t} |\kappa|\, d\mathcal{H}^1 = -2 \sum_{\{s\in s(\mathbb{S}^1,\, t)\,|\,\kappa(s,t)=0\}} \left|\frac{\partial\kappa}{\partial s}(s,t)\right| \leq 0 \tag{9.54}$$

for $t \in (0, T)$. Thus, the integral $\int_{\Sigma_t} |\kappa|\, d\mathcal{H}^1$ is decreasing in time on $(0, T)$. We deduce

$$\int_{\Sigma_t} |\kappa|\, d\mathcal{H}^1 \to C < \infty \tag{9.55}$$

for $t \to T$. Let $t_1, t_2 \in (0, T)$. Integrating (9.54) yields

$$\int_{\Sigma_{t_2}} |\kappa|\, d\mathcal{H}^1 - \int_{\Sigma_{t_1}} |\kappa|\, d\mathcal{H}^1 = -2\int_{t_1}^{t_2} \sum_{\{s\in s(\mathbb{S}^1,\, t)\,|\,\kappa(s,t)=0\}} \left|\frac{\partial\kappa}{\partial s}(s,t)\right| dt\,. \tag{9.56}$$

We rescale according to (9.11). Let $\tau_1, \tau_2 \in \mathbb{R}$ with $\tau_1 < \tau_2$ and $k_0 \in \mathbb{N}$ so that $[\tau_1, \tau_2] \subset J_k$ for all $k \geq k_0$ (see (9.47) for the definition of J_k). Then

$$\lim_{k\to\infty} \bar{t}_k + \frac{\tau_i}{\lambda_k^2} = T \tag{9.57}$$

for all $k \geq k_0$ and for $i = 1, 2$ (see (9.10) for the definition of $\bar{t}_k$). Since the integral

$$\int_{\Sigma_t} |\kappa|\, d\mathcal{H}^1 = \sum_{S\in\mathcal{S}_t} \left|\int_{\tilde{F}(S,t)} \kappa\, d\mathcal{H}^1\right| = \sum_{S\in\mathcal{S}_t} |\theta(s_1^S, s_2^S, t)|$$

is the sum over the absolute value of the angles between inflection points, it is scaling invariant and (9.56) also holds for the rescaled flow, that is,

$$\begin{aligned} &2\int_{\tau_1}^{\tau_2} \sum_{\{s\in s_k(\mathbb{S}^1,\tau)\,|\,\kappa_k(s,\tau)=0\}} \left|\frac{\partial\kappa_k}{\partial s}(s,\tau)\right| d\tau \\ &\overset{(9.56)}{=} \int_{\Sigma_{\tau_1}^k} |\kappa_k|\, d\mathcal{H}^1 - \int_{\Sigma_{\tau_2}^k} |\kappa_k|\, d\mathcal{H}^1 \\ &\overset{(9.11)}{=} \int_{\Sigma_{\bar{t}_k+\tau_1/\lambda_k^2}} |\kappa|\, d\mathcal{H}^1 - \int_{\Sigma_{\bar{t}_k+\tau_2/\lambda_k^2}} |\kappa|\, d\mathcal{H}^1 \end{aligned} \tag{9.58}$$

for all $k \geq k_0$. Taking the limit $k \to \infty$ in (9.58) yields

$$
\begin{aligned}
0 &\leq 2 \lim_{k\to\infty} \left[\int_{\tau_1}^{\tau_2} \sum_{\{s \in s_k(\mathbb{S}^1,\tau)\,|\,\kappa_k(s,\tau)=0\}} \left| \frac{\partial \kappa_k}{\partial s}(s,\tau) \right| d\tau \right] \\
&\overset{(9.58)}{=} \lim_{k\to\infty} \left[\int_{\Sigma_{t_k+\tau_1/\lambda_k^2}} |\kappa|\, d\mathcal{H}^1 - \int_{\Sigma_{t_k+\tau_2/\lambda_k^2}} |\kappa|\, d\mathcal{H}^1 \right] \\
&\overset{(9.55),(9.57)}{=} C - C = 0\,. \qquad (9.59)
\end{aligned}
$$

Let $0 \in I \subset \mathbb{R}$ be a bounded interval, $\tau_0 \in (-\infty, T_\infty)$ and $k_1 \geq k_0$ so that $I \subset I_k$ and $\tau_0 \in \mathcal{J}_k$ for all $k \geq k_1$. By Lemma 9.12,

$$
c_1(\Sigma_0,\tau_0,\tau_1,\tau_2)|I| \leq L(F_{k,\tau_0}(I,\tau)) \leq c_2(\Sigma_0,\tau_0,\tau_1,\tau_2)|I| \qquad (9.60)
$$

for all $\tau \in [\tau_1,\tau_2]$. By Theorem 9.13, there exists a subsequence $(p_k,t_k)_{k\geq k_1}$ so that the embeddings $(F_{k,\tau_0} : I_k \times J_k \to \mathbb{R}^2)_{k\geq k_1}$ converge smoothly on $I\times[\tau_1,\tau_2]$ along a subsequence to a smooth flow $F_{\infty,\tau_0} : \mathbb{R} \times (-\infty, T_\infty) \to \mathbb{R}^2$. Observe that

$$
\begin{aligned}
\bar{I} &:= \left[-\frac{c_1|I|}{2}, \frac{c_1|I|}{2} \right] \\
&\overset{(9.60)}{\subset} \left[-\frac{L\big(F_{k,\tau_0}(I,\tau)\big)}{2}, \frac{L\big(F_{k,\tau_0}(I,\tau)\big)}{2} \right] \overset{(9.15)}{=} s_{k,\tau_0}(I,\tau) \\
&\overset{(9.22)}{=} \left[-\frac{L\left(F_k\big(s_k^{-1}(I,\tau_0),\tau\big)\right)}{2}, \frac{L\left(F_k\big(s_k^{-1}(I,\tau_0),\tau\big)\right)}{2} \right] \overset{(9.15)}{\subset} s_k(\mathbb{S}^1,\tau) \qquad (9.61)
\end{aligned}
$$

for all $k \geq k_1$. Fatou's lemma, Lemma B.9, (9.59) and (9.61) yield

$$
\begin{aligned}
0 &\leq \int_{\tau_1}^{\tau_2} \sum_{\{s\in\bar{I}\,|\,\kappa_{\infty,\tau_0}(s,\tau)=0\}} \left| \frac{\partial \kappa_{\infty,\tau_0}}{\partial s}(s,\tau) \right| d\tau \\
&\overset{\text{Thm. 9.13}}{=} \int_{\tau_1}^{\tau_2} \liminf_{k\to\infty} \left(\sum_{\{s\in\bar{I}\,|\,\kappa_{k,\tau_0}(s,\tau)=0\}} \left| \frac{\partial \kappa_{k,\tau_0}}{\partial s}(s,\tau) \right| \right) d\tau \\
&\overset{\text{Lem. B.9}}{\leq} \liminf_{k\to\infty} \int_{\tau_1}^{\tau_2} \sum_{\{s\in\bar{I}\,|\,\kappa_{k,\tau_0}(s,\tau)=0\}} \left| \frac{\partial \kappa_{k,\tau_0}}{\partial s}(s,\tau) \right| d\tau \\
&\overset{(9.61)}{\leq} \liminf_{k\to\infty} \int_{\tau_1}^{\tau_2} \sum_{\{s\in s_k(\mathbb{S}^1,\tau)\,|\,\kappa_k(s,\tau)=0\}} \left| \frac{\partial \kappa_k}{\partial s}(s,\tau) \right| d\tau \overset{(9.59)}{=} 0\,.
\end{aligned}
$$

Since $I \subset \mathbb{R}$ was chosen arbitrarily,

$$
\int_{\tau_1}^{\tau_2} \sum_{\{s\in\mathbb{R}\,|\,\kappa_{\infty,\tau_0}(s,\tau)=0\}} \left| \frac{\partial \kappa_{\infty,\tau_0}}{\partial s}(s,\tau) \right| d\tau = 0\,.
$$

We send $\tau_1 \to -\infty$ and $\tau_2 \to T_\infty$ and obtain, for almost every $\tau \in (-\infty, T_\infty)$,

$$
\sum_{\{s\in\mathbb{R}\,|\,\kappa_{\infty,\tau_0}(s,\tau)=0\}} \left| \frac{\partial \kappa_{\infty,\tau_0}}{\partial s}(s,\tau) \right| = 0\,. \qquad (9.62)
$$

Thus, for almost every $\tau \in (-\infty, T_\infty)$, $\frac{\partial}{\partial s}\kappa_{\infty,\tau_0}(s,\tau) = 0$ whenever $\kappa_{\infty,\tau_0}(s,\tau) = 0$. Fix $\tau_3 \in (-\infty, T_\infty)$, where (9.62) holds. Since F_{∞,τ_0} satisfies (9.49) with $\kappa_{\infty,\tau_0} \not\equiv 0$, we can apply Corollary 5.7 to obtain that there exists no points $s \in \mathbb{R}$ with $\kappa_{\infty,\tau_0}(s,\tau) = 0$ for $\tau \in (\tau_3, T_\infty)$. By Lemma 9.15, strict positivity or negativity of the curvature is preserved under (9.49), so that $\kappa_{\infty,\tau_0} > 0$ or $\kappa_{\infty,\tau_0} < 0$ on $\Sigma_\tau^{\infty,\tau_0}$ for every $\tau \in (\tau_3, T_\infty)$. Now we can send $\tau_3 \to -\infty$ so that, for every time $\tau \in (-\infty, T_\infty)$, the curve $\Sigma_\tau^{\infty,\tau_0}$ is strictly convex or concave. □

9.4 Type-I singularities for the APCSF

In this section, we only consider the APCSF. We assume that a singularity develops in finite time and is of type I, that is, it satisfies condition (9.5). This setting has already been exploited in a similar fashion in [MB14, Chapter 7]. We refer also to [Whi97, Section 11] for a characterisation of singularities for almost Brakke flows with bounded global terms, using a monotonicity formula and a result of [Ilm95].

Proposition 9.17 (Structure equation for homothetically shrinking solutions of CSF, see [Eck04, p. 13] or [Man11, Proposition 1.4.1])**.** *Let $S \in \{\mathbb{S}^1, \mathbb{R}\}$. If an initial curve $F_0 : S \to \mathbb{R}^2$ satisfies*

$$\kappa(p) = \lambda \langle F_0(p) - x_0, \boldsymbol{\nu}_0(p)\rangle$$

at every point $p \in \Sigma$ for some constant $\lambda > 0$ and $x_0 \in \mathbb{R}^2$, then it generates a homothetically shrinking solution of CSF.

Proposition 9.18 (Shape of the limit flow for type-I singularities for the APCSF)**.** *Let $F : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ be a smooth, embedded solution of the APCSF with initial curve Σ_0 and $T < \infty$. For the type-I rescaling in case of a type-I singularity, each limit $F_{\infty,\tau_0} : \mathbb{R} \times (-\infty, 0) \to \mathbb{R}^2$, as obtained in Theorem 9.13, is a homothetically shrinking solution of CSF. Moreover, for all $\tau < 0$ and $R > 0$, there exists a constant $C = C(\Sigma_0, T, R, \tau)$ such that $\mathcal{H}^1(\Sigma_\tau^{\infty,\tau_0} \cap B_R(0)) \le C$.*

Proof. Let $x_0 \in \mathbb{R}^2$ be arbitrary. For $t \in [0,T)$, define the monotonicity quantity

$$\Theta_{(x_0,T)}(t) := f(t) \int_{\Sigma_t} \Phi_{(x_0,T)}(x,t)\, d\mathcal{H}^1 . \tag{9.63}$$

The monotonicity formula, Theorem 8.5, yields

$$\frac{d}{dt}\Theta_{(x_0,T)}(t) \le 0 \tag{9.64}$$

for $t \in (0,T)$. Hence, the monotonicity quantity is monotonically decreasing and strictly positive, so that the limit

$$\lim_{t\to T} \Theta_{(x_0,T)}(t)$$

exists and for any sequence $(t_k)_{k\in\mathbb{N}}$ with $t_k \nearrow T$ for $k \to \infty$,

$$\lim_{t\to T} \Theta_{(x_0,T)}(t) = \lim_{k\to\infty} \Theta_{(x_0,T)}(t_k) . \tag{9.65}$$

Since the global term h_{ap} is bounded by (3.10), we can apply Lemma 8.3 to conclude that

$$f \in [c, 1] \tag{9.66}$$

on $[0, T)$ with $c = c(\Sigma_0, T) > 0$ and

$$\lim_{k\to\infty} f(t_k) = \lim_{t\to T} f(t) =: c_0 \overset{(9.66)}{\in} [c, 1]\,. \tag{9.67}$$

Since the right-hand side of (9.67) is independent of the chosen sequence, the left-hand side is. For $k \in \mathbb{N}$, $y = \lambda_k(x - x_0) \in \mathbb{R}^2$ and $\tau = \lambda_k^2(t - T) \in [\alpha_k, 0)$, Lemma 8.2 rescales the backward heat kernel according to

$$\Phi_{(0,0)}(y, \tau) = \frac{1}{\lambda_k}\Phi_{(x_0+0/\lambda_k, T+0/\lambda_k^2)}(x, t) = \frac{1}{\lambda_k}\Phi_{(x_0,T)}(x, t)\,. \tag{9.68}$$

The auxiliary function is scaling invariant, that is,

$$\begin{aligned} f_k(\tau) &:= \exp\left(-\frac{1}{2}\int_{\alpha_k}^{\tau} h_k^2(\sigma)\, d\sigma\right) \\ &= \exp\left(-\frac{1}{2}\int_{0}^{T+\tau/\lambda_k^2} h^2(\rho)\, d\rho\right) = f\left(T + \frac{\tau}{\lambda_k^2}\right) \overset{(9.66)}{\in} [c, 1]\,, \end{aligned} \tag{9.69}$$

where we substituted $\sigma = T + \rho/\lambda_k^2$ and $d\sigma = \lambda_k^2\, d\rho$. Let $\tau \in (-\infty, 0)$ and $k_0 \in \mathbb{N}$ so that $\tau \in [\alpha_k, 0)$ for $k \geq k_0$. For the sequence

$$\left(t_k := T + \frac{\tau}{\lambda_k^2}\right)_{k\in\mathbb{N}}\,,$$

we obtain

$$c_0 \overset{(9.67)}{=} \lim_{k\to\infty} f(t_k) = \lim_{k\to\infty} f\left(T + \frac{\tau}{\lambda_k^2}\right) \overset{(9.69)}{=} \lim_{k\to\infty} f_k(\tau)$$

Since the left-hand side is independent of τ, the right-hand side is and

$$c_0 = \lim_{k\to\infty} f(t_k) = \lim_{k\to\infty} f_k(\tau) = f_\infty(\tau) \tag{9.70}$$

for any sequence $(t_k)_{k\in\mathbb{N}}$ and any $\tau \in (-\infty, 0)$. Let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence of positive real numbers with $\lambda_k \to \infty$ for $k \to \infty$. We rescale the flow according to the type-I rescaling (9.7) with respect to the sequence $(\lambda_k)_{k\in\mathbb{N}}$ and consider the rescaled curves Σ_τ^k for $\tau \in [\alpha_k, 0)$. We observe that we receive a factor of λ_k from the scaling behaviour (9.13) of the length element, and a factor of $1/\lambda_k$ from the scaling behaviour (9.68) of the backward heat kernel, and no scaling factor from the auxiliary function by the scaling invariance (9.69). Hence, the monotonicity quantity translates, for $t_k := T + \tau/\lambda_k^2$, by

$$\begin{aligned} \Theta_{(x_0,T)}(t_k) &\overset{(9.63)}{=} f(t_k)\int_{\Sigma_{t_k}} \Phi_{(x_0,T)}(x, t_k)\, d\mathcal{H}^1 \\ &= f_k(\tau)\int_{\Sigma_\tau^k} \Phi_{(0,0)}(y, \tau)\, d\mathcal{H}^1 =: \Theta_{(0,0)}^k(\tau)\,. \end{aligned} \tag{9.71}$$

Lemma 9.2 implies that there exists a singular point $x_0 \in \mathbb{R}^2$ and a blow-up sequence $(p_k, t_k)_{k\in\mathbb{N}}$ with

$$F(p_k, t_k) \to x_0 \qquad \text{and} \qquad |\kappa(p_k, t_k)| = |\kappa|_{\max}(t_k) \to \infty$$

for $k \to \infty$. Let $\tau_0 \in (-\infty, 0)$ and $k_0 \in \mathbb{N}$ so that $\tau_0 \in [\alpha_k, 0)$ for $k \geq k_0$. We rescale according to Definition 9.7 with respect to x_0 and $(p_k, t_k)_{k\in\mathbb{N}}$ and consider the rescaled embeddings $F_{k,\tau_0} : I_k \times [\alpha_k, 0) \to \mathbb{R}^2$ (see (9.22)). Let $0 \in I \subset \mathbb{R}$ be a bounded interval, $[\tau_1, \tau_2] \subset (-\infty, 0)$ and $k_1 \geq k_0$ so that $I \subset I_k$ and $[\tau_1, \tau_2] \subset [\alpha_k, 0)$ for all $k \geq k_1$. Since $\kappa_{k,\tau_0} = \kappa_k$ on $F_{k,\tau_0}(I, \tau)$,

$$F_{k,\tau_0}(I, \tau) \subset F_k(\mathbb{S}^1, \tau) = \Sigma^k_\tau \tag{9.72}$$

and the embeddings F_k suffice (2.15) (see (9.20)), we can apply the monotonicity formula 8.5 and estimate similar to [Bak10, Proposition 6.6] or [Coo11, Proposition 5.8],

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2} \frac{f_k}{2} \int_{F_{k,\tau_0}(I,\tau)} \left(\left\| (\kappa_{k,\tau_0} - h_k)\boldsymbol{\nu}_{k,\tau_0} + \frac{y^\perp}{-2\tau} \right\|^2 + \left\| \boldsymbol{\kappa}_{k,\tau_0} + \frac{y^\perp}{-2\tau} \right\|^2 \right) \Phi_{(0,0)}\, d\mathcal{H}^1 d\tau \\
&\overset{(9.72)}{\leq} \int_{\tau_1}^{\tau_2} \frac{f_k}{2} \int_{\Sigma^k_\tau} \left(\left\| (\kappa_k - h_k)\boldsymbol{\nu}_k + \frac{y^\perp}{-2\tau} \right\|^2 + \left\| \boldsymbol{\kappa}_k + \frac{y^\perp}{-2\tau} \right\|^2 \right) \Phi_{(0,0)}\, d\mathcal{H}^1 d\tau \\
&\overset{\text{Thm. 8.5}}{=} \Theta^k_{(0,0)}(\tau_1) - \Theta^k_{(0,0)}(\tau_2) \\
&\overset{(9.71)}{=} \Theta_{(x_0,T)}\left(T + \frac{\tau_1}{\lambda_k^2}\right) - \Theta_{(x_0,T)}\left(T + \frac{\tau_2}{\lambda_k^2}\right)
\end{aligned} \tag{9.73}$$

for all $k \geq k_1$. Since

$$T + \frac{\tau_i}{\lambda_k^2} \to T$$

for $k \to \infty$ and $i = 1, 2$, and by the existence of the limit (9.65), the right-hand side of (9.73) converges to 0 for $k \to \infty$. By Theorem 9.13, the sequence $(F_{k,\tau_0})_{k\in\mathbb{N}}$ converges smoothly on $I \times [\tau_1, \tau_2]$ to a smooth flow $F_{\infty,\tau_0} : \mathbb{R} \times (-\infty, 0) \to \mathbb{R}^2$. For $\tau \in [\tau_1, \tau_2]$, Lemma 9.12 and Fatou's lemma, Lemma B.9, imply

$$\begin{aligned}
&\liminf_{k\to\infty} \int_{F_{k,\tau_0}(I,\tau)} \left(\left\| (\kappa_{k,\tau_0} - h_k)\boldsymbol{\nu}_{k,\tau_0} + \frac{y^\perp}{-2\tau} \right\|^2 + \left\| \boldsymbol{\kappa}_{k,\tau_0} + \frac{y^\perp}{-2\tau} \right\|^2 \right) \Phi_{(0,0)}\, d\mathcal{H}^1 \\
&= \liminf_{k\to\infty} \int_I \left(\left\| (\kappa_{k,\tau_0} - h_k)\boldsymbol{\nu}_{k,\tau_0} + \frac{y^\perp}{-2\tau} \right\|^2 + \left\| \boldsymbol{\kappa}_{k,\tau_0} + \frac{y^\perp}{-2\tau} \right\|^2 \right) \Phi_{(0,0)} v_{k,\tau_0} dp \\
&\overset{\text{Lem. B.9}}{\geq} \int_I \liminf_{k\to\infty} \left[\left(\left\| (\kappa_{k,\tau_0} - h_k)\boldsymbol{\nu}_{k,\tau_0} + \frac{y^\perp}{-2\tau} \right\|^2 + \left\| \boldsymbol{\kappa}_{k,\tau_0} + \frac{y^\perp}{-2\tau} \right\|^2 \right) \Phi_{(0,0)} v_{k,\tau_0} \right] dp \\
&\overset{\substack{\text{Lem. 9.12} \\ \text{Thm. 9.13}}}{=} 2 \int_I \left\| \boldsymbol{\kappa}_{\infty,\tau_0} + \frac{y^\perp}{-2\tau} \right\|^2 \Phi_{(0,0)} v_{\infty,\tau_0}\, dp \\
&= 2 \int_{F_{\infty,\tau_0}(I,\tau)} \left\| \boldsymbol{\kappa}_{\infty,\tau_0} + \frac{y^\perp}{-2\tau} \right\|^2 \Phi_{(0,0)}\, d\mathcal{H}^1 \geq 0\,.
\end{aligned} \tag{9.74}$$

Thus, (9.70), (9.73) for $k \to \infty$ and (9.74) yield

$$0 \leq \int_{\tau_1}^{\tau_2} c_0 \int_{F_{\infty,\tau_0}(I,\tau)} \left\| \boldsymbol{\kappa}_{\infty,\tau_0} + \frac{y^\perp}{-2\tau} \right\|^2 \Phi_{(0,0)}\, d\mathcal{H}^1 d\tau \leq 0\,.$$

Since $I \subset \mathbb{R}$ was chosen arbitrarily, we deduce

$$\int_{\tau_1}^{\tau_2} \int_{\Sigma_\tau^{\infty,\tau_0}} \left\| \boldsymbol{\kappa}_{\infty,\tau_0} + \frac{y^\perp}{-2\tau} \right\|^2 \Phi_{(0,0)}\, d\mathcal{H}^1 d\tau = 0\,.$$

Hence, for almost every $\tau \in [\tau_1, \tau_2]$ and for almost every $y \in \Sigma_\tau^{\infty,\tau_0}$,

$$\left\| \boldsymbol{\kappa}_{\infty,\tau_0}(y,\tau) + \frac{y^\perp}{-2\tau} \right\|^2 = 0\,.$$

By Theorem 9.13, F_{∞,τ_0} is smooth in space and time, so the above equation holds for every $\tau \in [\tau_1, \tau_2]$ and every $y \in \Sigma_\tau^{\infty,\tau_0}$. By Proposition 9.17, the limit curve $\Sigma_\tau^{\infty,\tau_0}$ is a homothetically shrinking solution on the interval $(\tau_1, 0)$. Sending $\tau_1 \to -\infty$ yields the claim for all negative times.

For the area estimate, let again be $I \subset \mathbb{R}$ a bounded interval, $R > 0$ and $\tau \in (-\infty, 0)$. Then there exists again $k_1 \geq k_0$ so that $I \subset I_k$, $\tau \in [\alpha_k, 0)$ and

$$T - \frac{\tau}{\lambda_k^2} \geq \frac{T}{2} \tag{9.75}$$

for all $k \geq k_1$. By (9.64), the monotonicity quantity is decreasing in time and we can estimate with the definition (8.1) of the backward heat kernel and the behaviour of the length of the curve (see Corollary 3.5),

$$\begin{aligned}
\int_{F_{k,\tau_0}(I,\tau)} \Phi_{(0,0)}(y,\tau)\, d\mathcal{H}^1 &\overset{(9.72)}{\leq} \int_{\Sigma_\tau^k} \Phi_{(0,0)}(y,\tau)\, d\mathcal{H}^1 \\
&\underset{(9.71)}{\overset{(9.66),(9.69)}{\leq}} \frac{1}{c} \int_{\Sigma_{T-\tau/\lambda_k^2}} \Phi_{(x_0,T)}\left(x, T - \frac{\tau}{\lambda_k^2}\right) d\mathcal{H}^1 \\
&\overset{(9.64),(9.75)}{\leq} \frac{1}{c} \int_{\Sigma_{T/2}} \Phi_{(x_0,T)}\left(x, \frac{T}{2}\right) d\mathcal{H}^1 \\
&\overset{(8.1)}{=} \frac{1}{c\sqrt{4\pi(T-T/2)}} \int_{\Sigma_{T/2}} \exp\left(-\frac{\|x-x_0\|^2}{4(T-T/2)}\right) d\mathcal{H}^1 \\
&\leq \frac{L_{T/2}}{c\sqrt{2\pi T}} \overset{\text{Cor. 3.5}}{\leq} \frac{L_0}{c\sqrt{2\pi T}} =: C(\Sigma_0, T)\,.
\end{aligned} \tag{9.76}$$

Like before, Theorem 9.13 and Fatou's lemma, Lemma B.9, imply

$$\begin{aligned}
C &\overset{(9.76)}{\geq} \liminf_{k\to\infty} \int_{F_{k,\tau_0}(I,\tau)} \Phi_{(0,0)}\, d\mathcal{H}^1 = \liminf_{k\to\infty} \int_I \Phi_{(0,0)} v_{k,\tau_0}\, dp \\
&\overset{\text{Lem. B.9}}{\geq} \int_I \liminf_{k\to\infty} \left(\Phi_{(0,0)} v_{k,\tau_0}\right) dp \overset{\text{Thm. 9.13}}{=} \int_I \Phi_{(0,0)} v_{\infty,\tau_0}\, dp \\
&= \int_{F_{\infty,\tau_0}(I,\tau)} \Phi_{(0,0)}\, d\mathcal{H}^1
\end{aligned} \tag{9.77}$$

and the definition (8.1) of the backward heat kernel yields

$$\begin{aligned} C &\overset{(9.77)}{\geq} \int_{F_{\infty,\tau_0}(I,\tau)\cap B_R(0)} \Phi_{(0,0)}(y,\tau)\, d\mathcal{H}^1 \\ &\overset{(8.1)}{=} \frac{1}{\sqrt{-4\pi\tau}} \int_{F_{\infty,\tau_0}(I,\tau)\cap B_R(0)} \exp\left(-\frac{\|y\|^2}{-4\tau}\right) d\mathcal{H}^1 \\ &\geq \frac{1}{\sqrt{-4\pi\tau}} \int_{F_{\infty,\tau_0}(I,\tau)\cap B_R(0)} \exp\left(-\frac{R^2}{-4\tau}\right) d\mathcal{H}^1 \\ &= \frac{1}{\sqrt{-4\pi\tau}} \exp\left(-\frac{R^2}{-4\tau}\right) \mathcal{H}^1(F_{\infty,\tau_0}(I,\tau)\cap B_R(0)) \,. \end{aligned}$$

Since $I \subset \mathbb{R}$ was chosen arbitrarily,

$$\mathcal{H}^1(\Sigma_\tau^{\infty,\tau_0} \cap B_R(0)) \leq C\sqrt{-4\pi\tau}\exp\left(\frac{R^2}{-4\tau}\right) \leq C(\Sigma_0, T, R, \tau)$$

holds for all $\tau \in (-\infty, 0)$. □

Theorem 9.19 (Homothetically shrinking solutions of CSF, Abresch–Langer [AL86, Theorem A] see also [Hal12, Theorem 5.1] and [Man11, Proposition 3.4.1]). *Let $S \in \{\mathbb{R}/\mathbb{Z}, \mathbb{R}\}$ and let $F : S \to \mathbb{R}^2$ be a unit speed curve representing a homothetic solution of the curve shortening flow. If $S = \mathbb{R}/\mathbb{Z}$, then $F(\mathbb{R}/\mathbb{Z})$ is*

(i) an m-covered circle, or

(ii) a member of the family of Abresch–Langer curves.

If $S = \mathbb{R}$, then $F(\mathbb{R})$ is

(iii) a line $\mathbb{R} \times \{0\}$, or

(iv) a curve whose image is dense in an annulus of $\mathbb{R}^2$.

Theorem 9.20 (Nonexistence of type-I singularities for the APCSF). *Let $F : \mathbb{S}^1 \times (0,T) \to \mathbb{R}^2$ be a smooth, embedded solution of the APCSF with $T < \infty$. Then a type-I singularity cannot form at T.*

Proof. We follow the lines of [MB15, Proposition 4.12]. Assume, that a type-I singularity occurs at time T. Lemma 9.2 implies that there exists a singular point $x_0 \in \mathbb{R}^2$ and a corresponding blow-up sequence $(p_k, t_k)_{k\in\mathbb{N}}$. We rescale according to the type-I rescaling 9.7 with respect to x_0 and $(p_k, t_k)_{k\in\mathbb{N}}$. For $\tau_0 \in (-\infty, 0)$, Theorem 9.13 yields that the rescaled embedding $F_{k,\tau_0} : I_k \times J_k \to \mathbb{R}^2$ (see (9.22)) converge on compact subsets in the domain of definition and the ambient space to a curve shortening flow $F_{\infty,\tau_0} : \mathbb{R} \times (\infty, 0) \to \mathbb{R}^2$ with

1. $L(\Sigma_\tau^{\infty,\tau_0}) = \infty$ for all $\tau \in (-\infty, 0)$, and
2. $\kappa_{\infty,\tau_0} \not\equiv 0$ on $\mathbb{R} \times (-\infty, 0)$.

By Theorem 9.18, F_{∞,τ_0} is

3. a homothetically shrinking solution, and

4. $\mathcal{H}^1(\Sigma_\tau^{\infty,\tau_0} \cap B_R(0)) \le C(\Sigma_0, T, R, \tau)$ for all $\tau < 0$ and $R > 0$.

Property 3 implies that $\Sigma_\tau^{\infty,\tau_0}$, $\tau \in (-\infty, 0)$, is one of the four types of homothetically shrinking solutions listed in Theorem 9.19. But property 1 contradicts (i) and (ii), property 2 contradicts (iii), and property 4 contradicts (iv). Thus, the singularity could not have been of type I. □

9.5 Collapsed singularities

We say that a singularity is *collapsed*, if $\inf_{p,q\in\mathbb{R}}(d_{\infty,\tau_0}/l_{\infty,\tau_0})(p,q,\tau) = 0$ for every time $\tau \in (-\infty, T_\infty)$. In this section, we rule out collapsed singularities.

Theorem 9.21 (Non-existence of collapsed singularities)**.** *Let Σ_0 be a smooth, embedded curve satisfying $\theta_{\min} \ge -\pi$, and let $F : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ be a solution of* (2.15) *with initial curve Σ_0 and $T < \infty$. Then every limit flow $F_{\infty,\tau_0} : \mathbb{R} \times (-\infty, T_\infty) \to \mathbb{R}^2$, as obtained in Theorem 9.13, satisfies $\inf_{p,q\in\mathbb{R}}(d_{\infty,\tau_0}/l_{\infty,\tau_0})(p,q,\tau) \ge C^*$, where C^* is given in Theorem 7.21, at any time $\tau \in (-\infty, T_\infty)$. Hence, the singularity cannot be collapsed.*

Proof. We follow the idea of [Hui95, Theorem 2.4]. By Proposition 4.9, the curvature blows up for time approaching $T < \infty$. We rescale according to (9.11) and consider the rescaled embedding $F_k : \mathbb{S}^1 \times \mathcal{J}_k \to \mathbb{R}^2$. For $k \in \mathbb{N}$, $p, q \in \mathbb{S}^1$ and $\tau \in \mathcal{J}_k$, we obtain

$$\begin{aligned} d_k(p,q,\tau) &= \lambda_k d\left(p, q, \bar{t}_k + \frac{\tau}{\lambda_k^2}\right) \\ l_k(p,q,\tau) &= \lambda_k l\left(p, q, \bar{t}_k + \frac{\tau}{\lambda_k^2}\right) \\ L(\Sigma_\tau^k) &= \lambda_k L(\Sigma_{\bar{t}_k+\tau/\lambda_k^2}) \end{aligned}$$

and, by definition (7.1) of ψ,

$$\psi_k(p,q,\tau) = \lambda_k \psi\left(p, q, \bar{t}_k + \frac{\tau}{\lambda_k^2}\right) .$$

Hence, the ratio d/ψ is scaling invariant and Theorem 7.21 implies

$$\min_{\mathbb{S}^1\times\mathbb{S}^1\times\mathcal{J}_k} \frac{d_k}{\psi_k} \overset{(9.10)}{\ge} \min_{\mathbb{S}^1\times\mathbb{S}^1\times[0,T)} \frac{d}{\psi} \overset{\text{Thm. 7.21}}{\ge} C^* \tag{9.78}$$

for all $k \in \mathbb{N}$. Let $\tau_0 \in (-\infty, T_\infty)$ and $k_0 \in \mathbb{N}$ so that $\tau_0 \in \mathcal{J}_k$ for $k \ge k_0$. We consider the embeddings $F_{k,\tau_0} : I_k \times \mathcal{J}_k \to \mathbb{R}^2$ (see (9.22)). Since

$$F_{k,\tau_0}(I_k, \tau) \overset{(9.22)}{=} F_k\big(s_k^{-1}(I_k, \tau_0), \tau\big) \subset F_k\big(\mathbb{S}^1, \tau\big) = \Sigma_\tau^k$$

for all $\tau \in \mathcal{J}_k$ and $k \ge k_0$, we obtain

$$\min_{I_k\times I_k\times\mathcal{J}_k} \frac{d_{k,\tau_0}}{\psi_{k,\tau_0}} \ge \min_{\mathbb{S}^1\times\mathbb{S}^1\times\mathcal{J}_k} \frac{d_k}{\psi_k} \overset{(9.78)}{\ge} C^* \tag{9.79}$$

for all $k \geq k_0$, where

$$\psi_{k,\tau_0}(p,q,t) \overset{(7.1)}{=} \frac{L(\Sigma_\tau^k)}{\pi} \sin\left(\frac{\pi l_{k,\tau_0}(p,q,t)}{L(\Sigma_\tau^k)}\right). \tag{9.80}$$

Let $[p,q] \subset \mathbb{R}$ and $\tau \in (-\infty, T_\infty)$. Then there exists $k_1 \geq k_0$ so that $[p,q] \subset I_k$ and $\tau \in \mathcal{J}_k$ for all $k \geq k_1$. For $k_1 \leq k \leq \infty$, $F_{k,\tau_0}(\,\cdot\,,\tau_0)$ is an arc length parametrisation (see also (9.23) and Theorem 9.13), that is,

$$\begin{aligned}
l_{k,\tau_0}(p,q,\tau_0) &\overset{(9.22)}{=} l_k\big(s_k^{-1}(p,\tau_0), s_k^{-1}(q,\tau_0),\tau_0\big) = \int_{s_k^{-1}(p,\tau_0)}^{s_k^{-1}(q,\tau_0)} v_k(r,\tau_0)\,dr \\
&= \int_0^{s_k^{-1}(q,\tau_0)} v_k(r,\tau_0)\,dr - \int_0^{s_k^{-1}(p,\tau_0)} v_k(r,\tau_0)\,dr \\
&\overset{(A.2)}{=} s_k\big(s_k^{-1}(q,\tau_0),\tau_0\big) - s_k\big(s_k^{-1}(p,\tau_0),\tau_0\big) = q - p\,.
\end{aligned}$$

Lemma 9.12 implies

$$l_{k,\tau_0}(p,q,\tau) \leq c_2(\Sigma_0,\tau_0,\tau)|q-p|$$

for $k \geq k_1$ and

$$\frac{\pi l_{k,\tau_0}(p,q,\tau)}{L(\Sigma_\tau^k)} \to 0$$

for $k \to \infty$. Since $\sin(x)/x \to 1$ for $x \to 0$,

$$\frac{\psi_{k,\tau_0}}{l_{k,\tau_0}}(p,q,\tau) \overset{(9.80)}{=} \frac{L(\Sigma_\tau^k)}{\pi l_{k,\tau_0}(p,q,\tau)} \sin\left(\frac{\pi l_{k,\tau_0}(p,q,\tau)}{L(\Sigma_\tau^k)}\right) \to 1 \tag{9.81}$$

for $k \to \infty$. By Theorem 9.13, the embeddings $(F_{k,\tau_0})_{k \geq k_2}$ converge uniformly and smoothly along a subsequence on $[p,q] \times \{\tau\}$ to a smooth flow F_{∞,τ_0}. In particular,

$$l_{k,\tau_0}(p,q,\tau) \to l_{\infty,\tau_0}(p,q,\tau) \qquad \text{and} \qquad d_{k,\tau_0}(p,q,\tau) \to d_{\infty,\tau_0}(p,q,\tau) \tag{9.82}$$

for $k \to \infty$. Hence

$$C^* \overset{(9.79)}{\leq} \frac{d_{k,\tau_0}}{\psi_{k,\tau_0}}(p,q,\tau) = \frac{d_{k,\tau_0}}{l_{k,\tau_0}}(p,q,\tau)\frac{l_{k,\tau_0}}{\psi_{k,\tau_0}}(p,q,\tau) \overset{(9.82),(9.81)}{\longrightarrow} \frac{d_{\infty,\tau_0}}{l_{\infty,\tau_0}}(p,q,\tau)\,.$$

for $k \to \infty$. Since $p, q \in \mathbb{R}$ and $\tau \in (-\infty, T_\infty)$ were chosen arbitrarily, the claim follows. □

9.6 Type-II singularities for the APCSF

In this section, we only consider the APCSF. We assume that a singularity develops in finite time and that it is of type II, that is, it satisfies property (9.6). We know from Section 9.2 that the limit flow of the rescaling is an eternal solution of CSF (that is, it exists for all $\tau \in \mathbb{R}$) which curvature is never zero. Hence, we can proceed as in [Hui95, Theorem 2.4].

Theorem 9.22 (Hamilton [Ham95b, Main Theorem B])**.** *Any strictly convex eternal solution of MCF where the mean curvature assumes its maximum value at a fixed point in space time, must be a translating solution of MCF.*

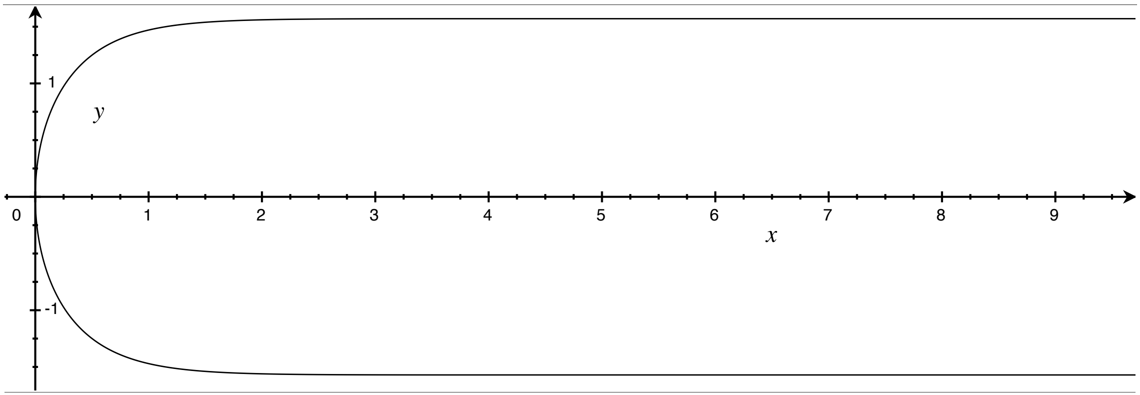

Figure 9.1: The rotated grim reaper $x = -\log(\cos(y))$ for $y \in (-\pi/2, \pi/2)$.

Proposition 9.23 (Structure equation for translating solution of CSF, see [Man11, Proposition 1.4.2])**.** *If $F : \mathbb{R} \times (0,T) \to \mathbb{R}^2$ is a translating solution of CSF, then there exists a vector $\mathbf{v} \in \mathbb{R}^2$ such that*

$$\kappa(p,\tau) = \langle \mathbf{v}, \boldsymbol{\nu}(p,\tau) \rangle \tag{9.83}$$

for every point $p \in \mathbb{R}$ and every $\tau \in (0,T)$.

The following name was introduced by Calabi (see also [Gra87, p. 298]).

Definition 9.24 (Grim reaper)**.** We call the graph of the function $u(\,\cdot\,,\tau)\colon(-\pi/2,\pi/2) \to \mathbb{R}$, where $\tau \in \mathbb{R}$ and

$$u(\sigma,\tau) := \tau - \log(\cos(\sigma))$$

the *grim reaper* (see Figure 9.1 for an illustration).

Lemma 9.25 (Characterisation of translating solutions of the CSF, see [Man11, p. 15])**.** *A curve $\Sigma = F(\mathbb{R})$, parametrised by arc length, that satisfies $\kappa(s) = \langle \mathbf{v}, \boldsymbol{\nu}(s) \rangle$ for some vector $\mathbf{v} \in \mathbb{R}^2$ and for all $s \in \mathbb{R}$, is the grim reaper for a fixed $\tau \in \mathbb{R}$.*

Remark 9.26. The above statement has also been proved in [Alt91, Proof of Theorem 8.16].

Proposition 9.27 (Shape of the limit flow for type-II singularities of the APCSF)**.** *Let $F : \mathbb{S}^1 \times (0,T) \to \mathbb{R}^2$ be a smooth, embedded solution of the APCSF with $T < \infty$. For the type-II rescaling in case of a type-II singularity, each limit $F_{\infty,\tau_0} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$, as obtained in Theorem 9.13, is the grim reaper up to rotation.*

Proof. Theorem 9.13 yields that the limit flow satisfies

$$\sup_{\mathbb{R}\times\mathbb{R}} |\kappa_{\infty,\tau_0}| = |\kappa_{\infty,\tau_0}(0,0)| = 1\,.$$

By Proposition 9.16 it consists of strictly convex or concave curves $\Sigma_\tau^{\infty,\tau_0}$ for $\tau \in \mathbb{R}$. If $\kappa_{\infty,\tau_0} < 0$, we change the direction of parametrisation so that $\kappa_{\infty,\tau_0} > 0$. Since the curvature attains its maximum at the point $(0,0) \in \mathbb{R}\times\mathbb{R}$, Theorem 9.22 yields that F_{∞,τ_0} is a translating solution. By Proposition 9.23, $\Sigma_\tau^{\infty,\tau_0}$ satisfies the structure equation (9.83) for each $\tau \in \mathbb{R}$. Lemma 9.25 implies that $\Sigma_\tau^{\infty,\tau_0}$ is the grim reaper for every $\tau \in \mathbb{R}$. $\square$

Corollary 9.28 (Non-existence of type-II singularities for the APCSF)**.** *Let Σ_0 be a smooth, embedded curve satisfying $\theta_{\min} \geq -\pi$, and let $F : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ be a solution of the APCSF with initial curve Σ_0 and $T < \infty$. Then a type-II singularity cannot form at T.*

Proof. By Proposition 9.27, any blow-up limit curve $\Sigma_\tau^{\infty,\tau_0}$, $\tau \in \mathbb{R}$, is given (up to rotation) by the grim reaper. The grim reaper is asymptotic to two parallel lines of distance π. Let $\tau \in \mathbb{R}$. We can find a sequence of points $(p_j, q_j)_{j \in \mathbb{N}}$ in $\mathbb{R} \times \mathbb{R}$ with

$$d_{\infty,\tau_0}(p_j, q_j, \tau) \leq \pi$$

for all $j \in \mathbb{N}$ and

$$l_{\infty,\tau_0}(p_j, q_j, \tau) \to \infty$$

for $j \to \infty$. Hence,

$$\inf_{\mathbb{R} \times \mathbb{R}} \frac{d_{\infty,\tau_0}}{l_{\infty,\tau_0}}(\,\cdot\,, \,\cdot\,, \tau) = 0\,.$$

Theorem 9.21 excludes these kinds of singularities. □

9.7 $T = \infty$ for the APCSF

Theorem 9.29. *Let Σ_0 be a smooth embedded, closed curve satisfying $\theta_{\min} \geq -\pi$. Then there exists a unique embedded solution $F : \mathbb{S}^1 \times [0, \infty) \to \mathbb{R}^2$ to the APCSF with initial curve Σ_0 and $F \in C^\infty(\mathbb{S}^1 \times (0, \infty))$.*

Proof. By the short time existence, Theorem 2.3, there exists a unique solution $F \in C^\infty(\mathbb{S}^1 \times (0, T))$ to the initial value problem (2.15) with $T \leq \infty$. By the bound (3.10) for h_{ap} and Lemma 3.13, the curves stays in a bounded region on $[0, T)$. Theorem 6.5 implies that the curves remain embedded on $(0, T)$. Assume that $T < \infty$. By Theorem 9.20 and Corollary 9.28 neither a type-I nor a type-II singularity can form in the interval $[0, T)$ so that curvature stays bounded on $[0, T]$ by a constant $C(\Sigma_0, T)$. Like in the proof of Proposition 4.9, we can extend the flow beyond T and repeat the above argument. Hence, for every time $T' < \infty$, there exists a constant $C(\Sigma_0, T') < \infty$ so that

$$|\kappa|_{\max}(t) \leq C$$

for all $t \in [0, T')$. Applying again Proposition 4.9 yields that the short time solution can be extended to a smooth solution on $(0, \infty)$. □

Chapter 10

Convexity in finite time

In this chapter, we show that a smooth, embedded solution $F : \mathbb{S}^1 \times (0, \infty) \to \mathbb{R}^2$ of (2.15) becomes convex in finite time. Like in [MB15, Section 7], we use the following Gagliardo–Nirenberg interpolation inequality.

Theorem 10.1 (Gagliardo–Nirenberg interpolation inequality, [Nir59, pp. 125], see also [Aub98, Theorem. 3.70])**.** *Let $f \in C^\infty(\mathbb{S}^1)$. Let $q, r \in \mathbb{R}$ with $1 \leq q, r \leq \infty$ and $j, m \in \mathbb{N} \cup \{0\}$ with $0 \leq j < m$. Let $\mu > 0$ and $\sigma \in [j/m, 1]$ with*

$$\frac{1}{p} := j + \sigma \left(\frac{1}{r} - m \right) + \frac{(1-\sigma)}{q} > 0$$

so that p is non-negative. Then there exist constants $c_1 = c_1(m, j, p, q, r, \sigma)$ and $c_2 = c_2(m, j, p, q, r, \sigma, \mu)$ such that

$$\left(\int_{\mathbb{S}^1} \left| \frac{d^j f}{dx^j} \right|^p dx \right)^{1/p} \leq c_1 \left(\int_{\mathbb{S}^1} \left| \frac{d^m f}{dx^m} \right|^r dx \right)^{\sigma/r} \left(\int_{\mathbb{S}^1} |f|^q \, dx \right)^{(1-\sigma)/q} + c_2 \left(\int_{\mathbb{S}^1} |f|^\mu \, dx \right)^{1/\mu} . \tag{10.1}$$

If $r = 1/(m-j) \neq 1$, then (10.1) is not valid for $\sigma = 1$. If $\int_{\mathbb{S}^1} f \, dx = 0$, then the last integral term in (10.1) can be omitted.

Corollary 10.2. *Let $f \in C^\infty(\mathbb{S}^1)$. Let $p > 2$ and $\sigma \in [0, 1)$ with*

$$\sigma = \frac{1}{2} - \frac{1}{p} .$$

Then there exist constants $c_1 = c_1(p, \sigma)$ and $c_2 = c_2(p, \sigma)$ such that

$$\left(\int_{\mathbb{S}^1} |f|^p \, dx \right)^{1/p} \leq c_1 \left(\int_{\mathbb{S}^1} \left(\frac{df}{dx} \right)^2 dx \right)^{\sigma/2} \left(\int_{\mathbb{S}^1} f^2 \, dx \right)^{(1-\sigma)/2} + c_2 \left(\int_{\mathbb{S}^1} f^2 \, dx \right)^{1/2} . \tag{10.2}$$

If $\int_{\mathbb{S}^1} f \, dx = 0$, then the last integral term in (10.2) can be omitted.

Proof. We chose $j = 0$, $m = 1$ and $q = 2$ in Theorem 10.1. □

Lemma 10.3. *Let $f \in C^\infty((0,\infty)) \cap L^1((0,\infty))$ with $f \geq 0$ and $\frac{d}{dt} f \leq c(1+f)^3$ for $c \geq 0$. Then $f(t) \to 0$ for $t \to \infty$.*

Proof. We follow the lines of the proof of [MB15, Corollary 7.5]. Assume that there exists $\delta > 0$ and a sequence $(t_k)_{k\in\mathbb{N}}$ with $t_k \to \infty$ for $k \to \infty$ and

$$f(t_k) \geq \delta \tag{10.3}$$

for all $k \in \mathbb{N}$. The assumption on f yields

$$-\frac{d}{dt}(1+f)^{-2} = 2(1+f)^{-3}\frac{df}{dt} \leq c\,.$$

Fix $k \in \mathbb{N}$. We integrate from $t \in (0, t_k)$ to t_k and obtain

$$-(1+f(t_k))^{-2} + (1+f(t))^{-2} \leq c(t_k - t)\,.$$

The assumption (10.3) implies

$$(1+f(t))^{-2} \leq c(t_k - t) + (1+\delta)^{-2}$$

so that

$$(1+f(t))^2 \geq \frac{1}{c(t_k - t) + (1+\delta)^{-2}} = \frac{(1+\delta)^2}{c(t_k - t)(1+\delta)^2 + 1}$$

for $t \in (0, t_k)$. Choose $\varepsilon > 0$ so that

$$c\varepsilon \leq \frac{\delta}{(1+\delta)^2(2+\delta)}\,.$$

Then

$$2\big(\delta - c\varepsilon(1+\delta)^2\big) = 2\delta - 2c\varepsilon(1+\delta)^2 \geq \delta + \delta c\varepsilon(1+\delta)^2 = \delta\big(1 + c\varepsilon(1+\delta)^2\big)$$

and, for all $t \in (t_k - \varepsilon, t_k)$,

$$f(t) > \frac{1+\delta}{(c\varepsilon(1+\delta)^2+1)^{1/2}} - 1 \geq \frac{1+\delta}{c\varepsilon(1+\delta)^2+1} - 1 = \frac{\delta - c\varepsilon(1+\delta)^2}{c\varepsilon(1+\delta)^2+1} \geq \frac{\delta}{2}\,. \tag{10.4}$$

Since δ and ε are independent of k, inequality (10.4) holds on $(t_k - \varepsilon, t_k)$ for every $k \in \mathbb{N}$, so that

$$\int_0^\infty f\,dt = \infty\,.$$

This contradicts $f \in L^1((0,\infty))$. □

Lemma 10.4. *Let $F : \mathbb{S}^1 \times [0,\infty) \to \mathbb{R}^2$ be a smooth, embedded solution of the LPCF with initial curve Σ_0. Then*

$$\int_{\mathbb{S}^1_{R_t}} (h_{\mathrm{lp}} - \kappa)\,ds_t \to 0 \qquad \textit{and} \qquad h_{\mathrm{lp}} \to \frac{2\pi}{L_0}$$

for $t \to \infty$, and there exists constants $c = c(\Sigma_0)$ such that

$$\sup_{[0,\infty)} \frac{dh}{dt} \leq c \qquad \textit{and} \qquad \int_0^\infty \int_{\mathbb{S}^1_{R_t}} (h - \kappa)\,ds_t\,dt \leq c\,.$$

Proof. By Remark 3.7, $h_{\mathrm{lp}} = \int_{\mathbb{S}^1_{R_0}} \kappa^2 \, ds_t/(2\pi)$ and by Cauchy–Schwarz (B.3),

$$\frac{dA}{dt} \overset{(3.9)}{=} \int_{\mathbb{S}^1_{R_0}} (h-\kappa)\, ds_t \overset{(B.3)}{\geq} 0$$

for $t \in (0,\infty)$, with equality only on the circle. We integrate from $\varepsilon > 0$ to $\tau < \infty$ and use the isoperimetric inequality, Lemma 3.4,

$$\frac{L_0^2}{4\pi} \overset{\text{Cor. 3.6}}{\equiv} \frac{L_\tau^2}{4\pi} \overset{\text{Lem. 3.4}}{\geq} A_\tau > A_\tau - A_\varepsilon = \int_\varepsilon^\tau \int_{\mathbb{S}^1_{R_0}} (h-\kappa)\, ds_t\, dt \geq 0\,.$$

Sending $\varepsilon \to 0$ and $\tau \to \infty$ yields

$$\frac{L_0^2}{4\pi} \geq \int_0^\infty \int_{\mathbb{S}^1_{R_0}} (h-\kappa)\; ds_t\, dt \geq 0\,. \tag{10.5}$$

We deduce with Corollary 10.2 for $p = 4$ and $\sigma = 1/4$ and Young's inequality (B.1) for $p = 4/3$ and $q = 4$ as well as for $p = q = 2$,

$$\begin{aligned}
\int_{\mathbb{S}^1_{R_0}} \kappa^4\, ds_t &\overset{\text{Cor. 10.2}}{\leq} \left[c \left(\int_{\mathbb{S}^1_{R_0}} \left(\frac{\partial \kappa}{\partial s}\right)^2 ds_t \right)^{1/8} \left(\int_{\mathbb{S}^1_{R_0}} \kappa^2\, ds_t \right)^{3/8} + c \left(\int_{\mathbb{S}^1_{R_0}} \kappa^2\, ds_t \right)^{1/2} \right]^4 \\
&\overset{(B.1)}{\leq} \frac{1}{2\pi} \int_{\mathbb{S}^1_{R_0}} \left(\frac{\partial \kappa}{\partial s}\right)^2 ds_t + c \left(\int_{\mathbb{S}^1_{R_0}} \kappa^2\, ds_t \right)^3 + c \left(\int_{\mathbb{S}^1_{R_0}} \kappa^2\, ds_t \right)^2
\end{aligned} \tag{10.6}$$

for a constant $c > 0$. Again, by Corollary 10.2 for $p = 3$ and $\sigma = 1/6$,

$$\int_{\mathbb{S}^1_{R_0}} \kappa^3\, ds_t \leq \left[c \left(\int_{\mathbb{S}^1_{R_0}} \left(\frac{\partial \kappa}{\partial p}\right)^2 ds_t \right)^{1/12} \left(\int_{\mathbb{S}^1_{R_0}} \kappa^2\, ds_t \right)^{5/12} + c \left(\int_{\mathbb{S}^1_{R_0}} \kappa^2\, ds_t \right)^{1/2} \right]^3 \tag{10.7}$$

so that with Young's inequality (B.1) for $p = 3/2$ and $q = 3$ as well as for $p = 4$ and $q = 4/3$,

$$\begin{aligned}
&\int_{\mathbb{S}^1_{R_0}} \kappa^3\, ds_t \int_{\mathbb{S}^1_{R_0}} \kappa^2\, ds_t \\
&\overset{(10.7),(B.1)}{\leq} c \left(\int_{\mathbb{S}^1_{R_0}} \left(\frac{\partial \kappa}{\partial s}\right)^2 ds_t \right)^{1/4} \left(\int_{\mathbb{S}^1_{R_0}} \kappa^2\, ds_t \right)^{9/4} + c \left(\int_{\mathbb{S}^1_{R_0}} \kappa^2\, ds_t \right)^{5/2} \\
&\overset{(B.1)}{\leq} \frac{1}{2\pi} \int_{\mathbb{S}^1_{R_0}} \left(\frac{\partial \kappa}{\partial s}\right)^2 ds_t + c \left(\int_{\mathbb{S}^1_{R_0}} \kappa^2\, ds_t \right)^3 + c \left(\int_{\mathbb{S}^1_{R_0}} \kappa^2\, ds_t \right)^{5/2}.
\end{aligned} \tag{10.8}$$

We use the evolution equations (3.2) and (4.1) for the length element and for the curvature, Remark 3.7 and integration by parts (B.4) to calculate

$$\begin{aligned}
\frac{dh}{dt} &\overset{\text{Rem. 3.7}}{=} \frac{1}{2\pi}\frac{d}{dt}\left(\int_{\mathbb{S}^1_{R_0}} \kappa^2\, ds_t\right) = \frac{1}{2\pi}\left(\int_{\mathbb{S}^1_{R_0}} \frac{\partial \kappa^2}{\partial t}\, ds_t + \int_{\mathbb{S}^1} \kappa^2 \frac{\partial v}{\partial t} dp\right) \\
&\overset{(3.2),(4.1)}{=} \frac{1}{\pi}\int_{\mathbb{S}^1_{R_0}} \left(\kappa \frac{\partial^2 \kappa}{\partial s^2} - (h-\kappa)\kappa^3\right) ds_t + \frac{1}{2\pi}\int_{\mathbb{S}^1} \kappa^3 (h-\kappa)\, v\, dp \\
&\overset{(B.4)}{=} -\frac{1}{\pi}\int_{\mathbb{S}^1_{R_0}} \left(\frac{\partial \kappa}{\partial s}\right)^2 ds_t - \frac{1}{4\pi^2}\int_{\mathbb{S}^1_{R_0}} \kappa^3\, ds_t \int_{\mathbb{S}^1_{R_0}} \kappa^2\, ds_t + \frac{1}{2\pi}\int_{\mathbb{S}^1_{R_0}} \kappa^4\, ds_t \\
&\overset{(10.6),(10.8)}{\leq} c\left(\int_{\mathbb{S}^1_{R_0}} \kappa^2\, ds_t\right)^3 + c\left(\int_{\mathbb{S}^1_{R_0}} \kappa^2\, ds_t\right)^{5/2} + c\left(\int_{\mathbb{S}^1_{R_0}} \kappa^2\, ds_t\right)^2 \\
&\leq c(h^3 + h^{5/2} + h^2)
\end{aligned} \tag{10.9}$$

for all $t \in (0,\infty)$. With Young's inequality, we estimate

$$h^\alpha \leq h^3 + 1 \leq (h+1)^3$$

for $\alpha \in \{2, 5/2\}$ and conclude

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{S}^1_{R_0}} (h-\kappa)\, ds_t = L_0 \frac{dh}{dt} &\overset{(10.9)}{\leq} c(h+1)^3 = \frac{c}{L_0^3}\left(L_0 h - 2\pi + 2\pi + L_0\right)^3 \\
&\leq c\left(\int_{\mathbb{S}^1_{R_0}} (h-\kappa)\, ds_t + 1\right)^3
\end{aligned}$$

for all $t \in (0,\infty)$. With (10.5) we can apply Lemma 10.3 to the non-negative function

$$f(t) := \int_{\mathbb{S}^1_{R_0}} (h-\kappa)\, ds_t$$

to obtain

$$\int_{\mathbb{S}^1_{R_0}} (h-\kappa)\, ds_t \to 0 \qquad \text{and therefore} \qquad h \to \frac{2\pi}{L_0}$$

for $t \to \infty$. Together with (10.9) this yields that $\sup_{[0,\infty)} \frac{d}{dt} h$ is bounded from above. □

Lemma 10.5. *Let $F : \mathbb{S}^1 \times (0,\infty) \to \mathbb{R}^2$ be a smooth, embedded solution of* (2.15)*. Then there exists a constant $c > 0$ such that*

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2\, ds_t \leq &-\int_{\mathbb{S}^1_{R_t}} \left(\frac{\partial \kappa}{\partial s}\right)^2 ds_t + 2\Lambda \frac{dh}{dt}\int_{\mathbb{S}^1_{R_t}} (h-\kappa)\, ds_t \\
&+ c\left(\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2\, ds_t\right)^3 + c\left(\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2\, ds_t\right)^{5/3} \\
&+ \Lambda c\left(\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2\, ds_t\right)^2 + \Lambda c\left(\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2\, ds_t\right)^{3/2}
\end{aligned}$$

for all $t \in (0,\infty)$, where $\Lambda = 0$ for the APCSF and $\Lambda = 1$ for the LPCF.

Proof. We follow the lines of [MB15, Lemmata 7.3 and 7.4]. Write

$$\kappa = h - (h-\kappa)\,.$$

Then

$$(h-\kappa)^3\kappa = h(h-\kappa)^3 - (h-\kappa)^4 \tag{10.10}$$

and

$$\begin{aligned}(h-\kappa)^2\kappa^2 &= (h-\kappa)^2\left(h^2 - 2h(h-\kappa) + (h-\kappa)^2\right)\\ &= h^2(h-\kappa)^2 - 2h(h-\kappa)^3 + (h-\kappa)^4\,.\end{aligned} \tag{10.11}$$

The evolution equations (3.2) and (4.1) for the length element and for the curvature, and integration by parts (B.4) yield

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2\,ds_t\\
&\overset{(3.2)}{=} \int_{\mathbb{S}^1_{R_t}} (h-\kappa)^3\kappa\,ds_t + 2\int_{\mathbb{S}^1_{R_t}} (h-\kappa)\left(\frac{dh}{dt} - \frac{\partial\kappa}{\partial t}\right) ds_t\\
&\overset{(4.1)}{=} \int_{\mathbb{S}^1_{R_t}} (h-\kappa)^3\kappa\,ds_t + 2\frac{dh}{dt}\int_{\mathbb{S}^1_{R_t}} (h-\kappa)\,ds_t + 2\int_{\mathbb{S}^1_{R_t}} (h-\kappa)\left(-\frac{\partial^2\kappa}{\partial s^2} + (h-\kappa)\kappa^2\right) ds_t\\
&\overset{(B.4)}{=} \int_{\mathbb{S}^1_{R_t}} (h-\kappa)^3\kappa\,ds_t + 2\frac{dh}{dt}\int_{\mathbb{S}^1_{R_t}} (h-\kappa)\,ds_t - 2\int_{\mathbb{S}^1_{R_t}} \left(\frac{\partial\kappa}{\partial s}\right)^2 ds_t + 2\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2\kappa^2\,ds_t\\
&\underset{(10.11)}{\overset{(10.10)}{=}} h\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^3\,ds_t - \int_{\mathbb{S}^1_{R_t}} (h-\kappa)^4\,ds_t\\
&\qquad + 2\frac{dh}{dt}\int_{\mathbb{S}^1_{R_t}} (h-\kappa)\,ds_t - 2\int_{\mathbb{S}^1_{R_t}} \left(\frac{\partial\kappa}{\partial s}\right)^2 ds_t\\
&\qquad + 2h^2\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2\,ds_t - 4h\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^3\,ds_t + 2\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^4\,ds_t\\
&= -2\int_{\mathbb{S}^1_{R_t}} \left(\frac{\partial\kappa}{\partial s}\right)^2 ds_t + 2\frac{dh}{dt}\int_{\mathbb{S}^1_{R_t}} (h-\kappa)\,ds_t + \int_{\mathbb{S}^1_{R_t}} (h-\kappa)^4\,ds_t\\
&\qquad - 3h\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^3\,ds_t + 2h^2\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2\,ds_t\,.
\end{aligned} \tag{10.12}$$

For the APCSF, by definition of the global term,

$$\int_{\mathbb{S}^1_{R_t}} (h-\kappa)\,ds_t = L_t h - \int_{\mathbb{S}^1_{R_t}} \kappa\,ds_t = L_t\frac{2\pi}{L_t} - 2\pi = 0\,. \tag{10.13}$$

Like in [MB15, Corollary 7.4], we use Corollary 10.2 with $p = 4$ and $\sigma = 1/4$ and Young's inequality (B.1) with $p = 4/3$ and $q = 4$ as well as for $p = q = 2$, to estimate

$$\begin{aligned}
&\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^4 \, ds_t \\
&\overset{\text{Cor. 10.2}}{\underset{\text{(B.1)}}{\leq}} \left[c \left(\int_{\mathbb{S}^1_{R_t}} \left(\frac{\partial \kappa}{\partial s} \right)^2 ds_t \right)^{1/8} \left(\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t \right)^{3/8} + \Lambda c \left(\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t \right)^{1/2} \right]^4 \\
&\overset{\text{(B.1)}}{\leq} \frac{1}{2} \int_{\mathbb{S}^1_{R_t}} \left(\frac{\partial \kappa}{\partial s} \right)^2 ds_t + c \left(\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t \right)^3 + \Lambda c \left(\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t \right)^2 , \qquad (10.14)
\end{aligned}$$

where we defined

$$\Lambda := \begin{cases} 0 & \text{for the APCSF} \\ 1 & \text{for the LPCF} \end{cases}$$

in view of (10.13) and the last sentence in Corollary 10.2. Again by Corollary 10.2 with $p = 3$ and $\sigma = 1/6$ and Young's inequality (B.1) for $p = 3/2$ und $q = 3$ as well as for $p = 4$ and $q = 4/3$, we obtain

$$\begin{aligned}
&\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^3 \, ds_t \\
&\overset{\text{Cor. 10.2}}{\underset{\text{(B.1)}}{\leq}} \left[c \left(\int_{\mathbb{S}^1_{R_t}} \left(\frac{\partial \kappa}{\partial s} \right)^2 ds_t \right)^{1/12} \left(\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t \right)^{5/12} + \Lambda c \left(\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t \right)^{1/2} \right]^3 \\
&\overset{\text{(B.1)}}{\leq} \frac{1}{2} \int_{\mathbb{S}^1_{R_t}} \left(\frac{\partial \kappa}{\partial s} \right)^2 ds_t + c \left(\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t \right)^{5/3} + \Lambda c \left(\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t \right)^{3/2} . \qquad (10.15)
\end{aligned}$$

Altogether, with the bounds on the global terms (see Lemma 10.4 for the LPCF),

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t \\
&\overset{(10.12),(10.13)}{\underset{(10.14),(10.15)}{\leq}} - \int_{\mathbb{S}^1_{R_t}} \left(\frac{\partial \kappa}{\partial s} \right)^2 ds_t + 2\Lambda \frac{dh}{dt} \int_{\mathbb{S}^1_{R_t}} (h-\kappa) \, ds_t \\
&\qquad + c \left(\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t \right)^3 + c \left(\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t \right)^{5/3} \\
&\qquad + \Lambda c \left(\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t \right)^2 + \Lambda c \left(\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t \right)^{3/2} . \qquad \square
\end{aligned}$$

Lemma 10.6. *Let $F : \mathbb{S}^1 \times [0, \infty) \to \mathbb{R}^2$ be a smooth, embedded solution of (2.15) with initial curve Σ_0. Then*

$$\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t \to 0$$

for $t \to \infty$ and

$$\int_0^\infty \int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t \, dt < \infty \, .$$

Proof. We treat both flows separately.

For the **APCSF**, $h = 2\pi/L_t$ so that

$$\int_{\mathbb{S}^1_{R_t}} h\kappa \, ds_t = \frac{2\pi}{L_t} \int_{\mathbb{S}^1_{R_t}} \kappa \, ds_t = \frac{L_t (2\pi)^2}{L_t^2} = \int_{\mathbb{S}^1_{R_t}} h^2 \, ds_t \, .$$

Like in [Hui87, p. 47], the evolution equation (3.8) for the length of the curve yields

$$\begin{aligned} \frac{dL}{dt} &\overset{(3.8)}{=} \int_{\mathbb{S}^1_{R_t}} \left(h\kappa - \kappa^2\right) \, ds_t = \int_{\mathbb{S}^1_{R_t}} \left(-h\kappa + 2h\kappa - \kappa^2\right) \, ds_t \\ &= \int_{\mathbb{S}^1_{R_t}} \left(-h^2 + 2h\kappa - \kappa^2\right) \, ds_t = - \int_{\mathbb{S}^1_{R_t}} \left(h - \kappa\right)^2 \, ds_t \end{aligned}$$

for all $t \in (0, \infty)$. We integrate from $\varepsilon > 0$ to $\tau < \infty$ and deduce that

$$L_0 \overset{\text{Cor. } 3.5}{\geq} L_\varepsilon \geq L_\varepsilon - L_\tau = \int_\varepsilon^\tau \int_{\mathbb{S}^1_{R_t}} \left(h - \kappa\right)^2 \, ds_t \, dt \geq 0 \, .$$

Send $\varepsilon \to 0$ and $\tau \to \infty$ yields

$$L_0 \geq \int_0^\infty \int_{\mathbb{S}^1_{R_t}} \left(h - \kappa\right)^2 \, ds_t \, dt \geq 0 \, . \tag{10.16}$$

Lemma 10.5 and Young's inequality (B.1) imply

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t &\overset{\text{Lem. } 10.5}{\leq} c \left(\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t \right)^3 + c \left(\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t \right)^{5/3} \\ &\overset{(B.1)}{\leq} c \left(1 + \int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t \right)^3 , \end{aligned}$$

where $c = c(\Sigma_0)$. Like in [MB15, Corollary 7.5], we define

$$f(t) := \int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t \geq 0$$

so that, with the bound (10.16), Lemma 10.3 yields the claim for the APCSF.

For the **LPCF**, we calculate with Remark 3.7,

$$\begin{aligned} \int_{\mathbb{S}^1_{R_0}} (h-\kappa)^2 \, ds_t &= \int_{\mathbb{S}^1_{R_0}} (h^2 - 2h\kappa + \kappa^2) \, ds_t = L_0 h^2 - 4\pi h + 2\pi h \\ &= L_0 h^2 - 2\pi h = h(L_0 h - 2\pi) \\ &= h \int_{\mathbb{S}^1_{R_0}} (h - \kappa) \, ds_t \overset{(3.9)}{=} h \frac{dA_t}{dt} \end{aligned} \tag{10.17}$$

for all $t \in (0, \infty)$. Lemma 10.4 yields

$$\int_{\mathbb{S}^1_{R_0}} (h-\kappa)^2 \, ds_t \overset{(10.17)}{=} L_0 h^2 - 2\pi h \overset{\text{Lem. } 10.4}{\longrightarrow} \frac{2\pi^2}{L_0} - \frac{2\pi^2}{L_0} = 0$$

for $t \to \infty$. Since

$$0 \overset{(3.11)}{<} h \overset{\text{Lem. 10.4}}{<} \infty\,, \qquad \frac{dA_t}{dt} \overset{\text{Cor. 3.6}}{\geq} 0 \qquad \text{and} \qquad A_0 \leq A_t \overset{\text{Cor. 3.6}}{\leq} \frac{L_0^2}{4\pi}$$

for all $t \in (0, \infty)$, we also obtain

$$\int_0^\infty \int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t \, dt \overset{(10.17)}{=} \int_0^\infty h \frac{dA_t}{dt} \, dt \leq \sup_{[0,\infty)} h \left[\frac{L_0}{2\pi} - A_0 \right] < \infty\,. \qquad \square$$

Lemma 10.7. *Let* $F : \mathbb{S}^1 \times [0,\infty) \to \mathbb{R}^2$ *be a smooth, embedded solution of* (2.15) *with initial curve* Σ_0. *Then*

$$\int_0^\infty \int_{\mathbb{S}^1_{R_t}} \left(\frac{\partial \kappa}{\partial s} \right)^2 ds_t \, dt < \infty\,.$$

Proof. For the **APCSF**,

$$\int_{\mathbb{S}^1_{R_t}} (h - \kappa) \, ds_t = 0 \tag{10.18}$$

for all $t \in [0, \infty)$. For the **LPCF**, by Cauchy–Schwarz,

$$\int_{\mathbb{S}^1_{R_t}} (h - \kappa) \, ds_t \geq 0$$

so that Lemma 10.4 yields

$$\int_0^\infty \frac{dh}{dt} \int_{\mathbb{S}^1_{R_t}} (h-\kappa) \, ds_t \, dt \leq \sup_{[0,\infty)} \frac{dh}{dt} \int_0^\infty \int_{\mathbb{S}^1_{R_t}} (h-\kappa) \, ds_t \, dt \overset{\text{Lem. 10.4}}{\leq} c \tag{10.19}$$

where $c = c(\Sigma_0)$. For both flows, Lemma 10.6 implies that there exists a time $t_0 \geq 0$ so that

$$\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t < 1$$

for all $t > t_0$, and thus

$$\left(\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t \right)^p \leq \int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t \tag{10.20}$$

for all $p \geq 1$. From Lemmata 10.5 and 10.6, we obtain

$$\begin{aligned}
&\int_{t_0}^\infty \int_{\mathbb{S}^1_{R_t}} \left(\frac{\partial \kappa}{\partial s} \right)^2 ds_t \, dt \\
&\overset{\text{Lem. 10.5}}{\leq} - \int_{t_0}^\infty \frac{d}{dt} \int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t \, dt + 2\Lambda \int_{t_0}^\infty \frac{dh}{dt} \int_{\mathbb{S}^1_{R_t}} (h-\kappa) \, ds_t \, dt \\
&\qquad + c \int_{t_0}^\infty \left(\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t \right)^3 dt + c \int_{t_0}^\infty \left(\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t \right)^{5/3} dt \\
&\qquad + \Lambda c \int_{t_0}^\infty \left(\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t \right)^{3/2} dt + \Lambda c \int_{t_0}^\infty \left(\int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t \right)^2 dt \\
&\underset{(10.19),(10.20)}{\overset{\text{Lem. 10.6},(10.18)}{\leq}} - \int_{\mathbb{S}^1_{R_{t_0}}} (h-\kappa)^2 \, ds_t \, dt + c\Lambda + c \int_{t_0}^\infty \int_{\mathbb{S}^1_{R_t}} (h-\kappa)^2 \, ds_t \, dt \overset{\text{Lem. 10.6}}{<} \infty\,.
\end{aligned}$$

Since Σ_t is smooth for $t \in [0, t_0]$,

$$\int_0^{t_0} \int_{\mathbb{S}^1_{R_t}} \left(\frac{\partial \kappa}{\partial s}\right)^2 ds_t \, dt < \infty$$

as well. □

For the next result, we need the following theorem.

Theorem 10.8 (See [GT83, Theorem 7.26(ii)]). *Let $\Omega \subset \mathbb{R}^n$ be open. For*

$$0 \leq m < k - \frac{n}{p} < m + 1 \qquad \text{and} \qquad \alpha = k - \frac{n}{p} - m$$

the space $W^{k,p}(\Omega)$ is continuously embedded in $C^{m,\alpha}(\bar{\Omega})$, and compactly embedded in $C^{m,\beta}(\bar{\Omega})$ for any $\beta < \alpha$.

Theorem 10.9 (Convexity in finite time). *Let $F : \mathbb{S}^1 \times [0, \infty) \to \mathbb{R}^2$ be a smooth, embedded solution of (2.15) with initial curve Σ_0. Then there exists a time $T_0 \geq 0$ such that Σ_t is strictly convex for $t > T_0$.*

Proof. Lemma 10.7 implies that there exists a sequence $(t_k)_{k \in \mathbb{N}}$ with $t_k \to \infty$ for $k \to \infty$ so that

$$\int_{\mathbb{S}^1_{R_{t_k}}} \left(\frac{\partial \kappa}{\partial s}\right)^2 ds_{t_k} \to 0 \tag{10.21}$$

for $k \to \infty$. Hence, there exists $k_0 \in \mathbb{N}$ so that for all $k \geq k_0$

$$\int_{\mathbb{S}^1_{R_{t_k}}} \left(\frac{\partial \kappa}{\partial s}\right)^2 ds_{t_k} < 1 \,. \tag{10.22}$$

We employ Theorem 10.8 for $n = 1$, $m = 0$, $p = 2$ and $\beta = 0$ to obtain that $W^{1,2}(\mathbb{S}^1)$ is compactly embedded in $C^0(\mathbb{S}^1)$. Furthermore, $C^0(\mathbb{S}^1) \subset L^2(\mathbb{S}^1)$, and $\|f\|_{L^2(\mathbb{S}^1)} \leq \sqrt{2\pi}\|f\|_{C^0(\mathbb{S}^1)}$ for every $f \in C^0(\mathbb{S}^1)$. Hence, $C^0(\mathbb{S}^1)$ is continuously embedded in $L^2(\mathbb{S}^1)$ and

$$W^{1,2}(\mathbb{S}^1) \underset{\text{compact}}{\hookrightarrow} C^0(\mathbb{S}^1) \underset{\text{continuous}}{\hookrightarrow} L^2(\mathbb{S}^1) \,.$$

Let $f \in W^{1,2}(\mathbb{S}^1)$. By Ehrling's lemma, Theorem B.15, for all $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ so that

$$\|f\|_{C^0(\mathbb{S}^1)} \leq \varepsilon \|f\|_{W^{1,2}(\mathbb{S}^1)} + C(\varepsilon)\|f\|_{L^2(\mathbb{S}^1)} \,. \tag{10.23}$$

Lemma 10.6 and (10.21) yield

$$h(t_k) - \kappa(\cdot, t_k) \in W^{1,2}\big(\mathbb{S}^1_{R_{t_k}}\big)$$

for each $k \in \mathbb{N}$. Hence, we can use (10.23) to estimate

$$\begin{aligned}
&\max_{s\in\mathbb{S}^1_{R_{t_k}}} \left|h(t_k) - \kappa(s,t_k)\right| \\
&\overset{(10.23)}{\leq} \varepsilon \left(\int_{\mathbb{S}^1_{R_{t_k}}} \left(\frac{\partial\kappa}{\partial s}\right)^2 ds_{t_k} \right)^{1/2} + \varepsilon \left(\int_{\mathbb{S}^1_{R_{t_k}}} (h-\kappa)^2 \, ds_{t_k} \right)^{1/2} \\
&\qquad + C(\varepsilon) \left(\int_{\mathbb{S}^1_{R_{t_k}}} (h-\kappa)^2 \, ds_{t_k} \right)^{1/2} \\
&\overset{(10.22)}{\leq} \varepsilon + C(\varepsilon) \left(\int_{\mathbb{S}^1_{R_{t_k}}} (h-\kappa)^2 \, ds_{t_k} \right)^{1/2}
\end{aligned} \tag{10.24}$$

for all $k \geq k_0$. Choose

$$\varepsilon = \varepsilon(\Sigma_0) = \frac{\pi}{2L_0}$$

to deduce

$$\max_{s\in\mathbb{S}^1_{R_{t_k}}} \left|h(t_k) - \kappa(s,t_k)\right| \overset{(10.24)}{\leq} \frac{\pi}{2L_0} + C(\Sigma_0) \left(\int_{\mathbb{S}^1_{R_{t_k}}} (h-\kappa)^2 \, ds_{t_k} \right)^{1/2} . \tag{10.25}$$

Lemma 10.6 implies that there exists $k_1 \geq k_0$ so that for all $k \geq k_1$

$$\int_{\mathbb{S}^1_{R_{t_k}}} (h-\kappa)^2 \, ds_{t_k} < \left(\frac{\pi}{2C(\Sigma_0)L_0} \right)^2 .$$

Thus,

$$\max_{s\in\mathbb{S}^1_{R_{t_{k_1}}}} \left|h(t_{k_1}) - \kappa(s,t_{k_1})\right| \overset{(10.25)}{\leq} \frac{\pi}{L_0}$$

Since $h \geq 2\pi/L_0 > 0$ (see (3.10) and (3.11)), we conclude that $\kappa > 0$ on $\mathbb{S}^1$ at t_{k_1}. From Corollary 4.3 it follows that $\kappa > 0$ for all $t > t_{k_1}$. Hence, the claim holds for $T_0 = t_{k_1}$. □

Chapter 11

Longtime behaviour

In the previous chapters we proved that if Σ_0 is a smooth, embedded, closed curve satisfying $\theta_{\min} \geq -\pi$, then unique short time solution of the APCSF can be extended to a smooth embedded solution $(\Sigma_t)_{t\in[0,\infty)}$ which becomes convex at finite time $T_0 < \infty$. For the LPCF, we showed that an immortal solution $(\Sigma_t)_{t\in[0,\infty)}$ becomes convex at finite time $T_0 < \infty$. In this chapter we derive global higher derivative estimates for the curvature of convex curves and show that convex solutions $(\Sigma_t)_{t\in[T_0,\infty)}$ converge exponentially and smoothly in time to a round circle. This was already shown in [Gag86] for the APCSF and in [Pih98] for the LPCF. We repeat and extend the arguments here for the sake of completeness. We mostly follow the lines of [GH86, Section 5] for rescaled convex CSF, [Gag86] for convex APCSF, and [Pih98, Chapter 7] for convex LPCF. Note that Pihan considered the speed $1 - h_{\mathrm{lp}}^{-1}\kappa$ (compare Remark 2.2(i)), so his calculations are somewhat different.

11.1 Uniform C^0-convergence

In this section, we show that convex initial curves evolving under (2.15) converge to a circle in C^0.

Lemma 11.1 (Isoperimetric inequality, [Gag83]). *For a closed, convex C^2-curve in the plane,*

$$\int_{\mathbb{S}^1_R} \kappa^2\, ds \geq \frac{\pi L}{A}$$

with equality if and only if the curve is a circle.

Lemma 11.2 ([Gag86, Corollary 2.4] and [Pih98, Lemma 7.7]). *Let Σ_{T_0} be a smooth, embedded, convex curve of area $A_{T_0} = \pi R_0^2$ for the APCSF and length $L_{T_0} = 2\pi R_0$ for the LPCF. Let $F : \mathbb{S}^1 \times [T_0, \infty) \to \mathbb{R}^2$ be a solution of* (2.15) *with initial curve Σ_{T_0}. Then there exists a constant $D = D(\Sigma_{T_0}) > 0$, such that*

$$\left(\frac{L^2}{A} - 4\pi\right) \leq D \exp\left(-\frac{2t}{R_0^2}\right)$$

for all $t \geq T_0$.

Proof. For the **APCSF**, we follow the lines of [Gag86, Corollary 2.4] and use the evolution equation (3.8) for the length of the curve, the fact that the area is constant in time (see Corollary 3.5), $h_{\text{ap}} = 2\pi/L$, and the isoperimetric inequality, Lemma 11.1, to estimate

$$\begin{aligned}\frac{d}{dt}\left(\frac{L^2}{A} - 4\pi\right) &= \frac{2L}{A}\frac{dL}{dt} \overset{(3.8)}{=} \frac{2L}{A}\left(\frac{(2\pi)^2}{L} - \int_{\Sigma_t} \kappa^2 \, d\mathcal{H}^1\right) \\ &\overset{\text{Lem. 11.1}}{\leq} \frac{2L}{A}\left(\frac{4\pi^2}{L} - \frac{\pi L}{A}\right) = \frac{2\pi}{A}\left(4\pi - \frac{L^2}{A}\right)\end{aligned}$$

for $\tau > T_0$. By Lemma B.1,

$$\left(\frac{L_t^2}{A_t} - 4\pi\right) \leq \exp\left(-\frac{2\pi}{A_{T_0}} t\right)\left(\frac{L_{T_0}^2}{A_{T_0}} - 4\pi\right) = D(\Sigma_{T_0}) \exp\left(-\frac{2t}{R_0^2}\right)$$

where we used $A_{T_0} \equiv A_t$ for all $t \geq T_0$.

For the **LPCF**, we follow the lines of [Pih98, Lemma 7.7] and employ the evolution equation (3.9) for the area, the fact that the length is constant in time (see Corollary 3.6), $h_{\text{lp}} = \int_{\Sigma_t} \kappa^2 \, ds_t/(2\pi)$, and the isoperimetric inequalities, Lemmata 3.4 and 11.1, to estimate

$$\begin{aligned}\frac{d}{dt}\left(\frac{L^2}{A} - 4\pi\right) &= -\frac{L^2}{A^2}\frac{dA}{dt} \overset{(3.9)}{=} -\frac{L^2}{A^2}\left(\frac{L}{2\pi}\int_{\Sigma_t} \kappa^2 \, d\mathcal{H}^1 - 2\pi\right) \\ &\overset{\text{Lem. 11.1}}{\leq} -\frac{L^2}{2A^2}\left(\frac{L}{\pi}\frac{\pi L}{A} - 4\pi\right) \overset{\text{Lem. 3.4}}{\leq} -\frac{4\pi^2}{L^2}\left(\frac{L^2}{A} - 4\pi\right)\end{aligned}$$

for $\tau > T_0$. Again by Lemma B.1,

$$\left(\frac{L_t^2}{A_t} - 4\pi\right) \leq \exp\left(-\frac{4\pi^2}{L_{T_0}^2} t\right)\left(\frac{L_{T_0}^2}{A_{T_0}} - 4\pi\right) = D(\Sigma_{T_0}) \exp\left(-\frac{2t}{R_0^2}\right)$$

where we used $L_{T_0} \equiv L_t$ for all $t \geq T_0$. □

Proposition 11.3 (Bonnesen isoperimetric inequality, [Oss79, Theorem 4 (21)])**.** *For an embedded, closed curve Σ in the plane,*

$$\frac{L^2}{A} - 4\pi \geq \frac{\pi^2}{A}(r_{\text{circ}} - r_{\text{in}})^2 \geq 0\,,$$

where r_{circ} and r_{in} are the circumscribed and inscribed radius of Σ.

Corollary 11.4 ([Pih98, Corollary7.8])**.** *Let Σ_{T_0} be a smooth, embedded, convex curve of area $A_{T_0} = \pi R_0^2$ for the APCSF and length $L_{T_0} = 2\pi R_0$ for the LPCF. Let $F : \mathbb{S}^1 \times [T_0, \infty) \to \mathbb{R}^2$ be a solution of (2.15) with initial curve Σ_{T_0}. Then there exists a constant $D = D(\Sigma_{T_0}) > 0$ such that*

$$\frac{\pi^2}{A_t}(r_{\text{circ}}(t) - r_{\text{in}}(t))^2 \leq D \exp\left(-\frac{2t}{R_0^2}\right)$$

for all $t \geq T_0$.

Proof. The claim follows directly from Lemma 11.2 and Proposition 11.3. □

Proposition 11.5 (C^0-convergence of convex curves, [Pih98, Proposition 7.9])**.** *Let Σ_{T_0} be a smooth, embedded, convex curve of area $A_{T_0} = \pi R_0^2$ for the APCSF and length $L_{T_0} = 2\pi R_0$ for the LPCF. Let $F : \mathbb{S}^1 \times [T_0, \infty) \to \mathbb{R}^2$ be a solution of* (2.15) *with initial curve Σ_{T_0}. Then $\Sigma_t = F(\mathbb{S}^1, t)$ converges in C^0 to a circle of radius R_0 for $t \to \infty$.*

Proof. We can then apply Corollary 11.4 for $t \geq T_0$ to conclude that the inscribed and circumscribed radius converge towards each other, that is,

$$r_{\text{circ}}(t) - r_{\text{in}}(t) \to 0 \tag{11.1}$$

for $t \to \infty$. For embedded, closed, convex curves, we can estimate

$$r_{\text{in}} = \sqrt{\frac{A(B_{r_{\text{in}}})}{\pi}} \leq \sqrt{\frac{A}{\pi}}\,,$$

where $A(B_{r_{\text{in}}})$ is the area of the inscribed ball, and

$$\frac{L}{2\pi} \leq \frac{L(B_{r_{\text{circ}}})}{2\pi} = r_{\text{circ}}\,,$$

where $L(B_{r_{\text{circ}}})$ is the length of the circumscribed circle. The above two inequalities and the isoperimetric inequality, Lemma 3.4, imply

$$r_{\text{in}}(t) \leq \sqrt{\frac{A_t}{\pi}} \overset{\text{Lem. 3.4}}{\leq} \frac{L_t}{2\pi} \leq r_{\text{circ}}(t)$$

for all $t \geq T_0$ and the convergence (11.1) yields

$$\frac{L_t}{2\pi} - \sqrt{\frac{A_t}{\pi}} \to 0$$

for $t \to \infty$. For the **APCSF**, the enclosed area is constant so that

$$\frac{L_t}{2\pi} \to \sqrt{\frac{A_0}{\pi}} = R_0\,,$$

for the **LPCF**, the length is constant so that

$$\sqrt{\frac{A_t}{\pi}} \to \frac{L_0}{2\pi} = R_0$$

for $t \to \infty$. For a circle of radius R,

$$R = \frac{L}{2\pi} = \sqrt{\frac{A}{\pi}}\,.$$

Hence, Σ_t converges to a circle of radius R_0 for $t \to \infty$. □

11.2 Uniform C^2-convergence

In this section, we again assume that Σ_{T_0} is convex. By Corollary 4.3, Σ_t is strictly convex for all $t > T_0$. Like introduced in Chapter 5, let $\vartheta : \mathbb{S}^1 \times [T_0, \infty) \to [0, 2\pi)$ be the angle between the x_1-axis and the tangential vector at the point $F(p,t)$. Since Σ_t is strictly convex on (T_0, ∞), $\vartheta(\,\cdot\,, t)$ is injective for each $t \in (T_0, \infty)$. We want to use ϑ as spatial coordinate and define τ to be a new time variable so that $\tau = t$ as well as

$$\frac{d\tau}{dt} = 1 \qquad \text{and} \qquad \frac{\partial \vartheta}{\partial \tau} = 0\,. \tag{11.2}$$

For a C^1-function $f : \mathbb{S}^1 \to \mathbb{R}$, we then have,

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial \vartheta}\frac{\partial \vartheta}{\partial s} \overset{\text{Lem. 5.1}}{=} \frac{\partial f}{\partial \vartheta}\kappa\,.$$

Hence, the spatial derivatives transforms according to

$$\frac{1}{v}\frac{\partial}{\partial p} = \frac{\partial}{\partial s} = \kappa \frac{\partial}{\partial \vartheta}\,. \tag{11.3}$$

In the following sections, we use the coordinates (ϑ, τ) on $\mathbb{S}^1 \times (T_0, \infty)$.

Lemma 11.6 ([GH86, Lemma 4.1.3] and [Pih98, Lemma 6.12]). *Let $F : \mathbb{S}^1 \times (T_0, \infty) \to \mathbb{R}^2$ be a smooth, strictly convex solution of* (2.15). *Then*

$$\frac{\partial \kappa}{\partial \tau} = \kappa^2 \frac{\partial^2 \kappa}{\partial \vartheta^2} - (h - \kappa)\kappa^2\,. \tag{11.4}$$

Proof. The proof can be found in [GH86, Lemma 4.1.3] for CSF and [Pih98, Lemma 6.12]. For $\kappa = \kappa(\vartheta, \tau)$, by the evolution equation (5.2) of the angle and the transformation (11.3) of the derivatives,

$$\frac{\partial \vartheta}{\partial t} \overset{(5.2)}{=} \frac{\partial \kappa}{\partial s} \overset{(11.3)}{=} \kappa \frac{\partial \kappa}{\partial \vartheta}$$

so that, by (11.2),

$$\frac{\partial \kappa}{\partial t} = \frac{\partial \kappa}{\partial \vartheta}\frac{\partial \vartheta}{\partial t} + \frac{\partial \kappa}{\partial \tau}\frac{\partial \tau}{\partial t} \overset{(11.2)}{=} \kappa \left(\frac{\partial \kappa}{\partial \vartheta}\right)^2 + \frac{\partial \kappa}{\partial \tau}\,.$$

Furthermore,

$$\frac{\partial^2 \kappa}{\partial s^2} \overset{(11.3)}{=} \kappa \frac{\partial}{\partial \vartheta}\left(\kappa \frac{\partial \kappa}{\partial \vartheta}\right) = \kappa \left(\frac{\partial \kappa}{\partial \vartheta}\right)^2 + \kappa^2 \frac{\partial^2 \kappa}{\partial \vartheta^2}\,.$$

Subtracting the last two equations and using the evolution equation (4.1) of the curvature with respect to t yields

$$\frac{\partial \kappa}{\partial \tau} - \kappa^2 \frac{\partial^2 \kappa}{\partial \vartheta^2} = \frac{\partial \kappa}{\partial t} - \frac{\partial^2 \kappa}{\partial s^2} \overset{(4.1)}{=} -(h - \kappa)\kappa^2\,. \qquad \square$$

We define, for $\tau \geq T_0$,

$$m(\tau) := \begin{cases} \max\limits_{\sigma \in [T_0, \tau]} |\kappa|_{\max}(\sigma) & \text{for the APCSF} \\ 1 & \text{for the LPCF.} \end{cases} \tag{11.5}$$

Lemma 11.7 ([Gag86, Lemma 3.4 and Corollary 3.5] and [Pih98, Lemma 6.9])**.** *Let Σ_{T_0} be a smooth, embedded, convex curve. Let $F : \mathbb{S}^1 \times [T_0, \infty) \to \mathbb{R}^2$ be a solution of* (2.15) *with initial curve Σ_{T_0}. Then there exists a constant $D_0 = D_0(\Sigma_{T_0}) > 0$ such that*

$$\int_{\mathbb{S}^1} \left(\frac{\partial \kappa}{\partial \vartheta}\right)^2 d\vartheta \leq \int_{\mathbb{S}^1} \kappa^2 \, d\vartheta + D_0 m$$

for all $\tau > T_0$, where m is defined in (11.5)*.*

Proof. For the **APCSF**, we follow [Gag86, Lemma 3.4 and Corollary 3.5]. By Corollary 3.5, the length of the curve is decreasing so that $\frac{\partial}{\partial \tau} L \leq 0$ and we observe

$$0 < \int_{\mathbb{S}^1_{R_t}} \kappa^2 \, ds_t = \int_{\mathbb{S}^1} \kappa \, d\vartheta \leq 2\pi \kappa_{\max} \,. \tag{11.6}$$

We use the evolution equation (11.4) of the curvature, the time-independency (11.2) of ϑ, $h_{\mathrm{ap}} = 2\pi/L$, and integration by parts (B.4) to estimate

$$\begin{aligned}
&\frac{d}{d\tau} \int_{\mathbb{S}^1} \left(\kappa^2 - \left(\frac{\partial \kappa}{\partial \vartheta}\right)^2 - 2h\kappa\right) d\vartheta \\
&\overset{(11.2)}{=} \int_{\mathbb{S}^1} \left(2\kappa \frac{\partial \kappa}{\partial \tau} - 2 \frac{\partial \kappa}{\partial \vartheta} \frac{\partial}{\partial \tau} \frac{\partial \kappa}{\partial \vartheta} - 2h \frac{\partial \kappa}{\partial \tau}\right) d\vartheta + \frac{2\pi}{L^2} \frac{\partial L}{\partial \tau} \int_{\mathbb{S}^1_{R_t}} \kappa^2 \, ds_t \\
&\overset{(B.4)}{=} \int_{\mathbb{S}^1} 2 \left(\kappa + \frac{\partial^2 \kappa}{\partial^2 \vartheta} - h\right) \frac{\partial \kappa}{\partial \tau} \, d\vartheta - \frac{2\pi}{L^2} \left|\frac{\partial L}{\partial \tau}\right| \int_{\mathbb{S}^1_{R_t}} \kappa^2 \, ds_t \\
&\overset{(11.4)(11.6)}{\geq} \int_{\mathbb{S}^1} 2\kappa^2 \left(\kappa + \frac{\partial^2 \kappa}{\partial^2 \vartheta} - h\right)^2 d\vartheta - \frac{4\pi^2 \kappa_{\max}}{L^2} \left|\frac{\partial L}{\partial \tau}\right| \\
&\geq \frac{4\pi^2 \kappa_{\max}}{L^2} \frac{\partial L}{\partial \tau}
\end{aligned} \tag{11.7}$$

for all $\tau > T_0$. For $\tau_0, \tau \in (T_0, \infty)$, $\tau_0 < \tau$, with the definition (11.5) of m, $\frac{\partial}{\partial \tau} L \leq 0$ and $L_\tau \geq \sqrt{A_{T_0}/(4\pi)}$ (see Corollary 3.5),

$$\begin{aligned}
\int_{\tau_0}^{\tau} \frac{4\pi^2 \kappa_{\max}}{L^2} \frac{\partial L}{\partial \bar\tau} \, d\bar\tau &\geq 4\pi^2 \max_{[T_0, \tau]} \kappa_{\max} \int_{T_0}^{\tau} \frac{1}{L^2} \frac{\partial L}{\partial \bar\tau} \, d\tau \overset{(11.5)}{=} -4\pi^2 m(\tau) \left(\frac{1}{L_\tau} - \frac{1}{L_{T_0}}\right) \\
&\overset{\text{Cor. } 3.5}{\geq} -4\pi^2 m(\tau) \left(\sqrt{\frac{4\pi}{A_{T_0}}} - \frac{1}{L_{T_0}}\right) = -c(\Sigma_{T_0}) m(\tau) \,.
\end{aligned} \tag{11.8}$$

We integrate (11.7) from τ_0 to τ and use (11.8) to obtain

$$\begin{aligned}
&\int_{\mathbb{S}^1} \left(\kappa^2(\vartheta, \tau) - \left(\frac{\partial \kappa}{\partial \vartheta}(\vartheta, \tau)\right)^2 - 2h(\tau)\kappa(\vartheta, \tau)\right) d\vartheta \\
&\geq \int_{\mathbb{S}^1} \left(\kappa^2(\vartheta, \tau_0) - \left(\frac{\partial \kappa}{\partial \vartheta}(\vartheta, \tau_0)\right)^2 - 2h(\tau_0)\kappa(\vartheta, \tau_0)\right) d\vartheta - c(\Sigma_{T_0}) m(\tau) \\
&\geq -c(\Sigma_{T_0}) m(\tau) \,.
\end{aligned}$$

Rearranging terms yields

$$\int_{\mathbb{S}^1}\left(\frac{\partial\kappa}{\partial\vartheta}\right)^2 d\vartheta \le \int_{\mathbb{S}^1}\kappa^2\,d\vartheta - 2h\int_{\mathbb{S}^1}\kappa\,d\vartheta + c(\Sigma_{T_0})m$$
$$\overset{(11.6),(11.6)}{\le} \int_{\mathbb{S}^1}\kappa^2\,d\vartheta + c(\Sigma_{T_0})m\,.$$

For the **LPCF**, we follow [Pih98, Lemma 6.9]. We use the evolution equation (11.4) of the curvature, $h_{\mathrm{lp}} = \int_{\mathbb{S}^1}\kappa\,d\vartheta/(2\pi)$, and integration by parts (B.4) to calculate

$$\begin{aligned}
&\frac{d}{d\tau}\left[\int_{\mathbb{S}^1}\left(\kappa^2 - \left(\frac{\partial\kappa}{\partial\vartheta}\right)^2\right)d\vartheta - h^2\right]\\
&\overset{(11.2)}{=} \int_{\mathbb{S}^1}\left(2\kappa\frac{\partial\kappa}{\partial\tau} - 2\frac{\partial\kappa}{\partial\vartheta}\frac{\partial}{\partial\tau}\frac{\partial\kappa}{\partial\vartheta}\right)d\vartheta - 2h\int_{\mathbb{S}^1}\frac{\partial\kappa}{\partial\tau}\,d\vartheta\\
&\overset{(B.4)}{=} \int_{\mathbb{S}^1} 2\left(\kappa + \frac{\partial^2\kappa}{\partial^2\vartheta} - h\right)\frac{\partial\kappa}{\partial\tau}\,d\vartheta\\
&\overset{(11.4)}{=} \int_{\mathbb{S}^1} 2\kappa^2\left(\kappa + \frac{\partial^2\kappa}{\partial^2\vartheta} - h\right)^2 d\vartheta \ge 0
\end{aligned} \tag{11.9}$$

for all $\tau > T_0$. For $\tau_0, \tau \in (T_0, \infty)$, $\tau_0 < \tau$, we integrate (11.9) from τ_0 to τ and conclude with the convergence of the global term from Lemma 10.4,

$$\begin{aligned}
&\int_{\mathbb{S}^1}\left[\kappa^2(\vartheta,\tau) - \left(\frac{\partial\kappa}{\partial\vartheta}(\vartheta,\tau)\right)^2\right]d\vartheta - 2\pi h^2(\tau)\\
&\ge \int_{\mathbb{S}^1}\left[\kappa^2(\vartheta,\tau_0) - \left(\frac{\partial\kappa}{\partial\vartheta}(\vartheta,\tau_0)\right)^2\right]d\vartheta - 2\pi h^2(\tau_0) \overset{\text{Lem. 10.4}}{\ge} -c(\Sigma_{T_0})\,.
\end{aligned}$$

Hence,

$$\int_{\mathbb{S}^1}\left(\frac{\partial\kappa}{\partial\vartheta}\right)^2 d\vartheta \le \int_{\mathbb{S}^1}\kappa^2\,d\vartheta + c(\Sigma_{T_0})$$

where we estimated $-h^2(\tau) \le 0$. □

Let Σ_{T_0} be a smooth curve of length L_{T_0}. For $\tau \ge T_0$, define

$$\begin{aligned}
&D^*(\tau) = D^*(\Sigma_{T_0},\tau)\\
&:= \begin{cases}
\sqrt{2\pi} + \sqrt{\dfrac{2\pi D_0}{L_{T_0}}} & \text{for the APCSF in case } m(\tau) = \kappa_{\max}(\tau)\\
\sqrt{2\pi} + \sqrt{D_0} & \text{for the APCSF in case } m(\tau) \le \kappa_{\max}^2(\tau)\\
\sqrt{2\pi} + 2\pi\dfrac{\sqrt{D_0 m(\tau)}}{L_{T_0}} & \text{for the APCSF in case } m(\tau) > \kappa_{\max}^2(\tau)\\
\sqrt{2\pi} + 2\pi\dfrac{\sqrt{D_0}}{L_{T_0}} & \text{for the LPCF,}
\end{cases}
\end{aligned} \tag{11.10}$$

where m is defined in (11.5) and D_0 in Lemma 11.7.

Lemma 11.8 ([GH86, Paragraph 4.3.6] and [Pih98, Lemma 7.1])**.** *Let Σ_{T_0} be a smooth, embedded, convex curve. Let $F : \mathbb{S}^1 \times [T_0, \infty) \to \mathbb{R}^2$ be a solution of* (2.15) *with initial curve Σ_{T_0}. Let $\tau > T_0$, $\vartheta_1, \vartheta_2 \in \mathbb{S}^1$ and $\delta \in (0, \pi/2]$. If $|\vartheta_1 - \vartheta_2| < \delta$, then*

$$|\kappa(\vartheta_1, \tau) - \kappa(\vartheta_2, \tau)| < D^*(\tau)\sqrt{\delta}\,\kappa_{\max}(\tau)\,,$$

where D^* is defined in (11.10)*.*

Proof. We follow the lines of [GH86, Paragraph 4.3.6] and [Pih98, Lemma 7.1]. Corollaries 3.5 and 3.6 provide

$$\kappa_{\max}(\tau) \geq \frac{L_\tau}{2\pi} \overset{\text{Cors. 3.5,3.6}}{\geq} \frac{L_{T_0}}{2\pi}\,. \tag{11.11}$$

Let $\delta \in (0, \pi/2]$. For $|\vartheta_1 - \vartheta_2| < \delta$, Cauchy–Schwarz (B.3), Lemma 11.7 and $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ imply

$$\begin{aligned}
|\kappa(\vartheta_1, \tau) - \kappa(\vartheta_2, \tau)| &\leq \int_{\vartheta_1}^{\vartheta_2} \left|\frac{\partial\kappa}{\partial\vartheta}(\vartheta, \tau)\right| d\vartheta \\
&\overset{\text{(B.3)}}{\leq} |\vartheta_1 - \vartheta_2|^{1/2} \left(\int_{\vartheta_1}^{\vartheta_2} \left(\frac{\partial\kappa}{\partial\vartheta}(\vartheta, \tau)\right)^2 d\vartheta\right)^{1/2} \\
&\overset{\text{Lem. 11.7}}{\leq} \sqrt{\delta}\left(\int_{\mathbb{S}^1} \kappa^2(\vartheta, \tau)\, d\vartheta + D_0 m(\tau)\right)^{1/2} \\
&\leq \sqrt{\delta}\left(2\pi\kappa_{\max}^2(\tau) + D_0 m(\tau)\right)^{1/2} \\
&\leq \sqrt{\delta}\left(\sqrt{2\pi}\kappa_{\max}(\tau) + \sqrt{D_0 m(\tau)}\right) \\
&\leq \sqrt{\delta}\kappa_{\max}(\tau)\left(\sqrt{2\pi} + \frac{\sqrt{D_0 m(\tau)}}{\kappa_{\max}(\tau)}\right)\,,
\end{aligned} \tag{11.12}$$

where we used $\kappa_{\max}(\tau) > 0$ for $\tau > T_0$. For the **APCSF**, the definition (11.10) of D^* yields, that in case $m(\tau) = \kappa_{\max}(\tau)$,

$$\sqrt{2\pi} + \frac{\sqrt{D_0 \kappa_{\max}(\tau)}}{\kappa_{\max}(\tau)} \overset{(11.11)}{\leq} \sqrt{2\pi} + \sqrt{\frac{2\pi D_0}{L_{T_0}}} \overset{(11.10)}{=} D^*$$

in case $m(\tau) \leq \kappa_{\max}^2(\tau)$,

$$\sqrt{2\pi} + \frac{\sqrt{D_0 m(\tau)}}{\kappa_{\max}(\tau)} \leq \sqrt{2\pi} + \sqrt{D_0} \overset{(11.10)}{=} D^*$$

and in case $m(\tau) > \kappa_{\max}^2(\tau)$,

$$\sqrt{2\pi} + \frac{\sqrt{D_0 m(\tau)}}{\kappa_{\max}(\tau)} \overset{(11.11)}{\leq} \sqrt{2\pi} + \frac{2\pi\sqrt{D_0 m(\tau)}}{L_{T_0}} \overset{(11.10)}{=} D^*(\tau)\,.$$

For the **LPCF**, we deduce with $m = 1$,

$$\sqrt{2\pi} + \frac{\sqrt{D_0}}{\kappa_{\max}(\tau)} \overset{(11.11)}{\leq} \sqrt{2\pi} + \frac{2\pi\sqrt{D_0}}{L_{T_0}} \overset{(11.10)}{=} D^*\,.$$

All together, this yields the claim

$$|\kappa(\vartheta_1, \tau) - \kappa(\vartheta_2, \tau)| \overset{(11.12)}{\leq} \sqrt{\delta}\kappa_{\max}(\tau) D^*\,.$$

□

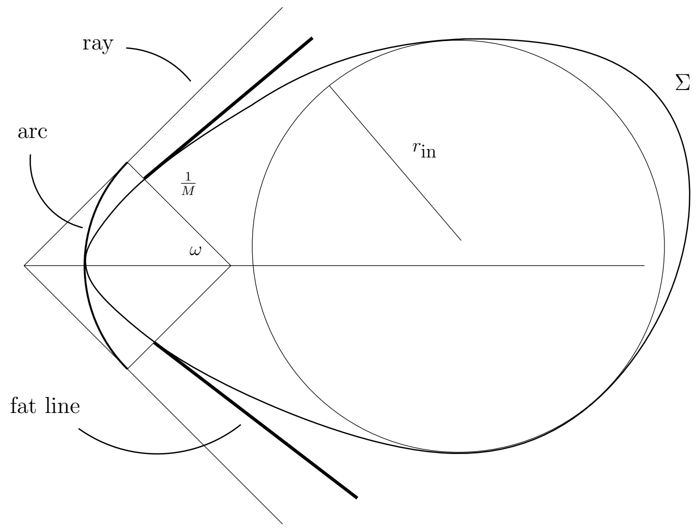

Figure 11.1: The arc of length ω with curvature greater than M.

Definition 11.9 (Generalised median curvature)**.** For for $\tau > T_0$ and $\omega \in (0, \pi)$, define the *generalised median curvature*

$$\kappa^*_\omega(\tau) := \sup \left\{ b \;\middle|\; |\kappa(\,\cdot\,, \tau)| > b \text{ on some interval of length } \omega \right\}.$$

Lemma 11.10 (Gage–Hamilton [GH86, Lemma 5.1])**.** *Let $\Sigma \subset \mathbb{R}^2$ be a, smooth, embedded, closed, convex curve. Then*

$$\kappa^*_\omega r_{\text{in}} \leq \left[1 - K(\omega) \left(\frac{r_{\text{circ}}}{r_{\text{in}}} - 1 \right) \right]^{-1},$$

where $K : (0, \pi] \to [0, \infty)$ is a positive decreasing function with $K(\omega) \to \infty$ for $\omega \searrow 0$ and $K(\pi) = 0$.

Proof. We repeat the proof here for the sake of completeness. Fix $M < \kappa^*_\omega$. By Definition 11.9 of the generalised median curvature, the set $S := \{\vartheta \in \mathbb{S}^1 \mid \kappa(\vartheta) > M\}$ contains an interval of length at least ω. By changing the parametrisation we can assume that $(-\omega/2, \omega/2) \subset S$. We construct a circular arc of curvature M and of angle ω which is tangent to the curve Σ at $\vartheta = 0$ (see Figure 11.1). Since Σ is convex, Σ must lie in the region bounded by the arc and the rays tangent to the ends of the arc. Moreover, the convexity assumption insures that Σ lies within the fat lines shown in Figure 11.1, while the estimate $\kappa(\vartheta) > M$ on $(-\omega/2, \omega/2)$ ensures that the fat lines lie within the cone formed by the circular arc and the rays. Since the inscribed circle lies within the cone and the circumscribed circle must encircle every point on the curve, we see that, for given M, ω and r_{in}, the smallest r_{circ} is obtained for the configuration shown in Figure 11.2. From Figure 11.2 and trigonometry, we determine that $|b| = 1/M$,

$$\frac{r_{\text{in}}}{|a| + |d|} = \cos\left(\frac{\omega}{2}\right) = \frac{1/M}{|b| + |d|} = \frac{1/M}{1/M + |d|} \tag{11.13}$$

and

$$2 r_{\text{circ}} \geq r_{\text{in}} + |a|\,. \tag{11.14}$$

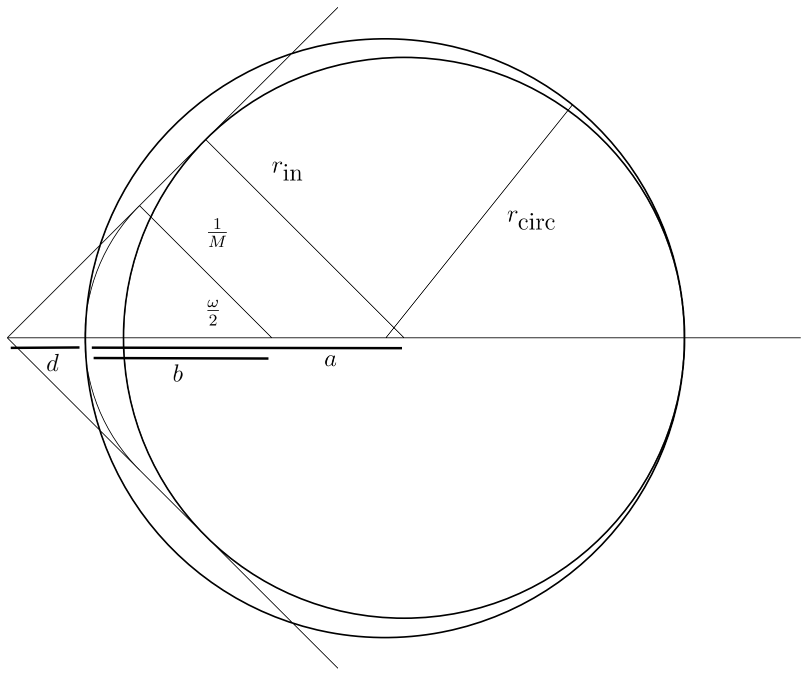

Figure 11.2: Inscribed and circumscribed radius.

The identity (11.13) implies

$$|d| = \frac{1}{M}\left(\frac{1}{\cos(\omega/2)} - 1\right)$$

as well as

$$|a| = \frac{r_{\text{in}}}{\cos(\omega/2)} - |d| = \frac{r_{\text{in}}}{\cos(\omega/2)} - \frac{1}{M}\left(\frac{1}{\cos(\omega/2)} - 1\right). \tag{11.15}$$

Hence, by (11.14) and (11.15),

$$\begin{aligned}\frac{r_{\text{circ}}}{r_{\text{in}}} - 1 &\overset{(11.14)}{\geq} -\frac{1}{2} + \frac{|a|}{2r_{\text{in}}} \\ &\overset{(11.15)}{=} -\frac{1}{2} + \frac{1}{2r_{\text{in}}}\left[\frac{r_{\text{in}}}{\cos(\omega/2)} - \frac{1}{M}\left(\frac{1}{\cos(\omega/2)} - 1\right)\right] \\ &= \frac{1}{2}\left(1 - \frac{1}{Mr_{\text{in}}}\right)\left(\frac{1}{\cos(\omega/2)} - 1\right).\end{aligned}$$

Rearranging terms yields

$$Mr_{\text{in}} \leq \frac{1}{1 - K(\omega)(r_{\text{circ}}/r_{\text{in}} - 1)},$$

where we defined

$$K(\omega) := 2\left(\frac{1}{\cos(\omega/2)} - 1\right)^{-1} = \frac{2\cos(\omega/2)}{1 - \cos(\omega/2)}.$$

Since M can be chosen arbitrarily close to κ_ω^*, this proves the lemma. □

Lemma 11.11 ([GH86, Corollary 5.2] and [Pih98, Lemma 7.11])**.** *Let Σ_{T_0} be a smooth, embedded, convex curve. Let $F : \mathbb{S}^1 \times [T_0, \infty) \to \mathbb{R}^2$ be a solution of* (2.15) *with initial curve Σ_{T_0}. Let $\varepsilon \in (0,1)$ and $\tau > T_0$. Then*

$$\kappa_{\max}(\tau) r_{\text{in}}(\tau) \leq \left\{ (1-\varepsilon) \left[1 - K\left(\left(\frac{\varepsilon}{D^*(\tau)} \right)^2 \right) \left(\frac{r_{\text{circ}}(\tau)}{r_{\text{in}}(\tau)} - 1 \right) \right] \right\}^{-1},$$

where K is defined in Lemma 11.10 and D^* in (11.10).

Proof. The proof can be found in [GH86, Corollary 5.2] and in [Pih98, Lemma 7.11]. Let $\varepsilon \in (0,1)$ and $\tau \in (T_0, \infty)$. Let $\vartheta_0 \in \mathbb{S}^1$ be a point with

$$\kappa(\vartheta_0, \tau) = \kappa_{\max}(\tau)\,. \tag{11.16}$$

By definition (11.10), $D^* = D^*(\Sigma_{T_0}, \tau) > \sqrt{2\pi}$. Set

$$\delta := \left(\frac{\varepsilon}{D^*} \right)^2 < \frac{1}{2\pi} \tag{11.17}$$

and let $\vartheta \in (\vartheta_0 - \delta/2, \vartheta_0 + \delta/2)$. Lemma 11.8 yields

$$0 < \kappa_{\max}(\tau) - \kappa(\vartheta, \tau) \overset{(11.16)}{=} |\kappa(\vartheta_0, \tau) - \kappa(\vartheta, \tau)| \overset{\text{Lem. 11.8}}{<} D^* \sqrt{\delta}\, \kappa_{\max}(\tau)\,. \tag{11.18}$$

By Definition 11.9 of the generalised curvature,

$$0 < \left(1 - D^*\sqrt{\delta}\right) \kappa_{\max}(\tau) \overset{(11.18)}{\leq} \kappa(\vartheta, \tau) \overset{\text{Def. 11.9}}{\leq} \kappa^*_\delta(\tau)\,. \tag{11.19}$$

Since Σ_τ is embedded, closed and convex for $\tau \in (T_0, \infty)$, Lemma 11.10 yields

$$\begin{aligned} 0 < \left(1 - D^*\sqrt{\delta}\right) \kappa_{\max}(\tau) r_{\text{in}}(\tau) &\overset{(11.19)}{\leq} \kappa^*_\delta(\tau) r_{\text{in}}(\tau) \\ &\overset{\text{Lem. 11.10}}{\leq} \left[1 - K(\delta) \left(\frac{r_{\text{circ}}(\tau)}{r_{\text{in}}(\tau)} - 1 \right) \right]^{-1}. \end{aligned}$$

The definition (11.17) of δ yields the claim. □

Corollary 11.12 ([GH86, Proposition 5.3] and [Pih98, Corollary 7.12])**.** *Let Σ_{T_0} be a smooth, embedded, convex curve. Let $F : \mathbb{S}^1 \times [T_0, \infty) \to \mathbb{R}^2$ be a solution of* (2.15) *with initial curve Σ_{T_0}. For every $\varepsilon \in (0,1)$, there exists a time $\tau_0 \in (T_0, \infty)$ such that*

$$\kappa_{\max}(\tau) r_{\text{in}}(\tau) \leq \frac{1}{(1-\varepsilon)^2}$$

for all $\tau \geq \tau_0$.

Proof. We extend ideas of [GH86, Proposition 5.3] and [Pih98, Corollary 7.12]. Corollary 11.4 implies that, for every $\delta > 0$, there exists a time $\tau_0(\delta) > T_0$ so that

$$r_{\text{circ}}(\tau) - r_{\text{in}}(\tau) \leq \frac{\sqrt{D(\Sigma_{T_0}) A_\tau}}{\pi} \exp\left(-\frac{\tau}{R_0^2} \right) \leq \delta$$

for all $\tau \geq \tau_0$, and thus

$$\frac{r_{\text{circ}}(\tau)}{r_{\text{in}}(\tau)} - 1 \leq \frac{\delta}{r_{\text{in}}(\tau)}\,. \tag{11.20}$$

Recall the definitions (11.5) and (11.10) of m and D^*. For the **APCSF**, we define

$$I_1 := \{\tau \in [\tau_0, \infty) \mid m(\tau) = \kappa_{\max}(\tau)\}$$

and

$$I_2 := \{\tau \in [\tau_0, \infty) \mid m(\tau) > \kappa_{\max}(\tau)\}\,.$$

Then m is monotone increasing on I_1 and constant on every connected subinterval of I_2. We distinguish between three cases.

(i) Assume that $\sup_{[\tau_0,\infty)} m < \infty$. Then $D^* = D^*(\Sigma_{T_0}, \sup_{[\tau_0,\infty)} m)$ is independent of time.

(ii) Assume that $\sup_{[\tau_0,\infty)} m = \infty$ and $\sup\{\tau \in I_2\} =: \tau_3 < \infty$. Then $[\tau_3, \infty) \subset I_1$ and $D^* = D^*(\Sigma_{T_0}, m(\tau_3))$ is independent of time.

(iii) Assume that $\sup_{[\tau_0,\infty)} m = \infty$ and $\sup\{\tau \in I_2\} = \infty$. Assume there exists $\tau_1 \in [\tau_0, \infty)$ so that $(\tau_1, \infty) \subset I_2$, then $m(\tau) = m(\tau_1) < \infty$ for all $\tau \in (\tau_1, \infty)$. This contradicts $\sup_{[\tau_0,\infty)} m = \infty$. Hence, I_2 consists of infinitely many disjoint open intervals $I_{2,k}$, $k \in \mathbb{N}$. And, for all $\tau \in I_1$, $D^*(\tau) = D^*(\Sigma_{T_0})$ is independent of time.

For the **LPCF**, $D^* = D^*(\Sigma_{T_0})$ is independent of time by definition. Further recall that K, as defined in Lemma 11.10, is a positive decreasing function that satisfies $K(\omega) \to \infty$ for $\omega \searrow 0$ and $K(\pi) = 0$. Thus, for any $\tau \geq \tau_0$ for the LPCF and for the APCSF in cases *(i)* and *(ii)*, and for $\tau \in I_1$ in case *(iii)*, D^* is independent of time. By Proposition 11.5, $r_{\text{in}}(\tau) \geq c(\Sigma_{T_0}) > 0$ for all $\tau \geq T_0$. Hence, for given $\varepsilon \in (0, 1)$, we can choose $\delta > 0$ and $\tau_0(\delta) > T_0$ so that

$$\frac{\delta}{r_{\text{in}}(\tau)} \leq \frac{\varepsilon}{K\big((\varepsilon/D^*)^2\big)} \tag{11.21}$$

for all $\tau \geq \tau_0$. Combining (11.20) and (11.21) yields

$$\frac{r_{\text{circ}}(\tau)}{r_{\text{in}}(\tau)} - 1 \leq \frac{\varepsilon}{K\big((\varepsilon/D^*)^2\big)}$$

and

$$1 - \varepsilon \leq 1 - K\left(\left(\frac{\varepsilon}{D^*}\right)^2\right)\left(\frac{r_{\text{circ}}(\tau)}{r_{\text{in}}(\tau)} - 1\right)$$

for all $\tau \geq \tau_0$. This and Lemma 11.11 imply

$$\begin{aligned} \kappa_{\max}(\tau) r_{\text{in}}(\tau) &\overset{\text{Lem. 11.11}}{\leq} \left\{(1-\varepsilon)\left[1 - K\left(\left(\frac{\varepsilon}{D^*}\right)^2\right)\left(\frac{r_{\text{circ}}(\tau)}{r_{\text{in}}(\tau)} - 1\right)\right]\right\}^{-1} \\ &\leq \frac{1}{(1-\varepsilon)^2} \end{aligned} \tag{11.22}$$

for any $\tau \geq \tau_0$ for the LPF and for APCSF in cases *(i)* and *(ii)*, and for $\tau \in I_1$ in case *(iii)*. In case *(iii)*, define the sequence

$$\big(\tau_k := \sup\{\tau \in I_{2,k}\} \in I_1\big)_{k\in\mathbb{N}}\,.$$

Then $\tau_k \to \infty$ for $k \to \infty$. By Proposition 11.5, $r_{\text{in}}(\tau_k) \geq c(\Sigma_{T_0}) > 0$ for all $k \in \mathbb{N}$. Thus, for given $\varepsilon \in (0,1)$ and for all $k \in \mathbb{N}$,

$$m(\tau) = m(\tau_k) = \kappa_{\max}(\tau_k) \overset{(11.22)}{\leq} \frac{1}{c(1-\varepsilon)^2} \tag{11.23}$$

for all $\tau \in I_{2,k}$, since $\tau_k \in I_1$. Hence,

$$\infty \overset{(iii)}{=} \sup_{\tau\in[\tau_0,\infty)} m(\tau) = \sup_{\tau\in I_1\cup I_2} m(\tau) \overset{(11.22),(11.23)}{\leq} \frac{1}{c(1-\varepsilon)^2}$$

which is a contradiction. So case *(iii)* could not have happened and we are in case *(i)* or *(ii)*. □

Proposition 11.13 ([GH86, Theorem 5.4] and [Pih98, Proposition 7.13])**.** *Let Σ_{T_0} be a smooth, embedded, convex curve. Let $F : \mathbb{S}^1 \times [T_0,\infty) \to \mathbb{R}^2$ be a solution of* (2.15) *with initial curve Σ_{T_0}. Then $\kappa(\,\cdot\,,\tau)r_{\text{in}}(\tau) \to 1$ uniformly for $\tau \to \infty$.*

Proof. We follow the lines of [GH86, Theorem 5.4] and in [Pih98, Proposition 7.13]. Fix $\varepsilon_0 \in (0,1/2)$. Corollary 11.12 implies that there exists a time $\tau_0 > T_0$ so that

$$\kappa(\vartheta,\tau)r_{\text{in}}(\tau) \leq \kappa_{\max}(\tau)r_{\text{in}}(\tau) \overset{\text{Cor. 11.12}}{\leq} \frac{1}{1-\varepsilon_0} \tag{11.24}$$

for all $\vartheta \in \mathbb{S}^1$ and for all $\tau \geq \tau_0$. Hence, the set of functions

$$\mathcal{F} := \big\{\kappa(\,\cdot\,,\tau)r_{\text{in}}(\tau) : \mathbb{S}^1 \to \mathbb{R} \;\big|\; \tau \geq \tau_0\big\}$$

is uniformly bounded. By definition (11.10), $D^* > \sqrt{2\pi}$. Lemma 11.8 yields that, for all $\delta \leq \pi/2$ and $|\vartheta_1 - \vartheta_2| < \delta$,

$$|\kappa(\vartheta_1,\tau)r_{\text{in}}(\tau) - \kappa(\vartheta_2,\tau)r_{\text{in}}(\tau)| \overset{\text{Lem. 11.8}}{<} D^*\sqrt{\delta}\,r_{\text{in}}(\tau)\kappa_{\max}(\tau) \overset{(11.24)}{\leq} \frac{D^*\sqrt{\delta}}{1-\varepsilon_0}$$

for $\tau \geq \tau_0$. Thus, $\mathcal{F}$ is equicontinuous and we can apply the Arzelà–Ascoli theorem, Theorem B.12, to deduce that, for all sequences $(\tau_k)_{k\in\mathbb{N}}$ with $\tau_0 \leq \tau_k \to \infty$ for $k \to \infty$, there exists a subsequence $(\tau_k)_{k\in\mathbb{N}}$ so that the sequence

$$\big(\,f_k(\,\cdot\,) := \kappa(\,\cdot\,,\tau_k)r_{\text{in}}(\tau_k) \in \mathcal{F}\,\big)_{k\in\mathbb{N}} \tag{11.25}$$

converges uniformly to a continuous function $f : \mathbb{S}^1 \to \mathbb{R}$ (which may depend on the subsequence). By the preservation of strict convexity (see Corollary 4.3), and the lower bound on r_{in} (see Proposition 11.5), there exists a constant so that $f_k \geq c > 0$ on $\mathbb{S}^1$ for all $k \in \mathbb{N}$. We employ Corollary 11.12 again to conclude that, for every $\varepsilon \in (0,1)$, there exists $k_0 \in \mathbb{N}$ so that

$$f_k(\vartheta) \overset{(11.25)}{\leq} \kappa_{\max}(\tau_k)r_{\text{in}}(\tau_k) \overset{\text{Cor. 11.12}}{\leq} \frac{1}{(1-\varepsilon)^2}$$

for all $\vartheta \in \mathbb{S}^1$ and for all $k \geq k_0$. We let $\varepsilon \to 0$ and obtain

$$c \leq f(\vartheta) \leq 1$$

for all $\vartheta \in \mathbb{S}^1$. Furthermore, the sequence $\big(1/f_k\big)_{k\in\mathbb{N}}$ is uniformly bounded between 1 and $1/c$ on $\mathbb{S}^1$ for every $k \in \mathbb{N}$, and converges to

$$\frac{1}{f} : \mathbb{S}^1 \to \left[1\,,\ \frac{1}{c}\right].$$

Since $\mathbb{S}^1$ is compact, we can apply Fatou's lemma, Lemma B.9, and obtain

$$\begin{aligned}\int_{\mathbb{S}^1} \frac{1}{f(\vartheta)}\,d\vartheta &= \int_{\mathbb{S}^1} \liminf_{k\to\infty} \frac{1}{f_k(\vartheta)}\,d\vartheta \overset{\text{Lem. B.9}}{\leq} \liminf_{k\to\infty} \int_{\mathbb{S}^1} \frac{1}{f_k(\vartheta)}\,d\vartheta \\ &\overset{(11.25)}{=} \liminf_{k\to\infty} \frac{1}{r_{\text{in}}(\tau_k)} \int_{\mathbb{S}^1} \frac{1}{\kappa(\vartheta,\tau_k)}\,d\vartheta \overset{(11.3)}{=} \liminf_{k\to\infty} \frac{1}{r_{\text{in}}(\tau_k)} \int_{s(\mathbb{S}^1,\tau_k)} ds_{\tau_k} \\ &= \liminf_{k\to\infty} \frac{L_{\tau_k}}{r_{\text{in}}(\tau_k)} \overset{\text{Prop. 11.5}}{=} 2\pi\,,\end{aligned}$$

where we used in the last step that $L_{\tau_k} \to 2\pi R_0$ and $r_{\text{in}}(\tau_k) \to R_0$ for $k \to \infty$ (see Proposition 11.5). Since $f \leq 1$, the above estimate yields

$$1 \geq \frac{1}{2\pi} \int_{\mathbb{S}^1} \frac{1}{f(\vartheta)}\,d\vartheta \geq \min_{\mathbb{S}^1} \frac{1}{f} = \frac{1}{\max_{\mathbb{S}^1} f} \geq 1$$

so that $f \equiv 1$ on $\mathbb{S}^1$. It follows that every sequence $(\tau_k)_{k\in\mathbb{N}}$ has a subsequence so that $f_k \to 1$ uniformly on $\mathbb{S}^1$. Furthermore, every subsequence has a subsubsequence so that $f_k \to 1$ uniformly on $\mathbb{S}^1$. Hence, for all sequences $(\tau_k)_{k\in\mathbb{N}}$ with $\tau_0 \leq \tau_k \to \infty$, $\kappa(\,\cdot\,,\tau_k) r_{\text{in}}(\tau_k) \to 1$ uniformly for $k \to \infty$ and thus $\kappa(\,\cdot\,,\tau) r_{\text{in}}(\tau) \to 1$ uniformly for $\tau \to \infty$. □

Corollary 11.14 (C^2-convergence of convex curves, [Pih98, Corollary 7.14])**.** *Let Σ_{T_0} be a smooth, embedded, convex curve of area $A_{T_0} = \pi R_0^2$ for the APCSF and length $L_{T_0} = 2\pi R_0$ for the LPCF. Let $F : \mathbb{S}^1 \times [T_0,\infty) \to \mathbb{R}^2$ be a solution of* (2.15) *with initial curve Σ_{T_0}. Then*

$$\frac{\kappa_{\max}(\tau)}{\kappa_{\min}(\tau)} \to 1\,, \qquad \kappa(\vartheta,\tau) \to \frac{1}{R_0} \qquad \textit{and} \qquad h(\tau) \to \frac{1}{R_0}$$

for every $\vartheta \in \mathbb{S}^1$ and for $\tau \to \infty$. Hence, the flow converges uniformly in C^2 to a circle of radius R_0 which solves the corresponding isoperimetric problem.

Proof. By Proposition 11.5, Σ_τ is strictly convex for $\tau \in (T_0,\infty)$. By Proposition 11.13,

$$\kappa(\vartheta,\tau) r_{\text{in}}(\tau) \to 1$$

for all $\vartheta \in \mathbb{S}^1$ and for $\tau \to \infty$. Hence, it also holds that

$$\kappa_{\max}(\tau) r_{\text{in}}(\tau) \to 1 \qquad \text{and} \qquad \kappa_{\min}(\tau) r_{\text{in}}(\tau) \to 1$$

for $\tau \to \infty$ and the first claim follows. By Propositions 11.5, the curve converges to a circle of radius R_0. This yields the second claim. For the APCSF, $h_{\text{ap}} = 2\pi/L$, hence the third claim, follows from Proposition 11.5. For the LPCF, the third claim is given by Lemma 10.4 and follows also from the above curvature convergence using $1/R_0 \leq h_{\text{lp}} \leq \kappa_{\max}$ (see (3.11) and Corollary 5.9). □

Corollary 11.15 (Boundedness of curvature derivatives I)**.** *Let Σ_0 be a smooth, embedded curve of area $A_0 = \pi R_0^2$ for the APCSF and length $L_0 = 2\pi R_0$ for the LPCF. Let $F : \mathbb{S}^1 \times [0,\infty) \to \mathbb{R}^2$ be a solution of (2.15) with initial curve Σ_0. Then, for every $n \in \mathbb{N}$, there exists a constant $C_n = C_n(n, \Sigma_0)$ such that*

$$\max_{\mathbb{S}^1_{R_t}} \left| \frac{\partial^n \kappa}{\partial s^n} \right| \leq C_n$$

for all $t \geq 0$.

Proof. By Theorem 10.9, there exists a time $T_0 > 0$ so that the curves are strictly convex on (T_0, ∞). Proposition 4.9 implies that the curvature is bounded on $(0, T_0]$ and Corollary 11.14 implies that the curvature is bounded on (T_0, ∞). Hence, we can apply Corollary 4.7 with $T = \infty$ to find that, for all $n \in \mathbb{N}$, the arc length derivatives are bounded. □

Corollary 11.16 (Boundedness of curvature derivatives II, [Pih98, Corollary 7.15])**.** *Let Σ_{T_0} be a smooth, embedded, convex curve of area $A_{T_0} = \pi R_0^2$ for the APCSF and length $L_{T_0} = 2\pi R_0$ for the LPCF. Let $F : \mathbb{S}^1 \times [T_0, \infty) \to \mathbb{R}^2$ be a solution of (2.15) with initial curve Σ_{T_0}. Then, for all $n \in \mathbb{N}$, there exists a constant $\bar{C}_n = \bar{C}_n(n, \Sigma_{T_0})$ and time $\tau_0 = \tau_0(\Sigma_0) > T_0$ such that*

$$\max_{\mathbb{S}^1} \left| \frac{\partial^n \kappa}{\partial \vartheta^n} \right| \leq \bar{C}_n$$

on $[\tau_0, \infty)$.

Proof. By Corollary 4.3, the curves are strictly convex on (T_0, ∞). By Corollary 11.14, for every $c_0 \in (0, 1/R_0)$ there exists $\tau_0(c_0) > T_0$ such that

$$\kappa_{\min} \geq c_0$$

on $[\tau_0, \infty)$. For $c_0 = 1/(2R_0)$, the transformation (11.3) of the derivatives and Corollary 11.15 imply

$$\left| \frac{\partial \kappa}{\partial \vartheta}(\vartheta(s,t), \tau(t)) \right| \overset{(11.3)}{=} \left| \frac{1}{\kappa(s,t)} \frac{\partial \kappa}{\partial s}(s,t) \right| \overset{\text{Cor. 11.15}}{\leq} \frac{C_1(\Sigma_0)}{c_0} =: \bar{C}_1(\Sigma_0)$$

for all $t \in [\tau_0, \infty)$, and, for $n \geq 2$,

$$\begin{aligned} \max_{\vartheta \in \mathbb{S}^1} \left| \frac{\partial^n \kappa}{\partial \vartheta^n}(\vartheta, \tau) \right| &= \max_{\mathbb{S}^1_{R_t}} \left| \left(\frac{1}{\kappa(s,t)} \frac{\partial}{\partial s} \right)^n \kappa(s,t) \right| \\ &\leq c(n, c_0, C_1, \ldots, C_n) =: \bar{C}_n(n, \Sigma_0) \, . \end{aligned}$$ □

11.3 Uniform C^∞-convergence

In this section, we show that, for convex solutions of (2.15), the derivatives of the curvature with respect to ϑ converge to zero.

Theorem 11.17 (C^∞-convergence of convex curves, [Pih98, Proposition 7.17])**.** *Let Σ_{T_0} be a smooth, embedded, convex curve of area $A_{T_0} = \pi R_0^2$ for the APCSF and length $L_{T_0} = 2\pi R_0$ for the LPCF. Let $F : \mathbb{S}^1 \times [T_0, \infty) \to \mathbb{R}^2$ be a solution of* (2.15) *with initial curve Σ_{T_0}. Then, for all $n \in \mathbb{N}$, $\frac{\partial^n}{\partial \vartheta^n}\kappa \to 0$ uniformly for $\tau \to \infty$. Hence, the curves converge uniformly in C^∞ to a circle of radius R_0.*

Proof. We follow the lines of [Pih98, Proposition 7.17]. Corollary 11.14 implies that the curvature and the global terms are uniformly bounded by a constant $C_0 > 0$ on $[T_0, \infty)$. We abbreviate in the following

$$\kappa' := \frac{\partial \kappa}{\partial \vartheta} \qquad \text{and} \qquad \kappa^{(n)} := \frac{\partial^n \kappa}{\partial \vartheta^n}\,.$$

Corollary 11.16 implies that there exists a time $\tau_0(\Sigma_0) > T_0$ so that, for every $n \in \mathbb{N}$, $\vartheta_1, \vartheta_2 \in \mathbb{S}^1$, $\varepsilon > 0$ and $\tau \geq \tau_0$,

$$\left|\kappa^{(n)}(\vartheta_1,\tau) - \kappa^{(n)}(\vartheta_2,\tau)\right| \leq \left|\int_{\vartheta_1}^{\vartheta_2} \kappa^{(n+1)}(\vartheta,\tau)\,d\vartheta\right| \leq \bar{C}_{n+1}(n,\Sigma_0)|\vartheta_1 - \vartheta_2|^{1/2}.$$

Thus, $\kappa^{(n)}$ is uniformly continuous in space for each $n \in \mathbb{N}$ and fixed time. Let $(\tau_k)_{k\in\mathbb{N}}$ be a sequence with $\tau_k \to \infty$ for $k \to \infty$. Then, for fixed $n \in \mathbb{N}$, the sequence

$$\left(\kappa^{(n)}(\,\cdot\,,\tau_k) : \mathbb{S}^1 \to \mathbb{R}\right)_{k\in\mathbb{N}}$$

is bounded and equicontinuous. The Arzelà–Ascoli theorem, Theorem B.12, implies that, for each $n \in \mathbb{N}$, there exists a subsequence $\left(\kappa^{(n)}(\,\cdot\,,\tau_k)\right)_{k\in\mathbb{N}}$ that converges uniformly to a continuous function $f_n : \mathbb{S}^1 \to \mathbb{R}$ which may depend on the subsequence. We proceed by induction over n. For $n = 0$, we use Corollary 11.14 to conclude that the sequence

$$\left(\kappa(\,\cdot\,,\tau_k) : \mathbb{S}^1 \to \mathbb{R}\right)_{k\in\mathbb{N}}$$

converges pointwise to $1/R_0$ for $k \to \infty$. Furthermore, as stated above, there exists a subsequence $(\tau_k)_{k\in\mathbb{N}}$ so that

$$\left(\kappa'(\,\cdot\,,\tau_k) : \mathbb{S}^1 \to \mathbb{R}\right)_{k\in\mathbb{N}}$$

converges uniformly to a continuous function $f_1 : \mathbb{S}^1 \to \mathbb{R}$ which may depend on the subsequence. Since the convergence is uniform, there exists $k_0 \in \mathbb{N}$ so that

$$\left|\kappa'(\,\cdot\,,\tau_k)\right| \leq 2f_1 \in L^1(\mathbb{S}^1) \tag{11.26}$$

for all $k \geq k_0$. Fix $\vartheta_0 \in \mathbb{S}^1$. By (11.26) and the uniform convergence, we can apply the dominated convergence theorem, Theorems B.10, and Theorem B.14 which allow us to interchange the limit with the integral bounds and differentiation. Thus, for all $\vartheta \in \mathbb{S}^1$,

$$\begin{aligned} f_1(\vartheta) &= \frac{\partial}{\partial\vartheta}\int_{\vartheta_0}^{\vartheta} f_1(\sigma)\,d\sigma = \frac{\partial}{\partial\vartheta}\int_{\vartheta_0}^{\vartheta} \lim_{k\to\infty} \kappa'(\sigma,\tau_k)\,d\sigma \\ &\overset{\text{Thm. B.10}}{=} \frac{\partial}{\partial\vartheta}\lim_{k\to\infty}\int_{\vartheta_0}^{\vartheta} \kappa'(\sigma,\tau_k)\,d\sigma \overset{\text{Thm. B.14}}{=} \lim_{k\to\infty}\frac{\partial}{\partial\vartheta}\int_{\vartheta_0}^{\vartheta} \kappa'(\sigma,\tau_k)\,d\sigma \\ &= \lim_{k\to\infty}\frac{\partial}{\partial\vartheta}\kappa(\vartheta,\tau_k) \overset{\text{Thm. B.14}}{=} \frac{\partial}{\partial\vartheta}\lim_{k\to\infty}\kappa(\vartheta,\tau_k) \overset{\text{Cor. 11.14}}{=} \frac{\partial}{\partial\vartheta}\frac{1}{R_0} = 0\,. \end{aligned}$$

By the same argument, every subsubsequence $(\kappa'(\,\cdot\,,\tau_k))_{k\in\mathbb{N}}$ converges uniformly to 0. Hence, κ' converges uniformly to 0 as $\tau\to\infty$. For the induction step, we assume that $\kappa^{(n)}\to 0$ uniformly for $\tau\to\infty$. Again, since the convergences are uniform and there exists $k_0\in\mathbb{N}$ so that

$$\left|\kappa^{(n)}(\,\cdot\,,\tau_k)\right| \le 2f_n \in L^1(\mathbb{S}^1)$$

for all $k\ge k_0$, Theorems B.10 and B.14 allow us to interchange the limit with the integral bounds and differentiation. Thus, for all $\vartheta\in\mathbb{S}^1$,

$$\begin{aligned} f_{n+1}(\vartheta) &= \frac{\partial}{\partial\vartheta}\int_{\vartheta_0}^{\vartheta} f_{n+1}(\sigma)\,d\sigma = \frac{\partial}{\partial\vartheta}\int_{\vartheta_0}^{\vartheta} \lim_{k\to\infty}\kappa^{(n+1)}(\sigma,\tau_k)\,d\sigma \\ &\overset{\text{Thm. B.10}}{=} \frac{\partial}{\partial\vartheta}\lim_{k\to\infty}\int_{\vartheta_0}^{\vartheta}\kappa^{(n+1)}(\sigma,\tau_k)\,d\sigma \overset{\text{Thm. B.14}}{=} \lim_{k\to\infty}\frac{\partial}{\partial\vartheta}\int_{\vartheta_0}^{\vartheta}\kappa^{(n+1)}(\sigma,\tau_k)\,d\sigma \\ &= \lim_{k\to\infty}\frac{\partial}{\partial\vartheta}\kappa^{(n)}(\vartheta,\tau_k) \overset{\text{Thm. B.14}}{=} \frac{\partial}{\partial\vartheta}\lim_{k\to\infty}\kappa^{(n)}(\vartheta,\tau_k) = \frac{\partial}{\partial\vartheta}0 = 0\,. \end{aligned}$$

The same argument as for $n=0$ above yields that $\kappa^{(n+1)}$ converges uniformly to 0 as $\tau\to\infty$. □

11.4 Exponential convergence

In this section, we show that, once the curve is convex, the curvature converges exponentially to $1/R_0$ and all curvature derivatives converge exponentially to zero. In the end of the section we state the main results of this thesis. In the following, let again be $T_0\ge 0$ be the time where the curve is convex.

Lemma 11.18 (Wirtinger's inequality, see e. g. [AE06, p. 91])**.** *Let $f:[a,b]\to\mathbb{R}$ be in C^1 with $b-a\le\pi$ and $f(a)=f(b)=0$. Then*

$$\int_a^b f^2\,d\vartheta \le \frac{(b-a)^2}{\pi^2}\int_a^b\left(\frac{df}{d\vartheta}\right)^2 d\vartheta\,.$$

Lemma 11.19 ([GH86, Lemma 5.7.9] and [Pih98, Lemma 7.23])**.** *Let Σ_{T_0} be a smooth, embedded, convex curve of area $A_{T_0}=\pi R_0^2$ for the APCSF and length $L_{T_0}=2\pi R_0$ for the LPCF. Let $F:\mathbb{S}^1\times[T_0,\infty)\to\mathbb{R}^2$ be a solution of (2.15) with initial curve Σ_{T_0}. Let $\beta\in(0,1)$. Then there exists a time $\tau_0>T_0$ such that*

$$4\beta\int_{\mathbb{S}^1}\left(\frac{\partial\kappa}{\partial\vartheta}\right)^2 d\vartheta \le \int_{\mathbb{S}^1}\left(\frac{\partial^2\kappa}{\partial\vartheta^2}\right)^2 d\vartheta$$

for all $\tau\ge\tau_0$.

Proof. We follow the lines of [GH86, Lemma 5.7.9] and [Pih98, Lemma 7.23]. The system

$$\left\{1,\sqrt{2}\cos(n\vartheta),\sqrt{2}\sin(n\vartheta)\right\}_{n\in\mathbb{Z}} \tag{11.27}$$

forms an orthonormal basis of the periodic functions in the Hilbert space $C^2([0,2\pi])$ with respect to the L^2-inner product (see e. g. [HL99, p. 124]). By the transformation (11.3) of the derivatives, we have $ds_t = d\vartheta/\kappa$ so that

$$\int_{\mathbb{S}^1} \frac{\sin(\vartheta)}{\kappa}\, d\vartheta = \int_{\mathbb{S}^1_{R_t}} \sin\left(\frac{s}{R_t}\right) ds_t = \cos(2\pi) - \cos(0) = 1 - 1 = 0 \tag{11.28}$$

and

$$\int_{\mathbb{S}^1} \frac{\cos(\vartheta)}{\kappa}\, d\vartheta = \int_{\mathbb{S}^1_{R_t}} \cos\left(\frac{s}{R_t}\right) ds_t = \sin(2\pi) - \sin(0) = 0\,. \tag{11.29}$$

Integration by parts (B.4) yields

$$\begin{aligned} 0 &\overset{(11.28)}{=} \int_{\mathbb{S}^1} \frac{\sin(\vartheta)}{\kappa}\, d\vartheta = \int_{\mathbb{S}^1} \frac{1}{\kappa}\frac{\partial \cos}{\partial\vartheta}(\vartheta)\, d\vartheta \\ &\overset{(B.4)}{=} -\int_{\mathbb{S}^1} \cos(\vartheta)\frac{\partial}{\partial\vartheta}\left(\frac{1}{\kappa}\right) d\vartheta = \int_{\mathbb{S}^1} \cos(\vartheta)\frac{1}{\kappa^2}\frac{\partial\kappa}{\partial\vartheta}\, d\vartheta \end{aligned}$$

and

$$\begin{aligned} 0 &\overset{(11.29)}{=} -\int_{\mathbb{S}^1} \frac{\cos(\vartheta)}{\kappa}\, d\vartheta = \int_{\mathbb{S}^1} \frac{1}{\kappa}\frac{\partial \sin}{\partial\vartheta}(\vartheta)\, d\vartheta \\ &\overset{(B.4)}{=} -\int_{\mathbb{S}^1} \sin(\vartheta)\frac{\partial}{\partial\vartheta}\left(\frac{1}{\kappa}\right) d\vartheta = \int_{\mathbb{S}^1} \sin(\vartheta)\frac{1}{\kappa^2}\frac{\partial\kappa}{\partial\vartheta}\, d\vartheta\,. \end{aligned}$$

Additionally, we have

$$0 = -\int_{\mathbb{S}^1} \frac{\partial}{\partial\vartheta}\left(\frac{1}{\kappa}\right) d\vartheta = \int_{\mathbb{S}^1} \frac{1}{\kappa^2}\frac{\partial\kappa}{\partial\vartheta}\, d\vartheta\,.$$

Hence, $1/\kappa^2 \frac{\partial}{\partial\vartheta}\kappa$ is orthogonal to the first five basis functions of the basis (11.27). Since all the other basis functions are zero at at least four points in $[0, 2\pi]$ with distance $\le \pi/2$, there exists a number $i_0 \ge 4$ and points $\vartheta_i \in \mathbb{S}^1$, $i \in \{1, \dots, i_0\}$, so that

$$\left(\frac{1}{\kappa^2}\frac{\partial\kappa}{\partial\vartheta}\right)(\vartheta_i, \tau) = 0$$

and

$$|\vartheta_i - \vartheta_{i+1}| \le \frac{\pi}{2}$$

for $i \in \{1, \dots, i_0 - 1\}$ and

$$|\vartheta_{i_0} - (2\pi + \vartheta_1)| \le \frac{\pi}{2}\,.$$

Since $1/\kappa^2 \frac{\partial}{\partial\vartheta}\kappa$ is periodic on $[0, 2\pi]$, i_0 is even. Define the intervals

$$I_i := [\theta_i, \theta_{i+1}]$$

for $i \in \{1, \dots, i_0 - 1\}$ and

$$I_{i_0} := [0, \theta_1] \cup [\theta_{i_0}, 2\pi]\,.$$

Then $|I_i| \le \pi/2$ for all $i \in \{1, \dots, i_0\}$. We apply Wirtinger's inequality, Lemma 11.18, to the function $1/\kappa^2 \frac{\partial}{\partial\vartheta}\kappa$ on the intervals I_i, where we identify $I_{i_0} = [\theta_{i_0}, 2\pi + \theta_1]$. This yields

$$\int_{I_i} \left(\frac{1}{\kappa^2}\frac{\partial\kappa}{\partial\vartheta}\right)^2 d\vartheta \le \frac{1}{4}\int_{I_i} \left[\frac{\partial}{\partial\vartheta}\left(\frac{1}{\kappa^2}\frac{\partial\kappa}{\partial\vartheta}\right)\right]^2 d\vartheta$$

for all $i \in \{1, \dots, i_0\}$ and thus

$$\int_{\mathbb{S}^1} \left(\frac{1}{\kappa^2}\frac{\partial\kappa}{\partial\vartheta}\right)^2 d\vartheta \le \frac{1}{4}\int_{\mathbb{S}^1} \left[\frac{\partial}{\partial\vartheta}\left(\frac{1}{\kappa^2}\frac{\partial\kappa}{\partial\vartheta}\right)\right]^2 d\vartheta\,. \tag{11.30}$$

For $\varepsilon > 0$, Peter–Paul (B.2) implies

$$\begin{aligned} 4\int_{\mathbb{S}^1} \left(\frac{1}{\kappa^2}\frac{\partial\kappa}{\partial\vartheta}\right)^2 d\vartheta &\overset{(11.30)}{\le} \int_{\mathbb{S}^1} \left[\frac{\partial}{\partial\vartheta}\left(\frac{1}{\kappa^2}\frac{\partial\kappa}{\partial\vartheta}\right)\right]^2 d\vartheta \\ &= \int_{\mathbb{S}^1} \left[\frac{1}{\kappa^2}\frac{\partial^2\kappa}{\partial\vartheta^2} - \frac{2}{\kappa^3}\left(\frac{\partial\kappa}{\partial\vartheta}\right)^2\right]^2 d\vartheta \\ &= \int_{\mathbb{S}^1} \left[\frac{1}{\kappa^4}\left(\frac{\partial^2\kappa}{\partial\vartheta^2}\right)^2 - \frac{4}{\kappa^5}\frac{\partial^2\kappa}{\partial\vartheta^2}\left(\frac{\partial\kappa}{\partial\vartheta}\right)^2 + \frac{4}{\kappa^6}\left(\frac{\partial\kappa}{\partial\vartheta}\right)^4\right] d\vartheta \\ &\overset{(B.2)}{\le} \int_{\mathbb{S}^1} \left[(1+4\varepsilon)\frac{1}{\kappa^4}\left(\frac{\partial^2\kappa}{\partial\vartheta^2}\right)^2 + \left(4+\frac{1}{\varepsilon}\right)\frac{1}{\kappa^6}\left(\frac{\partial\kappa}{\partial\vartheta}\right)^4\right] d\vartheta\,. \end{aligned} \tag{11.31}$$

For arbitrary $\beta \in (0,1)$, it is possible to choose $\varepsilon(\beta) > 0$ small enough so that

$$\begin{aligned} &\left[\frac{4}{(1/R_0+\varepsilon)^4} - \frac{\varepsilon^2(4+1/\varepsilon)}{(1/R_0-\varepsilon)^4}\right]\frac{(1/R_0-\varepsilon)^4}{1+4\varepsilon} \\ &\quad= \left[\left(\frac{1/R_0-\varepsilon}{1/R_0+\varepsilon}\right)^4 - \varepsilon\left(\varepsilon+\frac{1}{4}\right)\right]\frac{4}{1+4\varepsilon} \ge 4\beta\,. \end{aligned} \tag{11.32}$$

By Corollary 11.14, we can choose $\tau_0(\beta) > T_0$ large enough so that, for $\tau \ge \tau_0$,

$$\frac{1}{R_0} - \varepsilon \le \kappa \le \frac{1}{R_0} + \varepsilon \tag{11.33}$$

and, by the uniform convergence in Theorem 11.17, for $\tau \ge \tau_0$,

$$\max_{\mathbb{S}^1}\left|\frac{\partial\kappa}{\partial\vartheta}\right| \le \varepsilon\left(\frac{1}{R_0} - \varepsilon\right)\,. \tag{11.34}$$

We use (11.31), (11.33) and (11.34) to estimate, for $\tau \ge \tau_0$,

$$\begin{aligned} \frac{4}{(1/R_0+\varepsilon)^4}\int_{\mathbb{S}^1}\left(\frac{\partial\kappa}{\partial\vartheta}\right)^2 d\vartheta &\overset{(11.33)}{\le} 4\int_{\mathbb{S}^1}\left(\frac{1}{\kappa^2}\frac{\partial\kappa}{\partial\vartheta}\right)^2 d\vartheta \\ &\overset{(11.31)}{\le} \int_{\mathbb{S}^1} \left[(1+4\varepsilon)\frac{1}{\kappa^4}\left(\frac{\partial^2\kappa}{\partial\vartheta^2}\right)^2 + \left(4+\frac{1}{\varepsilon}\right)\frac{1}{\kappa^6}\left(\frac{\partial\kappa}{\partial\vartheta}\right)^4\right] d\vartheta \\ &\overset{(11.33)}{\le} \frac{1+4\varepsilon}{(1/R_0-\varepsilon)^4}\int_{\mathbb{S}^1}\left(\frac{\partial^2\kappa}{\partial\vartheta^2}\right)^2 d\vartheta + \frac{4+1/\varepsilon}{(1/R_0-\varepsilon)^6}\int_{\mathbb{S}^1}\left(\frac{\partial\kappa}{\partial\vartheta}\right)^4 d\vartheta \\ &\le \frac{1+4\varepsilon}{(1/R_0-\varepsilon)^4}\int_{\mathbb{S}^1}\left(\frac{\partial^2\kappa}{\partial\vartheta^2}\right)^2 d\vartheta + \frac{4+1/\varepsilon}{(1/R_0-\varepsilon)^6}\max_{\mathbb{S}^1}\left(\frac{\partial\kappa}{\partial\vartheta}\right)^2\int_{\mathbb{S}^1}\left(\frac{\partial\kappa}{\partial\vartheta}\right)^2 d\vartheta \\ &\overset{(11.34)}{\le} \frac{1+4\varepsilon}{(1/R_0-\varepsilon)^4}\int_{\mathbb{S}^1}\left(\frac{\partial^2\kappa}{\partial\vartheta^2}\right)^2 d\vartheta + \frac{\varepsilon^2(4+1/\varepsilon)}{(1/R_0-\varepsilon)^4}\int_{\mathbb{S}^1}\left(\frac{\partial\kappa}{\partial\vartheta}\right)^2 d\vartheta\,. \end{aligned}$$

Rearranging terms yields

$$\left[\frac{4}{(1/R_0+\varepsilon)^4} - \frac{\varepsilon^2(4+1/\varepsilon)}{(1/R_0-\varepsilon)^4}\right]\frac{(1/R_0-\varepsilon)^4}{1+4\varepsilon}\int_{\mathbb{S}^1}\left(\frac{\partial\kappa}{\partial\vartheta}\right)^2 d\vartheta \le \int_{\mathbb{S}^1}\left(\frac{\partial^2\kappa}{\partial\vartheta^2}\right)^2 d\vartheta$$

for $\tau \ge \tau_0$, which, together with (11.32), proves the claim. □

Lemma 11.20 ([GH86, Lemma 5.7.10] and [Pih98, Lemma 7.24]). *Let Σ_{T_0} be a smooth, embedded, convex curve of area $A_{T_0} = \pi R_0^2$ for the APCSF and length $L_{T_0} = 2\pi R_0$ for the LPCF. Let $F : \mathbb{S}^1 \times [T_0, \infty) \to \mathbb{R}^2$ be a solution of (2.15) with initial curve Σ_{T_0}. Let $\beta \in (0,1)$. Then there exists a time $\tau_0 > T_0$ and a constant $D = D(\Sigma_{T_0}, \Sigma_{\tau_0}, \tau_0) > 0$ such that*

$$\int_{\mathbb{S}^1} \left(\frac{\partial \kappa}{\partial \vartheta}\right)^2 d\vartheta \leq D \exp\left(-\frac{2\beta\tau}{R_0^2}\right)$$

for all $\tau \geq \tau_0$.

Proof. Analogous results can be found in [GH86, Lemma 5.7.10] and [Pih98, Lemma 7.24]. We use the evolution equation (11.4) of the curvature and integration by parts (B.4) to calculate

$$\begin{aligned}
\frac{\partial}{\partial \tau} \int_{\mathbb{S}^1} \left(\frac{\partial \kappa}{\partial \vartheta}\right)^2 d\vartheta &= 2 \int_{\mathbb{S}^1} \frac{\partial \kappa}{\partial \vartheta} \frac{\partial}{\partial \vartheta} \left(\frac{\partial \kappa}{\partial \tau}\right) d\vartheta \\
&\overset{(11.4)}{=} 2 \int_{\mathbb{S}^1} \frac{\partial \kappa}{\partial \vartheta} \frac{\partial}{\partial \vartheta} \left(\kappa^2 \frac{\partial^2 \kappa}{\partial \vartheta^2} - (h - \kappa)\kappa^2\right) d\vartheta \\
&= 2 \int_{\mathbb{S}^1} \frac{\partial \kappa}{\partial \vartheta} \left[\frac{\partial}{\partial \vartheta} \left(\kappa^2 \frac{\partial^2 \kappa}{\partial \vartheta^2}\right) - \frac{\partial}{\partial \vartheta} \left(h\kappa^2 - \kappa^3\right)\right] d\vartheta \\
&\overset{(B.4)}{=} 2 \int_{\mathbb{S}^1} \left[-\kappa^2 \left(\frac{\partial^2 \kappa}{\partial \vartheta^2}\right)^2 - \left(2h\kappa - 3\kappa^2\right) \left(\frac{\partial \kappa}{\partial \vartheta}\right)^2\right] d\vartheta \,. \qquad (11.35)
\end{aligned}$$

Let $\beta \in (0,1)$ be arbitrary. Choose

$$\gamma(\beta) \in \left(\frac{1}{4}, 1\right) \qquad \text{and} \qquad \varepsilon(\gamma, \beta) \in \left(0, \frac{1}{6R_0}\right)$$

so that

$$(4\gamma + 2)\,(1 - \varepsilon R_0)^2 - 3\,(1 + \varepsilon R_0)^2 \geq \beta \,. \qquad (11.36)$$

Corollary 11.14, implies that there exists $\tau_0(\varepsilon, \gamma, \beta) = \tau_0(\beta) > T_0$ so that, for all $\tau \geq \tau_0$,

$$\frac{1}{R_0} - \varepsilon \leq \kappa \leq \frac{1}{R_0} + \varepsilon \qquad (11.37)$$

as well as

$$\frac{1}{R_0} - \varepsilon \leq h \leq \frac{1}{R_0} + \varepsilon \qquad (11.38)$$

and, by Lemma 11.19,

$$\int_{\mathbb{S}^1} \left(\frac{\partial^2 \kappa}{\partial \vartheta^2}\right)^2 d\vartheta \geq 4\gamma \int_{\mathbb{S}^1} \left(\frac{\partial \kappa}{\partial \vartheta}\right)^2 d\vartheta \,. \qquad (11.39)$$

Hence,

$$
\begin{aligned}
\frac{\partial}{\partial\tau}\int_{\mathbb{S}^1}\left(\frac{\partial\kappa}{\partial\vartheta}\right)^2 d\vartheta &\overset{(11.35)}{=} 2\int_{\mathbb{S}^1}\left[-\kappa^2\left(\frac{\partial^2\kappa}{\partial\vartheta^2}\right)^2-(2h\kappa-3\kappa^2)\left(\frac{\partial\kappa}{\partial\vartheta}\right)^2\right]d\vartheta\\
&\overset{(11.37),(11.38)}{\leq} -2\left(\frac{1}{R_0}-\varepsilon\right)^2\int_{\mathbb{S}^1}\left(\frac{\partial^2\kappa}{\partial\vartheta^2}\right)^2 d\vartheta\\
&\qquad -2\left[2\left(\frac{1}{R_0}-\varepsilon\right)^2-3\left(\frac{1}{R_0}+\varepsilon\right)^2\right]\int_{\mathbb{S}^1}\left(\frac{\partial\kappa}{\partial\vartheta}\right)^2 d\vartheta\\
&\overset{(11.39)}{\leq} -8\gamma\left(\frac{1}{R_0}-\varepsilon\right)^2\int_{\mathbb{S}^1}\left(\frac{\partial\kappa}{\partial\vartheta}\right)^2 d\vartheta\\
&\qquad -2\left[2\left(\frac{1}{R_0}-\varepsilon\right)^2-3\left(\frac{1}{R_0}+\varepsilon\right)^2\right]\int_{\mathbb{S}^1}\left(\frac{\partial\kappa}{\partial\vartheta}\right)^2 d\vartheta\\
&= -\frac{2}{R_0^2}\left[(4\gamma+2)\left(1-\varepsilon R_0\right)^2-3\left(1+\varepsilon R_0\right)^2\right]\int_{\mathbb{S}^1}\left(\frac{\partial\kappa}{\partial\vartheta}\right)^2 d\vartheta\\
&\overset{(11.36)}{\leq} -\frac{2\beta}{R_0^2}\int_{\mathbb{S}^1}\left(\frac{\partial\kappa}{\partial\vartheta}\right)^2 d\vartheta
\end{aligned}
$$

for all $\tau \geq \tau_0$. We apply Lemma B.1 to solve the above ODE inequality and estimate, for all $\tau \geq \tau_0$,

$$\int_{\mathbb{S}^1}\left(\frac{\partial\kappa}{\partial\vartheta}(\vartheta,\tau)\right)^2 d\vartheta \leq \exp\!\left(-\frac{2\beta}{R_0^2}(\tau-\tau_0)\right)\int_{\mathbb{S}^1}\left(\frac{\partial\kappa}{\partial\vartheta}(\vartheta,\tau_0)\right)^2 d\vartheta\,. \qquad \square$$

Lemma 11.21 (See also [Pih98, Theorem 7.28])**.** *Let Σ_{T_0} be a smooth, embedded, convex curve of area $A_{T_0} = \pi R_0^2$ for the APCSF and length $L_{T_0} = 2\pi R_0$ for the LPCF. Let $F : \mathbb{S}^1 \times [T_0,\infty) \to \mathbb{R}^2$ be a solution of (2.15) with initial curve Σ_{T_0}. Let $\beta \in (0,1)$ and $n \in \mathbb{N}$. Then there exists a time $\tau_0 > T_0$ and a constant $\bar{D}_n = \bar{D}_n(n,\Sigma_{T_0},\Sigma_{\tau_0},\tau_0) > 0$ such that*

$$\max_{\vartheta\in\mathbb{S}^1}\left|\frac{\partial^n\kappa}{\partial\vartheta^n}(\vartheta,\tau)\right| \leq \bar{D}_n\exp\left(-\frac{\beta\tau}{(n+1)R_0^2}\right)$$

for all $\tau \geq \tau_0$.

Proof. For $n \in \mathbb{N}\cup\{0\}$, $C^{n+1}(\mathbb{S}^1) \subset C^n(\mathbb{S}^1) \subset L^2(\mathbb{S}^1)$. Furthermore, every bounded sequence in $C^{n+1}(\mathbb{S}^1)$ has a convergent subsequence in $C^n(\mathbb{S}^1)$ and $\|f\|_{L^2(\mathbb{S}^1)} \leq \sqrt{2\pi}\|f\|_{C^n(\mathbb{S}^1)}$ for every $f \in C^n(\mathbb{S}^1)$. Hence,

$$C^{n+1}(\mathbb{S}^1) \underset{\text{compact}}{\hookrightarrow} C^n(\mathbb{S}^1) \underset{\text{continuous}}{\hookrightarrow} L^2(\mathbb{S}^1)\,.$$

Let $f \in C^{n+1}(\mathbb{S}^1)$. Ehrling's lemma, Theorem B.15, yields that, for all $\delta > 0$, there exists a constant $C(\delta) > 0$ so that

$$\|f\|_{C^n(\mathbb{S}^1)} \leq \delta\|f\|_{C^{n+1}(\mathbb{S}^1)} + C(\delta)\|f\|_{L^2(\mathbb{S}^1)}\,.$$

For $\delta = 1/2$, we obtain

$$\max_{\mathbb{S}^1}|D^nf| \leq \max_{\mathbb{S}^1}|D^{n+1}f| + C\|f\|_{L^2(\mathbb{S}^1)}\,. \tag{11.40}$$

Let $\varepsilon > 0$ and define $\kappa_\varepsilon : [0, 2\pi/\varepsilon) \times (T_0, \infty) \to \mathbb{R}$ by

$$\kappa_\varepsilon(\vartheta, \tau) := \kappa(\varepsilon\vartheta, \tau)\,.$$

Then

$$\frac{\partial^n \kappa_\varepsilon}{\partial \vartheta^n}(\vartheta, \tau) = \varepsilon^n \frac{\partial \kappa}{\partial \vartheta}(\varepsilon\vartheta, \tau) \tag{11.41}$$

for $n \in \mathbb{N}$. Let $\beta \in (0,1)$. By Corollary 11.16 and Lemma 11.20, there exists a time $\tau_0(\beta, \Sigma_0) > T_0$ and constants $\bar{C}_{n+1} = \bar{C}_{n+1}(n, \Sigma_{T_0}, \Sigma_{\tau_0})$ and $D = D(\Sigma_{T_0}, \Sigma_{\tau_0}, \tau_0)$ so that

$$\begin{aligned}
\varepsilon^n \max_{\vartheta \in \mathbb{S}^1} \left| \frac{\partial^n \kappa}{\partial \vartheta^n}(\vartheta, \tau) \right| &= \varepsilon^n \max_{\vartheta \in [0, 2\pi/\varepsilon)} \left| \frac{\partial^n \kappa}{\partial \vartheta^n}(\varepsilon\vartheta, \tau) \right| \overset{(11.41)}{=} \max_{\vartheta \in [0, 2\pi/\varepsilon)} \left| \frac{\partial^n \kappa_\varepsilon}{\partial \vartheta^n}(\vartheta, \tau) \right| \\
&\overset{(11.40)}{\leq} \max_{\vartheta \in [0, 2\pi/\varepsilon)} \left| \frac{\partial^{n+1} \kappa_\varepsilon}{\partial \vartheta^{n+1}}(\vartheta, \tau) \right| + C \left(\int_0^{2\pi/\varepsilon} \left(\frac{\partial \kappa_\varepsilon}{\partial \vartheta}(\vartheta, \tau) \right)^2 d\vartheta \right)^{1/2} \\
&\overset{(11.41)}{=} \varepsilon^{n+1} \max_{\vartheta \in [0, 2\pi/\varepsilon)} \left| \frac{\partial^{n+1} \kappa}{\partial \vartheta^{n+1}}(\varepsilon\vartheta, \tau) \right| + \frac{C\varepsilon}{\sqrt{\varepsilon}} \left(\int_0^{2\pi/\varepsilon} \left(\frac{\partial \kappa}{\partial \vartheta}(\varepsilon\vartheta, \tau) \right)^2 d\vartheta \right)^{1/2} \\
&= \varepsilon^{n+1} \max_{\vartheta \in \mathbb{S}^1} \left| \frac{\partial^{n+1} \kappa}{\partial \vartheta^{n+1}}(\vartheta, \tau) \right| + C\sqrt{\varepsilon} \left(\int_{\mathbb{S}^1} \left(\frac{\partial \kappa}{\partial \vartheta}(\vartheta, \tau) \right)^2 d\vartheta \right)^{1/2} \\
&\underset{\text{Lem. } 11.20}{\overset{\text{Cor. } 11.16}{\leq}} \varepsilon^{n+1} \bar{C}_{n+1} + C\sqrt{\varepsilon}\sqrt{D} \exp\left(-\frac{\beta\tau}{R_0^2} \right)
\end{aligned}$$

for all $\tau \geq \tau_0$. Thus,

$$\max_{\vartheta \in \mathbb{S}^1} \left| \frac{\partial^n \kappa}{\partial \vartheta^n}(\vartheta, \tau) \right| \leq \bar{D}_n \left[\varepsilon + \varepsilon^{1/2-n} \exp\left(-\frac{\beta\tau}{R_0^2} \right) \right]\,, \tag{11.42}$$

where $\bar{D}_n = \bar{D}_n(n, \Sigma_{T_0}, \Sigma_{\tau_0}, \tau_0)$. We choose

$$\varepsilon = \exp\left(-\frac{\beta\tau}{(n+1/2)R_0^2} \right)\,,$$

then

$$\varepsilon^{1/2-n} \exp\left(-\frac{\beta\tau}{R_0^2} \right) = \exp\left(-\frac{(1/2-n)\beta\tau}{(n+1/2)R_0^2} - \frac{\beta\tau}{R_0^2} \right) = \exp\left(-\frac{\beta\tau}{(n+1/2)R_0^2} \right)$$

and

$$\max_{\vartheta \in \mathbb{S}^1} \left| \frac{\partial^n \kappa}{\partial \vartheta^n}(\vartheta, \tau) \right| \overset{(11.42)}{\leq} 2\bar{D}_n \exp\left(-\frac{\beta\tau}{(n+1/2)R_0^2} \right) \leq 2\bar{D}_n \exp\left(-\frac{\beta\tau}{(n+1)R_0^2} \right)$$

for all $\tau \geq \tau_0$. □

Corollary 11.22. *Let Σ_0 be a smooth, embedded curve of area $A_0 = \pi R_0^2$ for the APCSF and length $L_0 = 2\pi R_0$ for the LPCF. Let $F : \mathbb{S}^1 \times [0, \infty) \to \mathbb{R}^2$ be a solution of (2.15) with initial curve Σ_0. Let $\beta \in (0,1)$ and $m, n \in \mathbb{N} \cup \{0\}$, $m + n > 0$. Then there exists a constant $D_{n,m} > 0$ such that*

$$\max_{s \in \mathbb{S}^1_{R_t}} \left| \frac{\partial^m}{\partial t^m} \frac{\partial^n \kappa}{\partial s^n}(s, t) \right| \leq D_{n,m} \exp\left(-\frac{\beta t}{(n+2m+1)R_0^2} \right)$$

for all $t \geq 0$.

Proof. By Lemma 4.6, we can express time derivatives of the curvature in terms of spatial derivatives of the curvature. Hence, it suffices to show the claim for $m = 0$. By Theorem 10.9, there exists a time $T_0 \geq 0$ so that Σ_t is strictly convex for $t > T_0$. By the transformation (11.3) of the derivatives, the boundedness of the curvature, $t = \tau$, Corollary 11.16 and Lemma 11.21 implies that, for $\beta \in (0, 1)$, there exist a time $\tau_0(\beta, \Sigma_0) > T_0$ and constants $\bar{C}_0, \dots, \bar{C}_{n-2}, \bar{D}_1, \dots, \bar{D}_n > 0$ depending only on n, Σ_{T_0}, Σ_{τ_0} and τ_0 so that

$$\max_{s \in \mathbb{S}^1_{R_t}} \left| \frac{\partial^n \kappa}{\partial s^n}(s,t) \right| = \max_{\vartheta \in \mathbb{S}^1} \left| \left(\kappa(\vartheta,\tau) \frac{\partial}{\partial \vartheta} \right)^n \kappa(\vartheta,\tau) \right| \overset{\text{Lem. 11.21}}{\leq} c(n, \bar{C}_0, \dots, \bar{C}_{n-2}, \bar{D}_1, \dots, \bar{D}_n) \exp\left(-\frac{\beta t}{(n+1)R_0^2} \right)$$

for all $\tau = t \geq \tau_0$. On the other hand, Corollary 11.15 implies that $\left| \frac{\partial^n}{\partial s^n} \kappa \right|$ is bounded on $[0, \tau_0]$ by a constant $C_n(n, \Sigma_0)$. Choosing D_n sufficiently large, yields the claim. □

Remark 11.23. By repeating the proof of Lemma 11.20 for higher derivatives of the curvature and using a Sobolev inequality, we can also achieve a better exponential decay (see also [GH86, Lemmata 5.7.13–5.7.15] for a similar approach). More precisely, for $\beta \in (0, 1)$ and $m, n \in \mathbb{N} \cup \{0\}$, $m + n > 0$, there exists a time-independent constant $D_{n,m} > 0$ so that

$$\max_{s \in \mathbb{S}^1_{R_t}} \left| \frac{\partial^m}{\partial t^m} \frac{\partial^n \kappa}{\partial s^n}(s,t) \right| \leq D_{n,m} \exp\left(-\frac{2\beta t}{R_0^2} \right)$$

for all $t \geq 0$.

We summarise our results in the following and two theorems. We call a solution *immortal* if it exists for all positive times.

Theorem 11.24 (Exponential convergence for immortal solutions)**.** *Let Σ_0 be a smooth, embedded curve of area $A_0 = \pi R_0^2$ for the APCSF and length $L_0 = 2\pi R_0$ for the LPCF. Let $F : \mathbb{S}^1 \times [0, \infty) \to \mathbb{R}^2$ be a solution of (2.15) with initial curve Σ_0. Then the evolving surfaces $\Sigma_t = F(\mathbb{S}^1, t)$ are contained in a uniformly bounded region of the plane for all times. And, for all $\beta \in (0, 1)$, there exists a time-independent constant $C > 0$ such that, for all $t \geq 0$,*

(i) $|\kappa_{\max}(t) - \kappa_{\min}(t)| \leq C \exp\left(-\frac{\beta}{R_0^2} t \right)$,

(ii) $|\kappa(p,t) - 1/R_0| \leq C \exp\left(-\frac{\beta}{R_0^2} t \right)$ *for all* $p \in \mathbb{S}^1$,

(iii) $|h(t) - 1/R_0| \leq C \exp\left(-\frac{\beta}{R_0^2} t \right)$, *and*

(iv) $\left| \frac{\partial^n}{\partial t^m} \frac{\partial^n}{\partial p^n} \kappa(p,t) \right| \leq C \exp\left(-\frac{\beta}{(n+2m+1)R_0^2} t \right)$ *for all* $p \in \mathbb{S}^1$ *and all* $n, m \in \mathbb{N}$.

Hence, the solution converges smoothly and exponentially to a circle of radius R_0.

Remark 11.25. Following Remark 11.23, we can obtain exponential decay of $\exp\left(-\frac{2\beta}{R_0^2} t \right)$ in Theorem 11.24(i)–(iv).

Proof of Theorem 11.24. Since, by assumption, the solution exists for all times, Proposition 4.9 yields that the curvature is uniformly bounded on every finite time interval. By Lemma 3.13, the curves stay in a bounded region on every finite time interval. Theorem 6.5 implies that the curves remain embedded on $(0, \infty)$. By Theorem 10.9, there exists a time $T_0 > 0$ so that the curves are strictly convex on (T_0, ∞). For the first order curvature convergence, we partially follow the lines of [Pih98, Proposition 7.27]. Fix $t \geq 0$. Let $s_1, s_2 \in \mathbb{S}^1_{R_t}$ be the points where the curvature attains its maximum and minimum. By Corollary 11.22, there exists a time-independent constant $D_1 > 0$ so that

$$\begin{aligned}|\kappa_{\max}(t) - \kappa_{\min}(t)| = |\kappa(s_2, t) - \kappa(s_1, t)| &= \left| \int_{s_1}^{s_2} \frac{\partial \kappa}{\partial s} \, ds_t \right| \\ &\leq \int_{\mathbb{S}^1_{R_t}} \left| \frac{\partial \kappa}{\partial s} \right| ds_t \overset{\text{Cor. 11.22}}{\leq} L_0 D_1 \exp\left(-\frac{\beta t}{R_0^2} \right) ,\end{aligned}$$

where we used $L_t \leq L_0$ for both flows. This proves claim (i). For claim (ii), first consider a fixed time $t > T_0$, where Σ_t is strictly convex. We observe that for the **APCSF**,

$$\begin{aligned}\kappa_{\min}(t) \leq \frac{1}{r_{\mathrm{circ}}(t)} &= \sqrt{\frac{\pi}{A(B_{r_{\mathrm{circ}}(t)})}} \leq \sqrt{\frac{\pi}{A_t}} = \sqrt{\frac{\pi}{A_0}} = \frac{1}{R_0} \\ &\leq \sqrt{\frac{\pi}{A(B_{r_{\mathrm{in}}(t)})}} = \frac{1}{r_{\mathrm{in}}(t)} \leq \kappa_{\max}(t) .\end{aligned}$$

For the **LPCF**,

$$\kappa_{\min}(t) \leq \frac{1}{L_0} \int_{\mathbb{S}^1_{R_0}} \kappa \, ds_t = \frac{2\pi}{L_0} = \frac{1}{R_0} \leq \kappa_{\max}(t) .$$

The intermediate value theorem yields for both flows that there exists a point $s_0 \in \mathbb{S}^1_{R_t}$ with $\kappa(s_0, t) = 1/R_0$, and we can estimate, for $s \in \mathbb{S}^1_{R_t}$,

$$\begin{aligned}\left| \kappa(s, t) - \frac{1}{R_0} \right| = |\kappa(s, t) - \kappa(s_0, t)| &= \left| \int_{s_0}^{s} \frac{\partial \kappa}{\partial s} \, ds_t \right| \\ &\leq \int_{\mathbb{S}^1_{R_t}} \left| \frac{\partial \kappa}{\partial s} \right| ds_t \overset{\text{Cor. 11.22}}{\leq} L_0 D_1 \exp\left(-\frac{\beta t}{R_0^2} \right) . \end{aligned} \tag{11.43}$$

The boundedness of the curvature on $[0, T_0]$ yields the claim for all $t \geq 0$. For claim (iii), we estimate for the **APCSF** with the upper bound (3.10) on the global term and $\kappa_{\min} \leq \pi / L_t = h_{\mathrm{ap}}$,

$$0 \overset{(3.10)}{<} \sqrt{\frac{\pi}{A_0}} - h_{\mathrm{ap}}(t) = \frac{1}{R_0} - h_{\mathrm{ap}}(t) \leq \frac{1}{R_0} - \kappa_{\min}(t) \overset{(11.43)}{\leq} L_0 D_1 \exp\left(-\frac{\beta t}{R_0^2} \right) \tag{11.44}$$

and for the **LPCF**, by the lower bound (3.11) on the global term and Corollary 5.9,

$$0 \overset{(3.11)}{<} h_{\mathrm{lp}}(t) - \frac{2\pi}{L_0} = h_{\mathrm{lp}}(t) - \frac{1}{R_0} \overset{\text{Cor. 5.9}}{\leq} \kappa_{\max}(t) - \frac{1}{R_0} \overset{(11.43)}{\leq} L_0 D_1 \exp\left(-\frac{\beta t}{R_0^2} \right) \tag{11.45}$$

for all $t \geq 0$. For claim (iv), we use the evolution equation (3.2) of the length element and the above inequalities to estimate for $p \in \mathbb{S}^1$, $t \geq 0$ and $\beta \in (0, 1)$,

$$\frac{\partial v}{\partial t} \overset{(3.2)}{=} \kappa(h - \kappa) v \leq |\kappa| \left(\left| h - \frac{1}{R_0} \right| + \left| \frac{1}{R_0} - \kappa \right| \right) v \underset{(11.45)}{\overset{(11.43),(11.44)}{\leq}} c \exp\left(-\frac{\beta t}{R_0^2} \right) v$$

for a time-independent constant $c = c(\Sigma_0, D_1) > 0$. By Lemma B.1,

$$v(p,t) \overset{\text{Lem. B.1}}{\leq} v(p,0)\exp\left(c\int_0^t \exp\left(-\frac{\beta\tau}{R_0^2}\right)d\tau\right)$$
$$= c(\Sigma_0)\exp\left(-\frac{cR_0^2}{\beta}\exp\left(-\frac{\beta t}{R_0^2}\right) + \frac{c(\Sigma_0)}{\beta}\right) \leq c(\Sigma_0, D_1, \beta) \tag{11.46}$$

for every $p \in \mathbb{S}^1$ and $t \geq 0$. By identity (A.3) for the arc length differentiation and Corollary 11.22,

$$\left|\frac{\partial\kappa}{\partial p}(p,t)\right| \overset{\text{(A.3)}}{=} \left|v(p,t)\frac{\partial\kappa}{\partial s}(s(p,t),t)\right| \overset{\text{Cor. 11.22}}{\leq} cD_1\exp\left(-\frac{\beta t}{R_0^2}\right)$$

for every $p \in \mathbb{S}^1$ and $t \geq 0$. We observe that

$$\left|v\frac{\partial v}{\partial s}\right| = \left|\frac{\partial v}{\partial p}\right| = v^{-1}\left|\left\langle\frac{\partial F}{\partial p}, \frac{\partial^2 F}{\partial p^2}\right\rangle\right| \overset{\text{(A.8)}}{\leq} v^2|\kappa| \overset{\text{(11.43),(11.46)}}{\leq} c(\Sigma_0, D_1, \beta)\,,$$

so that we can estimate, for every $m, n \in \mathbb{N}\cup\{0\}$, $m + n > 0$,

$$\left|\frac{\partial^m}{\partial t^m}\frac{\partial^n\kappa}{\partial p^n}\right| \overset{\text{(A.3)}}{=} \left|\frac{\partial^m}{\partial t^m}\left(v\frac{\partial}{\partial s}\right)^n\kappa\right|$$
$$\overset{\text{Cor. 11.22}}{\leq} c(n, m, \beta, \Sigma_0, D_1, \dots, D_{n+2m})\exp\left(-\frac{\beta t}{(n+2m+1)R_0^2}\right).$$

To show that the curves stay in a bounded region, we estimate

$$\|F(p,t) - F(p,0)\| \leq \int_0^t |\kappa(p,\tau) - h(\tau)|\,d\tau \underset{\text{(11.45)}}{\overset{\text{(11.43),(11.44)}}{\leq}} c\int_0^t \exp\left(-\frac{\beta}{R_0^2}\tau\right)d\tau \leq c$$

for all $p \in \mathbb{S}^1$ and $t \in (0,\infty)$, where c is independent of time. □

Corollary 11.26 (Solutions to the APCSF)**.** *Let Σ_0 be a smooth, embedded, closed curve of area $A_0 = \pi R_0^2$, satisfying $\theta_{\min} \geq -\pi$. Then there exists a unique, smooth, embedded solution $F : \mathbb{S}^1\times[0,\infty) \to \mathbb{R}^2$ to the APCSF with initial curve Σ_0. Hence, Theorem 11.24 holds and the solution converges smoothly and exponentially in a bounded domain to a circle of radius R_0.*

Proof. By Theorem 9.29, there exists a unique embedded solution $F : \mathbb{S}^1\times[0,\infty) \to \mathbb{R}^2$ to the APCSF with initial curve Σ_0 and $F \in C^\infty(\mathbb{S}^1\times(0,\infty))$. Hence, we can apply Theorem 11.24. □

Appendix A

Notation and geometric definitions

In this appendix we give short introductions to curves in $\mathbb{R}^2$ and hypersurfaces in $\mathbb{R}^{n+m}$.

A.1 Curves in $\mathbb{R}^2$

Let $I = [a, b] \subseteq \mathbb{R}$ be an interval. A C^k-map, $k \in \mathbb{N} \cup \{\infty\}$,

$$F : I \to \mathbb{R}^2$$

is called *parametrised curve of class* C^k in $\mathbb{R}^2$. A parametrised curve is called *regular* or *immersed* if

$$\frac{dF}{dp}(p) \neq 0$$

for all $p \in I$. A *curve* is the equivalence class $[F]$ of regular parameter transformations with image

$$\Sigma := F(I) \subset \mathbb{R}^2 .$$

We say that a curve is *embedded* if the map $F : I \to F(I)$ is injective. The immersion or embedding is called *proper* if for all compact subsets $K \subset \mathbb{R}^2$ the pre-image $F^{-1}(K \cap \Sigma)$ is compact. A C^k-curve is called *closed* if

$$\frac{d^l F}{dp^l}(a) = \frac{d^l F}{dp^l}(b)$$

for all $0 \leq l \leq k$. We can identify a and b so that the parametrisation is given by $F : \mathbb{S}^1_\rho \to \mathbb{R}^2$, where $\rho = (b - a)/(2\pi)$. For $F : \mathbb{S}^1_\rho \to \mathbb{R}^2$, we can define the parametrisation $\bar{F} : \mathbb{S}^1 \to \mathbb{R}^2$ by $\bar{F}(p/\rho) := F(p)$. Hence, we can assume w. l. o. g. that $\rho = 1$ so that, for arbitrary curves, the pre-image is given by

$$S \in \{I = [0, 2\pi], \mathbb{S}^1, \mathbb{R}\} .$$

For a parametrisation $F : I \to \mathbb{R}^2$ of a regular curve Σ, we define the *length element* $v : I \to \mathbb{R}$ of the curve by

$$v(p) := \left\| \frac{dF}{dp}(p) \right\| . \tag{A.1}$$

The length of the curve is then given by

$$L := L(\Sigma) := \int_\Sigma d\mathcal{H}^1 = \int_I v\, dp\,.$$

The *arc length parameter* $s : I \to [0, L]$ is defined as

$$s(p) := \int_0^p v(r)\, dr\,, \tag{A.2}$$

so that we also have

$$L = \int_0^L ds\,.$$

The intrinsic distance between two points is

$$l(p, q) := \int_p^q v(r)\, dr\,.$$

Notice that

$$l(p, q) = s(q) - s(p) = \int_{s(p)}^{s(q)} ds = \int_{F(p)}^{F(q)} d\mathcal{H}^1\,.$$

Differentiating (A.2) yields

$$\frac{ds}{dp}(p) = v(p)\,.$$

Let $f : [0, L] \to \mathbb{R}$ with $f : s \mapsto f(s)$ be a C^1-function. Then

$$\frac{df}{dp}(s(p)) = \frac{df}{ds}(s(p))\frac{ds}{dp}(p) = v(p)\frac{df}{ds}(s(p))$$

so that

$$\frac{d}{ds(p)} = \frac{1}{v(p)}\frac{d}{dp}\,. \tag{A.3}$$

For a C^2-curve, s is differentiable and s' is positive, thus the inverse function theorem, Theorem B.4, yields that the inverse $s^{-1} : [0, L] \to I$ exists. We define the anti-clockwise *arc length parametrisation* $\tilde{F} := F \circ s^{-1} : [0, L] \to \mathbb{R}^2$ of Σ by

$$\tilde{F}(q) = F\big(s^{-1}(q)\big)$$

for $q \in [0, L]$ so that $F = \tilde{F} \circ s$ and

$$\tilde{F}(s(p)) = F(p)$$

for $p \in I$. If Σ is a closed curve parametrised by $F : \mathbb{S}^1 \to \mathbb{R}^2$ the arc length parameter is given by $s : \mathbb{S}^1 \to \mathbb{S}^1_{L/(2\pi)}$ and $\tilde{F} : \mathbb{S}^1_{L/(2\pi)} \to \mathbb{R}^2$ parametrises Σ by arc length (see Figure A.1). If Σ is parametrised by $F : \mathbb{R} \to \mathbb{R}^2$ the arc length parameter is given by $s : \mathbb{R} \to \mathbb{R}$ and $\tilde{F} : \mathbb{R} \to \mathbb{R}^2$ parametrises Σ by arc length. Define the image of the arc length parameter as

$$\tilde{S} := s(S) = \tilde{F}^{-1}(\Sigma) \in \Big\{[0, L],\, \mathbb{S}^1_{L/(2\pi)},\, \mathbb{R}\Big\}\,,$$

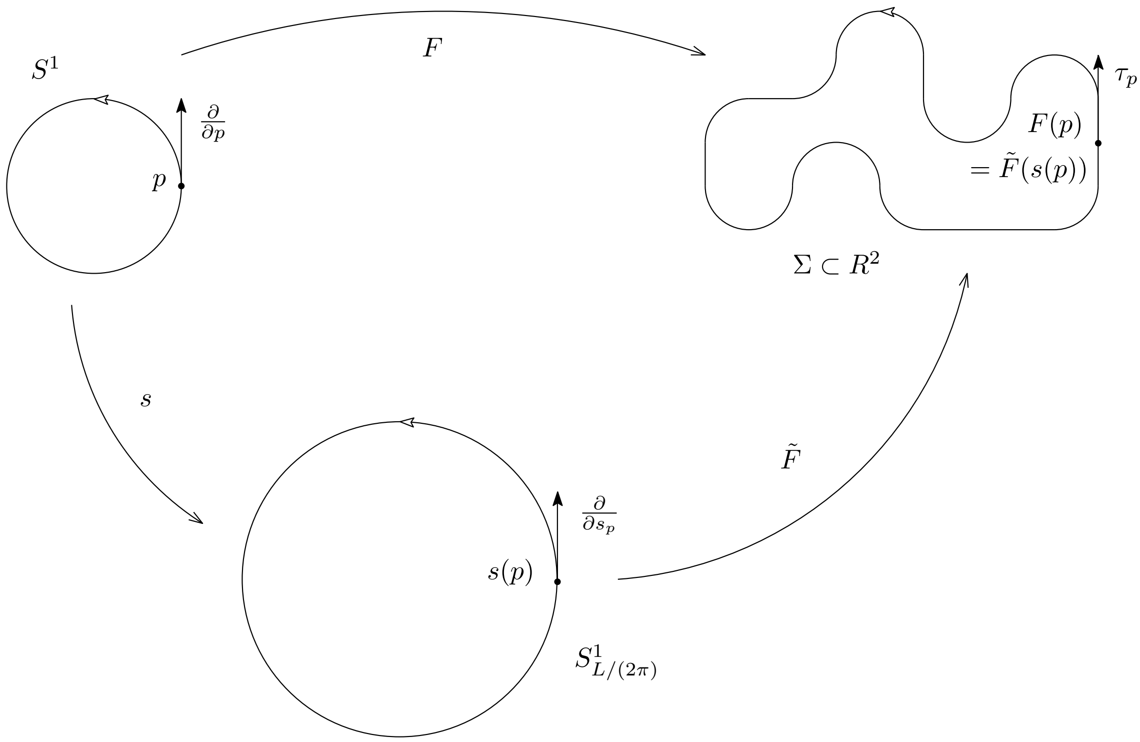

Figure A.1: The embeddings and the unit tangent vectors.

where $S \in \{I, \mathbb{S}^1, \mathbb{R}\}$. Differentiating $\tilde{F}$ at $s = s(p)$ yields

$$\left\| \frac{d\tilde{F}}{ds}(s) \right\| = \frac{1}{v(p)} \left\| \frac{dF}{dp}(p) \right\| = 1\,.$$

The *unit tangent vector field* $\boldsymbol{\tau}$ to Σ at $s = s(p)$ in direction of the arc length parametrisation is given by

$$\boldsymbol{\tau}_p := \boldsymbol{\tau}(p) = \frac{1}{v(p)} \frac{dF}{dp}(p) = \frac{d\tilde{F}}{ds}(s) = \boldsymbol{\tau}(s)\,. \tag{A.4}$$

For the *outward unit normal*,

$$\boldsymbol{\nu}_p := \boldsymbol{\nu}(p) = (\boldsymbol{\tau}_2(p), -\boldsymbol{\tau}_1(p)) = \boldsymbol{\nu}(s) \tag{A.5}$$

(see Figure A.2). The *curvature* $\kappa : \tilde{S} \to \mathbb{R}$ is defined as

$$\kappa(s) := -\langle \nabla_{\boldsymbol{\tau}} \boldsymbol{\tau}, \boldsymbol{\nu} \rangle = -\left\langle \frac{d\boldsymbol{\tau}}{ds}, \boldsymbol{\nu} \right\rangle \tag{A.6}$$

$$= \langle \boldsymbol{\tau}, \nabla_{\boldsymbol{\tau}} \boldsymbol{\nu} \rangle = \left\langle \boldsymbol{\tau}, \frac{d\boldsymbol{\nu}}{ds} \right\rangle\,. \tag{A.7}$$

Note that

$$\kappa \overset{(A.4),(A.6)}{=} -\left\langle \frac{d^2\tilde{F}}{ds^2}, \boldsymbol{\nu} \right\rangle \overset{(A.3)}{=} -\left\langle \frac{1}{v}\frac{d}{dp}\left(\frac{1}{v}\frac{dF}{dp}\right), \boldsymbol{\nu} \right\rangle$$

$$= -\frac{1}{v^2}\left\langle \frac{d^2F}{dp^2}, \boldsymbol{\nu} \right\rangle - \frac{1}{v}\frac{d}{dp}\left(\frac{1}{v}\right)\left\langle \frac{dF}{dp}, \boldsymbol{\nu} \right\rangle = -\frac{1}{v^2}\left\langle \frac{d^2F}{dp^2}, \boldsymbol{\nu} \right\rangle$$

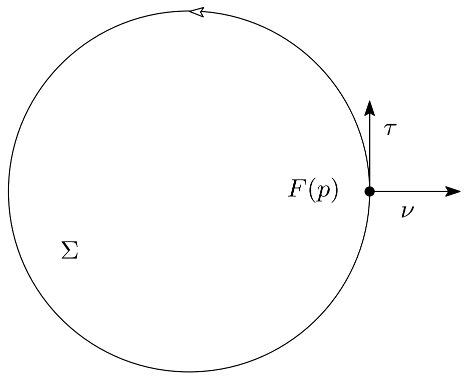

Figure A.2: The unit tangent vector and outward normal.

so that curvature $\kappa : S \to \mathbb{R}$ is given by

$$\kappa(p) = -\frac{1}{v^2}\left\langle \frac{d^2F}{dp^2}, \boldsymbol{\nu} \right\rangle \tag{A.8}$$

The *curvature vector* is defined as

$$\boldsymbol{\kappa} := -\kappa\boldsymbol{\nu}\,.$$

Lemma A.1 (Frenet–Serret Equations)**.** *Let $\tilde{F} : \tilde{S} \to \mathbb{R}^2$ be an arc length parametrisation of a C^2-curve Σ. Let $\boldsymbol{\tau} = \frac{d}{ds}\tilde{F}$ and $\boldsymbol{\nu} = (\boldsymbol{\tau}_2, -\boldsymbol{\tau}_1)$. Then*

$$\frac{d\boldsymbol{\tau}}{ds} = -\kappa\boldsymbol{\nu} \qquad \text{and} \qquad \frac{d\boldsymbol{\nu}}{ds} = \kappa\boldsymbol{\tau}\,.$$

Proof. The vector fields $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ are both of unit length. Since $\left\langle \frac{d}{ds}\boldsymbol{\tau}, \boldsymbol{\tau}\right\rangle = 0$, both $\frac{d}{ds}\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ are normal vectors and, by (A.6),

$$\frac{d\boldsymbol{\tau}}{ds} \overset{(A.3)}{=} \frac{d^2\tilde{F}}{ds^2} = \left\langle \frac{d^2\tilde{F}}{ds^2}, \boldsymbol{\nu} \right\rangle \boldsymbol{\nu} \overset{(A.6)}{=} -\kappa\boldsymbol{\nu}\,.$$

Since $\left\langle \frac{d}{ds}\boldsymbol{\nu}, \boldsymbol{\nu}\right\rangle = 0$, both $\frac{d}{ds}\boldsymbol{\nu}$ and $\boldsymbol{\tau}$ are tangent vectors and, by (A.7),

$$\frac{d\boldsymbol{\nu}}{ds} = \left\langle \frac{d\boldsymbol{\nu}}{ds}, \boldsymbol{\tau} \right\rangle \boldsymbol{\tau} \overset{(A.7)}{=} \kappa\boldsymbol{\tau}.$$ □

The first identity in Lemma A.1 can also be stated as

$$\Delta_\Sigma \tilde{F} = \frac{d^2\tilde{F}}{ds^2} = \boldsymbol{\kappa}\,. \tag{A.9}$$

Let $f : S \to \mathbb{R}$ be in C^2 and $\tilde{f} : \tilde{S} \to \mathbb{R}$ so that $f(p) = \tilde{f}(s(p))$. The arc length differentiation at $s = s(p)$ is then given by

$$\boldsymbol{\tau}_p(f) = \frac{d\tilde{f}}{ds}\,.$$

For a function $f : S \times S \to \mathbb{R}$ in C^2, the two-point arc length differentiation with respect to a vector $\xi_{(p,q)} \in T_{F(p)}\Sigma \bigoplus T_{F(q)}\Sigma$, $\xi_{(p,q)} = a\boldsymbol{\tau}_p \oplus b\boldsymbol{\tau}_q$ for $p, q \in \Sigma$ and $a, b \in \mathbb{R}$, is given by

$$\xi_{(p,q)}(f)(p,q) = \left(a\boldsymbol{\tau}_p \oplus b\boldsymbol{\tau}_q\right)(f)(p,q) = a\boldsymbol{\tau}_p(f)(p,q) + b\boldsymbol{\tau}_q(f)(p,q)\,. \tag{A.10}$$

Theorem A.2 (Theorem on turning tangents, [Küh06, Theorem 2.28])**.** *Let* $F : \mathbb{S}^1 \to \mathbb{R}^2$ *an embedded* C^2*-curve with outward pointing unit normal* (A.5)*. Then the* total curvature *of* Σ *is given by*

$$\int_\Sigma \kappa \, d\mathcal{H}^1 = 2\pi \,.$$

A.2 Hypersurfaces in $\mathbb{R}^{n+m}$

Let M^n, $n \geq 1$, be an abstract, smooth, compact, n-dimensional manifold without boundary and F a smooth embedding with

$$F : M^n \to \mathbb{R}^{n+m}$$

for $m \geq 1$. Set $M := F(M^n)$. For all $p \in M^n$ and $\mathbf{v}, \mathbf{w} \in T_pM^n$, the embedding F induces an isomorphism

$$dF_p : T_pM^n \to T_{F(p)}M \,,$$

and the first fundamental form or *metric* $G_p : T_pM^n \times T_pM^n \to \mathbb{R}$ with

$$G_p(\mathbf{v}, \mathbf{w}) := \langle dF_p(\mathbf{v}), dF_p(\mathbf{w}) \rangle \,.$$

If $\{p_i\}_{1 \leq i \leq n}$ are coordinates for M^n at p, then the matrix entries of the metric are

$$g_{ij}(p) = \left\langle dF_p\left(\frac{\partial}{\partial p_i}\right), dF_p\left(\frac{\partial}{\partial p_j}\right)\right\rangle = \left\langle \frac{\partial F}{\partial p_i}(p), \frac{\partial F}{\partial p_j}(p) \right\rangle \,,$$

where $\langle \cdot \,, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^{n+m}$. We define by g the determinant of the matrix $(g_{ij})_{ij}$ and by $(g^{ij})_{ij}$ its coordinate dependent inverse. The corresponding Levi–Cevita connection $\nabla := \nabla^M$ on M is given by

$$\nabla_{\mathbf{v}} \mathbf{w} = dF_p^{-1}\Big(\big(D_{dF_p(\mathbf{v})} dF_p(\mathbf{w})\big)^\top\Big) \,.$$

Here D is the standard connection in $\mathbb{R}^{n+m}$, and $^\top$ denotes the tangential component with respect to M, that is the orthogonal projection onto $dF(p)(T_pM^n) = T_{F(p)}M$. The connection can be evaluated in coordinates in terms of the Christoffel symbols Γ_{ij}^k defined by

$$\nabla_{\frac{\partial}{\partial p_i}} \frac{\partial}{\partial p_j} = \Gamma_{ij}^k \frac{\partial}{\partial p_k} \,,$$

where Γ_{ij}^k is explicitly given by

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial p_i} + \frac{\partial g_{il}}{\partial p_j} - \frac{\partial g_{ij}}{\partial p_k} \right) = g^{kl} \left\langle \frac{\partial^2 F}{\partial p_i \partial p_j}, \frac{\partial F}{\partial p_l} \right\rangle \,.$$

Here and in the following, we sum over repeated indices. Then,

$$\Gamma_{ij}^k \frac{\partial F}{\partial p_k} = \left\langle \frac{\partial^2 F}{\partial p_i \partial p_j}, \frac{\partial F}{\partial p_l} \right\rangle \frac{\partial F}{\partial p_l} \,. \tag{A.11}$$

For vectors $\mathbf{v} = \lambda^i \frac{\partial}{\partial p_i}$ and $\mathbf{w} = \mu^j \frac{\partial}{\partial p_j}$,

$$\nabla_{\mathbf{v}} \mathbf{w} = \lambda^i \left(\frac{\partial \mu^k}{\partial p_i} + \mu^j \Gamma^k_{ij} \right) \frac{\partial}{\partial p_k} .$$

For a 1-form $\omega = \omega^j dp^j$,

$$\nabla_{\mathbf{v}} \omega = \lambda^i \left(\frac{\partial \omega^k}{\partial p_i} - \omega^j \Gamma^k_{ij} \right) dp^k .$$

The tangential gradient, grad_M or ∇, of a function $f \in C^1(M)$ is given by

$$G(\operatorname{grad}_M f, \mathbf{v}) \equiv G(\nabla f, \mathbf{v}) = df(\mathbf{v}) = \mathbf{v}(f)$$

for all $\mathbf{v} \in TM$. In coordinates this reduces to

$$G\left(\nabla f, \frac{\partial}{\partial p_i}\right) = \frac{\partial f}{\partial p_i}$$

so that we obtain

$$\nabla f = g^{ij} \frac{\partial f}{\partial p_i} \frac{\partial}{\partial p_j}$$

and also

$$\nabla f = Df - \langle Df, \boldsymbol{\nu}_k \rangle \boldsymbol{\nu}_k = D_{\boldsymbol{\tau}_i} f \boldsymbol{\tau}_i = \boldsymbol{\tau}_i(f) \boldsymbol{\tau}_i \tag{A.12}$$

for an orthonormal tangent frame $\{\boldsymbol{\tau}_i\}_{1 \le i \le n}$ for $T_{F(p)}M$ and an orthonormal normal frame $\{\boldsymbol{\nu}_k\}_{1 \le k \le m}$ for $(T_{F(p)}M)^\perp$. We write the i-th component

$$\nabla_i f := \nabla_{\boldsymbol{\tau}_i} f = \langle \nabla f, \boldsymbol{\tau}_i \rangle = \langle \boldsymbol{\tau}_j(f) \boldsymbol{\tau}_j, \boldsymbol{\tau}_i \rangle = \boldsymbol{\tau}_j(f) \langle \boldsymbol{\tau}_j, \boldsymbol{\tau}_i \rangle = \boldsymbol{\tau}_i(f) .$$

We will use the abbreviation ∇_i for both $\nabla_{\frac{\partial}{\partial p_i}}$ and $\nabla_{\boldsymbol{\tau}_i}$. The tangential divergence $\operatorname{div}_M : T_pM \to \mathbb{R}$ of a tangent vector field $\mathbf{v} = \lambda^i \frac{\partial}{\partial p_i}$ is given by

$$\operatorname{div}_M \mathbf{v} = \frac{\partial \lambda^i}{\partial p_i} + \Gamma^i_{ij} \lambda^j$$

and also

$$\operatorname{div}_M \mathbf{v} = \operatorname{div}_{\mathbb{R}^{n+m}} \mathbf{v} - \langle D_{\boldsymbol{\nu}_k} \mathbf{v}, \boldsymbol{\nu}_k \rangle = \langle D_{\boldsymbol{\tau}_i} \mathbf{v}, \boldsymbol{\tau}_i \rangle .$$

For the embedding vector F, we therefore have

$$\operatorname{div}_M F = \langle D_{\boldsymbol{\tau}_i} F, \boldsymbol{\tau}_i \rangle = \langle \boldsymbol{\tau}_i, \boldsymbol{\tau}_i \rangle = n . \tag{A.13}$$

For $\omega = df = \frac{\partial f}{\partial p_i} dp^i$, we obtain the Hessian of the function f

$$(\operatorname{Hess}_M f)(\mathbf{v}, \mathbf{w}) := \left(\nabla^2 f\right)(\mathbf{v}, \mathbf{w}) = (\nabla_{\mathbf{v}} f)(\mathbf{w}) = \lambda^i \mu^j \left(\frac{\partial^2 f}{\partial p_i \partial p_j} - \Gamma^k_{ij} \frac{\partial f}{\partial p_k} \right) ,$$

or in coordinates

$$\nabla_i \nabla_j f \equiv \nabla_{\frac{\partial}{\partial p_i}} \nabla_{\frac{\partial}{\partial p_j}} f = \left\langle \nabla_{\frac{\partial}{\partial p_i}} \nabla f, \frac{\partial}{\partial p_j} \right\rangle = (\operatorname{Hess}_M f)\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j} \right) .$$

The Laplace–Beltrami operator Δ_M of a function $f \in C^2(M)$ is defined as

$$\Delta_M f := \operatorname{div}_M(\nabla f) = g^{ij}\nabla_i\nabla_j f$$

and also given by

$$\Delta_M f = \frac{1}{\sqrt{g}}\frac{\partial}{\partial p_j}\left(\sqrt{g}g^{ij}\frac{\partial}{\partial p_j}f\right).$$

We define the second fundamental form $\mathbf{A}_p : T_pM \times T_pM \to \left(T_{F(p)}M\right)^{\perp}$ by

$$\begin{aligned}\mathbf{A}_p(\mathbf{v},\mathbf{w}) &= -\left\langle D_{dF_p(\mathbf{v})}dF_p(\mathbf{w}), \boldsymbol{\nu}_k(p)\right\rangle \boldsymbol{\nu}_k(p)\\ &= \left\langle dF_p(\mathbf{w}), D_{dF_p(\mathbf{v})}\boldsymbol{\nu}_k(p)\right\rangle \boldsymbol{\nu}_k(p).\end{aligned}$$

In coordinates $\{p_i\}_{1\le i\le n}$,

$$\begin{aligned}\mathbf{A}_{ij} := \mathbf{A}_p\left(\frac{\partial}{\partial p_i},\frac{\partial}{\partial p_j}\right) &= \left\langle \frac{\partial F}{\partial p_i},\frac{\partial \boldsymbol{\nu}_k}{\partial p_j}\right\rangle \boldsymbol{\nu}_k\\ &= -\left\langle \frac{\partial^2 F}{\partial p_i\partial p_j},\boldsymbol{\nu}_k\right\rangle \boldsymbol{\nu}_k = -\frac{\partial^2 F}{\partial p_i\partial p_j} + \Gamma^k_{ij}\frac{\partial F}{\partial p_k}.\end{aligned}$$

The mean curvature vector $\mathbf{H} : M \to \left(T_{F(p)}M\right)^{\perp}$ is the trace of the second fundamental form

$$\mathbf{H} := -g^{ij}\mathbf{A}_{ij} = -g^{ij}\left\langle \frac{\partial F}{\partial p_i},\frac{\partial \boldsymbol{\nu}_k}{\partial p_j}\right\rangle \boldsymbol{\nu}_k = -\operatorname{div}(\boldsymbol{\nu}_k)\boldsymbol{\nu}_k. \tag{A.14}$$

Using (A.11) we can also calculate that

$$\begin{aligned}\Delta_M F = g^{ij}\left(\frac{\partial^2 F}{\partial p_i\partial p_j} - \Gamma^k_{ij}\frac{\partial F}{\partial p_k}\right) &\overset{(A.11)}{=} g^{ij}\left\langle \frac{\partial^2 F}{\partial p_i\partial p_j},\boldsymbol{\nu}_\gamma\right\rangle \boldsymbol{\nu}_\gamma\\ &= -g^{ij}\left\langle \frac{\partial F}{\partial p_i},\frac{\partial \boldsymbol{\nu}_\gamma}{\partial p_j}\right\rangle \boldsymbol{\nu}_\gamma = \mathbf{H},\end{aligned} \tag{A.15}$$

where $i,j,k = 1,\dots,n$ and $\gamma = 1,\dots,m$.

For a submanifold Σ of M, the mean curvature vector is given by

$$\mathbf{H}_\Sigma(p) = -\operatorname{div}_\Sigma(\boldsymbol{\nu}_k(p))\boldsymbol{\nu}_k(p) - \operatorname{div}_\Sigma(\boldsymbol{\nu}_\Sigma(p))\boldsymbol{\nu}_\Sigma(p),$$

where $\boldsymbol{\nu}_\Sigma$ is the unit co-normal of Σ. Since $\boldsymbol{\nu}_\Sigma$ tangential to M,

$$\langle \mathbf{H}_\Sigma, \boldsymbol{\nu}_\Sigma\rangle = -\operatorname{div}_\Sigma \boldsymbol{\nu}_\Sigma \tag{A.16}$$

and on Σ, using (A.11) restricted to Σ,

$$\begin{aligned}\Delta_\Sigma F_{|\Sigma} &\overset{(A.15)}{=} g^{ij}_\Sigma\left(\frac{\partial^2 F_{|\Sigma}}{\partial p_i\partial p_j} - {}^\Sigma\Gamma^k_{ij}\frac{\partial F_{|\Sigma}}{\partial p_l}\right)\\ &\overset{(A.11)}{=} g^{ij}_\Sigma\left\langle \frac{\partial^2 F_{|\Sigma}}{\partial p_i\partial p_j},\boldsymbol{\nu}_\gamma\right\rangle \boldsymbol{\nu}_\gamma + g^{ij}_\Sigma\left\langle \frac{\partial^2 F_{|\Sigma}}{\partial p_i\partial p_j},\boldsymbol{\nu}_\Sigma\right\rangle \boldsymbol{\nu}_\Sigma\\ &= -g^{ij}_\Sigma\left\langle \frac{\partial F_{|\Sigma}}{\partial p_i},\frac{\partial \boldsymbol{\nu}_\gamma}{\partial p_j}\right\rangle \boldsymbol{\nu}_\gamma - g^{ij}_\Sigma\left\langle \frac{\partial F_{|\Sigma}}{\partial p_l},\frac{\partial \boldsymbol{\nu}_\Sigma}{\partial p_j}\right\rangle \boldsymbol{\nu}_\Sigma = \mathbf{H}_\Sigma,\end{aligned} \tag{A.17}$$

where $i,j,k = 1,\dots,n-1$ and $\gamma = 1,\dots,m$.

If $m = 1$, M has only one unit normal $\boldsymbol{\nu}$. In this case the second fundamental form of M has the simpler form

$$\mathbf{A}(\mathbf{v}, \mathbf{w}) = -\langle D_{\mathbf{v}} \mathbf{w}, \boldsymbol{\nu} \rangle \boldsymbol{\nu} = h(\mathbf{v}, \mathbf{w}) \boldsymbol{\nu} \,.$$

The coefficients of the operator $h : TM \times TM \to \mathbb{R}$, with respect to an orthonormal frame $\{\boldsymbol{\tau}_i\}_{1 \leq i \leq n}$, are given by

$$h_{ij} = h(\boldsymbol{\tau}_i, \boldsymbol{\tau}_j) = \langle \mathbf{A}(\boldsymbol{\tau}_i, \boldsymbol{\tau}_j), \boldsymbol{\nu} \rangle = -\left\langle D_{\boldsymbol{\tau}_i} D_{\boldsymbol{\tau}_j} F, \boldsymbol{\nu} \right\rangle = \left\langle D_{\boldsymbol{\tau}_j} F, D_{\boldsymbol{\tau}_i} \boldsymbol{\nu} \right\rangle \,.$$

The norm of the second fundamental form is given by

$$\|\mathbf{A}\| = g^{ik} g^{lj} h_{kl} h_{ij} = h^{ij} h_{ij} \,,$$

and the mean curvature vector is given by

$$\mathbf{H} = -g^{ij} h_{ij} \boldsymbol{\nu} = -H \boldsymbol{\nu} \,,$$

where we define the mean curvature H of M as the trace of the second fundamental form. With (A.14) we conclude that

$$H = \operatorname{div} \boldsymbol{\nu} \,.$$

Appendix B

Useful theorems and equations

B.1 Background from analysis

Lemma B.1. *Let $f : [0,T) \to \mathbb{R}$ be in C^1 and $\alpha : [0,T) \to \mathbb{R}$ be in $C^0 \cap L^1$. Let $\frac{d}{dt}f(t) \leq \alpha(t)f(t)$. Then*

$$f(t) \leq \exp\left(\int_0^t \alpha(\tau)\,d\tau\right) f(0)$$

for all $t \in [0,T)$.

Proof. Set

$$g(t) := \exp\left(-\int_0^t \alpha(\tau)\,d\tau\right).$$

Then $\frac{d}{dt}g = -\alpha g$ and

$$\frac{d}{dt}(gf) = g\left(\frac{df}{dt} - \alpha f\right) \leq 0.$$

Integrating yields the claim. □

Lemma B.2 (Young's Inequality/Peter–Paul Inequality)**.** *For $a, b \in \mathbb{R}$ and $1/p + 1/q = 1$,*

$$ab \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}, \tag{B.1}$$

with equality if and only if $|a|^p = |b|^q$. For $\varepsilon > 0$,

$$ab \leq \frac{\varepsilon|a|^p}{p} + \frac{|b|^q}{\varepsilon^{q/p}q}. \tag{B.2}$$

Lemma B.3 (Cauchy–Schwarz Inequality)**.** *For $f, g : M \to \mathbb{R}$ in L^2,*

$$\int_M |fg|\,d\mathcal{H}^n \leq \left(\int_M f^2\,d\mathcal{H}^n\right)^{1/2} \left(\int_M g^2\,d\mathcal{H}^n\right)^{1/2}. \tag{B.3}$$

Theorem B.4 (Inverse function theorem)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be in C^1 and $f'(x) \neq 0$ for $x \in \mathbb{R}$. Then the inverse f^{-1} exists at $f(x)$ and $\left(f^{-1}\right)'(f(x)) = (f')^{-1}(x)$.*

Theorem B.5 (Rademacher's theorem, see [Fed69, Theorem 3.1.6])**.** *Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ be Lipschitz continuous. Then f is differentiable almost everywhere in U.*

Theorem B.6 (First variation of the area formula, [Sim83, p. 51]). *Let $M \subset \mathbb{R}^{n+m}$ be a smooth, compact, n-dimensional manifold with boundary. Let $U \subset \mathbb{R}^{n+m}$ be a open and bounded such that $M \subset U$. Let $\phi : U \times (-1,1) \to U$ be a one-parameter family of $C^{2;1}$-diffeomorphisms. Set $M_t := \phi(M,t)$ and $\mathbf{v}(p) := \frac{\partial \phi(p,t)}{\partial t}_{|t=0}$. Then*

$$\frac{d}{dt}_{|t=0} \mathcal{H}^n(M_t) = \int_M \operatorname{div}_M \mathbf{v} \, d\mathcal{H}^n \,.$$

Theorem B.7 (Divergence theorem, [Sim83, p. 43], [DHTK10, p. 304], [Eck04, p. 116]). *Let $M \subset \mathbb{R}^{n+m}$ be a smooth, compact, n-dimensional manifold with boundary. Let $\mathbf{v}$ be a C^1-vector field on M. Then*

$$\int_M \operatorname{div}_M \mathbf{v} \, d\mathcal{H}^n = -\int_M \langle \mathbf{v}, \mathbf{H}_M \rangle \, d\mathcal{H}^n + \int_{\partial M} \langle \mathbf{v}, \boldsymbol{\nu}_{\partial M} \rangle \, d\mathcal{H}^{n-1} \,,$$

where $\langle \cdot \, , \cdot \rangle$ is the Euclidean inner product.

Corollary B.8 (Integration by parts). *Let M be a manifold with boundary ∂M, $f, g : M \to \mathbb{R}$ in $W^{2,2} \cap C^2$. Then*

$$\int_M f \Delta_M g \, d\mathcal{H}^n = \int_{\partial M} f \left\langle \nabla^{\partial M} g, \boldsymbol{\nu}_{\partial M} \right\rangle d\mathcal{H}^{n-1} - \int_M \left\langle \nabla^M f, \nabla^M g \right\rangle d\mathcal{H}^n \,. \tag{B.4}$$

Lemma B.9 (Fatou's lemma, [AE06, Theorem 3.7]). *Let $(\Omega, \sigma, d\mu)$ be a measure space and let $(f_i : \Omega \to [0,\infty))_{i \in \mathbb{N}}$ be a sequence of non-negative integrable functions such that* $\liminf_{i\to\infty} \int_\Omega f_i \, d\mu < \infty$. *Then*

$$\int_\Omega \liminf_{i\to\infty} f_i \, d\mu \le \liminf_{i\to\infty} \int_\Omega f_i \, d\mu \,.$$

Theorem B.10 (Lebesgue dominated convergence). *Let $I \subseteq \mathbb{R}$ be an interval. Let $(f_i : I \to \mathbb{R})_{i\in\mathbb{N}}$ be a sequence of integrable functions with $f_i \to f$ pointwise almost everywhere on I and there exists an integrable function $g : I \to \mathbb{R}$ with $|f_i| \le g$ almost everywhere in I. Then f is integrable on I and*

$$\lim_{i\to\infty} \int_I f_i \, dx = \int_I f \, dx \,.$$

Theorem B.11 (Arzelà–Ascoli, [AMR93, Theorem 1.5.11]). *Let (M, d_M) and (N, d_N) be metric spaces, with M compact and N complete. A set $\mathcal{F} \subset C^0(M,N)$ is relatively compact if and only if it is equicontinuous and all the sets $\mathcal{F}(m) = \{f(m) \,|\, f \in \mathcal{F}\}$ are relatively compact in N.*

Corollary B.12. *Let $K \subset \mathbb{R}^n$ be compact and let $(f_i : K \to \mathbb{R}^m)_{i\in\mathbb{N}}$ be a sequence of bounded and equicontinuous functions. Then $(f_i)_{i\in\mathbb{N}}$ has a uniformly convergent subsequence.*

Theorem B.13 (C^0-convergence of function sequences, [Rud76, Theorem 7.12]). *Let $I \subset \mathbb{R}$ be an interval. Let $(f_i : I \to \mathbb{R})_{i\in\mathbb{N}}$ be a sequence of C^0-functions with $f_i \to f$ uniformly on I. Then $f \in C^0(I)$.*

Theorem B.14 (C^1-convergence of function sequences, [Rud76, Theorem 7.17]). *Let $I \subset \mathbb{R}$ be an interval. Let $(f_i : I \to \mathbb{R})_{i\in\mathbb{N}}$ be a sequence of C^1-functions such that the sequence $(f_i(x_0))_{i\in\mathbb{N}}$ converges for some point $x_0 \in I$. If $(f_i' : I \to \mathbb{R})_{i\in\mathbb{N}}$ converges uniformly on I, then $f_i \to f \in C^1(I)$ uniformly and $\lim_{i\to\infty} f_i' = f'$ on I.*

Theorem B.15 (Ehrling's lemma, [RR04, Theorem. 7.30]). *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be Banach spaces. Assume that X is compactly embedded in Y and that Y is continuously embedded in Z, that is,*

$$X \underset{compact}{\hookrightarrow} Y \underset{continuous}{\hookrightarrow} Z\,.$$

Then, for every $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that, for every $f \in X$,

$$\|f\|_Y \le \varepsilon\|f\|_X + C(\varepsilon)\|f\|_Z\,.$$

B.2 Parabolic maximum principles

Let $\Omega \subset \mathbb{R}^n$ be open and bounded. For $t \in (0, T]$, we define the parabolic cylinder

$$Q_t := \Omega \times (0, t)$$

and the parabolic boundary

$$\mathcal{P}Q_t := (\Omega \times \{0\}) \cup (\partial\Omega \times (0, t])\,.$$

Let $f : \bar\Omega \times [0, T] \to \mathbb{R}^{n+1}$ in $C^{2;1}(Q_T) \cap C^0(\overline{Q_T})$. We define the parabolic operator

$$L(f) := \frac{\partial f}{\partial t} - a^{ij}\nabla_i\nabla_j f - b^i\nabla_i f - cf\,,$$

where $a_{ij}, b_i, c \in L^\infty$ and where $(a^{ij})_{ij}$ is uniformly elliptic, that is, there exists $\lambda > 0$ so that $\lambda\|\xi\|^2 \le a_{ij}\xi_i\xi_j \le \Lambda\|\xi\|^2$ for all $\xi \in \mathbb{R}^{n+1}$.

Theorem B.16 (Weak maximum principle, see [Fri64, Chapter 1.1 and 1.2]). *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Let $f \in C^{2;1}(Q_T) \cap C^0(\overline{Q_T})$ be a solution of $Lf \le 0$ in Q_T.*

(i) If $c \equiv 0$, then $\sup_{Q_T} f \le \sup_{\mathcal{P}Q_T} f$.

(ii) If $c \ge 0$ and $f \le 0$ on $\mathcal{P}Q_T$, then $\sup_{Q_T} f \le \sup_{\mathcal{P}Q_T} f$.

(iii) If $c \in L^\infty$ and $\sup_{\mathcal{P}Q_T} f \le 0$, then $\sup_{Q_T} f \le 0$.

Theorem B.17 (Strong maximum principle, see [Fri64, Chapter 1.1 and 1.2]). *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and connected. Let $f \in C^{2;1}(Q_T) \cap C^0(\overline{Q_T})$ be a solution of $Lf \le 0$ in Q_T. Let $(p_0, t_0) \in \overline{Q_T} \setminus \mathcal{P}Q_T$ with $f(p_0, t_0) = \max_{\overline{Q_T}} f$ and either*

(i) $c \equiv 0$, or

(ii) $c \ge 0$ and $f(x_0, t_0) \ge 0$, or

(iii) $c \in L^\infty$ and $f(x_0, t_0) = 0$,

Then f is constant in $\overline{Q_{t_0}}$.

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