

LATTICE POLYTOPES —
APPLICATIONS AND
PROPERTIES

Ehrhart Theory, Graph Colorings, and Level Algebras

beim Fachbereich Mathematik und Informatik
der Freien Universität Berlin
eingereichte Dissertation

vorgelegt von
FLORIAN KOHL, Master of Arts in Mathematics
geboren in Erlenbach am Main

Berlin 2018

Florian Kohl: *Lattice Polytopes — Applications and Properties*
© July 7, 2018

BETREUER:
Prof. Dr. Christian Haase
Tag der Disputation: 6. Juli, 2018

GUTACHTER:
Prof. Dr. Christian Haase
Prof. Dr. Akihiro Higashitani

ACKNOWLEDGEMENTS

This chapter is a big THANK YOU to all of the people who supported me in my personal life and during my studies. Being explicitly mentioned below is by no means a necessary condition for my gratitude, but I can assure you that it is certainly sufficient.

As a student in school, I always enjoyed math and physics. Around eighth grade, I was sure that I wanted to become a (theoretical) physicist. This changed when **Hartmut Stapf** became my teacher in mathematics. He was the first person to show me the beauty of math. His love and playful joy for the subject still influence me to this day. Another lucky encounter was when I met **Matthias Beck**. **Matt** was the first to introduce me to lattice polytopes and Ehrhart theory. He not only sacrificed his time to teach me about the subject and to grade some exercises from his book, but he also advised my Bachelor's thesis and in the process became my first coauthor. Moreover, he supported my applications to the University of Kentucky and to the Berlin Mathematical School. A lot of this journey would not have been possible without him.

In Kentucky, I not only had the best time of my life, but I also had the fortune to be supported by **Benjamin Braun** and **Uwe Nagel**. Both gave me a lot of academic freedom while showing me a lot of exciting math. Both supported numerous applications to summer schools and conferences and thus helped me to frequently travel, and to get to know a lot of mathematicians. I am especially grateful to both of them for supporting my plan to apply to the Berlin Mathematical School. Furthermore, **Uwe** was my adviser in Kentucky and I truly enjoyed working with him. I still consider myself as part of his academic family.

I moved to Berlin, as I knew that it was a stimulating environment with a lot of exciting people working in my area. However, I did not expect to end up in a group which such amazing people. I want to thank my adviser **Christian Haase** for his constant support, his patience, all the things he taught me, and, of course, for all the tea he provided. **Christian** had a lot of work on his plate and he was often busy, but he always had time for me. I often came to his office for a “quick” question, before leaving his office after more than an hour filled with math, laughter, and interesting discussions. Also, he provided me with financial support for numerous trips, especially for the month that I spent in Helsinki. I am proud to be his student and I am glad to consider him as a friend. Moreover, I was fortunate to have two amazing office mates in **Jan Hofmann** and **Lena Walter**. We are not only colleagues, we are friends. I didn't just go to my office to work, but also to have a good time. Thanks for all of the fun, the help, the support, the movie nights, and all the laughter we had over the last years. Even though we did not share an office, I want to thank fellow Team Haase member **Carlos Améndola** for the good times we had and for his support, both mathematical and non-mathematical.

The **Berlin Mathematical School** is a great graduate school with students from

all over the world. I really enjoy being part of such a diverse and active school. Not only do they provide a stimulating environment with a lot of social events, but they also offer generous funding. I want to especially thank them for supporting me with a scholarship and for funding research stays in the US and in Colombia. In particular, I want to thank the entire **One-Stop Office** for making the bureaucratic things go as smoothly as possible.

This thesis is based on four different articles for which I had the pleasure to collaborate with seven people from five different countries. For me math is a collaborative effort and I want to thank **Alexander Engström**, **Christian Haase**, **Akiyoshi Tsuchiya**, **McCabe Olsen**, **Yanxi Li**, **Johannes Rauh**, and **Ruriko Yoshida**. You all showed me how much fun collaborating can be and I want to thank you for all the discussions and the fun we had together.

Furthermore, during these projects me and my coauthors got invaluable feedback and support from other mathematicians and institutions. I would like to thank the anonymous referee of [EK18] for helpful suggestions and constructive criticism. She/he had a big impact on the exposition and the structure of our article and consequently of Chapter 3. Moreover, I would like to express my deepest gratitude to **Alex** and **Aalto University** for their hospitality and financial support during a month-long research stay. I would like to explicitly thank **Takayuki Hibi** and **Akiyoshi** for organizing and funding the Workshop on Convex Polytopes for Graduate Students at **Osaka University** in January 2017, which is where [HKT] and [KO] began. While working on [KO], **McCabe** and I got invaluable feedback and support from **Matthias Beck**, **Benjamin Braun**, **Carla Savage**, and **Zafeirakis Zafeirakopoulos**. I want to especially thank **Christian Haase** for fruitful discussions about Gorenstein cones. **Christian** also pointed my attention to linear ordering polytopes while I was working on [KLR18]. Also, I would also like to express my gratitude to the **Normaliz** team for helpful discussions and suggestions.

For me one of the worst parts of mathematics is editing and proofreading articles and theses. That's why I am so grateful that **Dustin Hedmark**, **Jan Hofmann**, **Eric Kaper**, **Francisco Santos**, **Liam Solus**, and **Lena Walter** sacrificed their time to find mistakes and to give constructive criticism. I really appreciate *y'all's* help.

A big project like writing a thesis is plainly impossible without the help of a lot of friends — in- and outside of academia. There is no need to mention names, you know who you are.

Last but not least, I want to thank my parents, **Brigitte** and **Harald Kohl**, for their love, support, patience, and for everything they did for me. Growing up in a household where math and science are considered as interesting instead of as frightening makes a big difference. I want to especially thank you for all of your support and for the time you spent throughout my school and sports career. You did much more than you needed to and this thesis would not have been possible without you.

CONTENTS

| | | |
|-------|---|----|
| 1 | ACKNOWLEDGEMENTS | v |
| 2 | INTRODUCTION | 1 |
| 2.1 | The Ubiquity of Lattice Polytopes | 2 |
| 2.2 | Lattice Polytopes and Ehrhart Theory | 4 |
| 2.3 | Lattice Polytopes and Commutative Algebra | 7 |
| 2.4 | Lattice Polytopes and Graph Theory | 13 |
| 3 | TRANSFER-MATRIX METHODS MEET EHRHART THEORY | 19 |
| 3.1 | Introduction | 19 |
| 3.2 | Background and Notation | 22 |
| 3.3 | Prominent Special Cases | 24 |
| 3.3.1 | Transfer-Matrix Methods | 24 |
| 3.3.2 | Ehrhart Theory and Inside-Out Polytopes | 26 |
| 3.4 | Transfer-Matrix Methods Meet Ehrhart Theory | 29 |
| 3.4.1 | Enter Symmetry | 29 |
| 3.4.2 | Ehrhart Theory and Symmetry | 31 |
| 3.4.3 | Main Results | 37 |
| 3.5 | Open Questions | 44 |
| 4 | LEVELNESS OF ORDER POLYTOPES | 47 |
| 4.1 | Introduction | 47 |
| 4.2 | Background and Notation | 50 |
| 4.3 | Miyazaki's characterization | 52 |
| 4.4 | A New Characterization of Levelness | 54 |
| 4.5 | A Necessary Condition of Ene, Herzog, Hibi, and Saeedi Madani | 57 |
| 4.6 | Series-Parallel Posets | 59 |
| 4.7 | Connected components of level posets | 62 |
| 5 | LEVEL ALGEBRAS AND s -LECTURE HALL POLYTOPES | 69 |
| 5.1 | Introduction | 69 |
| 5.2 | Background | 71 |
| 5.3 | Gorenstein lecture hall polytopes | 73 |
| 5.4 | Characterization of level lecture hall polytopes | 77 |
| 5.4.1 | Proof of characterization | 77 |
| 5.4.2 | Consequences of the characterization | 82 |
| 5.5 | Concluding remarks and future directions | 85 |
| 6 | SEMIGROUPS — A COMPUTATIONAL APPROACH | 87 |
| 6.1 | Introduction | 87 |
| 6.2 | Computing holes | 89 |
| 6.3 | Performance of the algorithm | 90 |

| | | |
|-------|---|-----|
| 6.3.1 | Speeding up zsolve | 91 |
| 6.4 | Common diagonal effect models | 92 |
| 6.5 | Computational experiments | 96 |
| 6.5.1 | The Integer-Decomposition Property and Linear Order Polytopes | 96 |
| 6.5.2 | Lifting Markov bases and Gröbner bases | 96 |
| 6.6 | Discussion and open problems | 97 |
| 7 | HASE | 99 |
| 7.1 | The Basics | 99 |
| 7.1.1 | Notation | 99 |
| 7.1.2 | A First Example | 100 |
| 7.1.3 | Frobenius Revisited | 101 |
| 7.1.4 | A Bigger Example | 102 |
| 7.2 | More Options | 104 |
| 8 | APPENDIX | 107 |
| | Bibliography | 117 |
| | List of Figures | 125 |
| | List of Tables | 125 |

“It ain’t what you don’t know that gets you into trouble. It’s what you know for sure that just ain’t so.” (On-screen quote attributed to Mark Twain in the movie “The Big Short”)

The fact that I am now working in the area of discrete geometry would come as a big surprise to my high-school self. In high school, geometry did not intrigue me as much as the other mathematical areas. What converted me to this beautiful area was a free summer during my undergraduate studies, the excellent book [BR15], and Matthias Beck’s support. Over the course of this summer, I learned that lattice polytopes are ubiquitous and beautiful objects. While being interesting objects in their own right, they and their unbounded analogues have applications to algebraic geometry and commutative algebra [BH93, CLS11, Sta96], optimization [BP03, Stu96], number theory [BBK⁺15, BK14, Pom93], combinatorics [BR15, Sta96], and — for Chapter 3 most importantly — to proper graph colorings [BZ06]. What personally got me excited was the connection to number theory. Parts of Example 2.1.1 are a relict of this time.

In Section 2.1, I want to convince you of the beauty of lattice polytopes and discrete geometry by giving three explicit examples of how lattice polytopes relate to other areas. In Section 2.2, we will introduce the basics of lattice polytopes and Ehrhart theory. Most prominently, we will state Ehrhart’s theorem that the Ehrhart function of a lattice polytope is a polynomial and the related reciprocity result by Ehrhart and Macdonald, see Theorem 2.2.1 and Theorem 2.2.6. In Section 2.3, we introduce the relevant basics of combinatorial commutative algebra. In particular, we introduce the Gorenstein and the level property, see Definition 2.3.9. In Section 2.4, we review proper graph colorings, the deletion-contraction formula, and a transfer-matrix method to count walks in graphs.

Equipped with these basics, we turn to the main parts of this thesis. There are four main parts. Chapter 3 examines proper k -colorings of Cartesian graph products of the form $G \times P_n$ and $G \times C_n$. Here G is any simple graph and P_n and C_n are the path and cycle graph on n vertices, respectively. It is important to note that both k and n are treated as variables, so the size of the graphs is not fixed. We will combine transfer-matrix methods with Ehrhart theory and, in particular, inside-out polytopes to tackle this problem. Using the underlying symmetries will be a main ingredient. Chapter 4 is devoted to an examination of *level* posets, i.e., posets whose order polytope is level in the sense of Definition 2.3.9. This will be done by examining the Ehrhart ring of the associated order polytopes. We use the Bellman–Ford algorithm [Bel58] to test levelness and to make statements about the complexity class. Furthermore, we study the more general class of alcoved polytopes and see when they are level, i.e., when their associated Ehrhart rings are level algebras. In Chapter 5, the level property of

s-lecture hall polytopes will be at the center of our attention. Moreover, we will have a closer look at the Gorenstein property, which has desirable implications for an associated generating series. The Gorenstein property and the level property of lattice polytopes are closely related to semigroups and their associated semigroup ring. The subtle difference between the associated Ehrhart ring and the associated toric ring lies in the properties of the underlying semigroup. Affine semigroups and their holes will be object of study in Chapter 6. In the course of this project, Yanxi Li, Johannes Rauh, Ruriko Yoshida, and I developed a mathematical software called HASE. This software is available at <http://ehrhart.math.fu-berlin.de/People/fkohl/HASE/>. Chapter 7 can be seen as a brief manual of HASE. The source code of HASE can be found in Chapter 8.

2.1 THE UBIQUITY OF LATTICE POLYTOPES

Lattice polytopes can be described as the solution set of finitely many linear inequalities, and hence integer points in polytopes correspond to integral solutions to these inequalities. It is of special interest *how many* integral solutions there are. Therefore, for a (lattice) polytope $P \subset \mathbb{R}^d$, we define the counting function $\text{ehr}_P: \mathbb{Z}_{\geq 1} \rightarrow \mathbb{Z}_{\geq 0}$

$$\text{ehr}_P(t) := \#tP \cap \mathbb{Z}^d.$$

This function is called the *Ehrhart function* of P . Ehrhart famously proved that ehr_P is a polynomial in t . The complexity of computing this function for 3-dimensional lattice polytopes is related to number theory:

Example 2.1.1 (Connections to number theory [BK14, BR15]). Richard Dedekind, in the 1880's [Ded53], defined what we now call *Dedekind sums*. For positive integers a, b , we call

$$s(a, b) = \sum_{k=0}^{b-1} \left(\left(\frac{ka}{b} \right) \right) \left(\left(\frac{k}{b} \right) \right),$$

the Dedekind sum of a, b , where

$$\left(\left(x \right) \right) := \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

Historically, Dedekind sums first appeared in the study of the transformation properties of $\eta(z) := e^{\pi iz/12} \prod_{n \geq 1} (1 - e^{2\pi inz})$ under $\text{SL}_2(\mathbb{Z})$. While Dedekind sums have been the object of extensive research in analytic [Alm98, Die59] and algebraic number theory [Mey57, Sol98], they have also appeared in topology [HZ74, MS79], combinatorics [Bri88, Pom93], and in algorithmic complexity [Knu81]. Furthermore, they form one of the building blocks of the Ehrhart polynomials in dimension 3:

THEOREM 2.1.2 ([BR15, Thm 8.11]). Let P be the tetrahedron with vertices $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$, where $a, b, c \in \mathbb{Z}_{\geq 1}$ are pairwise relatively prime. Then

$$\text{ehr}_P(t) = \frac{abc}{6}t^3 + \frac{ab + ac + bc + 1}{4}t^2 + \left(\frac{3}{4} + \frac{a + b + c}{4} + \frac{1}{12} \left(\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} + \frac{1}{abc} - s(bc, a) - s(ac, b) - s(ab, c) \right) \right) t + 1.$$

This theorem tells us that understanding Dedekind sums leads to an understanding of the Ehrhart polynomial of P . In particular, using that Dedekind sums satisfy a reciprocity theorem of the form $s(a, b) + s(b, a) = \text{const}(a, b)$ for coprime a, b , where $\text{const}(a, b)$ denotes an explicit constant only depending on a and b , one can efficiently compute ehr_P . This reciprocity theorem can also be proven and generalized using and understanding the integer points in a 2-dimensional fan [BK14, Thm. 4] and [BHM08, Thm. 1].

The next example shows how geometry can be used to count the number of proper k -colorings of a graph. Proper k -colorings and the chromatic polynomial (see Def. 2.4.1) will be defined in Section 2.4. Loosely speaking, a proper k -coloring is a labeling of the vertices with labels from $\{1, 2, \dots, k\}$ such that connected vertices have different labels. The chromatic polynomial counts the number of such k -colorings.

Example 2.1.3 (Connections to combinatorics [BZ06]). Let $G = (\{1, 2, \dots, n\}, E)$ be a simple graph, i.e., a graph without multiple edges and loops. The number of proper k -colorings agrees with a polynomial in k , see Theorem 2.4.2. Beck and Zaslavsky showed that counting proper colorings is that same as counting integer points in dilates of what is called *inside-out polytopes*. Following [BZ06], we define the hyperplane arrangement

$$\mathcal{H}_G = \{x_i = x_j : \{i, j\} \in E\}.$$

This hyperplane arrangement subdivides $P = [0, 1]^n$ into full-dimensional closed regions R_1, R_2, \dots, R_m . The pair (P, \mathcal{H}_G) is called an *inside-out polytope*. Now we set

$$\text{ehr}_{P^\circ, \mathcal{H}_G}(k) := \sum_{i=0}^m \text{ehr}_{R_i^\circ}(k), \quad (2.1)$$

where Q° denotes the topological interior of a compact set Q and where $\text{ehr}_{R_i^\circ}(t) = \#tR_i^\circ \cap \mathbb{Z}^n$. Now every integer point counted by (2.1) is in bijection with a proper $k - 1$ -coloring of G . This implies:

THEOREM 2.1.4 ([BZ06, Thm. 5.1]). With the notation from above:

$$\text{ehr}_{P^\circ, \mathcal{H}_G}(t) = \chi_G(t - 1).$$

This geometric set-up has a lot of theoretical consequences and will be extensively used in Chapter 3. These consequences include the polynomiality of χ , that the leading coefficient is always 1, and it gives a geometric proof of Stanley's famous reciprocity theorem for chromatic polynomials, see Proposition 2.4.3.

In general, it is extremely challenging to compute the Ehrhart polynomial of a given lattice polytope. However, if the lattice polytope has a unimodular triangulation, then one can infer the Ehrhart polynomial from the combinatorics of this triangulation, see Theorem 2.2.5. We will give a precise definition in the next section (see Definition 2.2.3), but for now we will only say that a unimodular triangulation is a very desirable subdivision of a lattice polytope. In fact, it is so desirable that most lattice polytopes don't seem to have one. However:

Example 2.1.5 (Connections to algebraic geometry [KKMSD73, Ch. 3]). This theorem is due to Knudsen and Mumford, and one of the key steps in the proof is due to Alan Waterman.

THEOREM 2.1.6. Every lattice polytope P has a constant factor $c \in \mathbb{Z}_{\geq 1}$ such that the dilated polytope cP has a unimodular triangulation.

The proof relies on what is called *semi-stable reduction*, see [KKMSD73, Ch. 3]. This is a beautiful example of how algebraic geometric methods give rise to a powerful theorem in discrete geometry. Moreover, there is a close connection between toric varieties and lattice polytopes, see for instance [CLS11]. This connection gives rise to a *dictionary* with which one can translate properties of toric varieties to properties of polytopes and vice versa.

2.2 LATTICE POLYTOPES AND EHRHART THEORY

In this section, we want to give a brief introduction to Ehrhart theory and lattice polytopes. For a more detailed account of Ehrhart theory and (general) polytopes, we refer the reader to [BR15, Zie95]. A *lattice* Λ is a discrete, additive subgroup of \mathbb{R}^d . A *basis* of a lattice is a set of linearly independent vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m$, $\mathbf{a}_i \in \Lambda$, that generate Λ over \mathbb{Z} . A *lattice polytope* $P \subset \mathbb{R}^d$ is the convex hull of finitely many points in Λ , i.e.,

$$P = \text{conv}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r : \mathbf{v}_i \in \Lambda\} := \left\{ \sum_{i=1}^r \lambda_i \mathbf{v}_i : \lambda_i \geq 0, \sum_{i=1}^r \lambda_i = 1 \right\}.$$

The inclusion-minimal set $\{\mathbf{v}_{i_1}, \mathbf{v}_{i_2}, \dots, \mathbf{v}_{i_s}\}$ such that $P = \text{conv}\{\mathbf{v}_{i_1}, \mathbf{v}_{i_2}, \dots, \mathbf{v}_{i_s}\}$ is called the *vertex set of P* and its elements are called the *vertices*. Similarly, a *rational polytope* $P \subset \mathbb{R}^d$ is the convex hull of finitely many points in \mathbb{Q}^d , i.e.,

$$P = \text{conv}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r : \mathbf{v}_i \in \mathbb{Q}^d\} := \left\{ \sum_{i=1}^r \lambda_i \mathbf{v}_i : \lambda_i \geq 0, \sum_{i=1}^r \lambda_i = 1 \right\}.$$

The *dimension of P* is defined to be the dimension of its affine span, and if P has dimension d , we say that P is a *d -polytope*. A d -simplex $\Delta = \text{conv}\{\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_d\}$, $\mathbf{v}_i \in \Lambda$, is called *unimodular* if $\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \dots, \mathbf{v}_d - \mathbf{v}_0$ generate the lattice Λ . We will mostly be concerned with the case $\Lambda = \mathbb{Z}^d$.

Given a lattice polytope P , we recall that the Ehrhart polynomial ehr_P is defined as

$$\text{ehr}_P(t) := \#(tP \cap \mathbb{Z}^d).$$

Ehrhart proved the following:

THEOREM 2.2.1 ([BR15, Thm 3.8]). If P is a lattice d -polytope, then ehr_P is a polynomial of degree d with leading coefficient $\text{vol}(P)$, where $\text{vol}(P)$ is the Euclidean volume of P .

This also implies that the formal generating function

$$\text{Ehr}_P(z) := 1 + \sum_{k \geq 1} \text{ehr}_P(k) z^k = \frac{h_0^* + h_1^* z + \cdots + h_d^* z^d}{(1 - z)^{d+1}}$$

is a rational function with denominator $(1 - z)^{d+1}$ and that the degree of the numerator is at most d , see [BR15, Lem 3.9]. We call Ehr_P the *Ehrhart series of P* and the numerator is called the *h^* -polynomial of P* . If we want to emphasize the corresponding polytope P , we will write h_P^* . The coefficient vector $(h_0^*, h_1^*, \dots, h_d^*)$ is called the *h^* -vector*. Moreover, we define the *degree of P* , denoted $\deg(P)$, to be $\deg(P) := \deg h^*(z)$, and the *codegree of P* , denoted $\text{codeg}(P)$ to be $\text{codeg}(P) := d + 1 - \deg(P)$. Instead of studying the Ehrhart polynomial, it is often times more convenient to work with the Ehrhart series of a polytope. This is essentially due to the following result by Richard Stanley:

THEOREM 2.2.2 ([BR15, Thm. 3.12]). Let $P \subset \mathbb{R}^d$ be a d -dimensional lattice polytope. Then

$$\text{Ehr}_P(z) = \frac{h_0^* + h_1^* z + \cdots + h_d^* z^d}{(1 - z)^{d+1}}$$

and the coefficients h_i^* are non-negative integers.

We have previously hinted at a way of determining the Ehrhart polynomial from a unimodular triangulation, provided such a triangulation exists. To make this more precise, we first need to define what a unimodular triangulation is:

Definition 2.2.3. A *triangulation* of a lattice polytope is a subdivision into lattice simplices such that the intersection of any two simplices is a (possibly empty) face of both. A *unimodular triangulation* is a triangulation into simplices, where every full-dimensional simplex is unimodular. Let f_i be the number of i -dimensional simplices in a given triangulation T . Then the *f -vector* of a triangulation T is the vector $f_T := (f_{-1}, f_0, \dots, f_d)$, where we set $f_{-1} = 1$.

Remark 2.2.4. We note that every lattice polytope has a triangulation. However, most lattice polytopes do not seem to have a unimodular triangulation. We refer the interested reader to [HPPS14] for a state-of-the-art account of positive results .

Sometimes it is more convenient to encode the face structure of a triangulation T in the h -polynomial

$$h_T(z) := \sum_{k=-1}^{d-1} f_k z^{k+1} (1-z)^{d-1-k}.$$

Given the notation, the attentive reader might have already guessed that the h^* -vector might be related to the h -vector of a triangulation. This is indeed (sometimes) the case.

THEOREM 2.2.5 ([BR15, Thm. 10.3]). If P is an integral d -polytope that admits a unimodular triangulation T , then

$$\text{Ehr}_P(z) = \frac{h_T(z)}{(1-z)^{d+1}}.$$

In other words, the h^* -polynomial of the Ehrhart series is given by the h -polynomial of the triangulation T .

Moreover, the h^* -polynomial encodes important algebraic information about the polytope. The Gorenstein property, and thus reflexivity, is completely characterized in terms of the h^* -vector. For the definitions of the Gorenstein and reflexive properties, we refer the reader to Definition 2.3.9. For the characterization of these properties in terms of the h^* -vector, we refer the reader to Theorem 2.3.16. Many additional properties are known about Ehrhart h^* -polynomials (see e.g [BR15, Hib92]). In fact, classifying the set of h^* -vectors is one of the most important open problems in Ehrhart theory. Therefore, inequalities for the coefficients are of special interest, see [Hib90, Sta91, Sta09, Sta16]. Hofscheier, Katthän, and Nill prove a structural result about h^* -vectors, see [HKN16, Thm. 3.1], where they show that if the integer points of a lattice polytope span the integer lattice, then h^* cannot have internal zeros. There are even some universal inequalities for h^* -vectors, i.e., there are relations among the coefficients that are true independent of the degree and the dimension of the polytope, see [BH17].

For simplices, there is an easily-stated geometric interpretation for the coefficients of h^* . Suppose that P is a simplex with vertex set $\{\mathbf{v}_0, \dots, \mathbf{v}_d\}$. The (*half-open*) *fundamental parallelepiped*, Π_P , of P is the bounded region

$$\Pi_P := \left\{ \sum_{i=0}^d \eta_i (\mathbf{v}_i, 1) : 0 \leq \eta_i < 1 \right\}.$$

For simplicies, we can use the fundamental parallelepiped to compute the Ehrhart h^* -polynomial. In particular, the coefficients are given by

$$h_i^*(P) = \# \{(\mathbf{v}, i) \in \Pi_P \cap \mathbb{Z}^{n+1}\},$$

that is the number of lattice points at height i in Π_P . For more details and exposition, the reader should consult [BR15].

The polynomiality of ehr has another interesting consequence. A priori, the Ehrhart function ehr is defined as a counting function with domain $\mathbb{Z}_{\geq 1}$. However, since ehr

agrees with a polynomial, we can extend the domain to \mathbb{R} (or \mathbb{C}). This raises the question whether other evaluations have nice combinatorial or geometric interpretations. The following result was conjectured by Ehrhart, but first proven in full generality by Macdonald.

THEOREM 2.2.6 (Ehrhart-Macdonald reciprocity, [BR15, Thm 4.1]). Let P be a d -dimensional lattice polytope. Then

$$\text{ehr}_P(-t) = (-1)^d \text{ehr}_{P^\circ}(t), \quad (2.2)$$

where $\text{ehr}_{P^\circ}(t)$ counts the number of integer points in the interior of tP .

Remark 2.2.7. There are generalizations of Theorem 2.2.1 and Theorem 2.2.6 to rational polytopes, but since we will only work with lattice polytopes, we omit the precise statements and instead refer the reader to [BR15, Thm. 3.23, Thm. 4.1].

2.3 LATTICE POLYTOPES AND COMMUTATIVE ALGEBRA

In this section, we will see how commutative algebra can be used to capture interesting aspects of lattice polytopes. This section is based on [BG09, Ch. 6]. Let $P \subset \mathbb{R}^d$ be a lattice d -polytope with vertex set $V(P)$ and let \mathbb{k} be an algebraically closed field of characteristic zero. We define the *cone over P* as

$$\text{cone}(P) := \text{span}_{\mathbb{R}_{\geq 0}} \{(\mathbf{v}, 1) : \mathbf{v} \in V(P)\} \subset \mathbb{R}^d \times \mathbb{R}.$$

The set

$$\mathcal{M}(P) := \left\{ \mathbf{x} : \mathbf{x} = \sum_{i \in I} \lambda_i (\mathbf{v}_i, 1), \text{ for } \mathbf{v}_i \in P \cap \mathbb{Z}^d \text{ and } \lambda_i \in \mathbb{Z}_{\geq 0}, \forall i \in I \right\}$$

of integral linear combinations of integer points in $(P, 1)$ forms an additive *semigroup*, i.e., a set that is closed under addition with a neutral element, where addition is associative. We remark that some authors would call this set a monoid, since they don't require semigroups to contain the neutral element. Let $\mathcal{M} = \mathcal{M}(P)$ be such a semigroup. We call an element $x \in \mathcal{M} \setminus \{0\}$ *irreducible* if $x = z + y$ implies that either $y = 0$ or $z = 0$. The *Hilbert basis* $\mathcal{H}(\mathcal{M})$ is the unique set of irreducible elements of this semigroup. The integer points in $(P, 1)$ also generate a lattice Λ

$$\Lambda := \Lambda_P := \left\{ \mathbf{x} : \mathbf{x} = \sum_{i \in I} \lambda_i (\mathbf{v}_i, 1), \text{ for } \mathbf{v}_i \in P \cap \mathbb{Z}^d \text{ and } \lambda_i \in \mathbb{Z}, \forall i \in I \right\}.$$

We say that P has the *integer-decomposition property* (IDP) if $\mathcal{M}(P) = \text{cone}(P) \cap \mathbb{Z}^{d+1}$, and we say that P is *normal* if $\mathcal{M}(P) = \text{cone}(P) \cap \Lambda$. As not every polytope is normal, it is natural to define $\overline{\mathcal{M}}(P) := \text{cone}(P) \cap \Lambda$ and $\text{C}_{\mathbb{Z}}(P) := \text{cone}(P) \cap \mathbb{Z}^{d+1}$. We remark that if P has a unimodular triangulation, then P automatically has the IDP. Furthermore, we define the *affine semigroup ring* of P to be

$$\mathbb{k}[P] := \mathbb{k}[\text{C}_{\mathbb{Z}}(P)] := \mathbb{k}[\mathbf{x}^{\mathbf{p}} \cdot \mathbf{y}^m : (\mathbf{p}, m) \in \text{C}_{\mathbb{Z}}(P)] \subset \mathbb{k}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, y].$$

Remark 2.3.1. A few comments are in order, since this definition differs from parts of the literature. Often times, one considers the ring $\mathbb{k}[\mathcal{M}(P)]$ which is a normal semigroup ring by [MS05, Prop. 7.25]. However, we want to use commutative algebra to count lattice points in dilates of P , so it is more convenient to work with the normal, affine semigroup $C_{\mathbb{Z}}(P)$. In this context, $\mathbb{k}[P]$ is sometimes called the *Ehrhart ring of P* .

The ring $\mathbb{k}[P]$ is also a finitely generated \mathbb{k} -algebra of Krull dimension $d + 1$, which inherits a natural $\mathbb{Z}_{\geq 0}$ -grading given by the y -degree. Therefore, we can write it as

$$\mathbb{k}[P] = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} \mathbb{k}[P]_i,$$

where $\mathbb{k}[P]_i$ is the \mathbb{k} -vector space generated by the degree i monomials. Next, we define a graded version of locality:

Definition 2.3.2 ([BG09, Def. 6.15]). Let \mathcal{R} be a $\mathbb{Z}_{\geq 0}$ -graded ring. Then we say that \mathcal{R} is **local with *maximal ideal \mathfrak{m}* if the homogeneous nonunits of \mathcal{R} generate the proper ideal \mathfrak{m} .

Remark 2.3.3. For any lattice polytope P , the graded ring $\mathbb{k}[P]$ is *local with *maximal ideal $\mathfrak{m} = \bigoplus_{i \geq 1} \mathbb{k}[P]_i$.

We can consider the *Hilbert function* $\text{hilb}: \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ given by

$$\text{hilb}(t) := \dim_{\mathbb{k}} \mathbb{k}[P]_t.$$

As with the Ehrhart function, it makes sense to examine the formal power series

$$\text{Hilb}(\mathbb{k}[P]; z) := \sum_{t \geq 0} \text{hilb}(t) z^t,$$

which we call the *Hilbert series of $\mathbb{k}[P]$* . This Hilbert series is in fact a rational function:

THEOREM 2.3.4 ([BG09, Thm. 6.39, 6.40]). Let P be a lattice d -polytope. Then the Hilbert series of $\mathbb{k}[P]$ is of the form

$$\text{Hilb}(\mathbb{k}[P]; z) = \frac{n(z)}{(1-z)^{d+1}},$$

where n is a polynomial with non-negative, integral coefficients.

The reader might have noticed that $\text{ehr}_P(t) = \text{hilb}_P(t)$ and thus $\text{Ehr}_P(z) = \text{Hilb}(\mathbb{k}[P]; z)$, a fact that is used in [MS05, Sec. 12.1] to prove the polynomiality of ehr_P .

The non-negativity of the coefficients in Theorem 2.3.4 is a consequence of $\mathbb{k}[P]$ being Cohen–Macaulay. We will now follow [BG09, Sec. 6.A] to describe the basics of Cohen–Macaulay rings. For more about Cohen–Macaulay rings, we refer the reader to the excellent book [BH93].

Let \mathcal{R} be a ring and let M be an \mathcal{R} -module. Elements $x_1, x_2, \dots, x_r \in \mathcal{R}$ form a (*regular*) M -*sequence* if

1. $M/(x_1, x_2, \dots, x_r)M \neq 0$,
2. and x_i is a non-zero divisor of $M/(x_1, x_2, \dots, x_{i-1})M$ for all $i \in \{1, 2, \dots, r\}$.

It is a consequence of Rees's theorem [BG09, Thm. 6.1] that if \mathcal{R} is a noetherian ring, then all maximal M -sequences have the same, finite, length.

In what follows, we will always assume that \mathcal{R} is a *local, noetherian ring of Krull dimension d with *maximal ideal \mathfrak{m} and that M is a finitely generated \mathcal{R} -module. Elements $x_1, x_2, \dots, x_d \in \mathfrak{m}$ are called a *system of parameters* if $\mathfrak{m} = \text{Rad}(x_1, x_2, \dots, x_d)$, or equivalently, $\dim \mathcal{R}/(x_1, \dots, x_d) = 0$. A *system of parameters* of M is a sequence $x_1, x_2, \dots, x_e \in \mathfrak{m}$ such that $\dim M/(x_1, x_2, \dots, x_e)M = 0$, where $e = \dim M$. We note that the dimension of an \mathcal{R} -module M is defined as the Krull dimension of $\mathcal{R}/\text{Ann}(M)$ where $\text{Ann}(M) := \{x \in \mathcal{R} : xM = 0\}$.

There is a graded analogue of this definition, namely a homogeneous system of parameters. Let \mathcal{R} be a finitely generated, positively graded \mathbb{k} -algebra of Krull dimension d , i.e., $\mathcal{R} = \bigoplus_{i \geq 0} \mathcal{R}_i$, where $\mathcal{R}_0 = \mathbb{k}$, and set $\mathfrak{m} = \bigoplus_{i > 0} \mathcal{R}_i$. Then homogeneous elements x_1, x_2, \dots, x_d form a *homogeneous system of parameters* if $\text{Rad}(x_1, x_2, \dots, x_d) = \mathfrak{m}$. In this setting, there always exists a homogeneous system of parameters x_1, x_2, \dots, x_d . Moreover, if \mathcal{R} is generated in degree 1 and \mathbb{k} is infinite, then such x_1, x_2, \dots, x_d can be chosen to be of degree 1 [BG09, Thm. 6.3].

Example 2.3.5. Let $\Delta = \text{conv}\{\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_d\}$ be a lattice d -simplex with the integer-decomposition property. Let's consider the case $\mathcal{R} = \mathbb{k}[\Delta] = \bigoplus_{i \geq 0} \mathbb{k}[\Delta]_i$ for an infinite \mathbb{k} . Then there is a natural choice for a homogeneous system of parameters given by the monomials x_0, x_1, \dots, x_d corresponding to the integer points $(\mathbf{v}_0, 1), (\mathbf{v}_1, 1), \dots, (\mathbf{v}_d, 1)$. This is indeed a homogeneous system of parameters, as the quotient contains only finitely many equivalence classes, which are given by the integer points in the half-open fundamental parallelepiped Π_Δ and thus $\dim \mathcal{R}/(x_0, \dots, x_d) = 0$.

Furthermore, we define $\text{depth}(M)$ to be the length of a maximal M -sequence in \mathfrak{m} . We say that M is *Cohen–Macaulay* if $\text{depth}(M) = \dim M$. A noetherian ring \mathcal{R} is called Cohen–Macaulay if it is Cohen–Macaulay as a module over itself. To relate this definition to homogeneous systems of parameters, we mention the following result:

THEOREM 2.3.6 ([Sta96, Thm. 5.9]). Let M have a homogeneous system of parameters. Then M is Cohen–Macaulay if and only if

1. every homogeneous system of parameters is a regular sequence, if and only if
2. M is a finitely-generated and free $\mathbb{k}[x_1, \dots, x_d]$ -module for some (equivalently for every) homogeneous system of parameters x_1, x_2, \dots, x_d .

As mentioned above, in our case, there always exists a homogeneous system of parameters. We will now give an example of a ring which is *not* Cohen–Macaulay illustrating the previous theorem.

Example 2.3.7 ([ILL⁺07, Ex. 10.6, Ex. 10.12]). Let $\mathcal{R} = \mathbb{k}[s^4, s^3t, st^3, t^4]$. Then \mathcal{R} has Krull dimension 2 and there is a homogeneous system of parameters given by s^4 and t^4 .

We know that \mathcal{R} is a domain and thus s^4 is not a zero divisor. However, t^4 is a zero divisor in $\mathcal{R}/s^4\mathcal{R}$, since

$$t^4(s^3t)^2 = s^4(st^3)^2$$

and $(s^3t) \notin s^4\mathcal{R}$. Hence, \mathcal{R} is not Cohen–Macaulay. Another way to see this is noticing that \mathcal{R} is a finitely generated module over $\mathbb{k}[s^4, t^4]$. The minimal generating set is given by $1, s^3t, st^3, s^6t^2, s^2t^6$. However, this module is *not* free, since we have the following relation between the generators

$$t^4(s^3t)^2 = s^4(st^3)^2 \iff t^4s^6t^2 = s^4s^2t^6.$$

We now want to relate the rather abstract notion of Cohen–Macaulay rings/modules to our geometric setting. Let $P \subset \mathbb{R}^d$ be a lattice d -polytope and let $\mathbb{k}[P]$ be defined as above. By a seminal result of Melvin Hochster [Hoc72], $\mathbb{k}[P]$ is Cohen–Macaulay:

THEOREM 2.3.8 ([BG09, Thm. 6.10]). Let M be a normal, affine semigroup. Then $\mathbb{k}[M]$ is Cohen–Macaulay for every field \mathbb{k} .

From now on we will hence always assume that \mathcal{R} is Cohen–Macaulay. Several important properties of $\mathbb{k}[P]$ can be stated in terms of the *canonical module* $\omega_{\mathbb{k}[P]}$, which is also known as the dualizing module.

Definition 2.3.9. The *canonical module* of \mathcal{R} , $\omega_{\mathcal{R}}$, is the unique module (up to isomorphism) such that $\text{Ext}_{\mathcal{R}}^d(\mathbb{k}, \omega_{\mathcal{R}}) = \mathbb{k}$ and $\text{Ext}_{\mathcal{R}}^i(\mathbb{k}, \omega_{\mathcal{R}}) = 0$ when $i \neq d$. We say that \mathcal{R} is *Gorenstein* if $\omega_{\mathcal{R}} \cong \mathcal{R}$ as an \mathcal{R} -module, or equivalently if $\omega_{\mathcal{R}}$ is generated by a single element. Moreover, we say that \mathcal{R} is *level* if $\omega_{\mathcal{R}}$ as an \mathcal{R} -module is generated by elements of the same degree. A lattice polytope P is *Gorenstein (level)* if $\mathbb{k}[P]$ is Gorenstein (level). We say that P is *reflexive* if it is Gorenstein and has an interior lattice point.

Definition 2.3.9 will be illustrated in Example 2.3.13 and in Example 2.3.14. We will mostly be concerned with semigroup rings $\mathbb{k}[P]$ arising from lattice polytopes P . In this case, the canonical module has a particularly nice description, which is why we omit a proper definition of the Ext-functor:

THEOREM 2.3.10 (Danilov, Stanley, [BG09, Thm. 6.31]). Let the notation be as above. Then the ideal generated by the monomials corresponding to interior integer points, $\text{cone}(P)^\circ \cap \mathbb{Z}^{d+1}$, is the canonical module of $\mathbb{k}[P]$.

Motivated by this theorem, we make the following definition:

Definition 2.3.11. We say that $\mathbf{x} \in \text{cone}(P)^\circ \cap \mathbb{Z}^{d+1}$ is *minimal* if the corresponding monomial is a minimal generator of $\omega_{\mathbb{k}[P]}$.

Remark 2.3.12. Theorem 2.3.10 implies that $\mathbb{k}[P]$ being *Gorenstein* is equivalent to saying that there exists $\mathbf{c} \in \mathbb{Z}^{d+1}$ such that

$$\mathbf{c} + (\text{cone}(P) \cap \mathbb{Z}^{d+1}) = \text{cone}(P)^\circ \cap \mathbb{Z}^{d+1},$$

and \mathbf{c} is called the *Gorenstein point* of $\text{cone}(P)$.

We illustrate these definitions and results in the following example:

Example 2.3.13 (Gorenstein). Let $P = [0, 1]^2$ be the unit square. Then the cone over the polytope is

$$\text{cone}(P) = \{\lambda_1(0, 0, 1) + \lambda_2(1, 0, 1) + \lambda_3(0, 1, 1) + \lambda_4(1, 1, 1) : \lambda_i \geq 0\}.$$

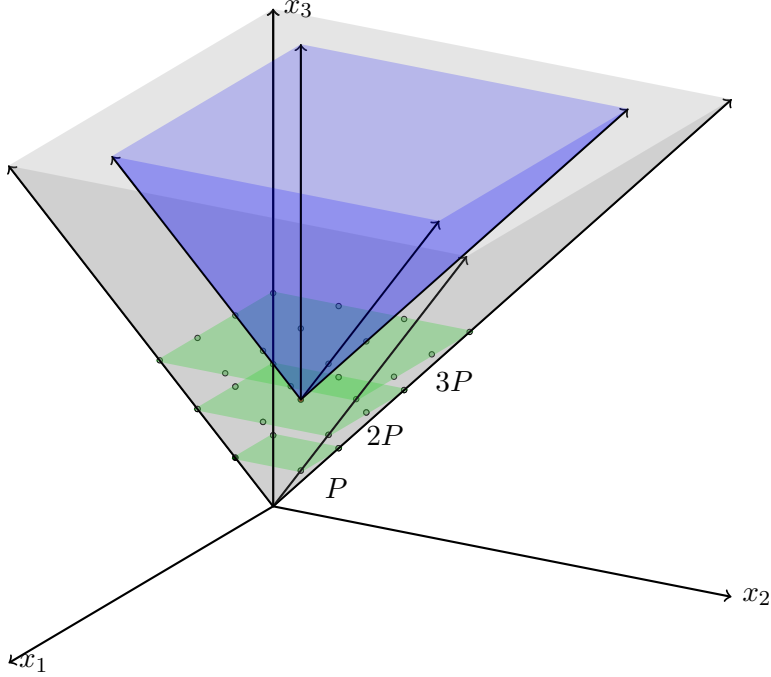


Figure 2.1: The unit square P and its dilates $2P$ and $3P$ (green), the cone over P (gray) and the (conical hull of the) canonical module (blue).

The semigroup is generated (over \mathbb{Z}) by the points $(0, 0, 1)$, $(1, 0, 1)$, $(0, 1, 1)$, and $(1, 1, 1)$. There are exactly $(t + 1)^2$ many monomials of degree t and thus $\text{hilb}(t) = \dim \mathbb{k}[P]_t = (t + 1)^2$. Figure 2.1 shows the monomials of degree less than four. Hence, the Hilbert series $\text{Hilb}(\mathbb{k}[P]; z)$ equals

$$\text{Hilb}(\mathbb{k}[P]; z) = 1 + \sum_{t \geq 1} (t + 1)^2 z^t = \frac{1 + z}{(1 - z)^3}.$$

Thus $[0, 1]^2$ is a polytope of degree 1 with codegree 2. The canonical module is generated by the interior lattice points of $\text{cone}(P)$. There is a unique interior lattice point of lowest degree, namely $(1, 1, 2)$. As it can be seen, this point has lattice distance one to all facets showing that it indeed is a Gorenstein point, see [BG09, Thm. 6.33].

Example 2.3.14 (Level). Let $P := [0, 2] \times [0, 1]$. Then the cone over the polytope is

$$\text{cone}(P) = \{\lambda_1(0, 0, 1) + \lambda_2(0, 1, 1) + \lambda_3(2, 0, 1) + \lambda_4(2, 1, 1) : \lambda_i \geq 0\}.$$

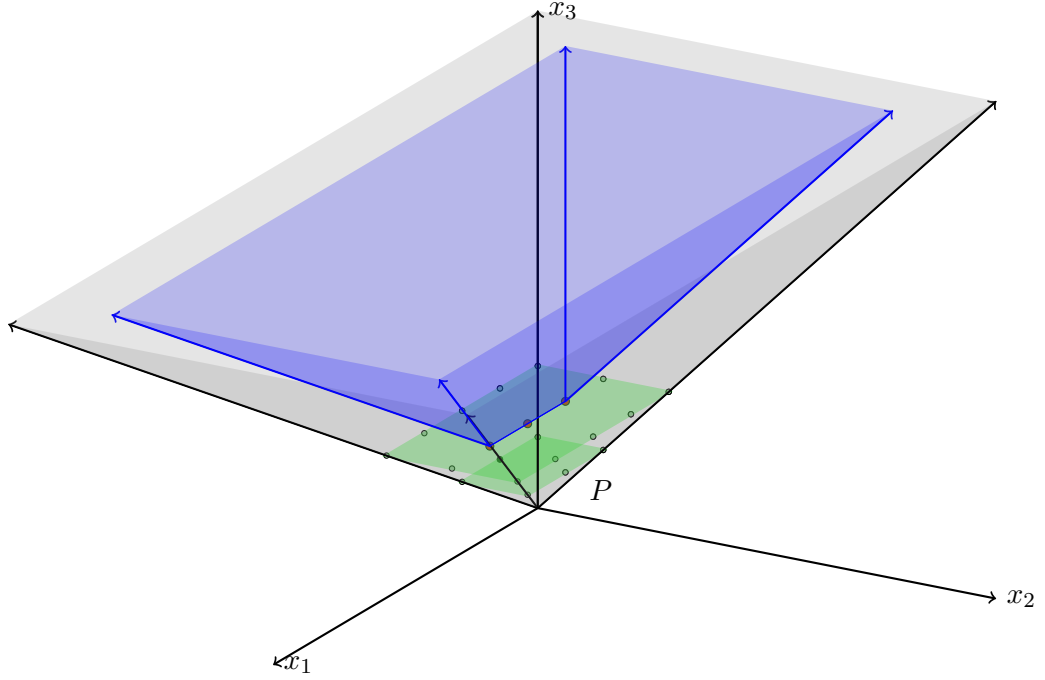


Figure 2.2: The box P and its dilate $2P$, the cone over P (gray) and the (conical hull of the) canonical module (blue).

The semigroup is generated (over \mathbb{Z}) by the points $(0, 0, 1)$, $(2, 0, 1)$, $(0, 1, 1)$, and $(2, 1, 1)$. There are exactly $(t + 1)(2t + 1)$ many monomials of degree t and thus $\text{hilb}(t) = \dim \mathbb{k}[P]_t = (t + 1)(2t + 1)$. Figure 2.2 shows the monomials of degree less than three. Hence, the Ehrhart series $\text{Ehr}_P(z)$ equals

$$\text{Ehr}_P(z) = 1 + \sum_{k \geq 1} (k + 1)(2k + 1)z^k = \frac{1 + 3z}{(1 - z)^3}.$$

Thus P is a polytope of degree 1 with codegree 2. The canonical module is generated by the three interior integer points of lowest degree (marked red), namely $(1, 1, 2)$, $(2, 1, 2)$, and $(3, 1, 2)$. This shows that P is level.

An equivalent formulation of the level property is often more fruitful for computational purposes. For any \mathcal{R} -module M , the *socle* of M is $\text{soc}(M) := \{u \in M : \mathfrak{m}u = 0\}$ where \mathfrak{m} is the unique \ast -maximal ideal of \mathcal{R} . It is equivalent to say that \mathcal{R} is level if for any homogeneous system of parameters x_1, \dots, x_d of \mathcal{R} , all the elements of the graded vector space $\text{soc}(\mathcal{R}/(x_1, \dots, x_d))$ are of the same degree (see [Sta96, Chapter III, Proposition 3.2]).

We have seen that the canonical module encodes interesting algebraic properties. To examine the canonical module, $\omega_{\mathbb{k}[P]}$, of the canonical module, it is natural to think about the Hilbert series of $\omega_{\mathbb{k}[P]}$. Richard Stanley proved the following.

THEOREM 2.3.15 ([BG09, Thm. 6.41]). With the notation from above, we have:

$$\text{Hilb}(\omega_{\mathbb{k}[P]}; t) = (-1)^{d+1} \text{Hilb}(\mathbb{k}[P]; t^{-1}),$$

where $d = \dim P$.

This implies the following result:

THEOREM 2.3.16 ([Sta78, Thm. 4.4]). A lattice polytope P is Gorenstein if and only if $h^*(z)$ is palindromic. Moreover, the Gorenstein index is given by the $\text{codeg}(P)$, i.e., the dilated polytope $\text{codeg}(P) \cdot P$ is reflexive.

Remark 2.3.17. In Example 2.3.13, we saw that the cone over the polytope $[0, 1]^2$ was indeed Gorenstein. Furthermore, we computed the corresponding h^* -polynomial $h^*(z) = 1 + z$, which in fact is palindromic as predicted by Theorem 2.3.16. Similarly, the h^* -polynomial of $[0, 2] \times [0, 1]$ is given by $h^*(z) = 1 + 3z$, which is not palindromic.

2.4 LATTICE POLYTOPES AND GRAPH THEORY

Given a graph $G = (V, E)$, we say that a function $c: V \rightarrow \{1, 2, \dots, k\}$ is a k -coloring. Furthermore, if it also satisfies $c(u) \neq c(v)$ for all $\{u, v\} \in E$, we call c a *proper k -coloring of G* . When we are talking about proper graph colorings, we will always assume that our graph is *simple*, i.e., it does not have any loops or multiple edges. Historically, most of the early results about proper colorings of graphs deal with *planar graphs*, i.e., graphs that can be embedded into \mathbb{R}^2 such that no two edges intersect except for at the nodes. The four-color conjecture stated that every planar has a proper 4-coloring. This conjecture was proven by Appel and Haken using extensive computer calculations, see [AHK77]. In an attempt to prove the four-color conjecture, Georg Birkhoff introduced for planar graphs what is now known as the *chromatic polynomial*. Whitney generalized this definition to arbitrary graphs, see [Whi32].

Definition 2.4.1. Let $G = (V, E)$ be a simple graph. For $k \in \mathbb{Z}_{\geq 1}$, we define the counting function

$$\chi_G(k) := \#\text{proper } k\text{-colorings of } G.$$

The function χ_G is called the *chromatic polynomial of G* .

As the name suggests, χ agrees with a polynomial.

THEOREM 2.4.2 ([Whi32]). Let G be a simple graph on n vertices. Then χ_G is a monic polynomial in k of degree n .

This statement has a beautiful proof based on deletion-contraction. Given a graph $G = (V, E)$ and an edge $e = \{u, v\} \in E$, we define that *deletion* of G with respect to e as $G \setminus e := (V, E \setminus \{e\})$. The *contraction* G/e of G along e is obtained by identifying the nodes u and v and removing all edges between them. Deletion-contraction now says

$$\chi_{G \setminus e}(k) = \chi_G(k) + \chi_{G/e}(k). \quad (2.3)$$

Equation (2.3) is based on the simple observation that there are two types of proper colorings of $G \setminus e$, namely proper colorings c where $c(u) = c(v)$ and proper colorings where $c(u) \neq c(v)$. Colorings, where $c(u) = c(v)$, are in bijection with proper colorings of G/e and colorings, where $c(u) \neq c(v)$, are in bijection with proper colorings of G .

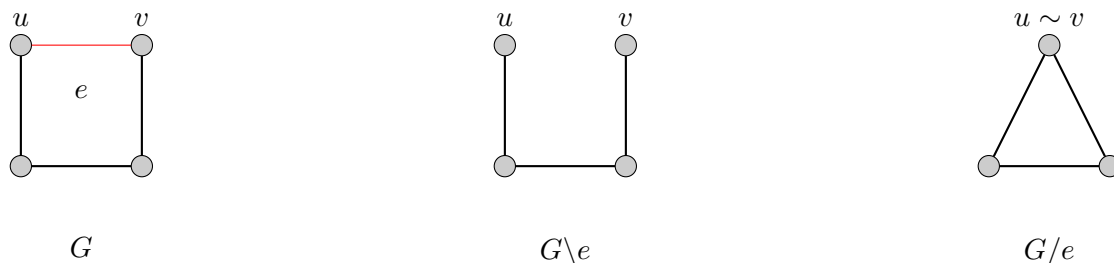


Figure 2.3: Deletion-contraction along e .

We now list some basic facts about the chromatic polynomial, which all except for the second can be proven with methods described in Chapter 3. Let G be a simple graph on n vertices.

- The chromatic polynomial χ_G is a polynomial of degree n with leading coefficient 1.
- The coefficients of χ_G alternate in sign and they form a log-concave sequence, see [Huh12, Cor. 27].
- The second highest coefficient equals $-\#E$.
- The chromatic polynomial is a product of the chromatic polynomials of the connected components of the graph.

The last remark implies that in order to understand the chromatic polynomial, it is enough to focus on the connected components of the graph. Therefore, we will always assume that our graphs are *connected* unless otherwise stated. As was the case with the Ehrhart polynomial, the polynomiality of χ extends the domain from $\mathbb{Z}_{\geq 1}$ to \mathbb{R} . This again raises the question whether we can find an interpretation for values at (say) negative integers. This question was answered by Richard Stanley [Sta73], where he famously showed that chromatic polynomials satisfy a reciprocity theorem by relating it to acyclic orientations. An *acyclic orientation* of a graph G is an orientation of the edges such that the directed graph does not contain any cycles, see Figure 2.4

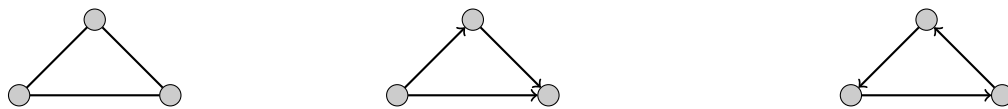


Figure 2.4: G , an acyclic orientation of G , and a cyclic orientation of G .

Given an acyclic orientation α of a graph G and given a (not necessarily proper) k -coloring c , we say that (α, c) are compatible if $c_i \geq c_j$, where $c_l := c(l)$, whenever the

orientation of the edge $\{i, j\}$ is (i, j) . We give Beck's and Zaslavsky's reformulation of [Sta73, Theorem 1.2]:

PROPOSITION 2.4.3. [BZ06, Cor. 5.5] The number of pairs (α, c) consisting of an acyclic orientation of a simple graph G and a compatible k -coloring equals $(-1)^n \chi(-k)$, where n is the number of vertices of G . In particular, $(-1)^n \chi(-1)$ equals the number of acyclic orientations of G .

Again, this result can be proven using *inside-out polytopes* and Ehrhart theory, see Section 3. We will prove a version of this theorem, where we partially color an induced subgraph, see Theorem 3.4.13.

Example 2.4.4. Let $G = C_3$ be the cycle graph on 3 vertices, see Figure 2.4. Then $\chi_G(k) = k(k-1)(k-2)$, since every node is connected to all of the other nodes. Now the evaluation $\chi_G(-1) = -6$ indicates that there are 6 acyclic orientations of the edges. There are $2^3 = 8$ possible orientations of the edges and only 2 give rise to a cyclic orientation illustrating Proposition 2.4.3.

The previous results were mainly about proper colorings of graphs. Now we will shift gears and talk about counting the number of walks in a graph. This can be done using a transfer-matrix method. We will follow [Sta12, Sec. 4.7] to introduce this method, which we will revisit in Chapter 3. We will state the procedure for directed graphs, as we will work with directed graphs in Chapter 4. This method can be adapted to undirected graphs by replacing an undirected edge $\{u, v\}$ by edges (u, v) and (v, u) . A directed graph D is a triple $D = (V, E, \varphi)$, where $V = \{v_1, v_2, \dots, v_p\}$ is the set of nodes, E is the set of (directed) edges, and $\varphi: E \rightarrow V \times V$ is a map determining the direction of the edges. If $\varphi(e) = (u, v)$, then we say that the edge e has *initial node* u and *target node* v , denoted $\text{init}(e) = u$ and $\text{fin}(e) = v$. A walk Γ from u to v of length n is a sequence of edges $e_1 e_2 \dots e_n$ satisfying $\text{init}(e_1) = u$, $\text{fin}(e_i) = \text{init}(e_{i+1})$ for $1 < i < n$, and $\text{fin}(e_n) = v$. If $u = v$, then Γ is called a *closed walk*. Moreover, we define a *weighted digraph* D to be a digraph D together with a weight function $w: E \rightarrow \mathbb{R}$. If $\Gamma = e_1 e_2 \dots e_n$, we set the weight of Γ to be $w(\Gamma) := \prod_{i=1}^n w(e_i)$. For every positive integer n , we define

$$A_{ij}(n) = \sum_{\Gamma} w(\Gamma),$$

where the sum ranges over all walks Γ of length n from node v_i to node v_j . We define the *transfer matrix* of G to be the matrix

$$A_{ij} = \sum_e w(e)$$

where the sum is taken over all edges with $\text{init}(e) = v_i$ and $\text{fin}(e) = v_j$. As it turns out, we can interpret the $A_{ij}(n)$ as the entries of a certain matrix:

THEOREM 2.4.5 ([Sta12, Thm. 4.7.1]). Let $n \in \mathbb{Z}_{>0}$. Then $A_{i,j}^n = A_{i,j}(n)$, where we define $A^0 := I$ and where $A_{i,j}^n$ is the (i, j) -entry of the matrix A^n .

Proving this theorem is a standard exercise in combinatorics classes, as the proof essentially is matrix multiplication. The matrix A is often referred to as the *adjacency matrix of D* . However, we will only use the term adjacency matrix for an *undirected*, simple graph G with $w \equiv 1$, i.e.,

$$A_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

In this case, we obtain the following corollary:

COROLLARY 2.4.6. Let G be an undirected, simple graph and let A be the adjacency matrix of G . Then $(A^n)_{ij}$ counts the number of walks of length n from v_i to v_j .

Remark 2.4.7. The number of *closed* walks is counted by trace of A^n . Since the trace is the sum of the eigenvalues, and since the (absolute value-wise) biggest eigenvalue λ_{\max} is positive by the Perron–Frobenius theorem [Mey00, Ch. 8], the trace is asymptotically dominated by λ_{\max}^n , where we implicitly use that G is connected.

We want to illustrate the previous result in a simple example:

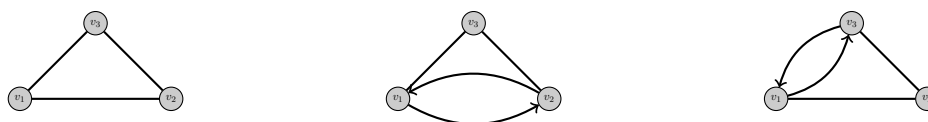


Figure 2.5: $G = C_3$ and the two walks of length 2 from v_1 to v_1 .

Example 2.4.8. Let $G = C_3$, see Figure 2.5. The adjacency matrix of G is given by

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

and

$$A^2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Now we see that $(A^2)_{12} = a_{11}a_{11} + a_{12}a_{21} + a_{13}a_{31} = 2$ is the number of walks of length 2 from vertex v_1 back to itself, as predicted by Theorem 2.4.5. The walks are shown in Figure 2.5. We specifically wrote out matrix multiplication to illustrate that $a_{1i}a_{i1}$ counts the number of walks from v_1 to v_1 via the node v_i . This product is 1 if and only if all factors are 1 and it is 0 otherwise. The sum is over all possible intermediate nodes.

TRANSFER-MATRIX METHODS MEET EHRHART
THEORY

“We learned to be happy. We danced ’round the hall. And learning to count was the key to it all.” (Sesame Street)

This chapter is joint work with Alexander Engström and based on [EK18].

3.1 INTRODUCTION

Graph colorings have been intriguing mathematicians and computer scientists for decades. Historically, graph colorings first appeared in the context of the 4-color conjecture. For planar graphs, Birkhoff — trying to prove this conjecture — introduced what is now called the chromatic polynomial. Whitney later generalized this notion from planar graphs to arbitrary graphs, see [Whi32]. Chromatic polynomials are one of the fundamental objects in algebraic graph theory with many questions about them still unanswered. For instance, in 1968 Read asked which polynomials arise as chromatic polynomials of some graph. This question remains wide open to this day. However, some progress has been made. In 2012, June Huh showed that the absolute values of the coefficients form a log-concave sequence, see [Huh12, Thm. 3], thus proving a conjecture by Rota, Heron, and Welsh. Not only classifying chromatic polynomials is extremely challenging, but also explicitly computing the coefficients turns out to be $\#P$ -hard, see [JVW90].

In the first part of this chapter, we examine proper k -colorings of Cartesian graph products of the form $G \times P_n$ and $G \times C_n$, where G is an arbitrary graph and P_n (C_n) is the path (cycle) graph on n nodes, respectively. The motivation for this problem is twofold:

First, this problem lies at the intersection of transfer-matrix methods and Ehrhart theory, both areas being interesting in their own right. Classically, transfer-matrix methods have been used to count the number of (possibly closed) walks on weighted graphs. However, transfer-matrix methods also made an appearance in seemingly unrelated areas such as calculating DNA-protein-drug binding in gene regulation [Tei07], the 3-dimensional dimer problem [Ciu98], counting graph homomorphisms [LM08], computing the partition function for some statistical physical models [FLS07], and determining the entropy in physical systems [FP05]. One of the big problems in these applications is that the size of the transfer matrices increases extremely fast as the size of the system increases. Therefore, one needs to either limit the size of the system or find a way of “compactifying” these transfer matrices. In [Ciu98], Ciucu uses symmetry to reduce the size of the matrix. Similar techniques have also been used by [LM08]. We follow and expand these ideas.

Second, this problem also has direct applications to physics: In [BEMPS10, Sec. 6], it is described how the chromatic polynomial of graphs of the form $G \times P_n$ is related to the

zero-temperature antiferromagnetic case of the k -state Potts model. If G represents a molecular structure, then $G \times P_n$ corresponds to several connected layers of that molecular structure. The k colors correspond to k different states of the atoms. Counting the number of possible combinations is the same as counting the number of colorings. If we furthermore assume that two adjacent atoms are not allowed to be in the same state, we arrive at a classical proper coloring problem. Since n is very large in physical systems and also k may vary, the (doubly or single) asymptotic behavior is of interest.

Our work combines transfer-matrix methods with Ehrhart theory. As an intermediate step, we examine proper colorings of a graph, where some nodes are already colored. We call these colorings *restricted colorings*. The associated counting function is a polynomial and it satisfies a reciprocity statement, i.e., there is a combinatorial interpretation for evaluations of this counting function at negative integers.

THEOREM (Theorem 3.4.13). Let $\Gamma = (\{1, 2, \dots, n\}, E)$ be a graph and fix a proper k' -coloring $c' : V' \rightarrow \{1, 2, \dots, k'\}$ on the induced subgraph $\Gamma|_{V'}$ for a subset $V' \subset V(\Gamma)$. Then, for $k \geq k'$, the restricted chromatic polynomial

$$\chi_{c', \Gamma}(k) = \#\text{proper } k\text{-colorings } c \text{ of } \Gamma \text{ such that } c|_{V'} = c'. \quad (3.1)$$

is a polynomial of degree $\#V - \#V' =: s$ with leading coefficient 1, whose coefficients a_i alternate in sign, and whose absolute values of the coefficient form a *log-concave sequence*, i.e., $a_i^2 \geq a_{i-1}a_{i+1}$ holds for $0 < i < s$. The second highest coefficient a_{n-1} is given by

$$-a_{n-1} = \#\text{edges } \{v_i, v_j\} \text{ such that } \{v_i, v_j\} \not\subseteq V'.$$

Moreover, we have the reciprocity statement

$$\begin{aligned} \chi_{c', \Gamma}(-k) &= (-1)^s \#(\alpha, c) \text{ of } \Gamma \text{ with } c|_{V'} = c' \\ &= (-1)^s \chi_{c', \Gamma}(k), \end{aligned}$$

where (α, c) is a pair of an acyclic orientation α and a compatible (*not necessarily proper*) k -coloring, c , of Γ .

If a graph is not connected, the chromatic polynomial is a product of the chromatic polynomials of its connected components. Therefore, we will — without loss of generality — from now on assume that all our graphs are *connected*.

With this result at hand, we turn to examine proper colorings of $G \times P_n$ and $G \times C_n$. In Definition 3.4.4, using a group action and restricted k -colorings, we define a compactified transfer matrix L . The rows and columns of L are labeled by orbits o_1, o_2, \dots, o_p of this group action, see Definition 3.4.1. As it turns out, all entries of L are polynomials in k :

THEOREM (Theorem 3.4.14). With the notation from Theorem 3.4.13 and with $k \geq N$, we have:

1. Every entry $L_{o_i, o_j}(k)$ equals the sum of Ehrhart polynomials of lattice inside-out polytopes of dimension $\#\text{colors of } o_j$ and hence is a polynomial of degree $\#\text{colors of } o_j$,
2. $L_{o_i, o_j}(k)$ is independent of the choice of the representative, i.e., it is well-defined, and
3. $\#o_i \cdot L_{o_i, o_j}(k) = \chi_{G_{o_i, o_j}}(k)$, where

$$\begin{aligned} \chi_{G_{o_i, o_j}}(k) = & \#\text{proper } k\text{-colorings of } G \times P_2 : \text{coloring of } G \times \{1\} \text{ is in } o_i \\ & \text{and coloring of } G \times \{2\} \text{ is in } o_j. \end{aligned}$$

The matrix L in fact behaves like a transfer matrix:

THEOREM (Theorem 3.4.17). Let $V(P_{n+1}) = \{1, 2, \dots, n+1\}$ and let G be any graph. Let o_1, o_2, \dots, o_p be the orbits as defined in Definition 3.4.1. Then, for $k \geq \#\text{colors used in } o_i$, the (o_i, o_j) -entry of L^n counts the number of proper k -colorings of $G \times P_{n+1}$, where $G \times \{1\}$ is fixed by a coloring in orbit o_i , and where the coloring of $G \times \{n+1\}$ lies o_j .

Moreover, L can be used to explicitly compute the chromatic polynomial of $G \times P_n$.

COROLLARY (Corollary 3.4.20). Let $G \times P_{n+1}$ and L be as above. Then

$$\chi_{G \times P_{n+1}}(k) = (w_1(k), \dots, w_p(k))L^n \mathbf{1}, \quad (3.2)$$

where $w_i(k)$ is the size of o_i and $\mathbf{1} := (1, \dots, 1)^t$.

The row sums of this matrix also satisfy a reciprocity statement:

PROPOSITION (Proposition 3.4.27). Let $L \in \mathbb{Z}^{p \times p}$ be as above, let $L_i^n := \sum_{k=1}^p (L^n)_{i,k}$ be the i^{th} row sum of L^n , and let $V(P_{n+1}) = \{1, 2, \dots, n+1\}$. Then, for $k \geq N = \#V(G)$, we have

$$L_i^n(-k) = (-1)^{Nn} \#(\alpha, c) \text{ of } G \times P_{n+1} \text{ where } G \times \{1\} \text{ is colored by repr. of } o_i, \quad (3.3)$$

where (α, c) is a pair of an acyclic orientation α and a compatible o_i -restricted k -coloring c .

Asymptotically, the power of the biggest eigenvalue λ_{\max}^{n-1} of L determines the number of proper colorings of $G \times C_n$. We give explicit bounds of this eigenvalue in terms of the row sums. Let $\delta(L)$ and $\Delta(L)$ be the smallest and biggest row sums of L , respectively.

PROPOSITION (Proposition 3.4.32). Let G be a graph and $N = \#V(G)$ and let $\delta(L)$ and $\Delta(L)$ be as above. Then the doubly asymptotic behavior of the number of proper k -colorings of $G \times C_n$ is dominated by λ_{\max} and

$$\delta(L) \leq \lambda_{\max} \leq \Delta(L),$$

where $\delta(L) = \sum_{i=0}^N a_i k^i$, $\Delta(L) = \sum_{i=0}^N b_i k^i$, $a_N = b_N$, and $a_{N-1} = b_{N-1}$.

This chapter is structured as follows. In Section 3.2, we introduce some basic notions about graphs. In particular, we introduce the Cartesian graph product and the automorphism group of a graph. In Section 3.3, we introduce a transfer-matrix method and show how one can use this to count the number of proper colorings when the number of colors is fixed. We then — following [BZ06] — introduce inside-out polytopes and show how to count proper colorings of $G \times P_n$ when n is fixed. In Section 3.4, we illustrate how one can use symmetry to define a compactified transfer matrix L , whose size does not depend on n . We state and prove our main results about this matrix L . We end this section, with a brief interlude on counting the number orbits under a group action, where Bell numbers make a surprising appearance.

3.2 BACKGROUND AND NOTATION

In this section, we introduce two prominent families of graphs, graph automorphisms, the Cartesian product of graphs, as well as restricted, proper k -colorings. We end this section by stating two explicit counting problems that we address in Section 3.4, see Problem 3.2.3. We assume all our graphs G to be finite and we globally set $N := \#V(G)$. Moreover, we define the *path graph* P_n to be the graph on the vertex set $[n]$ with edges $\{i, i+1\}$ for $i \in [n]$. We also define the *cycle graph* C_n to be the graph with vertex set $[n]$ and with edge $\{\{i, i+1\} : i \in [n-1]\} \cup \{1, n\}$. A *graph automorphism of a graph* $G = (V, E)$ is a permutation σ of the vertex set such that $\{i, j\}$ is an edge if and only if $(\sigma(i), \sigma(j))$ is an edge. The set of automorphisms of a graph G together with the composition operation forms a group, which is called the *automorphism group of G* .

Example 3.2.1 (Automorphism group of C_n). Let $G = C_n$ be the cycle graph on n nodes. The standard way of representing G is by drawing it as a regular n -gon. Now one notices that every graph automorphism has to be a symmetry of the n -gon and vice versa. Therefore, the automorphism group of C_n is the dihedral group D_n with $2n$ elements. In particular, if $n = 5$, the dihedral group D_5 is generated by the cycle (12345) and by the permutation $(1)(25)(34)$. The cycle (12345) corresponds to the rotation symmetry of the regular 5-gon and the permutation $(1)(25)(34)$ corresponds to a flip of the regular 5-gon.

This chapter mainly focuses on a special family of graphs, namely the Cartesian product of an arbitrary graph G with either the path graph P_n or the cycle graph C_n .

Definition 3.2.2. Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two graphs. The *Cartesian product* $G_1 \times G_2$ (sometimes in the literature also denoted $G_1 \square G_2$) is the

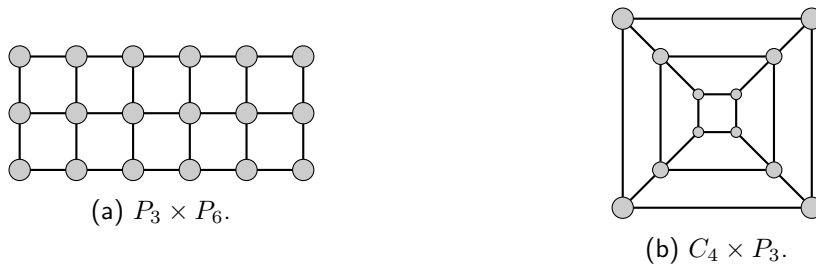


Figure 3.1: The Cartesian graph product illustrated.

graph with vertex set $V(G_1) \times V(G_2)$, and vertices (u_1, v_1) and (u_2, v_2) are connected by an edge if

- either $u_1 = u_2$ and $\{v_1, v_2\} \in E(G_2)$,
- or if $v_1 = v_2$ and $\{u_1, u_2\} \in E(G_1)$.

Figure 3.1 illustrates Definition 3.2.2.

We have already seen the notion of proper graph colorings and the related reciprocity statement in Section 2.4. In this chapter, we will come across what we call a c' -restricted, proper k -coloring. Let $\Gamma = (V, E)$ be a graph and fix a proper k' -coloring $c': V' \rightarrow \{1, 2, \dots, k'\}$ on the induced subgraph $\Gamma|_{V'}$ of a subset $V' \subset V(\Gamma)$, i.e., a proper k' -coloring on the graph $\Gamma|_{V'} = (V', E')$, where $E' = \{\{i, j\}: i, j \in V', \{i, j\} \in E\}$. Then, for $k \geq k'$, we define the c' -restricted chromatic polynomial

$$\chi_{c', \Gamma}(k) = \#\text{proper } k\text{-colorings } c \text{ of } \Gamma \text{ such that } c|_{V'} = c'.$$

This name will be justified in Theorem 3.4.13, where we will also state a restricted analogue of Proposition 2.4.3, which again is related to restricted acyclic orientations. A not necessarily proper c' -restricted k -coloring is *compatible* with an acyclic orientation α if

1. (α, c) are compatible in the usual sense,
2. and if $v \in V \setminus V'$ is adjacent to $u_1, u_2, \dots, u_s \in V'$ with $c'(u_1) = \dots = c'(u_s)$, then the orientations of the edges $\{v, u_i\}$ have to be the same for all i .

We are now ready to state the main problems of the first part of this chapter:

Problem 3.2.3. 1. How many proper k -colorings does the Cartesian graph product $G \times P_n$ have?

2. How many proper k -colorings does the Cartesian graph product $G \times C_n$ have?

We would like to remark that both k and n are variables. It is enough to consider the case where G is connected, since otherwise the chromatic polynomial is a product of the chromatic polynomials of the connected components. Hence, we will always assume that

G is *connected*, a fact we will use to properly apply the Perron–Frobenius theorem, see [Mey00, Ch. 8]. We answer Question 1 in Cor 3.4.20 and we will describe the doubly asymptotic behavior of Question 2 in Proposition 3.4.32.

There are two prominent special cases which we will address in Section 3.3:

1. k is fixed and n varies (see Section 3.3.1),
2. n is fixed and k varies (see Section 3.3.2).

3.3 PROMINENT SPECIAL CASES

3.3.1 Transfer-Matrix Methods

In this section, we want to introduce a transfer-matrix method and apply it to Problem 3.2.3 in the case where k is fixed. For a brief introduction to transfer-matrix methods, we refer the reader to Section 2.4. Let G be a connected graph with vertex set $V(G) = \{v_1, \dots, v_p\}$. To count the number of proper k -colorings of $G \times P_n$, we associate a new graph M_G to G . We define M_G to be the graph with

- vertex set \mathcal{C} , where \mathcal{C} is the set of proper k -colorings of G ,
- and where two vertices $(c_1(v_1), \dots, c_1(v_p))$ and $(c_2(v_1), \dots, c_2(v_p))$ are connected if $c_1(v_i) \neq c_2(v_i)$ for all $i \in [p]$.

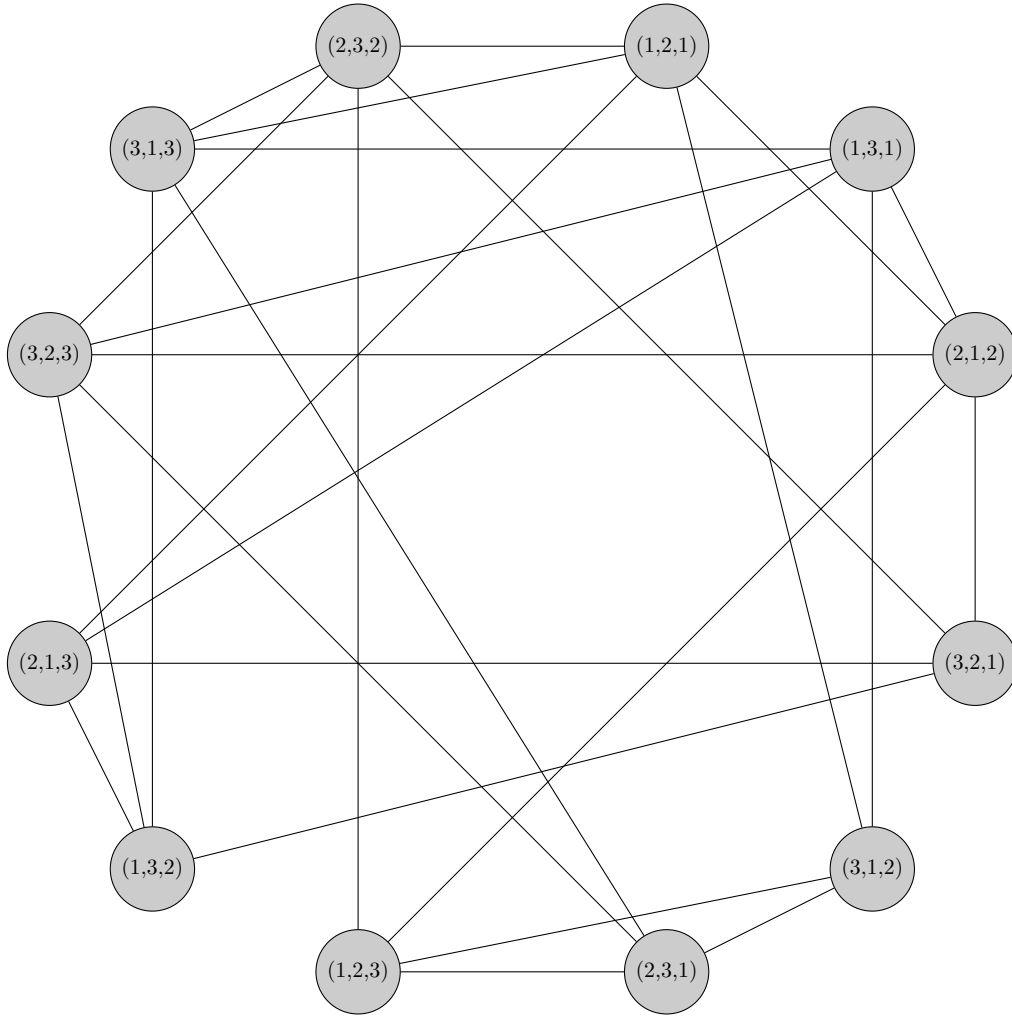


Figure 3.2: M_{P_3} .

Figure 3.2 shows M_{P_3} for three colors. This construction establishes a connection between k -colorings of $G \times P_n$ ($G \times C_n$) and walks (closed walks) of length n in M_G , respectively. By abuse of notation, let $u^1 u^2 \dots u^n$ be a walk on M_G . By construction, this walk corresponds to the proper k -coloring of $G \times P_n$, where $G \times \{i\}$ is colored by u^i for all i . Similarly, if the walk is closed and thus $u^1 = u^n$, we get a corresponding proper k -coloring of $G \times C_{n-1}$. Therefore, we can count the number of k -colorings of $G \times P_n$ by computing powers of the adjacency matrix A_{M_G} of M_G . Moreover, we can asymptotically count the number of colorings of $G \times C_n$ by analyzing the biggest eigenvalue of A_{M_G} .

The size of the transfer matrix A_{M_G} is $\chi_G(k) \times \chi_G(k)$, since the number of vertices of M_G is the number of proper k -colorings of G . Figure 3.3 shows $A_{M_{P_3}}$ for four colors.

| | 121 | 131 | 141 | 212 | 232 | 242 | 313 | 323 | 343 | 414 | 424 | 434 | 123 | 124 | 132 | 134 | 142 | 143 | 213 | 214 | 231 | 234 | 241 | 243 | 312 | 314 | 321 | 324 | 341 | 342 | 412 | 413 | 421 | 423 | 431 | 432 | | | |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|---|---|---|
| 121 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | | |
| 131 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | | |
| 141 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | | |
| 212 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | | |
| 232 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | | |
| 242 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | | |
| 313 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | |
| 323 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | |
| 343 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | |
| 414 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 424 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 434 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 123 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | | |
| 124 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | | |
| 132 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | |
| 134 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | |
| 142 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 |
| 143 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | |
| 213 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 1 | |
| 214 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | |
| 231 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | |
| 234 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | |
| 241 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 243 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| 312 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |
| 314 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 321 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| 324 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 1 |
| 341 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 1 |
| 342 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 |
| 412 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 413 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 421 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 423 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 431 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 432 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Figure 3.3: Adjacency matrix of M_{P_3} for 4 colors.

In Section 3.4, we identify a *compactified* transfer matrix L whose size does *not* depend on k and which can be used to count the number of proper k -colorings of $G \times P_n$ for *all* k . Furthermore, we show that the biggest eigenvalue of L equals the biggest eigenvalue of A_{M_G} for all k . We also give a combinatorial and a geometric interpretation for the entries of L .

3.3.2 Ehrhart Theory and Inside-Out Polytopes

Throughout this section, let n be fixed. This implies that the size of the graphs $G \times P_n$ and $G \times C_n$ is fixed, too. Under this assumption, Problem 3.2.3 reduces to computing the chromatic polynomial of a given graph. We will use the perspective of inside-out polytopes and Ehrhart theory developed by Beck and Zaslavsky [BZ06] to understand chromatic polynomials. For a brief introduction to Ehrhart theory and lattice polytopes, we refer to Section 2.2.

The following definitions are taken from [BZ06]. A *hyperplane arrangement* \mathcal{H} is a

set of finitely many linear or affine hyperplanes in \mathbb{R}^d . An *open region* is a connected component of $\mathbb{R}^d \setminus \bigcup_{H \in \mathcal{H}} H$. A *closed region* is the topological closure of an open region. Moreover, we define the *intersection semilattice*

$$\mathbb{L} := \left\{ \bigcap \mathcal{S} : \mathcal{S} \subset \mathcal{H}, \text{ and } \bigcap \mathcal{S} \neq \emptyset \right\},$$

where the order is given by reverse inclusion. The minimal element is $\hat{0} = \mathbb{R}^d$. The elements of \mathbb{L} are sometimes called *flats*. \mathbb{L} is therefore a partially ordered set — or poset for short — and we recursively define the *Möbius function* $\mu: \mathbb{L} \times \mathbb{L} \rightarrow \mathbb{Z}$ by

$$\mu(r, s) := \begin{cases} 0 & \text{if } r \not\leq s, \\ 1 & \text{if } r = s, \\ -\sum_{r \leq u < s} \mu(r, u) & \text{if } r < s. \end{cases}$$

The *characteristic polynomial* $p_{\mathcal{H}}$ of a hyperplane arrangement \mathcal{H} is defined as

$$p_{\mathcal{H}}(\lambda) := \begin{cases} 0 & \text{if } \mathcal{H} \text{ contains the degenerate hyperplane } \mathbb{R}^d, \\ \sum_{s \in \mathbb{L}} \mu(\hat{0}, s) \lambda^{\dim s} & \text{otherwise.} \end{cases}$$

Huh has shown that the coefficients of $p_{\mathcal{H}}$ are alternating in sign and the absolute value of the coefficients form a log-concave sequence, see [Huh12, Thm. 3].

Let $P \subset \mathbb{R}^d$ be a rational, closed, convex d -polytope and let \mathcal{H} be an arrangement of *rational* hyperplanes that meets P transversally, i.e., every flat

$$u \in \left\{ \bigcap \mathcal{S} : \mathcal{S} \subset \mathcal{H} \text{ and } \bigcap \mathcal{S} \neq \emptyset \right\}$$

that intersects the topological closure of P also intersects the interior P° . Here rational means that all vertices of P lie in \mathbb{Q}^d and all hyperplanes in \mathcal{H} are specified by equations with rational coefficients. Following [BZ06], we call the pair (P, \mathcal{H}) a *rational inside-out polytope*. A *region* of (P, \mathcal{H}) is one of the components of $P \setminus \bigcup \mathcal{H}$ or the closure of one such component. If the vertex set of P is a subset of \mathbb{Z}^d , we call (P, \mathcal{H}) a *lattice inside-out polytope*. The *multiplicity* of $x \in \mathbb{R}^d$ with respect to \mathcal{H} is

$$m_{\mathcal{H}}(x) := \#\text{closed regions of } \mathcal{H} \text{ containing } x.$$

The *multiplicity with respect to* (P, \mathcal{H}) is

$$m_{P, \mathcal{H}}(x) := \begin{cases} \#\text{closed regions of } (P, \mathcal{H}) \text{ that contain } x, & \text{if } x \in P, \\ 0 & \text{otherwise.} \end{cases}$$

We define the *closed Ehrhart quasipolynomial*

$$\text{ehr}_{P, \mathcal{H}}(t) := \sum_{x \in \mathbb{Z}^d} m_{tP, \mathcal{H}}(x),$$

where $t \in \mathbb{Z}_{\geq 1}$ and tP denotes the t^{th} dilate of P . Similarly, we define the *open Ehrhart quasipolynomial*

$$\text{ehr}_{P,\mathcal{H}}^{\circ}(t) := \# \left(\mathbb{Z}^d \cap t \left[P \setminus \bigcup H \right] \right).$$

With the notation from above, we have:

THEOREM 3.3.1 ([BZ06, Thm. 3.1]). Let $P \subset \mathbb{R}^d$ be a d -dimensional polytope and let \mathcal{H} be a hyperplane arrangement not containing the degenerate hyperplane \mathbb{R}^d . Then

$$\text{ehr}_{P,\mathcal{H}}^{\circ}(t) = \sum_{u \in \mathbb{L}} \mu(\hat{0}, u) \# \left(\mathbb{Z}^d \cap tP \cap u \right) \quad (3.4)$$

and if \mathcal{H} is transverse to P , we have

$$\text{ehr}_{P,\mathcal{H}}(t) = \sum_{u \in \mathbb{L}} |\mu(\hat{0}, u)| \# \left(\mathbb{Z}^d \cap tP \cap u \right), \quad (3.5)$$

where μ is the Möbius function of \mathbb{L} .

Since we assume P to be full-dimensional, the hyperplane arrangement subdivides P into closed regions R_1, R_2, \dots, R_m . Moreover, we have that

$$\text{ehr}_{P,\mathcal{H}}(t) = \sum_{i=1}^m \text{ehr}_{R_i}(t) \quad \text{and} \quad \text{ehr}_{P^{\circ},\mathcal{H}}^{\circ}(t) := \sum_{i=1}^m \text{ehr}_{R_i^{\circ}}(t),$$

where $\text{ehr}_{R_i}(t)$ is the classical Ehrhart quasipolynomial of the closed region R_i , and the interior of R_i is with respect to the topology of the ambient space \mathbb{R}^d , see [BZ06, (4.2)]. We remark that $\text{ehr}_{P^{\circ},\mathcal{H}}^{\circ}(t)$ does not count any integer point on the facets of P , whereas $\text{ehr}_{P,\mathcal{H}}^{\circ}(t)$ also counts integer points on facets. Furthermore, there is a reciprocity result:

THEOREM 3.3.2 ([BZ06, Theorem 4.1]). If (P, \mathcal{H}) is a d -dimensional lattice inside-out polytope, then $\text{ehr}_{P,\mathcal{H}}(t)$ and $\text{ehr}_{P^{\circ},\mathcal{H}}^{\circ}(t)$ are polynomials of degree d and they also satisfy the reciprocity theorem

$$\text{ehr}_{P^{\circ},\mathcal{H}}^{\circ}(t) = (-1)^d \text{ehr}_{P,\mathcal{H}}(-t). \quad (3.6)$$

In Section 3.4.2, we will intersect inside-out polytopes with hyperplanes, so we will need the following:

COROLLARY 3.3.3 ([BZ06, Corollary 4.3]). Let D be a discrete lattice in \mathbb{R}^d , let P be a D -fractional convex polytope, i.e., all vertices of P lie in $t^{-1}D$ for some $t \in \mathbb{Z}_{>0}$, and let \mathcal{H} be a hyperplane arrangement in $s := \text{aff}(P)$ that does not contain the degenerate hyperplane. Then $\text{ehr}_{P,\mathcal{H}}(t)$ and $\text{ehr}_{P^{\circ},\mathcal{H}}^{\circ}(t)$ are quasipolynomials in t that satisfy the reciprocity law

$$\text{ehr}_{P^{\circ},\mathcal{H}}^{\circ}(t) = (-1)^{\dim s} \text{ehr}_{P,\mathcal{H}}(-t).$$

The next result connects inside-out polytopes to proper colorings of graphs.

THEOREM 3.3.4 ([BZ06, Theorem 5.1]). Let G be an ordinary graph on n vertices and let $P = [0, 1]^n$. Moreover, we define

$$\mathcal{H}(G) := \{x_i = x_j : \{x_i, x_j\} \in E\}.$$

Then

$$\text{ehr}_{P^\circ, \mathcal{H}(G)}(t) = (-1)^n \text{ehr}_{P, \mathcal{H}(G)}(-t) = \chi_G(t-1). \quad (3.7)$$

Remark 3.3.5. The intuition behind this theorem is the following: Every integer point in the interior of tP corresponds to a (not necessarily proper) coloring of G . If G contains the edge $\{x_i, x_j\}$, then any proper coloring cannot have an integer point on the hyperplane $x_i = x_j$. Therefore, every integer point in $t(P^\circ \setminus \bigcup_{H \in \mathcal{H}(G)} H)$ corresponds to a proper coloring of G and vice versa.

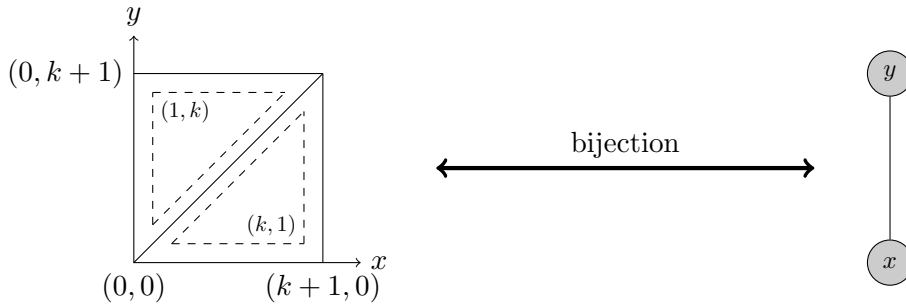


Figure 3.4: Integer points in dashed triangles correspond to proper k -colorings of P_2 .

One immediate consequence of Theorem 3.3.4 is that χ_G is a polynomial of degree n and the leading coefficient is 1, as the sum of the volumes of the regions of the subdivided cube equals 1.

3.4 TRANSFER-MATRIX METHODS MEET EHRHART THEORY

3.4.1 Enter Symmetry

In this section, we combine transfer-matrix methods with Ehrhart theory. Again, we will assume that all graphs are connected. We will use group actions and orbits to introduce a *compactified transfer matrix* L , whose size does not depend on k . There are two types of symmetries that appear. Firstly, the set of proper k -colorings in Problem 3.2.3 stays invariant under a permutation of colors. That is, simply renaming the colors does not change the graph M_G . For example, in Figure 3.2, the vertex $(1, 2, 1)$ has as many neighbors as the vertex $(3, 2, 3)$. Secondly, the graph G itself also has a symmetry group, called the *automorphism group of G* . We first quotient out by the group coming from permuting the colors and then we quotient out by a possibly trivial subgroup of the automorphism group.

Definition 3.4.1 (Orbit notation). Let G be a simple, connected graph on N vertices and let \mathcal{C} be the set of proper k -colorings, where $k \geq N$. Let \mathfrak{S}_k be the symmetric group on k elements and let \mathcal{G} be a possibly trivial subgroup of the automorphism group of G , see Subsection 3.2. The group \mathfrak{S}_k acts on \mathcal{C} by permuting the colors and it gives rise to orbits $\tilde{o}_1, \dots, \tilde{o}_q$. The group \mathcal{G} is acting on $\tilde{o}_1, \dots, \tilde{o}_q$ giving rise to orbits o_1, \dots, o_p .

Example 3.4.2. Let $G = C_5$. We first quotient by permutations of colors. This group action induces orbits $\tilde{o}_1, \tilde{o}_2, \dots, \tilde{o}_{11}$ (as computer generated in no particular order):

$$\begin{array}{c|c|c|c|c|c} 14\ 2\ 3\ 5 & 14\ 2\ 35 & 14\ 25\ 3 & 1\ 24\ 3\ 5 & 1\ 24\ 35 & 1\ 2\ 3\ 4\ 5 \\ \hline 1\ 2\ 35\ 4 & 1\ 25\ 3\ 4 & 13\ 24\ 5 & 13\ 2\ 4\ 5 & 13\ 25\ 4 & \end{array}$$

Here, for instance, 14 2 3 5 means that the vertices v_1 and v_4 are colored by the same color and that the vertices v_2, v_3 , and v_5 don't share a color with any other vertex. The automorphism group of C_5 is the dihedral group generated by (12345) and (1)(25)(34). The 11 partitions of the vertex set end up in 3 orbits o_1, o_2 , and o_3 after quotienting by the dihedral group. The classes are represented by:

$$1\ 2\ 3\ 4\ 5 \mid 1\ 24\ 35 \mid 1\ 2\ 4\ 35.$$

Remark 3.4.3. For $k \geq N$, quotienting by \mathfrak{S}_k always gives the same number of orbits. This is due to the fact that every orbit \tilde{o} can be seen as a partition of the vertex set of G into independent sets. In particular, the number of orbits is *finite*. Therefore, we also get that the number of orbits o_1, \dots, o_p is the same for all $k \geq N$. In Definition 3.4.1, we only assume that \mathcal{G} is a subgroup of the automorphism group, as it is in general difficult to determine the full automorphism group.

This enables us to define a matrix L encoding the necessary combinatorial information, whose size is independent of the number of colors k . Following the arguments described in [Ciu98], we give the following definition:

Definition 3.4.4. Let G be a graph on N nodes. Let o_1, o_2, \dots, o_p be orbits as defined in Definition 3.4.1 and let $k \geq N$ be any integer. Let A_{M_G} be the transfer matrix of the graph M_G . We define a $p \times p$ matrix L whose entries are given by

$$L_{i,j} = L_{o_i,o_j} = \sum_{m \in o_j} a_{i,m},$$

where i is *any* row of A_{M_G} that corresponds to a representative of orbit o_i . L is called the *compactified transfer matrix of G* .

We want to illustrate this definition for $k = 4$. The general case will be treated in Example 3.4.15.

Example 3.4.5. We illustrate this procedure in the following example, where $G = P_3$. The orbits are $o_1 = \{\{1, 3\}, \{2\}\}$ and $o_2 = \{\{1\}, \{2\}, \{3\}\}$, so we can expect a 2×2 matrix L . In this matrix, the (i, j) -entry counts the number of colorings where the first

P_3 is colored by a fixed representative of o_i and the second P_3 is colored by any element in o_j .

Figure 3.3 shows the transfer matrix for 4-colorings of P_3 . By following Definition 3.4.4 for $k = 4$, we can quotient out the orbits and we obtain the matrix

$$L = \begin{pmatrix} 7 & 10 \\ 5 & 11 \end{pmatrix},$$

which has the same maximal eigenvalue as the original transfer matrix in Figure 3.3, see Corollary 3.4.8. The entries of L equal the row sums within each orbit/rectangle of the matrix in Figure 3.3.

Remark 3.4.6. It also makes sense to define L for $1 \leq k < N$. However, the size of L will be smaller as not all orbits appear. For instance, the orbit, where all colors are different, cannot appear if $k < N$. As we will see in Corollary 3.4.20, it makes sense to talk about L even for $k < N$ provided we multiply L by appropriate row weights that are 0 if k is too small for an orbit to appear.

Since the entries of L are nonnegative integers, the Perron-Frobenius theorem ensures that the biggest eigenvalue is real and positive. Furthermore, the following lemma implies that the biggest eigenvalue of A_{M_G} and the biggest eigenvalue of L agree.

LEMMA 3.4.7 ([Ciu98], Lemma 3.2). If N is a nonnegative matrix that commutes with a group of permutation matrices G , then the largest eigenvalue of N is the same as the largest eigenvalue of N acting on the subspace of G -invariants.

COROLLARY 3.4.8. With the notation from above, the biggest eigenvalue of L and the biggest eigenvalue of A_{M_G} agree.

Remark 3.4.9. There is also a combinatorial reformulation of Definition 3.4.4. $L_{i,j}$ counts the number of colorings of $G \times P_2$, where $G \times \{1\}$ is colored by a fixed representative of o_i and the coloring of $G \times \{2\}$ is in o_j .

In Section 3.4.2, we will see that the entries of L are indeed polynomials in k . We also give a geometric interpretation of the entries.

3.4.2 Ehrhart Theory and Symmetry

As we have seen in Remark 3.4.9, the entries of L can be interpreted as counting proper k -colorings where one part of the graph is fixed by a coloring and another part has to lie in a given orbit. In this section, we will use inside-out polytopes to show that this counting function is indeed a polynomial. However, we first start with a small result concerning graph colorings that lie in a given orbit.

Let $G = (V, E)$ be a finite, simple graph and let \tilde{o} be an orbit as defined in Definition 3.4.1, i.e., we do *not* quotient out by graph automorphisms. The graph G determines a hyperplane arrangement

$$\mathcal{H}(G) = \{x_i = x_j : \{i, j\} \in E\}.$$

We define the \tilde{o} -restricted chromatic polynomial

$$\chi_{(G,\tilde{o})}(t) = \#\text{number of proper } t\text{-colorings of } G \text{ lying in orbit } \tilde{o}.$$

As mentioned above, every orbit can be described by a partition of the vertex set V into independent sets. Vertices in the same independent set are colored by the same color. Hence, for every independent set $I \in \tilde{o}$, we get an additional hyperplane arrangement

$$\mathcal{H}_I := \{x_i = x_j : i, j \in I\}.$$

Moreover, we also get a hyperplane arrangement

$$\mathcal{H}_{\tilde{o},I} := \{x_i = x_j : i \in I \text{ and } j \notin I\}.$$

of forbidden hyperplanes by requiring that elements in different independent sets have different colors. Lastly, we define the hyperplane arrangement

$$\mathcal{H}(G, \tilde{o}) = \mathcal{H}(G) \cup \bigcup_{I \in \tilde{o}} \mathcal{H}_{\tilde{o},I}.$$

With this set-up, we now have:

THEOREM 3.4.10. Let \mathcal{H}_I , $\mathcal{H}(G, \tilde{o})$, and $\mathcal{H}_{\tilde{o},I}$ be defined as above. Moreover, let

$$P_{\tilde{o}} := [0, 1]^n \cap \left(\bigcap_{I \in \tilde{o}} H_I \right).$$

Then

$$(-1)^s \text{ehr}_{P_{\tilde{o}}, \mathcal{H}(G, \tilde{o})}(-t) = \text{ehr}_{\text{relint}(P_{\tilde{o}}), \mathcal{H}(G, \tilde{o})}^{\circ}(t) = \chi_{G, \tilde{o}}(t-1), \quad (3.8)$$

where s is the number of colors used in orbit \tilde{o} . Furthermore, $\chi_{G, \tilde{o}}$ is a polynomial of degree s with leading coefficient 1.

Proof. We first note that $P_{\tilde{o}}$ is an s -dimensional lattice inside-out polytope with volume 1 (seen as an s -dimensional polytope). The hyperplane arrangement $\mathcal{H}(G, \tilde{o})$ subdivides $P_{\tilde{o}}$ into regions R_1, R_2, \dots, R_m , where $\dim R_i = \dim P_{\tilde{o}}$. We know that $\text{ehr}_{\text{relint}(P_{\tilde{o}}), \mathcal{H}(G, \tilde{o})}^{\circ}(t)$ counts the integer points in the relative interior of the t^{th} -dilate of these regions. The integer points in these regions correspond to proper $(t-1)$ -colorings of G that lie in \tilde{o} , and every proper $(t-1)$ -coloring in \tilde{o} corresponds to an integer point in a t relint R_i for some i , which proves (3.8). The claim about the degree of $\chi_{G, \tilde{o}}$ follows from the dimension and volume of $P_{\tilde{o}}$. \square

Remark 3.4.11. The same result could also be obtained by forming a quotient graph where all vertices in the independent sets of \tilde{o} get identified. Now counting proper k -colorings of this quotient graph is the same as counting proper k -colorings that lie in \tilde{o} . This however does not directly work if we also quotient out by graph automorphisms.

Remark 3.4.12. The statement and the proof still hold for an orbit o if we additionally quotient out by graph automorphisms, except that the leading coefficient will be the number of orbits \tilde{o} that are in the preimage of orbit o .

We now want to apply this geometric machinery to the transfer-matrix theory to interpret the entries of L in terms of Ehrhart polynomials. Recall that we want to color $\Gamma := G \times P_2$, where G is a graph on N nodes. In the geometric setting described in [BZ06], we have a subdivision of the $2N$ -dimensional unit cube stemming from the edges of the graph Γ . However, we want to further refine this subdivision according to the orbit structure. As we will see, this subdivision nicely resembles the symmetry of the orbits.

Let G be a graph with vertices $\{v_1, v_2, \dots, v_N\}$, let $o_i, o_j \in O_G$ be defined as in Definition 3.4.1, and let $V(P_2) = \{1, 2\}$. Pick a representative c of o_i such that $c(v_i) \leq N$ and color the first N vertices accordingly. This defines a $(G \times \{1\})$ -restricted coloring of $G \times P_2$, which we will call an o_i -restricted coloring. Recall that L_{o_i, o_j} counts the number of o_i -restricted colorings such that the coloring of $G \times \{2\}$ is an element of o_j .

Since the entries will correspond to restricted colorings, we first state a general result about restricted colorings.

THEOREM 3.4.13. Let $\Gamma = (\{1, 2, \dots, n\}, E)$ be a graph and fix a proper k' -coloring $c': V' \rightarrow \{1, 2, \dots, k'\}$ on the induced subgraph $\Gamma|_{V'}$ for a subset $V' \subset V(\Gamma)$. Then, for $k \geq k'$, the restricted chromatic polynomial

$$\chi_{c', \Gamma}(k) = \#\text{proper } k\text{-colorings } c \text{ of } \Gamma \text{ such that } c|_{V'} = c'. \quad (3.9)$$

is a polynomial of degree $\#V - \#V' =: s$ with leading coefficient 1, whose coefficients a_i alternate in sign, and whose absolute values of the coefficient form a *log-concave sequence*, i.e., $a_i^2 \geq a_{i-1}a_{i+1}$ holds for $0 < i < s$. The second highest coefficient a_{n-1} is given by

$$-a_{n-1} = \#\text{edges } \{v_i, v_j\} \text{ such that } \{v_i, v_j\} \not\subseteq V'.$$

Moreover, we have the reciprocity statement

$$\begin{aligned} \chi_{c', \Gamma}(-k) &= (-1)^s \#(\alpha, c) \text{ of } \Gamma \text{ with } c|_{V'} = c' \\ &= (-1)^s \chi_{c', \Gamma}(k), \end{aligned}$$

where (α, c) is a pair of an acyclic orientation α and a compatible (*not necessarily proper*) k -coloring, c , of Γ .

Remark. As we learned after publication of [EK18], the statement that the restricted counting function is a polynomial is [HM07, Thm. 8] and the combinatorial reciprocity theorem is [JS14, Thm. 4.2], where Jochemko and Sanyal also give a novel proof of the polynomiality result. We would like to thank Raman Sanyal for pointing this out.

Proof. Let $\mathcal{H} = \mathcal{H}(\Gamma)$ be the hyperplane arrangement coming from the edges of the graph Γ . Let $P = [0, 1]^n$. We intersect (P, \mathcal{H}) with the hyperplanes coming from the coloring

c' , i.e., we intersect (P, \mathcal{H}) with the hyperplanes $x_i = c'(v_i) =: c'_i$ for all $v_i \in V'$. This induces a new inside-out polytope $(\bar{P}, \bar{\mathcal{H}})$ of dimension s . This is illustrated in Figure 3.5. Using an affine, unimodular map we can assume that $(\bar{P}, \bar{\mathcal{H}}) \subset \mathbb{R}^s$ is full-dimensional.

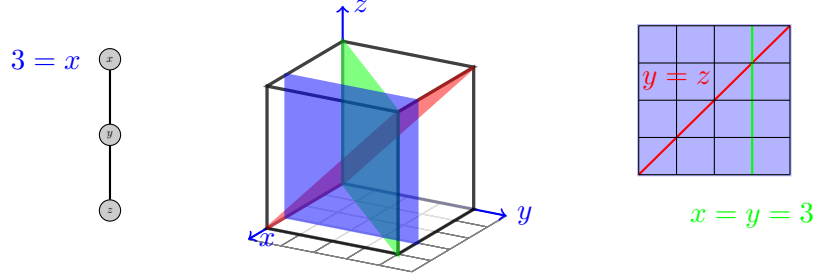


Figure 3.5: P_3 with vertex x colored by 3, the corresponding inside-out polytope (P, \mathcal{H}) , and the induced inside-out polytope $(\bar{P}, \bar{\mathcal{H}})$.

The integer points in $[1, k]^s$ that are not in $\bar{\mathcal{H}}$ are counted by $\chi_{c', \Gamma}(k)$, where we assume that $k \geq \max_i c'_i$.

Thus, by (3.4) we have that

$$\chi_{c', \Gamma}(k) = \sum_{u \in \mathcal{L}(\bar{\mathcal{H}})} \mu(\hat{0}, u) \# (\mathbb{Z}^s \cap u \cap k\bar{P}) = \sum_{u \in \mathcal{L}(\bar{\mathcal{H}})} \mu(\hat{0}, u) k^{\dim u}.$$

Note that this is actually the *characteristic polynomial*, denoted $p_{\bar{\mathcal{H}}}$, of the induced hyperplane arrangement $\bar{\mathcal{H}}$, since $\# (\mathbb{Z}^s \cap u \cap k\bar{P}) = k^{\dim u}$. The claim about the log-concavity now follows by a result of June Huh, see [Huh12, Cor. 27]. The statement about the second highest coefficient follows, since the only terms of dimension $s - 1$ come from flats of dimension $s - 1$, which are exactly the hyperplanes coming from edges $\{v_i, v_j\}$ such that $\{v_i, v_j\} \not\subseteq V'$. We remark that this polynomial is alternating in sign, again by [Huh12, Thm 3.1],

$$\chi_{c', \Gamma}(-k) = (-1)^s \sum_{u \in \mathcal{L}(\bar{\mathcal{H}})} |\mu(\hat{0}, u)| k^{\dim u}.$$

This — using (3.5) — is equivalent to

$$(-1)^s \chi_{c', \Gamma}(-k) = \text{ehr}_{\bar{P}, \bar{\mathcal{H}}}(k - 1) = (-1)^s \chi_{c', \Gamma}(k) = \sum_{x \in \mathbb{Z}^s} m_{((k-1)\bar{P}, \bar{\mathcal{H}})}(x).$$

Similar to [BZ06, Proof of Cor. 5.5], one can now observe that the right-hand side counts the number of compatible pairs (α, c) , where c is a —not necessarily proper— c' -restricted k -coloring and α is an acyclic orientation. Here we implicitly used that $k \geq \max_i c'_i$ while applying [BZ06, Thm 3.1]. \square

The following theorem states some basic facts about L_{o_i, o_j} :

THEOREM 3.4.14. With the notation from Theorem 3.4.13 and with $k \geq N$, we have:

1. Every entry $L_{o_i, o_j}(k)$ equals the sum of Ehrhart polynomials of lattice inside-out polytopes of dimension $\#\text{colors of } o_j$ and hence is a polynomial of degree $\#\text{colors of } o_j$,
2. $L_{o_i, o_j}(k)$ is independent of the choice of the representative, i.e., it is well-defined,
3. $\#o_i \cdot L_{o_i, o_j}(k) = \chi_{G_{o_i, o_j}}(k)$, where

$$\chi_{G_{o_i, o_j}}(k) = \#\text{proper } k\text{-colorings: coloring of } G \times \{1\} \text{ is in } o_i \text{ and} \\ \text{coloring of } G \times \{2\} \text{ is in } o_j.$$

Proof. We first prove the statement for the case, where we only quotient out by permutations of colors. Then we show that this implies the statement for orbits o when we also quotient out by a subgroup of the automorphism group.

Let G be a graph with vertices $\{v_1, v_2, \dots, v_N\}$, let P_2 be the path graph on 2 vertices, and let

$$\mathcal{H}' = \{x_i = x_j : \{i, j\} \in E(G \times P_2)\}.$$

Moreover, let $C = [0, 1]^{2N}$. Let \tilde{o}_i, \tilde{o}_j be given orbits of G after quotienting out by permutations of colors. Let \mathcal{I} be the collection of independent sets partitioning V that correspond to \tilde{o}_j . As above, for every $I \in \mathcal{I}$, we get

$$\mathcal{H}_I := \{x_i = x_j : i, j \in I\},$$

and we get a set of forbidden hyperplanes

$$\mathcal{H}_{\tilde{o}_j, I} := \{x_i = x_j : i \in I \text{ and } j \notin I\}.$$

Now let $P = C \cap (\bigcup_I \mathcal{H}_I)$ and let

$$\mathcal{H} := \mathcal{H}' \cup \bigcup_I \mathcal{H}_{\tilde{o}_j, I}.$$

Then (P, \mathcal{H}) is a lattice inside-out polytope and its dimension is

$$N + \text{number of colors in } \tilde{o}_j = N + \#\mathcal{I}.$$

We remark that this is the same hyperplane arrangement that one would obtain from a quotient graph of $G \times P_2$. This quotient graph can be obtained in the following way:

1. All vertices of $G \times \{2\}$ that are in the same independent set of \tilde{o}_j get identified, and
2. we turn all vertices in the image of $G \times \{2\}$ into a clique.

Therefore, one can now apply Theorem 3.4.13 to see that the entries of $L_{\tilde{o}_i, \tilde{o}_j}$ are polynomials. If we furthermore quotient by graph symmetries, we by definition fix a row and add all entries that are in columns indexed by orbits that get mapped to o_j . Thus, the the entries of (L_{o_i, o_j}) are also polynomials whose leading coefficient is the number of orbits in the preimage of o_j .

Now let c and c' be colorings in the same orbit \tilde{o}_i . Then there is a bijection of the colors mapping c to c' . This permutation gives rise to a bijection of the lattice points in $t \operatorname{relint}(R_i)$. Since $\chi_{G_{\tilde{o}_i, \tilde{o}_j}}(t)$ counts the number of proper t -colorings such that $G \times \{1\}$ is an element of orbit \tilde{o}_i and $G \times \{2\}$ is an element of orbit \tilde{o}_j and we therefore get

$$\#\tilde{o}_i \cdot L_{\tilde{o}_i, \tilde{o}_j}(t) = \chi_{G_{\tilde{o}_i, \tilde{o}_j}}(t).$$

The last statement follows, since we quotiented out by a graph symmetry. \square

Now that we have seen that the entries of L can be interpreted as Ehrhart polynomials of inside-out polytopes, we give an explicit example.

Example 3.4.15 (Example 3.4.5 continued). Again let $G = P_3$. Recall that the orbits are $o_1 = \{\{1, 3\}, \{2\}\}$ and $o_2 = \{\{1\}, \{2\}, \{3\}\}$, so we can expect a 2×2 matrix L .

We do the same calculation as in Example 3.4.5, but for k colors. For the matrix L the entries will be polynomial for $k \geq 3$, but for lower k the polynomials might not make sense, as for small k there are not enough colors for every orbit. On the other hand, the polynomials are of degree at most three. By explicit computer calculations for $k = 3, 4, 5, 6$ we infer that the matrix is

$$\begin{pmatrix} k^2 - 3k + 3 & k^3 - 6k^2 + 13k - 10 \\ k^2 - 4k + 5 & k^3 - 6k^2 + 14k - 13 \end{pmatrix}$$

for $k \geq 3$.

In Example 3.4.15, the graph P_3 was so small that there was no graph automorphism identifying two orbits \tilde{o} and \tilde{o}' . We now illustrate how the matrices are further reduced in size when the automorphisms of the underlying graphs are also considered. We first consider k -colorings of $G \times P_n$ with $G = C_5$. It should be noted that this question can be addressed with ad hoc methods adapted to this particular choice of G , but our method is completely general. In our method, we do not assume anything about G .

Example 3.4.16 (Example 3.4.2 continued). Let $G = C_5$ be the graph for which we want to calculate the chromatic polynomial of $G \times P_n$. We label the edges of C_5 by 12, 23, 34, 45 and 51. We first quotient by permutations of colors. The 11 partitions of the vertices into independent sets are (as computer generated in no particular order):

$$\begin{array}{c|c|c|c|c|c} 14\ 2\ 3\ 5 & 14\ 2\ 35 & 14\ 25\ 3 & 1\ 24\ 3\ 5 & 1\ 24\ 35 & 1\ 2\ 3\ 4\ 5 \\ \hline 1\ 2\ 35\ 4 & 1\ 25\ 3\ 4 & 13\ 24\ 5 & 13\ 2\ 4\ 5 & 13\ 25\ 4 & \end{array}$$

Since every partition corresponds to an orbit \tilde{o} as defined in Definition 3.4.1, we expect an 11×11 -matrix:

Figure 3.6: L matrix where we only quotient out by permutations of colors.

The automorphism group of C_5 is the dihedral group generated by (12345) and (1)(25)(34), see Example 3.2.1. The 11 partitions of the vertex set end up in 3 orbits after quotienting by the dihedral group. The classes are represented by:

$$1 \ 2 \ 3 \ 4 \ 5 \quad | \quad 1 \ 24 \ 35 \quad | \quad 1 \ 2 \ 4 \ 35$$

Adding up entries from columns — indexed by orbits \tilde{o}_i that get mapped to the same orbit o — of the 11×11 -matrix gives an even more compactified version, the 3×3 matrix L ,

$$\begin{pmatrix} k^5 - 15k^4 + 95k^3 - 325k^2 + 609k - 501 & 5k^3 - 40k^2 + 125k - 150 & 5k^4 - 55k^3 + 250k^2 - 565k + 535 \\ k^5 - 15k^4 + 93k^3 - 301k^2 + 510k - 360 & 5k^3 - 36k^2 + 96k - 93 & 5k^4 - 53k^3 + 224k^2 - 449k + 357 \\ k^5 - 15k^4 + 94k^3 - 313k^2 + 559k - 428 & 5k^3 - 38k^2 + 110k - 119 & 5k^4 - 54k^3 + 237k^2 - 506k + 441 \end{pmatrix}.$$

3.4.3 Main Results

In this section, we show that L , defined as in Definition 3.4.4, behaves like a transfer matrix. We then deduce that the chromatic polynomial of $G \times P_n$ can be determined by computing powers of L . Moreover, we will use geometry to find an Ehrhart-theoretic interpretation for the entries of L^n . This geometric interpretation allows us to deduce a reciprocity statement for the rows sums of L^n . We end this section by stating results about the biggest eigenvalue of L .

Let G be a graph let o_1, o_2, \dots, o_p be as defined in Definition 3.4.1. Then $L = (L_{i,j}(k))$ is a $p \times p$ matrix and the entry $(i, j) = (o_i, o_j)$ is given by

$$\begin{aligned} L_{i,j} &= \#o_i\text{-restricted colorings of } G \times \{1\} : \text{coloring of } G \times \{2\} \text{ lies in } o_j \\ &= \frac{\chi_{G_{o_i, o_j}}(k)}{\#o_i}. \end{aligned}$$

The next result shows that L behaves like a transfer matrix.

THEOREM 3.4.17. Let $V(P_{n+1}) = \{1, 2, \dots, n+1\}$ and let G be any graph. Let o_1, o_2, \dots, o_p be the orbits as defined in Definition 3.4.1. Then, for $k \geq \#\text{colors}$ used in o_i , the (o_i, o_j) -entry of L^n counts the number of proper k -colorings of $G \times P_{n+1}$, where $G \times \{1\}$ is fixed by a coloring in orbit o_i , and where the coloring of $G \times \{n+1\}$ lies o_j .

Before we prove Theorem 3.4.17, we illustrate the statement:

Example 3.4.18 (Example 3.4.5 continued). Let $G = P_3$ with orbits $o_1 = \{\{1, 3\}, \{2\}\}$ and $o_2 = \{\{1\}, \{2\}, \{3\}\}$. Recall that

$$L = \begin{pmatrix} k^2 - 3k + 3 & k^3 - 6k^2 + 13k - 10 \\ k^2 - 4k + 5 & k^3 - 6k^2 + 14k - 13 \end{pmatrix}.$$

Let us illustrate the combinatorial interpretation for L_{11} and L_{12} in the case where $k = 3$, see Figure 3.7.

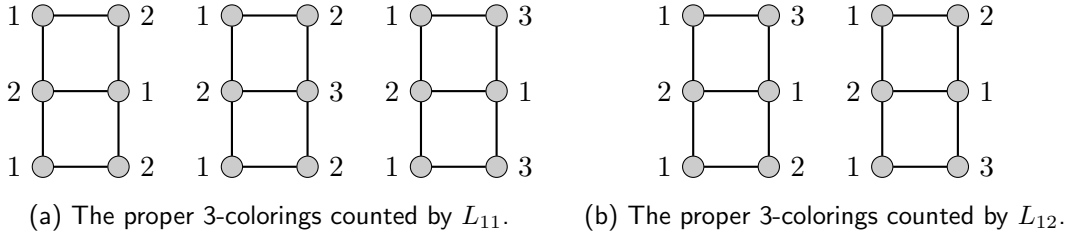


Figure 3.7: An illustration of the proper 3-colorings counted by the first row of L .

By Theorem 3.4.17, $(L^5)_{1,1}$ counts the number of colorings of $G \times P_6$, where the two black dots indicate that the two corresponding nodes need to be colored by the same color, see Figure 3.8.

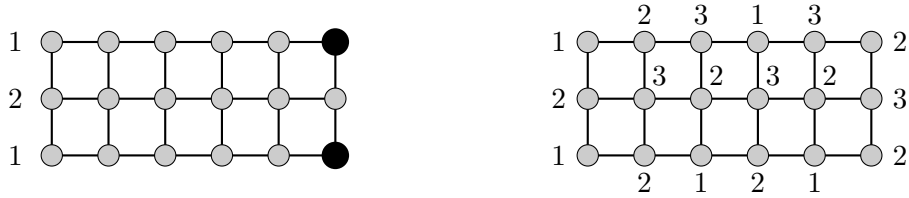


Figure 3.8: In (a), we illustrate the general form of a coloring counted by $(L^5)_{1,1}$. In (b), we give an explicit example of a 3-coloring counted by $(L^5)_{1,1}$.

Proof of Theorem 3.4.17. We induct on n . By construction, the statement is true for $n = 1$, so let the statement be true for $G \times P_m$ for all $m \leq n$. Now $V(P_{n+1}) = \{1, 2, \dots, n+1\}$. We denote the (o_i, o_j) -entry of $L^m \in \mathbb{Z}^{p \times p}$ by $L_{i,j}^m$. Then

$$L_{i,j}^n = (L^{n-1}L)_{i,j} = \sum_{k=1}^p L_{i,k}^{n-1} L_{k,j}.$$

By induction hypothesis, the entry $L_{i,k}^{n-1}$ counts the number of colorings where the coloring of $G \times \{1\}$ is fixed by a representative in o_i and the coloring of $G \times \{n\}$ lies in orbit o_k . Moreover, $L_{k,j}$ counts the colorings where the first G is fixed by a representative in o_k

and the coloring of the second G lies in o_j . Therefore, $L_{i,k}^{n-1}L_{k,j}$ counts the colorings where the coloring of $G \times \{1\}$ is fixed by a representative of o_i , the coloring of $G \times \{n\}$ lies in o_k , and the coloring of $G \times \{n+1\}$ lies in o_j . The sum is taken over all possible orbits and the claim follows. \square

Remark 3.4.19. This shows that the entries of L^n can be interpreted as sums of Ehrhart polynomials of (induced) inside-out polytopes assuming that the dilation factor is big enough. The inside-out polytopes can be explicitly described by following the construction from the proof of Proposition 3.4.14.

Moreover, this enables us to directly compute the chromatic polynomial of $G \times P_{n+1}$ from L^n .

COROLLARY 3.4.20. Let $G \times P_{n+1}$ and L be as above. Then

$$\chi_{G \times P_{n+1}}(k) = (w_1(k), \dots, w_p(k))L^n \mathbf{1}, \quad (3.10)$$

where $w_i(k)$ is the size of o_i and $\mathbf{1} := (1, \dots, 1)^t$.

Proof. Let $V(P_n) = \{1, 2, \dots, n\}$. The i^{th} entry L_i of $L^n \mathbf{1}$ counts the number of colorings where $G \times \{1\}$ is colored by a representative of o_i . By symmetry, the total number of colorings with the coloring of $G \times \{1\}$ being in o_i equals

$$w_i(k)L_i$$

by Theorem 3.4.14. Now $(w_1(k), \dots, w_p(k))L^n \mathbf{1}$ sums over all possible orbits and the claim now directly follows. \square

Remark 3.4.21. Even though the definition of L implicitly assumes that the number of colors $k \geq \#V(G)$, the corollary makes sense for all k . If $k \leq \#V(G)$, then the weights w_i of the orbits using more than k colors are 0.

Example 3.4.22 (Example 3.4.5 continued). The chromatic polynomial of $P_3 \times P_6$ is

$$\chi(k) = (w_1, w_2) \begin{pmatrix} k^2 - 3k + 3 & k^3 - 6k^2 + 13k - 10 \\ k^2 - 4k + 5 & k^3 - 6k^2 + 14k - 13 \end{pmatrix}^{6-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where $w_1 = k(k-1)$ and $w_2 = k(k-1)(k-2)$.

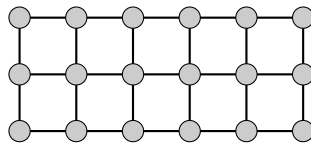


Figure 3.9: $P_3 \times P_6$.

In general, the chromatic polynomial of $P_3 \times P_n$ equals

$$\chi(k) = (w_1, w_2) \begin{pmatrix} k^2 - 3k + 3 & k^3 - 6k^2 + 13k - 10 \\ k^2 - 4k + 5 & k^3 - 6k^2 + 14k - 13 \end{pmatrix}^{n-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where $w_1 = k(k-1)$ and $w_2 = k(k-1)(k-2)$.

Example 3.4.23 (Example 3.4.16 continued). Recall that $G = C_5$ and that the matrix L was given by

$$\begin{pmatrix} k^5 - 15k^4 + 95k^3 - 325k^2 + 609k - 501 & 5k^3 - 40k^2 + 125k - 150 & 5k^4 - 55k^3 + 250k^2 - 565k + 535 \\ k^5 - 15k^4 + 93k^3 - 301k^2 + 510k - 360 & 5k^3 - 36k^2 + 96k - 93 & 5k^4 - 53k^3 + 224k^2 - 449k + 357 \\ k^5 - 15k^4 + 94k^3 - 313k^2 + 559k - 428 & 5k^3 - 38k^2 + 110k - 119 & 5k^4 - 54k^3 + 237k^2 - 506k + 441 \end{pmatrix}.$$

Define a row vector \mathbf{v} , where v_i is given by the size of the orbit o_i after quotienting by a permutation of the colors and the graph automorphism group:

$$\mathbf{v} = (k(k-1)(k-2)(k-3)(k-4), \quad 5k(k-1)(k-2), \quad 5k(k-1)(k-2)(k-3))^T$$

Let I be the 3×3 identity matrix and $\mathbf{1}$ the all-ones column vector of dimension 3. Then we have a nice formal generating function

$$\begin{aligned} \Xi_G(k, z) &= \sum_{n=0}^{\infty} \chi_{G \times P_{n+1}}(k) z^n \\ &= \sum_{n=0}^{\infty} \mathbf{v} L^n \mathbf{1} z^n \\ &= \mathbf{v} \left(\sum_{n=0}^{\infty} (zL)^n \right) \mathbf{1} \\ &= \mathbf{v} (I - zL)^{-1} \mathbf{1}. \end{aligned}$$

In general, if L is an $m \times m$ matrix, we expect that

$$\Xi_G(k, z) = \frac{\text{polynomial of } z\text{-degree } m-1}{\text{polynomial of } z\text{-degree } m}$$

by calculating $(I - zL)^{-1}$ using cofactors. For some G there is a mysterious cancellation and the z -degree of the denominator of $\Xi_G(k, z)$ is smaller than the size of the matrix. This is the case in our example, as

$$\Xi_{C_5}(k, z) = k(k-1)(k-2) \frac{p_1(k)z + p_0(k)}{q_2(k)z^2 + q_1(k)z + q_0(k)}$$

where

$$\begin{aligned} p_0(k) &= k^2 - 2k + 2, \\ p_1(k) &= -k^5 + 11k^4 - 44k^3 + 73k^2 - 42k + 14, \\ q_0(k) &= 1, \\ q_1(k) &= -k^5 + 10k^4 - 46k^3 + 124k^2 - 198k + 148, \\ q_2(k) &= k^8 - 19k^7 + 159k^6 - 767k^5 + 2339k^4 - 4627k^3 + 5800k^2 - 4212k + 1362. \end{aligned}$$

In the previous example, the degree of the denominator of $\Xi_G(k, z)$ is smaller than expected indicating some hidden symmetry.

Definition 3.4.24. A graph G has a *hidden symmetry* if the denominator of

$$\Xi_G(k, z) = \sum_{n=0}^{\infty} \chi_{G \times P_{n+1}}(k) z^n$$

has a z -degree less than the order of

$$\{c : V(G) \rightarrow \mathbb{Z}_{\geq 1} : c \text{ is a proper coloring of } G\} / \sim$$

where $c \sim c'$ if $c = \alpha c' \beta$ for a bijection α of \mathbb{N} and an automorphism β of G .

Example 3.4.25. We have done some computer calculations to tabulate graphs with hidden symmetries. The connected graphs with at most five vertices are in Figure 3.10.

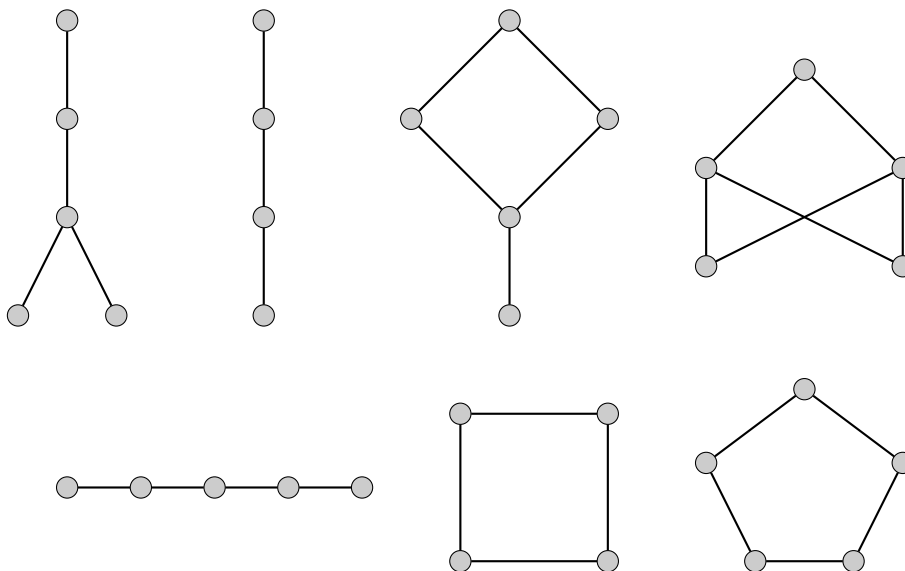


Figure 3.10: The connected graphs on at most five vertices with a hidden symmetry.

One could speculate that hidden symmetry is something fairly trivial, since the common factor of the numerator and denominator when calculating $\Xi_G(k, z)$ from cofactors is something straightforward. A piece of it usually seems to be a power of $(z - 1)$. But for example for $G = C_6$ the common factor is the non-trivial factor $(z - 1)^3(k^7 z^4 - 19k^6 z^4 + k^6 z^3 + 147k^5 z^4 - 20k^5 z^3 - 598k^4 z^4 + 157k^4 z^3 + 1381k^3 z^4 - 3k^4 z^2 - 627k^3 z^3 - 1821k^2 z^4 + 37k^3 z^2 + 1349k^2 z^3 + 1289k z^4 - 173k^2 z^2 - 1483k z^3 - 384z^4 + 3k^2 z + 364k z^2 + 659z^3 - 18kz - 309z^2 + 35z - 1)$.

CONJECTURE 3.4.26. All paths and cycles on at least four vertices have a hidden symmetry.

PROPOSITION 3.4.27. Let $L \in \mathbb{Z}^{p \times p}$ be as above, let $L_i^n := \sum_{k=1}^p (L^n)_{i,k}$ be the i^{th} row sum of L^n , and let $V(P_{n+1}) = \{1, 2, \dots, n+1\}$. Then, for $k \geq N = \#V(G)$, we have

$$L_i^n(-k) = (-1)^{Nn} \#(\alpha, c) \text{ of } G \times P_{n+1} \text{ where } G \times \{1\} \text{ is colored by rep. of } o_i, \quad (3.11)$$

where (α, c) is a pair of an acyclic orientation α and a compatible o_i -restricted k -coloring c .

Proof. $L_i^n(k)$ counts the number of colorings of $G \times P_{n+1}$, where $G \times \{1\}$ is fixed by a coloring c' . Now one can apply Theorem 3.4.13 and the claim follows. \square

COROLLARY 3.4.28. Let $k \geq N$ Then

$$\sum_{j=1}^m L_{i,j}(-k) = (-1) \cdot \#(\alpha, c) \text{ where first } G \text{ is fixed by a representative of } o_i, \quad (3.12)$$

where (α, c) is a pair of an acyclic orientation and compatible k -coloring.

Example 3.4.29 (3.4.5 continued). Figure 3.11 illustrates Corollary 3.4.28 for $P_3 \times P_2$, where $k = 3$, and where

$$L = \begin{pmatrix} k^2 - 3k + 3 & k^3 - 6k^2 + 13k - 10 \\ k^2 - 4k + 5 & k^3 - 6k^2 + 14k - 13 \end{pmatrix}.$$

The orientation of the dashed edges can be chosen arbitrarily. The red number gives the multiplicity of the given case. Therefore, the sum of the red numbers (up to a sign) equals the evaluation of the second row sum of L at -3

$$\begin{aligned} & (-3)^2 - 4 \cdot (-3) + 5 + (-4)^3 - 6 \cdot (-3)^2 + 14 \cdot (-3) - 13 = -110 = \\ & = (-1) \cdot (8 + 2 + 4 + 4 + 2 + 2 + 8 + 8 + 1 + 4 + 8 + 2 + 2 + \\ & + 1 + 1 + 2 + 4 + 1 + 8 + 2 + 2 + 8 + 2 + 4 + 4 + 8 + 8). \end{aligned}$$

If however one is interested in the asymptotic behavior of graphs $G \times C_n$, then we need to find good bounds for the biggest eigenvalue of L , as this dominates the asymptotics. The next theorem gives an upper and a lower bound and in particular it shows that the eigenvalue grows like a polynomial of degree $\#V(G)$.

LEMMA 3.4.30. Let λ_{\max} be the biggest eigenvalue of L (and thus of the adjacency matrix A_{M_G}). Then

$$\delta(L) \leq \lambda_{\max} \leq \Delta(L), \quad (3.13)$$

where $\delta(L)$ and $\Delta(L)$ are the smallest and biggest row sum of L , respectively.

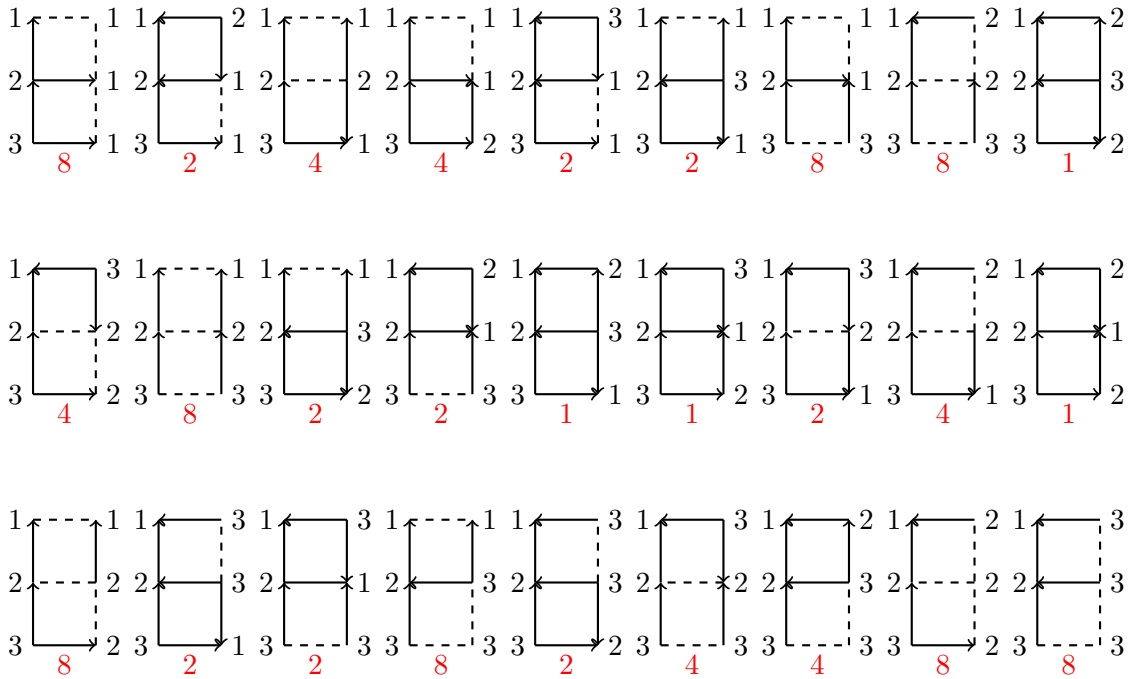


Figure 3.11: All $(1, 2, 3)$ -restricted, compatible pairs of acyclic orientations and (not necessary proper) colorings of $P_3 \times P_2$.

Proof. The biggest eigenvalue of the adjacency matrix of a graph is bounded above and below by the biggest and smallest degree of the graph, respectively, see [CR90, Thm. 1]. These are exactly the biggest and smallest row sums of L . \square

Now the question of determining the biggest eigenvalue reduces to determining the smallest and biggest row sum of L . This might be computationally challenging. Our next result gives a combinatorial interpretation of the two highest coefficients, which gives us a quicker way to obtain the two highest coefficients without computing L . One still needs to determine all of the orbits using $N - 1$ colors, which in general is computationally challenging, but one does not need to explicitly compute the entries of L .

LEMMA 3.4.31. Let $\tilde{o}_1, \tilde{o}_2, \dots, \tilde{o}_p$ be orbits as defined in Definition 3.4.1. Let \tilde{o}_p be the orbit using $N = \#V(G)$ colors. Then $\delta(L)$ and $\Delta(L)$ are polynomials of degree N , their leading coefficient is $a_N = 1$, and

$$a_{N-1} = -F + \#\text{orbits using } N - 1 \text{ colors,}$$

where

$$F = \#\text{edges of } G \times P_2 \text{ that are not edges of } G \times \{1\}.$$

In particular, the highest two terms of both polynomials agree.

Proof. This directly follows from Theorem 3.4.13. □

To summarize, we get:

PROPOSITION 3.4.32. Let G be a graph and $N = \#V(G)$ and let $\delta(L)$ and $\Delta(L)$ be as above. Then the doubly asymptotic behavior of the number of proper k -colorings of $G \times C_n$ is dominated by λ_{\max}^{n-1} and

$$\delta(L) \leq \lambda_{\max} \leq \Delta(L),$$

where $\delta(L) = \sum_{i=0}^N a_i k^i$, $\Delta(L) = \sum_{i=0}^N b_i k^i$, $a_N = b_N$, and $a_{N-1} = b_{N-1}$.

3.5 OPEN QUESTIONS

As we have briefly mentioned in the introduction, enumerating proper colorings of graphs $G \times P_n$ and $G \times C_n$ arises in statistical mechanics. In the excellent survey [BEMPS10], Beaudin, Ellis-Monaghan, and Pangborn describe how the partition function of the Potts model is related to the Tutte polynomial of a graph. In the case of the 0-temperature antiferromagnetic case, this partition function is equivalent to the chromatic polynomial of the underlying graph. Therefore, there are several potential directions for future work. First, one could explicitly compute the chromatic polynomials of $G \times P_n$ or $G \times C_n$ for (for physicists) interesting graphs G . It would be interesting to see how efficient our methods are in practice. Second, one could try to develop techniques to determine the Tutte polynomial of graphs like $G \times P_n$ and $G \times C_n$. This seems to be a fairly demanding challenge, since the Tutte polynomial carries a lot of information and complexity.

“Just play. Have fun. Enjoy the game.” (Michael Jordan)

This chapter is joint work with Christian Haase and Akiyoshi Tsuchiya and based on [HKT].

4.1 INTRODUCTION

Partially ordered sets — or posets for short — are ubiquitous objects in mathematics. One particularly nice way to study them was introduced by Richard Stanley. In [Sta86], he associated two geometric objects to every finite poset Π , namely the order polytope $\mathcal{O}(\Pi)$ and the chain polytope $\mathcal{C}(\Pi)$. These objects encode important information about the underlying poset Π . For instance, the vertices of the order polytope are given by the indicator vectors of order filters. This shows that the number of order filters equals the number of vertices of $\mathcal{O}(\Pi)$. Furthermore, for a poset Π on d elements, the Ehrhart polynomial $\text{ehr}_{\mathcal{O}(\Pi)}: \mathbb{Z}_{\geq 1} \rightarrow \mathbb{Z}_{\geq 0}$

$$\text{ehr}_{\mathcal{O}(\Pi)}(t) := \#t\mathcal{O}(\Pi) \cap \mathbb{Z}^d$$

equals the number of order-preserving maps $\Pi \rightarrow [t+1] := \{1, 2, \dots, t+1\}$, see Theorem 4.2.6.

In this chapter, we are particularly interested in the level property of order polytopes. As we have seen in Section 2.3, we say that a polytope is level if its canonical module is generated by elements of the same degree, see Definition 2.3.9. Historically, levelness was first introduced by Stanley [Sta77, p. 54] and it generalizes the Gorenstein property. Since the Gorenstein property nicely translates into the language of posets, it is natural to ask whether one can determine the level property in terms of the Hasse diagram.

Moreover, while the Gorenstein property can be completely classified by the h^* -vector, see Theorem 2.3.16, the same is not true for the level property, see also Remark 4.6.5. However, we have the following inequalities:

PROPOSITION 4.1.1 ([Sta96, 3.3 Prop]). Let $\mathcal{R} = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} \mathcal{R}_i$ be a standard level \mathbb{k} -algebra with Hilbert series

$$\text{Hilb}(\mathcal{R}, t) = \frac{h_0^* + h_1^*t + \dots + h_s^*t^s}{(1-t)^d},$$

where $h_s^* \neq 0$. Then, for all i, j with $h_{i+j}^* > 0$, we have $h_i^* \leq h_j^* h_{i+j}^*$.

While the Gorenstein property has been extensively studied in the past decades, the level property has only fairly recently been examined for certain classes of polytopes, with the exception of [Hib88]. Recent examples include [EHHSM15, HY18] and [KO], on which Chapter 5 is based on. In this chapter, we focus on the level property of order polytopes, i.e., the level property of the Ehrhart ring of the order polytope. Hibi [Hib87] was the first to examine minimal elements of the canonical module of this Ehrhart ring, which in this context is also known as the Hibi ring. In particular, he characterized Gorenstein posets. The biggest influence on this chapter comes from [Miy17], where Miyazaki examines and characterizes levelness of order polytopes. We provide an alternative characterization using weighted digraphs $\Gamma(\Pi')$ coming from subposets $\Pi' \subset \Pi \cup \{\pm\infty\}$, see Definition 4.4.1 for details.

COROLLARY (see Corollary 4.4.4). Let Π be a finite poset. Π is level if and only if for all $\Gamma(\Pi')$ that do not have a negative cycle, the digraph $\Gamma(\Pi' \cup \{\text{longest chains in } \overline{\Pi}\})$ does not have a negative cycle.

This corollary enables us to use the Bellman–Ford algorithm to check levelness. As a direct consequence, we get that determining levelness is in co-NP:

COROLLARY (see Corollary 4.4.6). Levelness of order polytopes is in co-NP.

We show that the necessary condition for levelness of order polytopes in [EHHSM15, Thm. 4.1] is indeed equivalent to a special case of our characterization. Furthermore, we give an example that was related to us by Alex Fink showing that this condition is not sufficient, see Remark 4.5.3 and Figure 4.4a.

THEOREM (see Theorem 4.5.2). Let Π be a finite poset and $r = \text{codeg}(\mathcal{O}(\Pi))$. The following are equivalent:

1. The inequality

$$\text{height}(j) + \text{depth}(i) \leq \text{rank}(\overline{\Pi}) + 1$$

is satisfied for all $j \succ i \in \Pi$.

2. for all Hasse edges $j \succ i \in \Pi$ there is an integer point $\mathbf{x} \in r\mathcal{O}(\Pi)^\circ$ such that $x_j = x_i + 1$.

In Section 4.6, we use Corollary 4.4.4 to describe an infinite family of level order polytopes. The main ingredient is the ordinal sum of two posets, denoted \triangleleft .

THEOREM (see Theorem 4.6.1). The ordinal sum $\Pi = \Pi_1 \triangleleft \Pi_2$ of two posets Π_1, Π_2 is level if and only if both Π_1 and Π_2 are level.

Moreover, the ordinal sum operation interacts nicely with the h^* -polynomial.

PROPOSITION (see Proposition 4.6.4). Let Π, Π_1, Π_2 , be finite posets such that $\Pi = \Pi_1 \triangleleft \Pi_2$. Moreover, let $h_{\Pi}^*, h_{\Pi_1}^*, h_{\Pi_2}^*$ be the h^* -polynomial of the Ehrhart series of the corresponding order polytopes. Then

$$h_{\Pi}^* = h_{\Pi_1}^* h_{\Pi_2}^*.$$

As we illustrate in Remark 4.6.5, Theorem 4.6.1 and Proposition 4.6.4 together can be used to create infinitely many examples of pairs of posets that have the same Ehrhart polynomial, but where one poset is level and the other one is not.

We then turn to the more general class of alcoved polytopes. We give a Minkowski sum characterization for levelness of alcoved polytopes.

PROPOSITION (see Proposition 4.7.8). Let $P \subset \mathbb{R}^d$ be an alcoved polytope and let $r = \text{codeg}(P)$. Then P is level if and only if for any integer $k \geq r$, it follows that $(kP)^{(1)} = (rP)^{(1)} + (k-r)P$, where $(lP)^{(1)} := \text{conv}(lP^\circ \cap \mathbb{Z}^d)$ for $l \in \{r, k\}$.

Then we examine when the Cartesian product of two alcoved polytopes is level. We arrive at the following result:

THEOREM (see Theorem 4.7.11). Let $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ be alcoved polytopes. Suppose that Q is level and $r = \text{codeg}(Q) \geq \dim P + 1$. Then $P \times Q \subset \mathbb{R}^{d+e}$ is level.

This results shows that — under the right assumptions — the product of a level polytope with a non-level polytope can indeed be guaranteed to be level.

THEOREM (see Theorem 4.7.12). Let Π be a poset on d elements and Π_1, \dots, Π_m the connected components of Π . If each Π_i is level, then Π is level.

This theorem tells us that in order to guarantee levelness of a poset it is sufficient to show that all components are level. More generally, for Cartesian products of level polytopes we have:

THEOREM (see Theorem 4.7.13). Let $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ be level polytopes. If either

1. $\text{codeg}(Q) < \text{codeg}(P)$ and Q has the integer-decomposition property,
2. $\text{codeg}(P) < \text{codeg}(Q)$ and P has the integer-decomposition property,
3. or if $\text{codeg}(Q) = \text{codeg}(P)$,

then $P \times Q$ is level.

The assumptions are indeed necessary. In Remark 4.7.14, we give an explicit example of two level polytopes whose product is not level.

The structure of this chapter is as follows: In Section 4.2, we recall the necessary background on order polytopes and chain polytopes. In Section 4.3, we recall Miyazaki's results on level posets. We then give an alternative characterization of level posets using

weighted digraphs in Section 4.4. In Section 4.5, we show that this characterization generalizes a necessary condition of Ene, Herzog, Hibi, and Saeedi. In Section 4.6, we use Corollary 4.4.4 to examine levelness of series-parallel posets. In the last section, we examine levelness of alcoved polytopes and examine when certain products of polytopes are again level.

4.2 BACKGROUND AND NOTATION

In this section, we will briefly introduce order polytopes and order-preserving maps. Furthermore, we will define the chain polytope and study the vertex description of both the order and the chain polytope.

A *partially ordered set* —or *poset* for short— Π is a set together with a binary relation \leq_{Π} that is reflexive, antisymmetric, and transitive, and \leq_{Π} is called a *partial order on Π* . If the poset is clear from the context, we will simply write \leq . All our posets will be *finite*.

Let Π be a finite poset. We recall that an element $j \in \Pi$ is said to *cover* an element $i \in \Pi$, denoted $j \succ i$, if $i \leq k \leq j$ implies that either $i = k$ or $j = k$. One can recover all partial orders from these cover relations. Therefore, it's convenient to illustrate the poset using these cover relations by a *Hasse diagram*, see Figure 4.1.

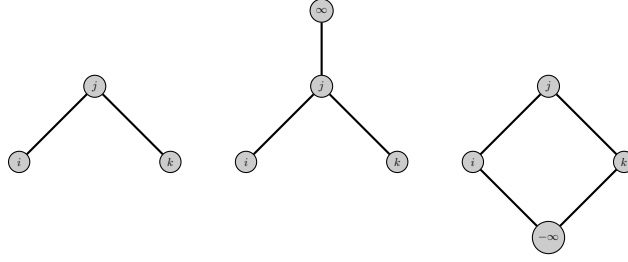


Figure 4.1: The Hasse diagram of the poset $\Pi = \{i < j > k\}$, $\Pi \cup \{\infty\}$, and $\Pi \cup \{-\infty\}$.

Given a poset Π , we define the poset $\bar{\Pi} := (\Pi \cup \{\infty\}, \leq_{\bar{\Pi}})$, where

$$i <_{\bar{\Pi}} j \iff \begin{cases} j = \infty \text{ and } i \in \Pi, \\ i <_{\Pi} j. \end{cases}$$

Similarly, we define $\underline{\Pi} := (\Pi \cup \{-\infty\}, \leq_{\underline{\Pi}})$, where

$$i <_{\underline{\Pi}} j \iff \begin{cases} i = -\infty \text{ and } j \in \Pi, \\ i <_{\Pi} j. \end{cases}$$

To every finite poset, Stanley associated two geometric objects, namely the order polytope and the chain polytope.

Definition 4.2.1 ([Sta86], Def. 1.1). The *order polytope* $\mathcal{O}(\Pi)$ of a finite poset Π is the subset of $\mathbb{R}^\Pi = \{f: \Pi \rightarrow \mathbb{R}\}$ defined by

$$\begin{aligned} 0 \leq f(i) \leq 1 & & \text{for all } i \in \Pi, \\ f(i) \leq f(j) & & \text{if } i \leq_\Pi j. \end{aligned}$$

Definition 4.2.2 ([Sta86], Def. 2.1). The *chain polytope* $\mathcal{C}(\Pi)$ of a finite poset Π is the subset of $\mathbb{R}^\Pi = \{g: \Pi \rightarrow \mathbb{R}\}$ defined by the conditions

$$\begin{aligned} 0 \leq g(i) & & \text{for all } i \in \Pi, \\ g(i_1) + g(i_2) + \dots + g(i_k) \leq 1 & & \text{for all chains } i_1 <_\Pi i_2 <_\Pi \dots <_\Pi i_k \text{ of } \Pi. \end{aligned}$$

Remark 4.2.3. In the following, we will use an isomorphism $\mathbb{R}^\Pi \cong \mathbb{R}^{\#\Pi}$ to make notation better.



Figure 4.2: The order and chain polytope of the poset described in Figure 4.1.

The definition of both the order and the chain polytope is illustrated in Figure 4.1. We define an *order filter* F of a poset Π to be a subset $F \subset \Pi$ such that if $i \in F$ and $i < j$, then $j \in F$. To every order filter F , one can associate a *characteristic function* $\mathbb{1}_F$ defined as

$$\mathbb{1}_F(i) := \begin{cases} 1 & \text{if } i \in F, \\ 0 & \text{otherwise.} \end{cases}$$

Stanley showed that vertices of $\mathcal{O}(\Pi)$ are given by the characteristic functions of order filters.

COROLLARY 4.2.4 ([Sta86, Cor. 1.3]). The vertices of $\mathcal{O}(\Pi)$ are the characteristic functions $\mathbb{1}_F$ of order filters F . In particular, the number of vertices equals the number of order filters.

Stanley also gave the vertex description of chain polytopes. We define an *antichain* A of a poset Π to be a subset $A \subset \Pi$ of pairwise incomparable elements. The characteristic function $\mathbb{1}_A$ of an antichain A is defined similarly to the characteristic function of an order filter.

THEOREM 4.2.5 ([Sta86, Thm 2.2]). The vertices of $\mathcal{C}(\Pi)$ are given by the characteristic functions $\mathbb{1}_A$ of antichains A . In particular, the number of vertices of $\mathcal{C}(\Pi)$ equals the number of antichains of Π .

Let Π be a d -element poset and let $m \in \mathbb{Z}_{\geq 1}$. We define $\Omega(\Pi, m)$ to be the number of *order-preserving maps* $\Pi \rightarrow \{1, 2, \dots, m\}$, where we say that a map f is *order preserving* if $i \leq_{\Pi} j$ implies $f(i) \leq f(j)$. These order-preserving maps correspond to integer points in dilates of the order polytope as the next theorem shows:

THEOREM 4.2.6 ([Sta86, Thm. 4.1]). The Ehrhart polynomials of $\mathcal{O}(\Pi)$ and $\mathcal{C}(\Pi)$ are given by

$$\text{ehr}_{\mathcal{O}(\Pi)}(k) = \text{ehr}_{\mathcal{C}(\Pi)}(k) = \Omega(\Pi, k + 1).$$

Remark 4.2.7. As is implicit in Stanley's proof, interior integer points are in bijection with *strictly order-preserving maps*, i.e., maps f that satisfy $i <_{\Pi} j$ implies $f(i) < f(j)$.

In a poset Π , maximal chains can have different lengths. A chain with maximum length is called a *longest chain*.

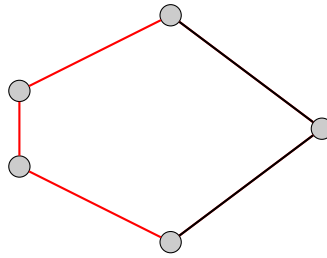


Figure 4.3: Edges belonging to longest chain are colored red.

Remark 4.2.8. The codegree of $\mathcal{O}(\Pi)$ equals the rank of $\overline{\Pi}$, i.e., it equals the number of edges in the longest chain of $\overline{\Pi}$.

4.3 MIYAZAKI'S CHARACTERIZATION

In this section, we recall the characterization for levelness of order polytopes which was introduced by Miyazaki, see [Miy17]. In order to give a characterization of the level property, Miyazaki defined sequences with condition N .

Definition 4.3.1 ([Miy17, Def. 3.1]). Let $i_1, j_1, i_2, j_2, \dots, i_t, j_t$ be a possible empty sequence of elements in a finite poset Π . We say the sequence satisfies condition N if

1. $i_1 < j_1 > i_2 < j_2 > \dots > i_t < j_t$ and
2. for any m, n with $1 \leq m < n \leq t$, $i_m \not\leq j_n$.

Definition 4.3.2. Let $i_1, j_1, i_2, j_2, \dots, i_t, j_t$ be a sequence of elements in a finite poset Π with condition N , and set $j_0 = \infty$ and $i_{t+1} = -\infty$. We set

$$r(i_1, j_1, \dots, i_t, j_t) := \sum_{s=1}^t (\text{rank}[i_s, j_{s-1}] - \text{rank}[i_s, j_s]) + \text{rank}[i_{t+1}, j_t].$$

Moreover, set

$$r_{\max} := \max\{r(i_1, j_1, \dots, i_t, j_t) : i_1, j_1, \dots, i_t, j_t \text{ is a sequence with condition } N\}.$$

Associated to every sequence with condition N , Miyazaki defines a special element of $\text{cone}(\mathcal{O}(\Pi))^\circ \cap \mathbb{Z}^{d+1}$, where d is the number of elements in the poset Π :

Definition 4.3.3 ([Miy17, Def 3.6]). Let $i_1, j_1, i_2, j_2, \dots, i_t, j_t$ be a sequence of elements in a finite poset Π with condition N , and set $j_0 = \infty$ and $i_{t+1} = -\infty$. We define

$$x(i_1, j_1, \dots, i_t, j_t)_{i_m} := \sum_{s=m}^t (\text{rank}[i_{s+1}, j_s] - \text{rank}[i_s, j_s])$$

for $1 \leq m \leq t+1$ and

$$y(i_1, j_1, \dots, i_t, j_t)_k := \max\{\text{rank}[i_s, k] + x(i_1, j_1, \dots, i_t, j_t)_{i_s} : k \geq i_s\}$$

for $k \in \overline{\Pi}$.

These elements give rise to an important class of minimal elements, as the next lemma shows.

LEMMA 4.3.4 ([Miy17, Lem. 3.8]). Let $i_1, j_1, i_2, j_2, \dots, i_t, j_t$ be a sequence of elements in a finite poset Π with condition N , and set $j_0 = \infty$ and $i_{t+1} = -\infty$. If $r(i_1, j_1, \dots, i_t, j_t) = r_{\max}$, then the element $\mathbf{y}(i_1, j_1, \dots, i_t, j_t)$ is minimal in the sense of Definition 2.3.11. In particular, it is an interior lattice point in the cone. Furthermore,

$$y(i_1, j_1, \dots, i_t, j_t)_{i_m} = x(i_1, j_1, \dots, i_t, j_t)_{i_m},$$

$$y(i_1, j_1, \dots, i_t, j_t)_{j_{m-1}} = \text{rank}[i_m, j_{m-1}] + x(i_1, j_1, \dots, i_t, j_t)_{i_m}$$

for $1 \leq m \leq t+1$. In particular, $y(i_1, j_1, \dots, i_t, j_t)_{j_0} = r_{\max}$.

Figure 4.4 illustrates Definition 4.3.1 and Lemma 4.3.4.

Now, we can introduce the characterization of Miyazaki:

LEMMA 4.3.5 ([Miy17, Thm 3.9]). Let Π be a poset and $r = \text{codeg}(\mathcal{O}(\Pi))$. Then Π is level is and only if $r_{\max} = r$.

In this chapter, we characterize levelness of order polytopes in terms of weighted digraphs. Given a poset Π , we define the *Hasse graph* $H(\Pi)$ of Π to be the digraph with nodes coming from $\overline{\Pi}$ and with directed, weighted edges $(i, j, -1)$ and $(j, i, 1)$, where

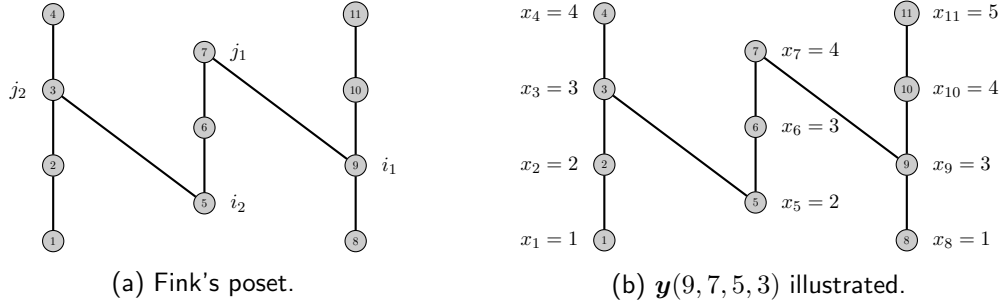


Figure 4.4: Fink's poset and the minimal element \mathbf{y} illustrated.

$i < j$. In our language, a sequence $i_1, j_1, \dots, i_t, j_t$ with condition N can be reinterpreted as a path \mathcal{P} in $H(\Pi)$ from ∞ to $-\infty$ with the up-edges (or the down-edges) coming from longest chains in $[i_m, j_m]$ (or $[i_{m+1}, j_m]$), where $j_0 = \infty$ and $i_{t+1} = -\infty$, and we say that such a path satisfies condition N . Moreover, we set $r(\mathcal{P}) := r(i_1, j_1, \dots, i_t, j_t)$ and $\mathbf{y}(\mathcal{P}) := \mathbf{y}(i_1, j_1, \dots, i_t, j_t)$. We chose the weights of the edges so that $-r(\mathcal{P})$ equals the weighted length.

Remark 4.3.6. This special minimal element $\mathbf{y}(\mathcal{P})$ has the property that for every up (or down) intervals $[i, j]$ in the path we have $\mathbf{y}(\mathcal{P})_i - \mathbf{y}(\mathcal{P})_j = \text{rank}[i, j]$. This is a direct consequence of Lemma 4.3.4 and a brief computation.

Now, we can characterize the level property as the following.

PROPOSITION 4.3.7. Let Π be a finite poset and $r = \text{codeg}(\mathcal{O}(\Pi))$. Then Π is not level if and only if there exists a path \mathcal{P} in $H(\Pi)$ with condition N such that $r(\mathcal{P}) > r$.

Proof. This directly follows from Lemmas 4.3.5 and 4.3.4. \square

Remark 4.3.8. Even if Π is level with $r = \text{codeg}(\mathcal{O}(\Pi))$, there may exist a path in $H(\Pi)$ of length $> r$. The interested reader might construct their favorite counter-example.

4.4 A NEW CHARACTERIZATION OF LEVELNESS

In this section, we introduce an algorithm for checking levelness of order polytopes. First, we need to associate a weighted digraph to a poset Π together with a subposet $\Pi' \subset \overline{\Pi}$, where we require that $i <_{\Pi'} j$ implies $i <_{\overline{\Pi}} j$.

Definition 4.4.1. Let Π be a finite poset and let Π' be a subposet of $\overline{\Pi}$ such that $i <_{\Pi'} j$ implies $i <_{\overline{\Pi}} j$. Let $\Gamma(\Pi, \Pi') = (\overline{\Pi}, E)$ be the weighted digraph with weighted, directed edges:

1. $(i, j, -1) \in E$ if and only if $j >_{\overline{\Pi}} i$;
2. $(j, i, 1) \in E$ if and only if $j >_{\Pi'} i$.

Clearly, $\Gamma(\Pi, \overline{\Pi}) = H(\Pi)$. If Π is clear from the context, we will write $\Gamma(\Pi')$.

A *negative cycle* is a directed cycle whose sum of weights is negative. A *wedge of cycles* is a closed directed path (where repetition is allowed) whose sum of weights is negative. An integer point $\mathbf{x} \in \text{cone}(\mathcal{O}(\Pi))^\circ \cap \mathbb{Z}^{d+1}$ is *sharp* along a covering pair $j \succ i$ if $x_j = x_i + 1$.

Next, we associate a weighted digraph to every integer point in $\text{cone}(\mathcal{O}(\Pi))^\circ$. The following lemma shows that the associated digraph does not have any negative cycles.

LEMMA 4.4.2. Let $\mathbf{b} \in \text{cone}(\mathcal{O}(\Pi))^\circ \cap \mathbb{Z}^{d+1}$ be given. Then the weighted digraph $\Gamma_{\mathbf{b}}$ whose nodes are given by the elements of $\overline{\Pi}$ and whose weighted, directed edges are

- $\{(i, j, -1) : i \leq j\} \cup \{(j, i, 1) : i \leq j, b_j - b_i = 1\}$,
- $\{(-\infty, i, -1) : i \succ -\infty\} \cup \{(i, -\infty, 1) : i \succ -\infty, b_i = 1\}$,
- and $\{(i, \infty, -1) : \infty \succ i\} \cup \{(\infty, i, 1) : \infty \succ i, b_i = \max_j b_j\}$.

does not have any negative cycles. In particular, every subgraph contains no negative cycles.

Proof. For $i, j \in \overline{\Pi}$, let $u(i, j)$ denote a directed path from i to j where $i < j$ and let $d(l, k)$ denote a directed path from l to k where $l > k$. Let $u(i_1, i_2), d(i_2, i_3), u(i_3, i_4), \dots, d(i_s, i_1)$ be a directed cycle in $\Gamma_{\mathbf{b}}$ with $i_1 < i_2, i_3 < i_2, \dots, i_1 < i_s$. We first remark that the weights of the down-paths $d(i, i+1)$ are given by

$$\overbrace{b_i - b^{(1)}}^{=1} + \overbrace{b^{(1)} - b^{(2)}}^{=1} + \dots - b^{(r)} + \overbrace{b^{(r)} - b_{i+1}}^{=1} = b_i - b_{i+1},$$

where we set $b_\infty = \max_k b_k + 1$ and $b_{-\infty} = 0$. Therefore, the sum of the weights in the cycle is equal to

$$\begin{aligned} & (b_{i_2} - b_{i_3}) + \dots + (b_{i_s} - b_{i_1}) - \text{length}(i_1, i_2) - \dots - \text{length}(i_{s-1}, i_s) \\ & = (b_{i_2} - b_{i_1}) + \dots + (b_{i_s} - b_{i_{s-1}}) - \text{length}(i_1, i_2) - \dots - \text{length}(i_{s-1}, i_s) \geq 0, \end{aligned}$$

since $\mathbf{b} \in \text{cone}(\mathcal{O}(\Pi))^\circ$ implies that $b_j - b_i \geq \text{length}(i, j)$ for all $i < j$, where $\text{length}(i, j)$ is the length of the path from i to j . \square

The following theorem uses the *Bellman–Ford algorithm*, which was introduced by Bellman and Ford, see for instance [Bel58]. We are using this algorithm as a black box. Instead of explicitly describing it, we will merely state some basic facts about it:

- The Bellman–Ford algorithm finds the shortest path from a sink to any other node in a weighted digraph. In contrast to other algorithms, it can also deal with negative weights assuming that the digraph does not contain any negative cycles (that can be reached from the starting node), see [Sch03, Thm. 8.5].
- If there is such a negative cycle, the Bellman–Ford algorithm can detect the negative cycle, [Sch03, Thm. 8.6].

- The Bellman–Ford algorithm runs in $O(\#V \cdot \#E)$, where V is the vertex set and E is the edge set of the underlying graph, see [Sch03, Thm. 8.5].

Given a path \mathcal{P} in $H(\Pi)$ with condition N , let $\Pi'(\mathcal{P})$ be the subposet of $\overline{\Pi}$ whose covering pairs are given by the up paths of \mathcal{P} . Now, we give a new characterization of level order polytopes.

THEOREM 4.4.3. Let Π be a finite poset on d elements and let $r = \text{codeg}(\mathcal{O}(\Pi))$. Then Π is level if and only if for any path \mathcal{P} in $H(\Pi)$ with condition N such that $\Gamma(\Pi'(\mathcal{P}))$ has no negative cycles, $\Gamma(\Pi'(\mathcal{P}) \cup \{\text{longest chains in } \overline{\Pi}\})$ has no negative cycles.

Proof. Let $\mathcal{M}(\mathcal{O}(\Pi)) := \text{cone}(\mathcal{O}(\Pi)) \cap Z^{d+1}$ and let $\mathcal{M}^\circ(\mathcal{O}(\Pi)) := \text{cone}(\mathcal{O}(\Pi))^\circ \cap Z^{d+1}$. To show the first direction, let's assume $\mathcal{O}(\Pi)$ is level. Associated to $\Gamma(\Pi'(\mathcal{P}))$, there is a $\mathbf{b} \in \mathcal{M}^\circ(\mathcal{O}(\Pi))$. To see this, run the Bellman–Ford algorithm on $\Gamma(\Pi'(\mathcal{P}))$. The Bellman–Ford algorithm minimizes the distance from $-\infty$ to any point i . After multiplying all entries by -1 , the algorithm will return a point \mathbf{b} such that for all covering pairs $j \succ i$ in $\overline{\Pi}$, we have $b_j \geq b_i + 1$ and moreover for any weighted edge $(j, i, 1)$ in $\Gamma(\Pi'(\mathcal{P}))$ we have $b_i \geq b_j - 1$. The first condition implies that $\mathbf{b} \in \mathcal{M}^\circ(\mathcal{O}(\Pi))$. Moreover, it follows from these conditions that for any weighted edge $(j, i, 1)$ in $\Gamma(\Pi'(\mathcal{P}))$, we have $b_j - b_i = 1$. Since Π is level, there exists a point $\tilde{\mathbf{b}} \in \mathcal{M}^\circ(\mathcal{O}(\Pi))$ on height r such that $\mathbf{b} - \tilde{\mathbf{b}} \in \mathcal{M}(\mathcal{O}(\Pi))$. This implies that for every covering pair $j \succ i$ in $\overline{\Pi}$, we have

$$b_j - b_i \geq \tilde{b}_j - \tilde{b}_i.$$

Hence for any weighted edge $(j, i, 1)$ in $\Gamma(\Pi'(\mathcal{P}))$, we have $\tilde{b}_j - \tilde{b}_i = 1$. Since $\tilde{\mathbf{b}}$ is on height r , we also know that $\tilde{\mathbf{b}}$ is sharp along the longest chains in $\overline{\Pi}$. Since $\mathcal{G} := \Gamma(\Pi' \cup \{\text{longest chains in } \overline{\Pi}\})$ is a subgraph of $\Gamma_{\tilde{\mathbf{b}}}$, using Lemma 4.4.2, we get that \mathcal{G} does not contain a negative cycle.

We prove the other direction by contraposition. Let's assume that Π is not level. Then there exists a path \mathcal{P} with condition N such that $r(\mathcal{P}) = r_{\max} > r$. Moreover, $\Gamma(\Pi'(\mathcal{P}))$ is a subgraph of $\Gamma_{\mathbf{y}(\mathcal{P})}$. Hence by using Lemma 4.4.2, it follows that $\Gamma(\Pi'(\mathcal{P}))$ has no negative cycle. On the other hand, since $\text{rank}(\overline{\Pi}) = r$, it follows that $\Gamma(\Pi'(\mathcal{P}) \cup \{\text{longest chains in } \overline{\Pi}\})$ has a negative cycle. \square

This directly implies the following:

COROLLARY 4.4.4. Let Π be a finite poset. Π is level if and only if for all $\Gamma(\Pi')$ that do not have a negative cycle, the digraph $\Gamma(\Pi' \cup \{\text{longest chains in } \overline{\Pi}\})$ does not have a negative cycle.

Proof. One direction directly follows from Theorem 4.4.3, so we only need to show that if Π is level and if $\Gamma(\Pi')$ does not contain a negative cycle, then $\Gamma(\Pi' \cup \{\text{longest chains in } \overline{\Pi}\})$ does not have a negative cycle. However, this follows from the proof of Theorem 4.4.3. \square

Remark 4.4.5. For practical purposes this corollary is more convenient than the previous characterization. This is due to the fact that it is hard to determine (all) paths with

condition N . We will use this corollary to give an infinite family of level posets, see Theorem 4.6.1.

Moreover, we get that — given the input Π — determining the levelness of order polytopes is in co-NP, where co-NP is the complexity class containing the problems of the complement of NP, i.e., the complexity class having a short certificate for rejection. For more about complexity classes, we refer to [Sch86, Sec. 2.5].

┆ COROLLARY 4.4.6. Levelness of order polytopes is in co-NP.

Proof. If $\mathcal{O}(\Pi)$ is not level, then there exists a short certificate $\Pi'(\mathcal{P})$ such that $\Gamma(\Pi'(\mathcal{P}))$ does not have a negative cycle but $\Gamma(\Pi'(\mathcal{P}) \cup \{\text{longest chains in } \overline{\Pi}\})$ has a negative cycle. This will be tested by the Bellman–Ford algorithm in polynomial time, since we need to run the Bellman–Ford algorithm twice, once for $\Gamma(\Pi'(\mathcal{P}))$ and once for $\Gamma(\Pi'(\mathcal{P}) \cup \{\text{longest chains in } \overline{\Pi}\})$. Therefore, we can verify non-levelness in polynomial time. \square

We now explicitly describe the algorithm underlying Corollary 4.4.4:

Algorithm 4.4.7.

For $\Gamma(\Pi') \subset H(\Pi)$:

 Run Bellman–Ford for $\Gamma(\Pi')$

 If negative cycle:

$1=1$

 Else:

 Run Bellman–Ford for $\Gamma(\Pi' \cup \{\text{longest chains in } \overline{\Pi}\})$

 If negative cycle:

 Return NOT LEVEL

 Else:

$1=1$

 Return LEVEL

┆ THEOREM 4.4.8. A poset Π is level if and only if Algorithm 4.4.7 returns level.

Proof. This directly follows from Corollary 4.4.4. \square

4.5 A NECESSARY CONDITION OF ENE, HERZOG, HIBI, AND SAEEDI MADANI

We now want to show that [EHHSM15, Thm 4.1] is a special case of Corollary 4.4.4. We first need to define the depth and the height of an element, where we follow again [EHHSM15]. The *height of an element* $i \in \Pi$, denoted $\text{height}(i)$ is the maximum length of a chain in $\overline{\Pi}$ descending from i . Similarly, we define the *depth of an element* i , denoted $\text{depth}(i)$, to be the maximum length of a chain in $\overline{\Pi}$ ascending from i .

They show that the following is a necessary condition for levelness:

THEOREM 4.5.1 ([EHSM15, Thm. 4.1]). Suppose Π is level. Then

$$\text{height}(j) + \text{depth}(i) \leq \text{rank}(\overline{\Pi}) + 1 \quad (4.1)$$

for all $j \succ i \in \Pi$.

Our next result shows that this is weaker than Corollary 4.4.4. In fact, it is equivalent to Corollary 4.4.4 where Π' is a single edge.

THEOREM 4.5.2. Let Π be a finite poset and $r = \text{codeg}(\mathcal{O}(\Pi))$. The following are equivalent:

- inequality (4.1) is satisfied by all covering pairs
- for all Hasse edges $j \succ i \in \Pi$ there is an integer point $\mathbf{x} \in r\mathcal{O}(\Pi)^\circ$ such that $x_j = x_i + 1$.

Proof. Let's assume that all covering pairs satisfy (4.1) and fix a covering pair $j \succ i$. We remark that $\text{rank}(\overline{\Pi})$ equals the codegree r of the order polytope $\mathcal{O}(\Pi)$. Thus, we need to show that there exists an integer point $\mathbf{x} \in r\mathcal{O}(\Pi)^\circ$ such that $x_j = x_i + 1$. To create such an \mathbf{x} , we can label the elements in Π using labels from $\{1, 2, \dots, r-1\}$. We first label $x_j = \text{height}(j)$ and hence $x_i = \text{height}(j) - 1$. For $k \in \Pi \setminus \{i, j\}$ we label $x_k = -\infty$, and then we recursively relabel by

$$x_k = \begin{cases} \max\{\text{height}(k), x_i + \text{length}([i, k])\} & \text{if } k > i, \\ \text{height}(k) & \text{otherwise.} \end{cases} \quad (4.2)$$

To show that this indeed gives an interior integer point in $r\mathcal{O}(\Pi)^\circ$, we need to show that $r > x_k \geq \text{height}(k)$ for all k . We say a label x_k is well-defined if it satisfies this condition. There are two cases:

1. $k > i$, then (4.1) ensures that (4.2) only yields well-defined labels;
2. $k \not> i$, then the recursive definition gives us $\text{height}(k)$, which by definition is well-defined.

This proves the first direction.

Now let assume that for all Hasse edges ($j \succ i$) in Π there exists an integer point $\mathbf{x} \in r\mathcal{O}(\Pi)^\circ$ such that $x_j = x_i + 1$. Let's fix a covering pair $j \succ i$. Then we have an integer point $\mathbf{x} \in r\mathcal{O}(\Pi)^\circ$ with $x_j = x_i + 1$ and it follows that $\text{height}(j) \leq x_j$. Since we have an integer point in the interior of $r\mathcal{O}(\Pi)$, we also get that

$$\text{depth}(i) \leq \text{rank}(\overline{\Pi}) - x_i.$$

Putting everything together, we obtain

$$\text{height}(j) + \text{depth}(i) \leq \text{rank}(\overline{\Pi}) + 1,$$

as desired. □

However, this result is not sufficient. The following example was related to us by Alex Fink, see Figure 4.4a.

Remark 4.5.3. Let Π be the poset from Figure 4.4a. We have that $\text{codeg}(\mathcal{O}(\Pi)) = 5$. Moreover, for any covering pair $i < j$ in Π , there is a minimal element x on height 5 with $x_i + 1 = x_j$. Thus, by Theorem 4.5.2 the condition of [EHHS15, Thm. 4.1] is satisfied. However, Π is *not* level. The minimal element $y(9, 7, 5, 3)$ is on height 6, see Figure 4.4b.

4.6 SERIES-PARALLEL POSETS

The goal of this section is to describe a new family of level posets. The main character of this section is the *ordinal sum*. We follow the notation of [Sta12, Sec. 3.2]. Let Π_1 and Π_2 be two posets. Then their ordinal sum $\Pi_1 \triangleleft \Pi_2$ is the poset with elements from the union $\Pi_1 \cup \Pi_2$ and with relations $s \leq t$ if

- $s, t \in \Pi_2$ with $s \leq_{\Pi_2} t$, or
- $s, t \in \Pi_1$ with $s \leq_{\Pi_1} t$, or
- $s \in \Pi_1$ and $t \in \Pi_2$.

Posets that can be built up as ordinal sums of posets are called *series-parallel posets*.

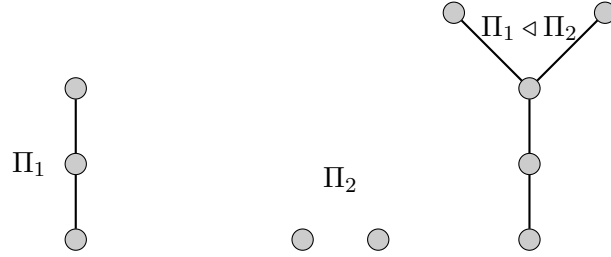


Figure 4.5: Ordinal sum of a chain of length 3 and an antichain of length 2.

We want to show the following result:

THEOREM 4.6.1. The ordinal sum $\Pi = \Pi_1 \triangleleft \Pi_2$ of two posets Π_1, Π_2 is level if and only if both Π_1 and Π_2 are level.

Proof. We prove the first direction by contraposition. So let's assume that Π_1 is not level. By Corollary 4.4.4, there exists a weighted digraph Γ_{Π_1} with nodes coming from $\overline{\Pi_1}$ which does not contain a negative cycle, but the weighted directed graph $\Gamma_{\Pi_1} \cup \{\text{longest chains in } \overline{\Pi_1}\}$ has a negative cycle. However, we also get that Γ_{Π} has a weighted digraph with up-edges of weight 1 only coming from up-edges of Γ_{Π_1} does not contain a negative cycle, but $\Gamma_{\Pi} \cup \{\text{longest chains in } \overline{\Pi}\}$ contains one, proving that $\Pi_1 \triangleleft \Pi_2$ is not level. The case where Π_2 is not level follows analogously.

We prove the other direction again by contraposition. So let's assume that $\Pi_1 \triangleleft \Pi_2$ is not level. By Corollary 4.4.4, there exists a weighted digraph Γ with nodes coming

from $\overline{\Pi}$ such that Γ does not have a negative cycle and $\Gamma \cup \{\text{longest chains in } \overline{\Pi}\}$ has a negative cycle, where $\Gamma \cup \{\text{longest chains in } \overline{\Pi}\}$ is the weighted digraph obtained from Γ by adding down edges of weight 1 along longest chains in $\overline{\Pi}$. In order to show that either Π_1 or Π_2 are not level, we will construct graphs Γ_{Π_1} and Γ_{Π_2} without negative cycles such that adding down edges of weight 1 along longest chains creates a negative cycle. The following two quotient maps will be essential for this:

$$\begin{aligned} \overline{\Pi_1 \triangleleft \Pi_2} &\xrightarrow{q_1} \overline{\Pi_1 \triangleleft \Pi_2} / (p_2 \sim p'_2 \sim \infty) \cong \overline{\Pi_1} \\ \overline{\Pi_1 \triangleleft \Pi_2} &\xrightarrow{q_2} \overline{\Pi_1 \triangleleft \Pi_2} / (p_1 \sim p'_1 \sim -\infty) \cong \overline{\Pi_2}, \end{aligned}$$

where $p_1, p'_1 \in \Pi_1$ and $p_2, p'_2 \in \Pi_2$.

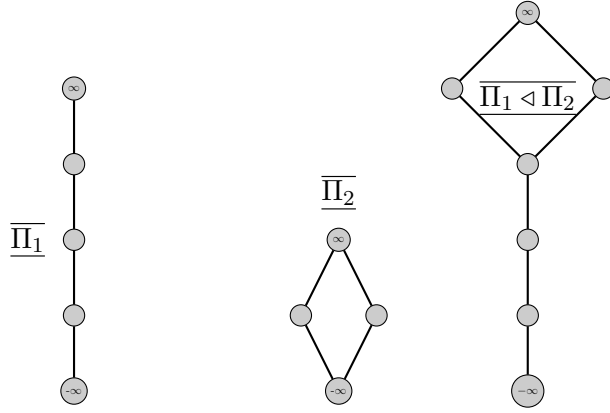


Figure 4.6: Original poset (on the right) and the two quotient posets (on the left and in the middle).

Note that these quotient maps also induce weighted directed graphs Γ_{Π_1} and Γ_{Π_2} on the underlying posets $\overline{\Pi_1}$ and $\overline{\Pi_2}$, respectively. We will show the following:

1. Both Γ_{Π_1} and Γ_{Π_2} do not have a negative cycle
2. Either $\Gamma_{\Pi_1} \cup \{\text{longest chains in } \overline{\Pi_1}\}$ or $\Gamma_{\Pi_2} \cup \{\text{longest chains in } \overline{\Pi_2}\}$ or both have a negative cycle.

This implies that either Π_1 or Π_2 or both cannot be level proving the claim. The first claim follows by contraposition. If either Γ_{Π_1} or Γ_{Π_2} had a negative cycle, then one can *lift* this cycle to obtain a negative cycle in Γ . This is due to the fact that every maximal element in Π_1 is comparable to every minimal element in Π_2 , together with the fact that every up-edge has the same weight, namely -1 .

Now let's prove the second claim. We remark that longest chains in $\overline{\Pi_1 \triangleleft \Pi_2}$ are concatenations of longest chains in $\overline{\Pi_1}$ and longest chains in $\overline{\Pi_2}$ and vice versa. This means that

$$\text{im}_{\Pi_1}(\Gamma \cup \{\text{longest chains in } \overline{\Pi_1 \triangleleft \Pi_2}\}) = \text{im}_{\Pi_1}(\Gamma) \cup \{\text{longest chains in } \overline{\Pi_1}\}$$

and

$$\text{im}_{\Pi_2}(\Gamma \cup \{\text{longest chains in } \overline{\Pi_1 \triangleleft \Pi_2}\}) = \text{im}_{\Pi_2}(\Gamma) \cup \{\text{longest chains in } \overline{\Pi_2}\},$$

where im_{Π_1} (or im_{Π_2}) denotes the image of the quotient map onto $\overline{\Pi_1}$ (or $\overline{\Pi_2}$). Moreover, if a negative cycle of $\Gamma \cup \{\text{longest chains in } \overline{\Pi_1 \triangleleft \Pi_2}\}$ is entirely contained in $q_1^{-1}(\overline{\Pi_1}) \cup \{\text{min. elt's of } \Pi_2\}$ or $q_2^{-1}(\overline{\Pi_2}) \cup \{\text{max. elt's of } \Pi_1\}$, then clearly the image also has a negative cycle. (Caveat: After forming the quotient map, the cycle might become a wedge of cycles. But since the total weight of the original cycle is the sum of the weights of the cycle in the image, at least one of these cycles in the wedge has to be negative.)

So we only need to consider the case where a negative cycle contains edges contained in Π_1 and in Π_2 . We can cover the cycle into the part, whose edges are entirely in $q_1^{-1}(\overline{\Pi_1}) \cup \{\text{min. elt's of } \Pi_2\}$, and a part whose edges are in $q_2^{-1}(\overline{\Pi_2}) \cup \{\text{max. elt's of } \Pi_1\}$. Note that the edges between Π_1 and Π_2 appear in both parts. Therefore, the total weight w of the cycle equals

$$0 > w = w_{\overline{\Pi_1}} + w_{\overline{\Pi_2}} - w_{\Pi_1 \Pi_2},$$

where $w_{\overline{\Pi_1}}$ and $w_{\overline{\Pi_2}}$ are the weights of the parts in the preimage of $\overline{\Pi_1}$ and $\overline{\Pi_2}$, respectively. The weight of the connecting edges between Π_1 and Π_2 is denoted $w_{\Pi_1 \Pi_2}$. This weight is 0, since there are as many up- as there are down-edges and the weights are -1 and 1 , respectively. Therefore, either $w_{\overline{\Pi_1}}$ or $w_{\overline{\Pi_2}}$ or both are negative. If $w_{\overline{\Pi_1}}$ is negative, applying the quotient map gives us a wedge of cycles in $\overline{\Pi_1}$ with negative weight. Hence it contains at least one negative cycle. The case where $w_{\overline{\Pi_2}}$ is negative is similar. Therefore, we have seen that either Π_1 or Π_2 is not level proving the claim. \square

For the remainder of this section, let $\Pi = \Pi_1 \triangleleft \Pi_2$. We will first give a geometric description of the order polytope and the chain polytope of Π in terms of the order and chain polytopes of Π_1 and Π_2 , respectively.

LEMMA 4.6.2. Let Π, Π_1, Π_2 be posets such that $\Pi = \Pi_1 \triangleleft \Pi_2$. Then

$$\mathcal{C}(\Pi) = \text{conv}\{\mathcal{C}(\Pi_1) \times \mathbf{0}_{\Pi_2} \cup \mathbf{0}_{\Pi_1} \times \mathcal{C}(\Pi_2)\} =: \mathcal{C}(\Pi_1) \oplus \mathcal{C}(\Pi_2), \quad (4.3)$$

where \oplus is the *free sum* of $\mathcal{C}(\Pi_1)$ and $\mathcal{C}(\Pi_2)$.

Proof. By Theorem 4.2.5, the vertices of the chain polytope are given by the indicator vectors of antichains. Now one notices that no antichain can contain elements from both Π_1 and Π_2 . \square

LEMMA 4.6.3. Let Π, Π_1, Π_2 be posets such that $\Pi = \Pi_1 \triangleleft \Pi_2$. Then

$$\mathcal{O}(\Pi) = \text{conv}\{\mathcal{O}(\Pi_1) \times \mathbf{1}_{\Pi_2} \cup \mathbf{0}_{\Pi_1} \times \mathcal{O}(\Pi_2)\}. \quad (4.4)$$

Proof. By Corollary 4.2.4, the vertices of the order polytope are given by the indicator vectors of filters. Now one notices that as soon as a filter contains an element of Π_1 , it contains *all* elements of Π_2 . \square

Moreover, we have:

PROPOSITION 4.6.4. Let Π, Π_1, Π_2 be posets such that $\Pi = \Pi_1 \triangleleft \Pi_2$. Moreover, let $h_{\Pi}^*, h_{\Pi_1}^*, h_{\Pi_2}^*$ be the h^* -polynomial of the Ehrhart series of the corresponding order polytopes. Then we have

$$h_{\Pi}^* = h_{\Pi_1}^* h_{\Pi_2}^*. \quad (4.5)$$

Proof. By Theorem 4.2.6 the Ehrhart series of the chain polytope of a poset Π is the same as the Ehrhart series of the order polytope of Π . In [HH16, Lem 3.2], Hibi and Higashitani show that if the free sum, $P \oplus Q$, of two lattice polytopes P, Q both containing the origin has the integer-decomposition property, then $h_{P \oplus Q}^* = h_P^* h_Q^*$. Now using Lemma 4.6.2 together with [HH16, Lem 3.2] implies the result. Note that every chain polytope and every order polytope has a unimodular triangulation and thus has the integer-decomposition property. For the order polytope, we directly get a regular, unimodular, flag triangulation by taking the standard triangulation of the cube and restricting it to the order polytope. For the chain polytope, Stanley [Sta86] constructs such a regular, unimodular, flag triangulation. \square

Remark 4.6.5. In [Hib88], Takayuki Hibi gives an example of two order polytopes $\mathcal{O}(\Pi_1), \mathcal{O}(\Pi_2)$ where both have the same h^* -polynomial, but Π_1 is level and Π_2 is not level. This shows that the level property cannot be characterized by the h^* -polynomial. We remark that Theorem 4.6.1 together with Proposition 4.6.4 gives a way to create infinitely many such examples $\Pi_1 \triangleleft \Pi_3$ and $\Pi_2 \triangleleft \Pi_3$, where Π_3 is any level poset.

4.7 CONNECTED COMPONENTS OF LEVEL POSETS

In this section, we discuss connected components of level posets. Any connected component of a Gorenstein poset is Gorenstein. This fact naturally leads us to consider whether any connected component of a level poset is level. However, this is not true in general. From the following result we know that there exists a level poset such that a connected component of the poset is not level.

THEOREM 4.7.1 ([EHSM15, Theorem 4.7]). Let Π be a poset on d elements and let C_s be a totally ordered set with s elements. Then the poset on the set $\Pi \cup C_s$, where elements from Π and C_s are incomparable, is level for all $s \gg 0$.

We give an explicit bound for s appearing in Theorem 4.7.1.

THEOREM 4.7.2. Let Π be a poset on d elements and let C_s be a totally ordered set with s elements. Then the poset on the set $\Pi \cup C_s$, where elements from Π and C_s are incomparable, is level for all $s \geq d$.

In order to prove this theorem, we consider a more general class of lattice polytopes containing any order polytope.

Definition 4.7.3. We say that a polytope $P \subset \mathbb{R}^d$ is *alcoved* if P is an intersection of some half-spaces bounded by the hyperplanes

$$H_{i,j}^m = \{(z_1, \dots, z_d) \in \mathbb{R}^d : z_i - z_j = m\} \text{ for } 0 \leq i < j \leq d, m \in \mathbb{Z},$$

where $z_0 = 0$.

It is known that any order polytope is alcoved. After a unimodular change of coordinates, every chain polytope is alcoved, too. Furthermore, any alcoved polytope possesses the integer-decomposition property.

For a lattice polytope $P = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq \mathbf{b}\} \subset \mathbb{R}^d$, we set $P^{(1)} = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq \mathbf{b} - 1\}$.

Remark 4.7.4. If $P = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq \mathbf{b}\} \subset \mathbb{R}^d$ is an alcoved polytope for some $m \times d$ integer matrix A and some integer vector $\mathbf{b} \in \mathbb{Z}^m$, then since A is a totally unimodular matrix, $P^{(1)}$ is a lattice polytope. In particular, one has $P^{(1)} = \text{conv}(P^\circ \cap \mathbb{Z}^d)$.

LEMMA 4.7.5. Let $P \subset \mathbb{R}^d$ be an alcoved polytope. Then for any positive integer k , kP and $P^{(1)}$ are alcoved.

Proof. Since P is alcoved, P is a polytope given by inequalities of the form $b_{ij} \leq z_i - z_j \leq c_{ij}$, for some collection of integer parameters b_{ij} and c_{ij} . Hence for any positive integer k , kP is a polytope given by inequalities of the form $kb_{ij} \leq z_i - z_j \leq kc_{ij}$. Moreover, $P^{(1)}$ is a polytope given by inequalities of the form $b_{ij} + 1 \leq z_i - z_j \leq c_{ij} - 1$. Therefore, both kP and $P^{(1)}$ are alcoved. \square

For two lattice polytopes P and Q in \mathbb{R}^d , set

$$\text{Cayley}(P, Q) = \text{conv}(P \times \{0\} \cup Q \times \{1\}) \subset \mathbb{R}^{d+1}.$$

We say that $\text{Cayley}(P, Q)$ is the *Cayley polytope* of P and Q .

LEMMA 4.7.6. Let P and Q be alcoved polytopes in \mathbb{R}^d . Then $\text{Cayley}(P, Q)$ has a regular unimodular triangulation. In particular, $\text{Cayley}(P, Q)$ has the integer-decomposition property.

Proof. This is [HPPS14, Lemma 4.15], since alcoved polytopes have a type A root system. \square

This directly implies the following result.

COROLLARY 4.7.7. If $P, Q \subset \mathbb{R}^d$ are alcoved polytopes, then the map

$$(P \cap \mathbb{Z}^d) \times (Q \cap \mathbb{Z}^d) \rightarrow (P + Q) \cap \mathbb{Z}^d.$$

is onto.

Proof. We have that

$$2 \text{Cayley}(P, Q) \cap \{(x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1} : x_{d+1} = 1\} = (P + Q) \times \{1\}.$$

Since $\text{Cayley}(P, Q)$ has the integer-decomposition property, it follows that every integer point in $2 \text{Cayley}(P, Q)$ can be written as a sum of two integer points in $\text{Cayley}(P, Q)$. However, the only way we can get an integer point at height 1 is if we add one integer point at height 0 and at height 1, i.e., one integer point belongs to P and one belongs to Q , proving the claim. \square

Now, we give a characterization on levelness of alcoved polytopes.

PROPOSITION 4.7.8. Let $P \subset \mathbb{R}^d$ be an alcoved polytope and let $r = \text{codeg}(P)$. Then P is level if and only if for any integer $k \geq r$, it follows that $(kP)^{(1)} = (rP)^{(1)} + (k-r)P$.

Proof. First, assume that P is level. Then from the definition of levelness, for any integer $k \geq r$, $(kP)^\circ \cap \mathbb{Z}^d = (rP)^\circ \cap \mathbb{Z}^d + (k-r)P \cap \mathbb{Z}^d$. Hence one has $(kP)^{(1)} = \text{conv}((rP)^\circ \cap \mathbb{Z}^d) + (k-r)P \cap \mathbb{Z}^d = \text{conv}((rP)^\circ \cap \mathbb{Z}^d) + \text{conv}((k-r)P \cap \mathbb{Z}^d) = (rP)^{(1)} + (k-r)P$.

Conversely, assume that for any integer $k \geq r$, $(kP)^{(1)} = (rP)^{(1)} + (k-r)P$. By Lemma 4.7.5, $(kP)^{(1)}$, $(rP)^{(1)}$ and $(k-r)P$ are alcoved polytopes. Hence by Corollary 4.7.7, P is level. \square

LEMMA 4.7.9. Let $P \subset \mathbb{R}^d$ be a lattice polytope and let $r' \geq \text{codeg}(P)$ be an integer. Assume that there exists an integer $k > r'$ such that $(kP)^{(1)} = (r'P)^{(1)} + (k-r')P$. Then for any integer $k > k' \geq r'$, we have $(k'P)^{(1)} = (r'P)^{(1)} + (k'-r')P$.

Proof. Assume that there exists an integer $k > k' \geq r'$ such that $(k'P)^{(1)} \supsetneq (r'P)^{(1)} + (k'-r')P$. In general, it follows that $(k'P)^{(1)} \supset (r'P)^{(1)} + (k'-r')P$. Then we have

$$(kP)^{(1)} \supset (k'P)^{(1)} + (k-k')P \supsetneq (r'P)^{(1)} + (k-r')P = (kP)^{(1)}.$$

Hence this is a contradiction. \square

On levelness of dilated polytopes, the following theorem is known.

THEOREM 4.7.10 ([BGT02, Theorem 1.3.3]). Let P be a lattice d -polytope. Then for any integer $k \geq d+1$, kP is level.

Now, we prove the following theorem about levelness of a product of alcoved polytopes.

THEOREM 4.7.11. Let $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ be alcoved polytopes. Suppose that Q is level and $r = \text{codeg}(Q) \geq \dim P + 1$. Then $P \times Q \subset \mathbb{R}^{d+e}$ is level.

Proof. By Theorem 4.7.10, rP is level. Hence for any positive integer k' , one has $(k'rP)^{(1)} = (rP)^{(1)} + (k'-1)rP$ from Proposition 4.7.8. Therefore, by Lemma 4.7.9, it follows that for any $k \geq r$, we obtain $(kP)^{(1)} = (rP)^{(1)} + (k-r)P$. Since Q is level, for

any $k \geq r$, we obtain $(kQ)^{(1)} = (rQ)^{(1)} + (k-r)Q$. Fix a positive integer $k \geq r$. Since $\text{int}(k(P \times Q)) \cap \mathbb{Z}^{d+e} = (\text{int}(kP) \cap \mathbb{Z}^d) \times (\text{int}(kQ) \cap \mathbb{Z}^e)$,

$$\begin{aligned} (k(P \times Q))^{(1)} &= (kP)^{(1)} \times (kQ)^{(1)} \\ &= ((rP)^{(1)} + (k-r)P) \times ((rQ)^{(1)} + (k-r)Q) \\ &\subset ((rP)^{(1)} \times (rQ)^{(1)}) + (k-r)(P \times Q) \\ &= (r(P \times Q))^{(1)} + (k-r)(P \times Q) \\ &\subset (k(P \times Q))^{(1)}. \end{aligned}$$

Hence $P \times Q$ is level. \square

Now, we prove Theorem 4.7.2.

Proof of Theorem 4.7.2. The order polytope of $\Pi \cup C_r$ is the Cartesian product of $\mathcal{O}(\Pi)$ and $\mathcal{O}(C_r)$, which is the r -dimensional unimodular simplex. This simplex has codegree $r+1$. Hence by Theorem 4.7.11, the claim now follows. \square

Conversely, we consider posets all of whose connected components are level. In fact, these posets are always level.

THEOREM 4.7.12. Let Π be a poset on d elements and Π_1, \dots, Π_m the connected components of Π . If each Π_i is level, then Π is level.

Theorem 4.7.12 follows from the following result:

THEOREM 4.7.13. Let $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ be level polytopes. If either

1. $\text{codeg}(Q) < \text{codeg}(P)$ and Q has the integer-decomposition property,
2. $\text{codeg}(P) < \text{codeg}(Q)$ and P has the integer-decomposition property,
3. or if $\text{codeg}(Q) = \text{codeg}(P)$,

then $P \times Q$ is level.

Proof. Let $r_P := \text{codeg}(P)$ and let $r_Q := \text{codeg}(Q)$. Without loss of generality, we assume $\text{codeg}(Q) \leq \text{codeg}(P)$. Then $r := \text{codeg}(P \times Q) = \max\{r_P, r_Q\} = r_P$, since $\mathbf{x} = (\mathbf{x}_P, \mathbf{x}_Q) \in (P \times Q)^\circ \cap \mathbb{Z}^{d+e}$ implies that $\mathbf{x}_P \in P^\circ \cap \mathbb{Z}^d$ and that $\mathbf{x}_Q \in Q^\circ \cap \mathbb{Z}^e$. Let $(\mathbf{x}_P, \mathbf{x}_Q, h) \in \text{cone}(P \times Q) \cap \mathbb{Z}^{d+e+1}$ with $h > r$. Then this point projects to points $(\mathbf{x}_P, h) \in \text{cone}(P) \cap \mathbb{Z}^{d+1}$ and $(\mathbf{x}_Q, h) \in \text{cone}(Q) \cap \mathbb{Z}^{e+1}$. Since P is level, we have that

$$(\mathbf{x}_P, h) = (\mathbf{x}_P^\circ, r) + (\tilde{\mathbf{x}}_P, h - r),$$

where $(\mathbf{x}_P^\circ, r) \in \text{cone}(P)^\circ \cap \mathbb{Z}^{d+1}$ and $(\tilde{\mathbf{x}}_P, h - r) \in \text{cone}(P) \cap \mathbb{Z}^{d+1}$. Similarly, since Q is level, we have that

$$(\mathbf{x}_Q, h) = (\mathbf{x}_Q^\circ, r_Q) + (\tilde{\mathbf{x}}_Q, h - r_Q),$$

where $(\mathbf{x}_Q^\circ, r_Q) \in \text{cone}(Q)^\circ \cap \mathbb{Z}^{e+1}$ and $(\tilde{\mathbf{x}}_Q, h - r_Q) \in \text{cone}(Q) \cap \mathbb{Z}^{e+1}$. If $\text{codeg}(P) = \text{codeg}(Q)$, we now get a decomposition

$$(\mathbf{x}_P, \mathbf{x}_Q, h) = (\mathbf{x}_P^\circ, \mathbf{x}_Q^\circ, r) + (\tilde{\mathbf{x}}_P, \tilde{\mathbf{x}}_Q, h - r),$$

where $(\mathbf{x}_P^\circ, \mathbf{x}_Q^\circ) \in r(P \times Q)^\circ \cap \mathbb{Z}^{d+e}$ and where $(\tilde{\mathbf{x}}_P, \tilde{\mathbf{x}}_Q) \in (h - r)(P \times Q) \cap \mathbb{Z}^{d+e}$ proving levelness of $P \times Q$.

So let's assume $\text{codeg}(Q) < \text{codeg}(P)$. Since Q has the integer-decomposition property, we can express $(\tilde{\mathbf{x}}_Q, h - r_Q)$ as a sum of height 1 elements, i.e., $(\tilde{\mathbf{x}}_Q, h - r_Q) = (\tilde{\mathbf{x}}_Q^{(1)}, r_P - r_Q) + (\tilde{\mathbf{x}}_Q^{(2)}, h - r_P)$ and thus obtain

$$(\mathbf{x}_Q, h) = (\mathbf{x}_Q^\circ, r_Q) + (\tilde{\mathbf{x}}_Q, h - r_Q) = (\mathbf{x}_Q^\circ + \tilde{\mathbf{x}}_Q^{(1)}, r_P) + (\tilde{\mathbf{x}}_Q^{(2)}, h - r_P),$$

where $(\mathbf{x}_Q^\circ + \tilde{\mathbf{x}}_Q^{(1)}, r_P) \in \text{cone}(Q)^\circ \cap \mathbb{Z}^{e+1}$ and $(\tilde{\mathbf{x}}_Q^{(2)}, h - r_P) \in \text{cone}(Q) \cap \mathbb{Z}^{e+1}$. Therefore, we can express $(\mathbf{x}_P, \mathbf{x}_Q, h)$ as

$$(\mathbf{x}_P, \mathbf{x}_Q, h) = (\mathbf{x}_P^\circ, \mathbf{x}_Q^\circ + \tilde{\mathbf{x}}_Q^{(1)}, r) + (\tilde{\mathbf{x}}_P, \tilde{\mathbf{x}}_Q^{(2)}, h - r)$$

where $(\mathbf{x}_P^\circ, \mathbf{x}_Q^\circ + \tilde{\mathbf{x}}_Q^{(1)}) \in r(P \times Q)^\circ \cap \mathbb{Z}^{d+e}$ and where $(\tilde{\mathbf{x}}_P, \tilde{\mathbf{x}}_Q^{(2)}) \in (h - r)(P \times Q) \cap \mathbb{Z}^{d+e}$. \square

Remark 4.7.14. In Theorem 4.7.13, we really need the assumption that the polytope of lower codegree has the integer-decomposition property. Consider the following example, where

$$P = \text{conv}\{(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 2)\}$$

and where

$$Q = \text{conv}\{(0, 0), (1, 0), (0, 1)\}.$$

Then P does not have the integer-decomposition property, but it is Gorenstein and thus level. Moreover, Q has the integer-decomposition property and it is level, since it is Gorenstein. However, the product $P \times Q$ is not level.

Proof. We first remark that $\text{codeg}(Q) = 3$ and $\text{codeg}(P) = 2$. Hence, $\text{codeg}(P \times Q) = 3$. There are exactly 4 integer points in the interior of $3(P \times Q)$, namely

$$3(P \times Q)^\circ \cap \mathbb{Z}^5 = \{(1, 1, 1, 1, 1), (1, 2, 1, 1, 1), (2, 1, 1, 1, 1), (2, 2, 3, 1, 1)\}.$$

However, the integer point $(2, 2, 2, 2, 1) \in (4(P \times Q)^\circ) \cap \mathbb{Z}^5$ cannot be written as a sum

$$(2, 2, 2, 2, 1) = \mathbf{x}_1 + \mathbf{x}_2$$

of points $\mathbf{x}_1 \in 3(P \times Q)^\circ \cap \mathbb{Z}^5$ and $\mathbf{x}_2 \in (P \times Q) \cap \mathbb{Z}^5$. Thus, $P \times Q$ is not level. \square

We end this article with the following criterion for levelness:

LEMMA 4.7.15. If a lattice d -polytope P has a covering by unimodular simplices such that every interior face of such a simplex contains an interior sub-face of dimension $\text{codeg}(P) - 1$, then P is level.

Proof. Let $\mathbf{b} \in \text{cone}(P)^\circ \cap \mathbb{Z}^{d+1}$ and let $r := \text{codeg}(P)$. We will prove levelness by showing that every such \mathbf{b} can be written as an integral combination of an interior integer point on height r with integer points on height 1.

The unimodular simplices in the covering of P give rise to a covering of $\text{cone}(P)$ by unimodular cones. Therefore, \mathbf{b} is in at least one such unimodular cone. Let $\Delta = \text{conv}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{d+1}\}$ be the corresponding unimodular simplex and let $\text{cone}(\Delta)$ be the cone over Δ . Then there is a unique representation

$$\mathbf{b} = \sum_{i=1}^{d+1} \lambda_i(\mathbf{v}_i, 1), \quad (4.6)$$

where $\lambda_i \in \mathbb{Z}_{\geq 0}$. There are two cases: If $\lambda_i = 0$ for some $i \in I$ in an index set I , then this means that \mathbf{b} is contained in a $(d + 1 - \#I)$ -dimensional face, which has to be an interior face. Then, for $J := \{1, \dots, d + 1\} \setminus I$, the face $\text{conv}\{\mathbf{v}_j\}_{j \in J}$ gives rise to a $(\#J)$ -dimensional cone containing \mathbf{b} . Hence, by assumption there is a subset $R \subset J$ with $\#R = r$, such that the point $\mathbf{x} = \sum_{i \in R}(\mathbf{v}_i, 1)$ is a point in $\text{cone}(P)^\circ \cap \mathbb{Z}^{d+1}$. Combining this with Equation (4.6), we obtain

$$\mathbf{b} = \mathbf{x} + (\mathbf{b} - \mathbf{x}),$$

where $\mathbf{b} - \mathbf{x} \in \text{cone}(P) \cap \mathbb{Z}^{d+1}$, which proves levelness,

If $\lambda_i \neq 0$ for all $i \in \{1, 2, \dots, d + 1\}$, then we can set $\mathbf{x} = \sum_{i \in J}(\mathbf{v}_i, 1)$, where J is an interior $(r - 1)$ -dimensional face. The claim now follows analogously. \square

It would be interesting to see a poset interpretation of this lemma.

“One and one and one is three.” (The Beatles (Come Together))

This chapter is based on joint work with McCabe Olsen, see also our paper [KO].

5.1 INTRODUCTION

Let $P \subset \mathbb{R}^n$ be a convex lattice polytope. It is a common question in Ehrhart theory to determine if P is a *Gorenstein* polytope, that is, to determine if the associated semigroup algebra of P is a Gorenstein algebra. Such polytopes are of interest in geometric combinatorics, because they have some integer dilate, cP , which is a reflexive polytope [DNH97]. Moreover, as we have seen in Chapter 2, Gorenstein polytopes have a palindromic Ehrhart h^* -polynomial, see Theorem 2.3.16. Gorenstein polytopes are also of interest in algebraic geometry for a variety of reasons, including connections to mirror symmetry (see e.g. [Bat94] and [CLS11, Section 8.3]). Roughly speaking, a pair of reflexive lattice polytopes gives rise to a mirror pair of Calabi–Yau manifolds. We recommend [Cox15] for an excellent survey article about reflexive polytopes and their connection to mirror symmetry. Subsequently, classifications of the Gorenstein property have been extensively studied and are known for many families including order polytopes [Hib87, Sta86], twinned poset polytopes [HM16], and r -stable (n, k) -hypersimplices [HS16].

As in the previous chapter, we say that P is a *level* polytope if its associated semigroup algebra is a *level algebra*. The question of detecting levelness has not been studied nearly to the degree as detecting the Gorenstein property (see e.g. [EHHSM15, HY18]). However, in addition to the independent interest in level algebras, if P is level, we obtain nontrivial inequalities on the coefficients of the Ehrhart h^* -polynomial, which are not satisfied for general lattice polytopes, see Proposition 4.1.1.

One family of well-studied polytopes are the s -lecture hall polytopes. For a given $s \in \mathbb{Z}_{\geq 1}^n$, the s -lecture hall polytope is the simplex defined by

$$\mathbf{P}_n^{(s)} := \left\{ \boldsymbol{\lambda} \in \mathbb{R}^n : 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \dots \leq \frac{\lambda_n}{s_n} \leq 1 \right\}.$$

These polytopes arise from the extensively investigated s -lecture hall partitions, introduced by Bousquet-Mélou and Eriksson [BME97a, BME97b]. To quote Savage and Schuster from [SS12]: “Since their discovery, lecture hall partitions and their generalizations have emerged as fundamental tools for interpreting classical partition identities and for discovering new ones.” Though many algebraic and geometric properties of s -lecture hall

polytopes are known (see e.g. [Sav16]), the known Gorenstein results are very limited and there are no known levelness results.

Our focus is to work towards a classification of the Gorenstein and level properties in \mathbf{s} -lecture hall polytopes. In particular, we provide a full characterization for the Gorenstein property. We also give another more geometric characterization in the case that \mathbf{s} has at least one index i , $1 < i \leq n$, such that $\gcd(s_{i-1}, s_i) = 1$. We also provide a characterization for levelness which applies to all \mathbf{s} -sequences in terms of \mathbf{s} -inversion sequences. These main results are as follows:

THEOREM 5.1.1. Let $\mathbf{s} = (s_1, s_2, \dots, s_n) \in \mathbb{Z}_{\geq 1}^n$. Then $\mathbf{P}_n^{(\mathbf{s})}$ is Gorenstein if and only if there exists a $\mathbf{c} \in \mathbb{Z}^{n+1}$ satisfying

$$c_j s_{j-1} = c_{j-1} s_j + \gcd(s_{j-1}, s_j)$$

for $j > 1$ and

$$c_{n+1} s_n = 1 + c_n$$

with $c_1 = 1$.

While the next result is not as general, it guarantees that, under the condition that $\gcd(s_{i-1}, s_i) = 1$ for some $1 < i \leq n$, the vertex cones of $\mathbf{P}_n^{(\mathbf{s})}$ at $(0, 0, \dots, 0)$ and at (s_1, s_2, \dots, s_n) being Gorenstein already implies that $\mathbf{P}_n^{(\mathbf{s})}$ is Gorenstein.

THEOREM 5.1.2. Let $\mathbf{s} = (s_1, s_2, \dots, s_n) \in \mathbb{Z}_{\geq 1}^n$ be a sequence such that there exists some $1 < i \leq n$ such that $\gcd(s_{i-1}, s_i) = 1$ and define $\overleftarrow{\mathbf{s}} = (\overleftarrow{s}_1, \dots, \overleftarrow{s}_n) := (s_n, s_{n-1}, \dots, s_1)$. Then $\mathbf{P}_n^{(\mathbf{s})}$ is Gorenstein if and only if there exist $\mathbf{c}, \mathbf{d} \in \mathbb{Z}^n$ satisfying

$$c_j s_{j-1} = c_{j-1} s_j + \gcd(s_{j-1}, s_j)$$

and

$$d_j \overleftarrow{s}_{j-1} = d_{j-1} \overleftarrow{s}_j + \gcd(\overleftarrow{s}_{j-1}, \overleftarrow{s}_j)$$

for $j > 1$ with $c_1 = d_1 = 1$.

Remark 5.1.3. As far as we know, Corollary 5.3.5 follows from Theorem 5.1.2, but not from Theorem 5.1.1.

In the following theorem, $\mathbf{I}_n^{(\mathbf{s})} = \{e : 0 \leq e_i < s_i\}$ is the set of \mathbf{s} -inversion sequences, and $\mathbf{I}_{n,k}$ is the set of \mathbf{s} -inversion sequences with exactly k ascents. We refer to Section 5.2 for details.

THEOREM 5.1.4. Let $\mathbf{s} = (s_1, s_2, \dots, s_n) \in \mathbb{Z}_{\geq 1}^n$ and let $r = \max \{ \text{asc}(e) : e \in \mathbf{I}_n^{(\mathbf{s})} \}$. Then $\mathbf{P}_n^{(\mathbf{s})}$ is level if and only if for any $e \in \mathbf{I}_{n,k}^{(\mathbf{s})}$ with $1 \leq k < r$ there exists some $e' \in \mathbf{I}_{n,1}^{(\mathbf{s})}$ such that $(e + e') \in \mathbf{I}_{n,k+1}^{(\mathbf{s})}$.

The structure of this chapter is as follows. In Section 5.2, we provide the necessary

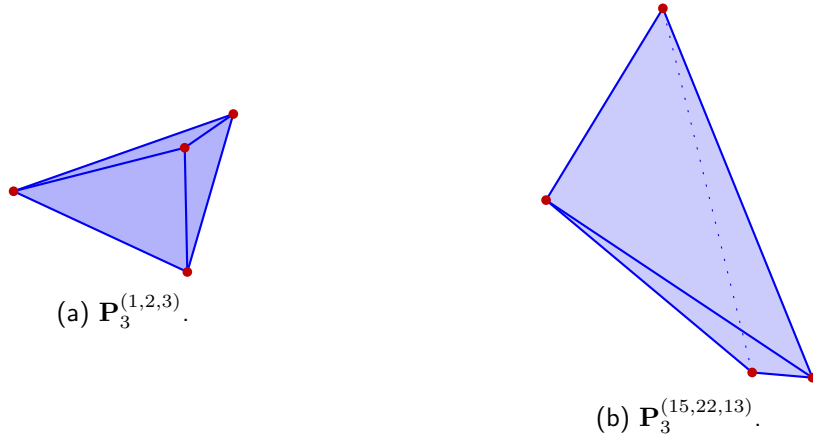


Figure 5.1: Two three-dimensional lecture hall polytopes (not drawn to scale)

background for lecture hall partitions including their vertex- and halfspace-descriptions. The focus of Section 5.3 is proving the Gorenstein classifications. In Section 5.4, we prove the characterization of the level property and use the characterization to arrive at several consequences of interest. We conclude in Section 5.5 with some potential ways to improve and extend these results and other future directions.

5.2 BACKGROUND

In this section, we provide the terminology and background literature necessary for our results. Specifically, we provide a review of the polyhedral geometry of lecture hall partitions. For the necessary background about Ehrhart theory and the levelness as well as the Gorenstein property, we refer the reader to Section 2.2 and Section 2.3, respectively. For a more in-depth overview of some of the results concerning \mathbf{s} -lecture hall polytopes and many others, the reader should consult the excellent survey of Savage [Sav16].

Let $\mathbf{s} = (s_1, s_2, \dots, s_n) \in \mathbb{Z}_{\geq 1}^n$ be a sequence. Given any \mathbf{s} -sequence, define the \mathbf{s} -lecture hall partitions to be the set

$$L_n^{(\mathbf{s})} := \left\{ \boldsymbol{\lambda} \in \mathbb{Z}^n : 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \dots \leq \frac{\lambda_n}{s_n} \right\}.$$

We can associate to the set of \mathbf{s} -lecture hall partitions several geometric objects. In particular, the \mathbf{s} -lecture hall polytope and the \mathbf{s} -lecture hall cone. For a given \mathbf{s} , the \mathbf{s} -lecture hall polytope is defined

$$\begin{aligned} \mathbf{P}_n^{(\mathbf{s})} &:= \left\{ \boldsymbol{\lambda} \in \mathbb{R}^n : 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \dots \leq \frac{\lambda_n}{s_n} \leq 1 \right\} \\ &= \text{conv}\{(0, \dots, 0), (0, \dots, 0, s_i, s_{i+1}, \dots, s_n) \text{ for } 1 \leq i \leq n\}. \end{aligned}$$

See Figure 5.1 for two three-dimensional examples of lecture hall polytopes.

The Ehrhart h^* -polynomials of $\mathbf{P}_n^{(\mathbf{s})}$ have been completely classified. Given \mathbf{s} , the set of \mathbf{s} -inversion sequences is defined as $\mathbf{I}_n^{(\mathbf{s})} := \{\mathbf{e} \in \mathbb{Z}_{\geq 0}^n : 0 \leq e_i < s_i\}$. For a given $\mathbf{e} \in \mathbf{I}_n^{(\mathbf{s})}$, the *ascent set* of \mathbf{e} is

$$\text{Asc}(\mathbf{e}) := \left\{ i \in \{0, 1, \dots, n-1\} : \frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}} \right\},$$

with the convention $s_0 := 1$ and $e_0 := 0$, and $\text{asc}(\mathbf{e}) := \#\text{Asc}(\mathbf{e})$. With these definitions, we can give the explicit formulation for the Ehrhart h^* -polynomials.

THEOREM 5.2.1 ([SS12, Theorem 8]). For a given $\mathbf{s} \in \mathbb{Z}_{\geq 1}^n$,

$$h_{\mathbf{P}_n^{(\mathbf{s})}}^*(z) = \sum_{\mathbf{e} \in \mathbf{I}_n^{(\mathbf{s})}} z^{\text{asc}(\mathbf{e})}.$$

The polynomials $h_{\mathbf{P}_n^{(\mathbf{s})}}^*(z)$ generalize Eulerian polynomials and are known as the \mathbf{s} -Eulerian polynomials, as the sequence $\mathbf{s} = (1, 2, \dots, n)$ gives rise to

$$\sum_{\mathbf{e} \in \mathbf{I}_n^{(1,2,\dots,n)}} z^{\text{asc}(\mathbf{e})} = \sum_{\pi \in \mathfrak{S}_n} z^{\text{des}(\pi)} = A_n(z)$$

which is the usual Eulerian polynomial. These \mathbf{s} -Eulerian polynomials are known to be real-rooted and, hence, unimodal [SV15].

\mathbf{s} -lecture hall polytopes have been the subject of much additional study (see e.g. [HOT16, HOT17, PS13, PS13, SV12]). Of particular interest are algebraic and geometric structural results such as Gorenstein and IDP results. The second author of [KO] along with Hibi and Tsuchiya in [HOT16] provides some Gorenstein results in particular circumstances. Additionally, the following theorem for IDP holds.

THEOREM 5.2.2 ([BS]). $\mathbf{P}_n^{(\mathbf{s})}$ has the IDP.

A proof for the case of monotonic \mathbf{s} -sequences was given by the second author of [KO] with Hibi and Tsuchiya in [HOT16] which Brändén and Solus [BS] show can be extended to any \mathbf{s} when they prove that all \mathbf{s} -lecture hall order polytopes have the IDP. Moreover, in [BL16, Conj 5.4] it is conjectured that for any \mathbf{s} , $\mathbf{P}_n^{(\mathbf{s})}$ possesses a regular, unimodular triangulation.

For a given \mathbf{s} , the \mathbf{s} -lecture hall cone is defined to be

$$\mathcal{C}_n^{(\mathbf{s})} := \left\{ \boldsymbol{\lambda} \in \mathbb{R}^n : 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \dots \leq \frac{\lambda_n}{s_n} \right\}.$$

These objects are related to \mathbf{s} -lecture hall polytopes in that $\mathcal{C}_n^{(\mathbf{s})}$ arises as the vertex cone of $\mathbf{P}_n^{(\mathbf{s})}$ at the origin $(0, \dots, 0)$. It is important to realize that $\mathcal{C}_n^{(\mathbf{s})}$ is not the same object as cone $(\mathbf{P}_n^{(\mathbf{s})})$. In fact, $\mathcal{C}_n^{(\mathbf{s})}$ arises as a nontrivial quotient of cone $(\mathbf{P}_n^{(\mathbf{s})})$. The \mathbf{s} -lecture hall cones have been studied extensively (see e.g. [BBK⁺15, BBK⁺16, Ols17]) and the

following Gorenstein results for the lecture hall cones are particularly of interest for our purposes.

THEOREM 5.2.3 ([BBK⁺15, Corollary 2.6], [BME97b, Proposition 5.4]). For a positive integer sequence \mathbf{s} , the \mathbf{s} -lecture hall cone $\mathcal{C}_n^{(\mathbf{s})}$ is Gorenstein if and only if there exists some $\mathbf{c} \in \mathbb{Z}^n$ satisfying

$$c_j s_{j-1} = c_{j-1} s_j + \gcd(s_{j-1}, s_j)$$

for $j > 1$, with $c_1 = 1$.

Moreover, in the case of \mathbf{s} -sequences where $\gcd(s_{i-1}, s_i) = 1$ holds for all i , we have a refinement to this theorem. We say that \mathbf{s} is \mathbf{u} -generated by a sequence \mathbf{u} of positive integers if $s_2 = u_1 s_1 - 1$ and $s_{i+1} = u_i s_i - s_{i-1}$ for $i > 1$.

THEOREM 5.2.4 ([BBK⁺15, Theorem 2.8], [BME97b, Proposition 5.5]). Let $\mathbf{s} = (s_1, \dots, s_n)$ be a sequence of positive integers such that $\gcd(s_{i-1}, s_i) = 1$ for $1 \leq i < n$. Then $\mathcal{C}_n^{(\mathbf{s})}$ is Gorenstein if and only if \mathbf{s} is \mathbf{u} -generated by some sequence $\mathbf{u} = (u_1, u_2, \dots, u_{n-1})$ of positive integers. When such a sequence exists, the Gorenstein point \mathbf{c} for $\mathcal{C}_n^{(\mathbf{s})}$ is defined by $c_1 = 1$, $c_2 = u_1$, and for $2 \leq i < n$, $c_{i+1} = u_i c_i - c_{i-1}$.

5.3 GORENSTEIN LECTURE HALL POLYTOPES

In this section, we will give a characterization of Gorenstein \mathbf{s} -lecture hall polytopes with the restriction that there exists some index i such that $1 < i \leq n$ satisfying $\gcd(s_{i-1}, s_i) = 1$. To give such a classification, we will analyze the structure of $\text{cone}(\mathbf{P}_n^{(\mathbf{s})})$. The following lemma gives a half-space inequality description of this cone:

LEMMA 5.3.1. With the notation from above, we have

$$\text{cone}(\mathbf{P}_n^{(\mathbf{s})}) = \{\boldsymbol{\lambda} \in \mathbb{R}^{n+1} : A\boldsymbol{\lambda} \geq \mathbf{0}\},$$

where

$$A = \begin{pmatrix} \frac{1}{s_1} & 0 & 0 & \dots & 0 \\ \frac{-1}{s_1} & \frac{1}{s_2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \frac{-1}{s_{n-1}} & \frac{1}{s_n} & 0 \\ 0 & \dots & 0 & \frac{-1}{s_n} & 1 \end{pmatrix}.$$

Moreover, this cone is simplicial.

Proof. This follows directly from the half space description of $\mathbf{P}_n^{(\mathbf{s})}$. Assume that $\mathbf{P}_n^{(\mathbf{s})} = \{\tilde{\boldsymbol{\lambda}} \in \mathbb{R}^n : M\tilde{\boldsymbol{\lambda}} \geq \mathbf{b}\}$, where $\mathbf{b} = (0, 0, \dots, 0, 1)^t$. Then on height λ_{n+1} , we have $M\tilde{\boldsymbol{\lambda}} \geq \lambda_{n+1}\mathbf{b}$. The statement now follows. \square

Equipped with the half-space description, we can now give a proof of Theorem 5.1.1.

Proof of Theorem 5.1.1. Let $\mathbf{s} = (s_1, s_2, \dots, s_n) \in \mathbb{Z}_{\geq 1}^n$. By Lemma 5.3.1, we can express cone $(\mathbf{P}_n^{(\mathbf{s})})$ as

$$\text{cone}(\mathbf{P}_n^{(\mathbf{s})}) = \left\{ \boldsymbol{\lambda} \in \mathbb{R}^{n+1} : 0 \leq \frac{\lambda_1}{s_1} \leq \dots \leq \dots \leq \frac{\lambda_n}{s_n} \leq \lambda_{n+1} \right\}.$$

Now one notices that cone $(\mathbf{P}_n^{(\mathbf{s})}) = \mathcal{C}_{n+1}^{(s_1, \dots, s_n, 1)}$, or in other words, that the cone over $\mathbf{P}_n^{(\mathbf{s})}$ equals the $\tilde{\mathbf{s}}$ -lecture hall cone, where $\tilde{\mathbf{s}} := (s_1, s_2, \dots, s_n, 1)$. By Theorem 5.2.3, this $\tilde{\mathbf{s}}$ -lecture hall cone is Gorenstein if and only if there exist $\mathbf{c} \in \mathbb{Z}^{n+1}$ satisfying

$$c_j s_{j-1} = c_{j-1} s_j + \gcd(s_{j-1}, s_j)$$

for $j > 1$ and

$$c_{n+1} s_n = 1 + c_n.$$

with $c_1 = 1$. □

We now recall a technical lemma which we will use in our characterization.

LEMMA 5.3.2 ([BBK⁺15, Lemma 2.5]). Let $\mathcal{C} = \{\boldsymbol{\lambda} \in \mathbb{R}^{n+1} : A\boldsymbol{\lambda} \geq \mathbf{0}\}$ be a full dimensional simplicial polyhedral cone where A is a rational matrix and denote the rows of A as linear functionals $\alpha^1, \dots, \alpha^{n+1}$ on \mathbb{R}^{n+1} . For $j = 1, \dots, n+1$, let the projected lattice $\alpha^j(\mathbb{Z}^{n+1}) \subset \mathbb{R}$ be generated by the number $q_j \in \mathbb{Q}_{>0}$, so $\alpha^j(\mathbb{Z}^{n+1}) = q_j \mathbb{Z}$.

1. The \mathcal{C} is Gorenstein if and only if there exists $\mathbf{c} \in \mathbb{Z}^{n+1}$ such that $\alpha^j(\mathbf{c}) = q_j$ for all $j = 1, \dots, n+1$.
2. Define a point $\tilde{\mathbf{c}} \in \mathcal{C} \cap \mathbb{Q}^{n+1}$ by $\alpha^j(\tilde{\mathbf{c}}) = q_j$ for all $j = 1, \dots, n+1$. Then \mathcal{C} is Gorenstein if and only if $\tilde{\mathbf{c}} \in \mathbb{Z}^{n+1}$ in which case we say that $\tilde{\mathbf{c}}$ is a *Gorenstein point*.

We can now prove our next result.

Proof of Theorem 5.1.2. Let $\mathbf{P}_n^{(\mathbf{s})}$ be Gorenstein with Gorenstein point $\mathbf{c} \in \text{cone}(\mathbf{P}_n^{(\mathbf{s})})$. Then \mathbf{c} has lattice distance 1 to all facets of cone $(\mathbf{P}_n^{(\mathbf{s})})$ by [BG09, Thm. 6.33]. For a vertex \mathbf{v} , the vertex cone $T_{\mathbf{v}}(\mathbf{P}_n^{(\mathbf{s})})$ of $\mathbf{P}_n^{(\mathbf{s})}$ at \mathbf{v} is obtained from cone $(\mathbf{P}_n^{(\mathbf{s})})$ by quotienting cone $(\mathbf{P}_n^{(\mathbf{s})})$ by $(\mathbf{v}, 1)$. The image of \mathbf{c} under this quotient map can be seen to be the Gorenstein point of $T_{\mathbf{v}}(\mathbf{P}_n^{(\mathbf{s})})$, since this point has lattice distance 1 to all facets of $T_{\mathbf{v}}(\mathbf{P}_n^{(\mathbf{s})})$ showing that all vertex cones are Gorenstein as well, again by [BG09, Thm. 6.33].

In particular, the vertex cone at the vertex $(0, 0, \dots, 0)$ is of the form

$$0 \leq \frac{\lambda_1}{s_1} \leq \dots \leq \frac{\lambda_n}{s_n}$$

and it is known by Theorem 5.2.3 that this cone is Gorenstein if and only if there exists a $\mathbf{c} \in \mathbb{Z}^n$ satisfying the recurrence above. Likewise, it is straightforward to see that $T_{\mathbf{s}}(\mathbf{P}_n^{(\mathbf{s})}) \cong T_{\mathbf{0}}(\mathbf{P}_n^{(\overleftarrow{\mathbf{s}})})$, where \cong means equivalence after an affine, unimodular transformation, and where $T_{\mathbf{0}}(\mathbf{P}_n^{(\overleftarrow{\mathbf{s}})})$ is of the form

$$0 \leq \frac{\lambda_1}{\overleftarrow{s}_1} \leq \cdots \leq \frac{\lambda_n}{\overleftarrow{s}_n}$$

which is Gorenstein if and only if there exists a $\mathbf{d} \in \mathbb{Z}^n$ satisfying the recurrence above. Therefore, this is certainly a necessary condition.

To show the other direction we employ Lemma 5.3.2. Since the characterization given in Lemma 5.3.2 essentially requires finding integer solutions to linear equations, we first deduce some divisibility conditions that will later prove useful.

Assume that we have $\mathbf{c}, \mathbf{d} \in \mathbb{Z}^n$ as described above and suppose that $\gcd(s_i, s_{i+1}) = 1$. Note that this gives us the following

$$c_n s_{n-1} = c_{n-1} s_n + \gcd(s_{n-1}, s_n)$$

and

$$d_2 \overleftarrow{s}_1 = d_1 \overleftarrow{s}_2 + \gcd(\overleftarrow{s}_1, \overleftarrow{s}_2) \iff d_2 s_n = d_1 s_{n-1} + \gcd(s_{n-1}, s_n)$$

where $d_1 = 1$. Subtracting both equalities, we get

$$(d_2 + c_{n-1})s_n = (1 + c_n)s_{n-1}.$$

Repeating the above process, we also have

$$(d_3 + c_{n-2})s_{n-1} = (d_2 + c_{n-1})s_{n-2}$$

and in general for some k , we have

$$(d_{k+1} + c_{n-k})s_{n-k+1} = (d_k + c_{n-k+1})s_{n-k}. \quad (5.1)$$

If we know that $i = n - k$, then $\gcd(s_{n-k}, s_{n-k+1}) = 1$ and we can deduce the division requirement $s_{n-k+1} | (d_k + c_{n-k+1})$.

By Lemma 5.3.2, we get that a cone of the form $A\lambda \geq 0$ is Gorenstein if and only if there is a point c such that $\alpha^i(c) = q_i$ for all i , where α^i is the i^{th} row of A and q_i is defined as in Lemma 5.3.2. Lemma 5.3.1 explicitly describes the rows. From this it follows that

$$q_1 = \frac{1}{s_1}, q_2 = \frac{1}{\text{lcm}(s_1, s_2)}, \dots, q_n = \frac{1}{\text{lcm}(s_{n-1}, s_{n-2})}, q_{n+1} = \frac{1}{s_n}.$$

Now we need to find a point $\mathbf{c} \in \mathbb{Z}^{n+1}$ such that $\alpha^i(c) = q_i$. This directly implies that $c_1 = 1$ and that

$$c_j s_{j-1} = c_{j-1} s_j + \gcd(s_{j-1}, s_j)$$

for $2 \leq j \leq n$. These conditions are all satisfied by assumption. However, we also need to satisfy the condition

$$\frac{-c_n}{s_n} + 1 \cdot c_{n+1} = \frac{1}{s_n} \iff c_{n+1}s_n = 1 + c_n.$$

Now, we note that from Equation (5.1) it follows that

$$s_n = \frac{(1 + c_n)}{(d_2 + c_{n-1})} s_{n-1},$$

so we can rewrite

$$c_{n+1}s_{n-1} = d_2 + c_{n-1}.$$

We can iterate these substitutions repeatedly to arrive at the equality

$$c_{n+1}s_{n-k+1} = d_k + c_{n-k+1}$$

However, since $s_{n-k+1} | (d_k + c_{n-k+1})$, c_{n+1} is an integer. Here we are implicitly using that $\mathbf{c}, \mathbf{d} \in \mathbb{Z}_{\geq 1}^n$, which follows from the recursive definition. So we are done. This also shows that $s_{n-j+1} | (d_j + c_{n-j+1})$ for all j . \square

Remark 5.3.3. We mentioned before that this theorem applies to a large subfamily of \mathbf{s} -lecture hall polytopes. This remark will make this statement more precise. Given two positive integers a and b , the probability that $\gcd(a, b) = 1$ converges to $\frac{1}{\zeta(2)} = \frac{6}{\pi^2}$, where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is the *Riemann ζ -function*. Heuristically, assuming that these events are independent (which they are not), we get that roughly $(1 - \frac{6}{\pi^2})^{n-1}$ -percent of sequences fall within the range of our theorem. Computer simulations suggest that this estimate is fairly precise for large dimensions.

Remark 5.3.4. In [BBK⁺15, Cor 2.7], the authors remark that if one truncates a sequences \mathbf{s} with corresponding $\mathbf{c} = (c_1, c_2, \dots, c_n)$, the truncated sequence (s_1, s_2, \dots, s_i) also has a corresponding point (c_1, c_2, \dots, c_i) . However, the direct analogue of this statement is not true in our case. The sequence $(8, 6, 10, 10, 5, 2, 4)$ gives rise to a Gorenstein lecture hall polytope, whereas $(8, 6, 10, 10, 5)$ does *not* give rise to a Gorenstein lecture hall polytope, since it has 39 interior lattice points. However, we can make the following statement: If (s_1, s_2, \dots, s_n) has corresponding integer points \mathbf{c}, \mathbf{d} , then the truncated sequence (s_1, s_2, \dots, s_i) has corresponding integer points $\tilde{\mathbf{c}} = (c_1, c_2, \dots, c_i)$, $\tilde{\mathbf{d}} = (d_{n-i+1}, \dots, d_n)$ provided $d_{n-i+1} = 1$.

This theorem along with Theorem 5.2.4 implies the following more specialized characterization.

COROLLARY 5.3.5. Let $\mathbf{s} = (s_1, s_2, \dots, s_n)$ be a sequence of positive integers satisfying $\gcd(s_i, s_{i+1}) = 1$ for all $1 \leq i < n$. Then $\mathbf{P}_n^{(\mathbf{s})}$ is Gorenstein if and only if \mathbf{s} and $\overleftarrow{\mathbf{s}}$ are \mathbf{u} -generated sequences.

We have the following corollary on the level of \mathbf{s} -Eulerian polynomials

COROLLARY 5.3.6. Let $\mathbf{s} = (s_1, s_2, \dots, s_n)$ be a sequence of positive integers. The \mathbf{s} -Eulerian polynomial is palindromic if and only if there exists a $\mathbf{c} \in \mathbb{Z}^{n+1}$ satisfying

$$c_j s_{j-1} = c_{j-1} s_j + \gcd(s_{j-1}, s_j)$$

for $j > 1$ and

$$c_{n+1} s_n = 1 + c_n.$$

with $c_1 = 1$.

Table 5.1 contains some examples of palindromic \mathbf{s} -Eulerian polynomials.

5.4 CHARACTERIZATION OF LEVEL LECTURE HALL POLYTOPES

We now give a characterization of which \mathbf{s} -sequences admit level $\mathbf{P}_n^{(\mathbf{s})}$, which is given in terms of the structure of \mathbf{s} -inversion sequences. It will be useful to define some new notation. Let $\mathbf{I}_{n,k}^{(\mathbf{s})} := \left\{ \mathbf{e} \in \mathbf{I}_n^{(\mathbf{s})} : \text{asc}(\mathbf{e}) = k \right\}$. Moreover, we define addition of inversion sequences as follows. Given $\mathbf{e}, \mathbf{e}' \in \mathbf{I}_n^{(\mathbf{s})}$ with $\mathbf{e} = (e_1, e_2, \dots, e_n)$ and $\mathbf{e}' = (e'_1, e'_2, \dots, e'_n)$, then $\mathbf{e} + \mathbf{e}' = (e_1 + e'_1 \bmod s_1, e_2 + e'_2 \bmod s_2, \dots, e_n + e'_n \bmod s_n)$.

5.4.1 Proof of characterization

Proving this characterization relies on understanding the link between the arithmetic structure of inversion sequences and the semigroup structure of lattice points in $\Pi_{\mathbf{P}_n^{(\mathbf{s})}}$. To fully understand and exploit this connection, we will need several lemmas.

LEMMA 5.4.1. Let $V(\mathbf{P}_n^{(\mathbf{s})}) = \{\mathbf{v}_0, \dots, \mathbf{v}_n\}$ denote the set of vertices of $\mathbf{P}_n^{(\mathbf{s})}$. Let $\mathcal{P}_n^{(\mathbf{s})} := (\mathbf{P}_n^{(\mathbf{s})} \cap \mathbb{Z}^n) - V(\mathbf{P}_n^{(\mathbf{s})})$. There is an explicit bijection

$$\varphi : \mathcal{P}_n^{(\mathbf{s})} \longrightarrow \mathbf{I}_{n,1}^{(\mathbf{s})}$$

where $\varphi(\lambda_1, \dots, \lambda_n) \mapsto (e_1, \dots, e_n)$ by $e_i = s_i - \lambda_i \pmod{s_i}$.

Proof. Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathcal{P}_n^{(\mathbf{s})}$. We have that

$$0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \dots \leq \frac{\lambda_n}{s_n} \leq 1$$

Note that this means that $\lambda_i \leq s_i$ for all i and if $\lambda_i = s_i$ then $\lambda_j = s_j$ for all $i \leq j \leq n$.

Additionally, note that the vertices of $\mathbf{P}_n^{(\mathbf{s})}$ are precisely the lattice points of the form

$$(0, \dots, 0, s_i, s_{i+1}, \dots, s_n)$$

So, then $\boldsymbol{\lambda}$ can be expressed as the following:

$$\boldsymbol{\lambda} = (0, \dots, 0, a_i, a_{i+1}, \dots, a_j, s_{j+1}, \dots, s_n)$$

| | sequence \mathbf{s} | corresponding \mathbf{c} | corresponding \mathbf{d} | \mathbf{s} -Eulerian polynomial |
|--------|-----------------------|----------------------------|----------------------------|---|
| (i) | (2, 1, 3, 2, 1) | (1, 1, 4, 3, 2) | (1, 3, 5, 2, 5) | $1 + 5z + 5z^2 + z^3$ |
| (ii) | (3, 2, 3, 1, 2) | (1, 1, 2, 1, 3) | (1, 1, 4, 3, 5) | $1 + 9z + 16z^2 + 9z^3 + z^4$ |
| (iii) | (1, 4, 3, 2, 3) | (1, 5, 4, 3, 5) | (1, 1, 2, 3, 1) | $1 + 16z + 38z^2 + 16z^3 + z^4$ |
| (iv) | (3, 5, 2, 3, 1) | (1, 2, 1, 2, 1) | (1, 4, 3, 8, 5) | $1 + 20z + 48z^2 + 20z^3 + z^4$ |
| (v) | (1, 2, 3, 4, 5) | (1, 3, 5, 7, 9) | (1, 1, 1, 1, 1) | $1 + 26z + 66z^2 + 26z^3 + z^4$ |
| (vi) | (1, 2, 5, 8, 3) | (1, 3, 8, 13, 5) | (1, 3, 2, 1, 1) | $1 + 50z + 138z^2 + 50z^3 + z^4$ |
| (vii) | (4, 3, 2, 5, 3) | (1, 1, 1, 3, 2) | (1, 2, 1, 2, 3) | $1 + 30z + 149z^2 + 149z^3 + 30z^4 + z^5$ |
| (viii) | (4, 7, 3, 2, 3) | (1, 2, 1, 1, 2) | (1, 1, 2, 5, 3) | $1 + 43z + 208z^2 + 208z^3 + 43z^4 + z^5$ |
| (ix) | (5, 9, 4, 3, 2) | (1, 2, 1, 1, 1) | (1, 2, 3, 7, 6) | $1 + 82z + 457z^2 + 457z^3 + 82z^4 + z^5$ |
| (x) | (3, 5, 12, 7, 2) | (1, 2, 5, 3, 1) | (1, 4, 7, 3, 2) | $1 + 175z + 1084z^2 + 1084z^3 + 175z^4 + z^5$ |
| (xi) | (3, 11, 8, 5, 2) | (1, 4, 3, 2, 1) | (1, 3, 5, 7, 2) | $1 + 180z + 1139z^2 + 1139z^3 + 180z^4 + z^5$ |
| (xii) | (2, 7, 5, 10, 4) | (1, 4, 3, 7, 3) | (1, 3, 2, 3, 1) | $1 + 181z + 1218z^2 + 1218z^3 + 181z^4 + z^5$ |
| (xiii) | (3, 8, 13, 5, 2) | (1, 3, 5, 2, 1) | (1, 3, 8, 5, 2) | $1 + 213z + 1346z^2 + 1346z^3 + 213z^4 + z^5$ |

Table 5.1: Palindromic \mathbf{s} -Eulerian Polynomials

where each $0 < a_k < s_k$.

If we apply our map $\lambda_i \mapsto s_i - \lambda_i \pmod{s_i}$, we get the inversion sequence

$$\mathbf{e} = (0, 0, \dots, 0, s_i - a_i, s_{i+1} - a_{i+1}, \dots, s_j - a_j, 0, \dots, 0).$$

It is left to verify that $\mathbf{e} \in \mathbf{I}_{n,1}^{(s)}$. Since we had that

$$0 < \frac{a_i}{s_i} \leq \frac{a_{i+1}}{s_{i+1}} \leq \dots \leq \frac{a_j}{s_j} < 1$$

which holds if and only if

$$1 > \frac{s_i - a_i}{s_i} \geq \frac{s_{i+1} - a_{i+1}}{s_{i+1}} \geq \dots \geq \frac{s_j - a_j}{s_j} > 0$$

which means that \mathbf{e} contains exactly one ascent at position $i - 1$.

This process is certainly reversible, so we have a bijection. \square

Note that $\mathcal{P}_n^{(s)}$ is precisely the elements at height 1 in $\Pi_{\mathbf{P}_n^{(s)}}$. We can extend this bijection to apply to elements of $\mathbf{I}_{n,k}^{(s)}$ and all elements of $\mathbf{P}_n^{(s)}$ in the following manner.

LEMMA 5.4.2. Let $\mathbf{e} = (e_1, e_2, \dots, e_n) \in \mathbf{I}_{n,k}^{(s)}$ and suppose that the k ascents are at positions i_1, i_2, \dots, i_k . There is a bijective correspondence between \mathbf{e} and lattice points $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n, k) \in \Pi_{\mathbf{P}_n^{(s)}} \cap \mathbb{Z}^{n+1}$. Suppose that $i_\ell < j \leq i_{\ell+1}$, then we map $e_j \mapsto \lambda_j$ by

$$\lambda_j = \ell \cdot s_j - e_j \tag{5.2}$$

and $\lambda_j = 0$ if $1 \leq j \leq i_1$. Moreover, addition in the semigroup corresponds to entry-wise addition of the inversion sequences modulo s_i in the i th position. That is, any decomposition of $\boldsymbol{\lambda}$ as a sum of elements of height one in $\Pi_{\mathbf{P}_n^{(s)}}$ is consistent with the sum of inversion sequences.

Remark 5.4.3. By entry-wise addition of the inversion sequences modulo s_i in the i th position, we mean that we pick the unique representative of this equivalence class in $\{0, 1, \dots, s_i - 1\}$.

Proof of Lemma 5.4.2. It is clear that this map is injective. We must verify the following:

- (A) The image of $\boldsymbol{\lambda}$ under this map is an element of $\Pi_{\mathbf{P}_n^{(s)}}$.
- (B) Entry-wise addition of inversion sequences is consistent with addition in the semigroup.

To show (A), note that it is clear that $\boldsymbol{\lambda}$ is at height k in \mathbb{R}^{n+1} . Moreover, it follows that

$$(\lambda_1, \dots, \lambda_n) \in k \cdot \mathbf{P}_n^{(s)} \cap \mathbb{Z}^n, \tag{5.3}$$

as if $i_\ell < t < i_{\ell+1}$ then $\frac{e_t}{s_t} \geq \frac{e_{t+1}}{s_{t+1}}$ implies that

$$\frac{\ell \cdot s_t - e_t}{k \cdot s_t} \leq \frac{\ell \cdot s_{t+1} - e_{t+1}}{k \cdot s_{t+1}}.$$

Moreover, if $t = i_{\ell+1}$ we have

$$\frac{\ell \cdot s_t - e_t}{k \cdot s_t} \leq \frac{(\ell + 1) \cdot s_{t+1} - e_{t+1}}{k \cdot s_{t+1}}$$

immediate from $e_{t+1} < s_{t+1}$, establishing (5.3).

To verify that $\boldsymbol{\lambda}$ is in fact in $\Pi_{\mathbf{P}_n^{(s)}}$, we must show that neither of the following hold:

- (i) $(\lambda_1, \dots, \lambda_n) \in (k-1) \cdot \mathbf{P}_n^{(s)} \cap \mathbb{Z}^n$
- (ii) $\boldsymbol{\lambda} = \boldsymbol{\lambda}' + \mathbf{v}$ where $(\lambda'_1, \dots, \lambda'_n) \in (k-1) \cdot \mathbf{P}_n^{(s)} \cap \mathbb{Z}^n$ and \mathbf{v} is a vertex of $\mathbf{P}_n^{(s)}$.

Note that (i) is impossible as we have $\lambda_n = k \cdot s_n - e_n > (k-1)s_n$ as $e_n < s_n$. For (ii), suppose that we write $\boldsymbol{\lambda} = \boldsymbol{\lambda}' + \mathbf{v}$, where $\mathbf{v} = (0, 0, \dots, 0, s_{j+1}, \dots, s_n)$ with $0 \leq j < n$. There are two possible cases: $j \in \text{Asc}(\mathbf{e})$ or $j \notin \text{Asc}(\mathbf{e})$.

If $j \in \text{Asc}(\mathbf{e})$, then we have $\frac{e_j}{s_j} < \frac{e_{j+1}}{s_{j+1}}$. Consider $\boldsymbol{\lambda}'$ and suppose that

$$(\lambda'_1, \dots, \lambda'_n) = (\lambda_1, \dots, \lambda_j, \lambda_{j+1} - s_{j+1}, \dots, \lambda_n - s_n) \in (k-1) \cdot \mathbf{P}_n^{(s)} \cap \mathbb{Z}^n$$

Given that $\lambda_j = (p-1) \cdot s_j - e_j$ and $\lambda_{j+1} = p \cdot s_{j+1} - e_{j+1}$ where j is the p th ascent, we have that the following inequality must hold:

$$\frac{(p-1) \cdot s_j - e_j}{(k-1)s_j} \leq \frac{p \cdot s_{j+1} - e_{j+1} - s_{j+1}}{(k-1)s_{j+1}} = \frac{(p-1) \cdot s_{j+1} - e_{j+1}}{(k-1)s_{j+1}}.$$

However, this is equivalent to $\frac{e_j}{s_j} \geq \frac{e_{j+1}}{s_{j+1}}$ so this cannot occur.

If $j \notin \text{Asc}(\mathbf{e})$, say that $j > i_p$, the location of the p th ascent, so $\lambda_j = p \cdot s_j - e_j$ and $\lambda_{j+1} = p \cdot s_{j+1} - e_{j+1}$. For $(\lambda'_1, \dots, \lambda'_n) \in (k-1) \cdot \mathbf{P}_n^{(s)} \cap \mathbb{Z}^n$, the following inequality must hold

$$\frac{p \cdot s_j - e_j}{(k-1)s_j} \leq \frac{p \cdot s_{j+1} - e_{j+1} - s_{j+1}}{(k-1)s_{j+1}} = \frac{(p-1) \cdot s_{j+1} - e_{j+1}}{(k-1)s_{j+1}}.$$

This inequality is equivalent to $\frac{e_j}{s_j} \geq \frac{e_{j+1}}{s_{j+1}} + 1$ which is a contradiction to $e_j < s_j$.

Therefore, we have shown (A). This is sufficient for showing the bijection, as the map is clearly injective and the sets are of the same cardinality by previous work of Savage and Schuster [SS12]. That said, the bijection can also be realized explicitly by reversing the map. In particular, suppose that $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n, k) \in \Pi_{\mathbf{P}_n^{(s)}}$, we get our inversion sequence \mathbf{e} by

$$e_i = -\lambda_i \pmod{s_i}.$$

Note that this inversion sequence will have precisely k ascents and moreover the p th ascent in the sequence will occur at i precisely when $(p-1) \cdot s_i \leq \lambda_i < p \cdot s_i$ and $p \cdot s_{i+1} \leq \lambda_{i+1} < (p+1) \cdot s_{i+1}$ for some $1 \leq p \leq k-1$. This is the exact reversal of the constructive map (5.2) from inversion sequences to lattice points is $\Pi_{\mathbf{P}_n^{(s)}}$.

To show (B), suppose that we have $\mathbf{f} \in \mathbf{I}_{n,k-1}^{(s)}$ and $\mathbf{g} \in \mathbf{I}_{n,1}^{(s)}$ such that $\mathbf{f} + \mathbf{g} = \mathbf{e} \in \mathbf{I}_{n,k}^{(s)}$. So we have

$$\mathbf{f} = (f_1, \dots, f_{j-1}, f_j, \dots, f_h, f_{h+1}, \dots, f_n)$$

and

$$\mathbf{g} = (0, \dots, 0, g_j, \dots, g_h, 0, \dots, 0)$$

and

$$\mathbf{e} = (f_1, \dots, f_{j-1}, (f_j + g_j) \bmod s_j, \dots, (f_h + g_h) \bmod s_h, f_{h+1}, \dots, f_n).$$

Consider the corresponding lattice points for \mathbf{f} and \mathbf{g} in $\Pi_{\mathbf{P}_n^{(s)}}$:

$$\boldsymbol{\lambda}_{\mathbf{f}} = (\lambda_{f_1}, \dots, \lambda_{f_{j-1}}, \lambda_{f_j}, \dots, \lambda_{f_h}, \lambda_{f_{h+1}}, \dots, \lambda_{f_n}, k-1)$$

and

$$\boldsymbol{\lambda}_{\mathbf{g}} = (0, \dots, 0, \lambda_{g_j}, \dots, \lambda_{g_h}, s_{h+1}, \dots, s_n, 1).$$

Adding these lattice points in the semigroup yields

$$\boldsymbol{\lambda}_{\mathbf{f}} + \boldsymbol{\lambda}_{\mathbf{g}} = (\lambda_{f_1}, \dots, \lambda_{f_{j-1}}, \lambda_{f_j} + \lambda_{g_j}, \dots, \lambda_{f_h} + \lambda_{g_h}, \lambda_{f_{h+1}} + s_{h+1}, \dots, \lambda_{f_n} + s_n, k).$$

We have two possible cases: either $\boldsymbol{\lambda}_{\mathbf{f}} + \boldsymbol{\lambda}_{\mathbf{g}} \in \Pi_{\mathbf{P}_n^{(s)}}$ or $\boldsymbol{\lambda}_{\mathbf{f}} + \boldsymbol{\lambda}_{\mathbf{g}} \notin \Pi_{\mathbf{P}_n^{(s)}}$.

If $\boldsymbol{\lambda}_{\mathbf{f}} + \boldsymbol{\lambda}_{\mathbf{g}} \in \Pi_{\mathbf{P}_n^{(s)}}$, we consider the reverse map which will give the inversion sequence

$$(\dots, \lambda_{f_{j-1}} \bmod s_{j-1}, -(\lambda_{f_j} + \lambda_{g_j}) \bmod s_j, \dots, -(\lambda_{f_h} + \lambda_{g_h}) \bmod s_h, -(\lambda_{f_{h+1}} + s_{h+1}) \bmod s_{h+1}, \dots)$$

and this inversion sequence is precisely $\mathbf{e} = \mathbf{f} + \mathbf{g}$, which is as desired.

Now suppose for contradiction that $\boldsymbol{\lambda}_{\mathbf{f}} + \boldsymbol{\lambda}_{\mathbf{g}} \notin \Pi_{\mathbf{P}_n^{(s)}}$. Note that we can express $\boldsymbol{\lambda}_{\mathbf{f}} + \boldsymbol{\lambda}_{\mathbf{g}} = \boldsymbol{\lambda}' + \sum_{i=1}^n \alpha_i v_i$ where $\boldsymbol{\lambda}' \in \Pi_{\mathbf{P}_n^{(s)}}$, with at least one $\alpha_i \neq 0$, $\alpha_i \in \mathbb{Z}_{\geq 1}$, and $\boldsymbol{\lambda}'$ is at height $r < k$. Additionally, given that $v_i = (0, \dots, 0, s_i, s_{i+1}, \dots, s_n)$, it is clear that $\boldsymbol{\lambda}_{\mathbf{f}} + \boldsymbol{\lambda}_{\mathbf{g}}$ maps to the same inversion sequence as $\boldsymbol{\lambda}'$ by definition of the inverse map. This implies that \mathbf{e} maps to $\boldsymbol{\lambda}'$ and thus $\mathbf{e} \in \mathbf{I}_{n,r}^{(s)}$ for $r < k$, which contradicts our initial assumption. \square

Remark 5.4.4. We should note that in the proof that addition is compatible, we only consider inversion sequences $\mathbf{f} \in \mathbf{I}_{n,k-1}^{(s)}$ and $\mathbf{g} \in \mathbf{I}_{n,1}^{(s)}$ such that $\mathbf{f} + \mathbf{g} \in \mathbf{I}_{n,k}^{(s)}$, as this is the requirement of staying inside the fundamental parallelepiped. However, this need not always be the case. If $\mathbf{f} + \mathbf{g} \in \mathbf{I}_{n,\ell}^{(s)}$ for some $\ell \leq k-1$, the addition of the sequences is still consistent with addition in the semigroup, but this occurrence is precisely when $\boldsymbol{\lambda}_{\mathbf{f}} + \boldsymbol{\lambda}_{\mathbf{g}} \notin \Pi_{\mathbf{P}_n^{(s)}}$. In particular, $\boldsymbol{\lambda}_{\mathbf{f}} + \boldsymbol{\lambda}_{\mathbf{g}} = \boldsymbol{\lambda}_{\mathbf{f}+\mathbf{g}} + (0, \dots, 0, k-\ell)$, which lies in the equivalence class $\boldsymbol{\lambda}_{\mathbf{f}+\mathbf{g}}$, but is not the representative in $\Pi_{\mathbf{P}_n^{(s)}}$.

Remark 5.4.5. One could rephrase the statement to say that addition of inversion sequences is compatible with addition of lattice points in the semigroup modulo the equivalence class given by the fundamental parallelepiped.

With this understanding of the arithmetic structure of $\mathbf{I}_n^{(s)}$, we can now give a proof of the characterization.

Proof of Theorem 5.1.4. First recall that if \mathcal{R} is a graded, *local, Cohen-Macaulay algebra with $\dim(\mathcal{R}) = d$, then \mathcal{R} is level if for some homogeneous system of parameters x_1, \dots, x_d of \mathcal{R} , all the elements of the graded vector space $\text{soc}(\mathcal{R}/(x_1, \dots, x_d))$ are of the same degree. Consider the semigroup algebra $\mathbb{k}[\mathbf{P}_n^{(s)}] := \mathbb{k}[\text{cone}(\mathbf{P}_n^{(s)}) \cap \mathbb{Z}^{n+1}]$. Notice that $\mathbf{P}_n^{(s)}$ is a simplex and let $\Pi_{\mathbf{P}_n^{(s)}}$ denote the (half-open) fundamental parallelepiped. Note that $\dim(\mathbb{k}[\mathbf{P}_n^{(s)}]) = n + 1$ and $\mathbb{k}[\mathbf{P}_n^{(s)}]$ has a natural homogeneous system of parameters, namely the monomials corresponding to the vertices, which we denote by x_0, x_1, \dots, x_n . The quotient $\mathbb{k}[\mathbf{P}_n^{(s)}]/(x_0, \dots, x_n)$ is precisely the equivalence classes of lattice point each in $\Pi_{\mathbf{P}_n^{(s)}}$. Let $m_1, \dots, m_\alpha \in \Pi_{\mathbf{P}_n^{(s)}}$ be the elements at height 1. The socle $\text{soc}(\mathbb{k}[\mathbf{P}_n^{(s)}]/(x_0, \dots, x_n))$ are precisely the lattice points in $\lambda \in \Pi_{\mathbf{P}_n^{(s)}}$ such that $\lambda + m_i \notin \Pi_{\mathbf{P}_n^{(s)}}$ for all m_i by Lemma 5.4.1 and Theorem 5.2.2. By Lemma 5.4.2, we know that semigroup addition corresponds to entry-wise addition on inversion sequences. Subsequently, this condition on inversion sequences is precisely the condition that only elements of highest degree in $\Pi_{\mathbf{P}_n^{(s)}}$ are in $\text{soc}(\mathbb{k}[\mathbf{P}_n^{(s)}]/(x_0, \dots, x_n))$, which then must contain elements which are all the same degree. □

5.4.2 Consequences of the characterization

First consider the following resulting inequalities given for the coefficients of \mathbf{s} -Eulerian polynomials.

COROLLARY 5.4.6. Let $\mathbf{s} = (s_1, s_2, \dots, s_n) \in \mathbb{Z}_{\geq 1}^n$ be a sequence such that $\mathbf{P}_n^{(s)}$ is level. Then the coefficients of the \mathbf{s} -Eulerian polynomial $h_{\mathbf{P}_n^{(s)}}^*(z) = 1 + h_1^*z + \dots + h_r^*z^r$ satisfies the the inequalities $h_i^* \leq h_j^*h_{i+j}^*$ for all pairs i and j such that $h_{i+j}^* > 0$.

These inequalities follow from [Sta96, Chapter III. Proposition 3.3] and provide additional information of the behavior of \mathbf{s} -Eulerian polynomials to complement the known log-concave inequalities from [SV15]. It is worth noting that these inequalities need not be satisfied for arbitrary \mathbf{s} . For example, the sequence $\mathbf{s} = (2, 3, 5, 9)$ does not give rise to a level polytope as there exists no element $\mathbf{f} \in \mathbf{I}_{4,1}^{(2,3,5,9)}$ such that $\mathbf{f} + \mathbf{e} \in \mathbf{I}_{4,4}^{(2,3,5,9)}$ for the inversion sequence $\mathbf{e} = (1, 1, 2, 4) \in \mathbf{I}_{4,3}^{(2,3,5,9)}$. Moreover, we have

$$h_{\mathbf{P}_4}^{*(2,3,5,9)}(z) = 1 + 48z + 154z^2 + 66z^3 + z^4$$

and we notice that $h_3^* > h_1^*h_4^*$.

In addition to the characterization of Gorenstein given in Section 5.3, we can also arrive at a different characterization by considering the following restriction of Theorem 5.1.4.

COROLLARY 5.4.7. Let $\mathbf{s} = (s_1, s_2, \dots, s_d) \in \mathbb{Z}_{\geq 1}^n$ and let $r = \max \{ \text{asc}(\mathbf{e}) : \mathbf{e} \in \mathbf{I}_n^{(\mathbf{s})} \}$. Then $\mathbf{P}_n^{(\mathbf{s})}$ is Gorenstein if and only if for any $\mathbf{e} \in \mathbf{I}_{n,k}^{(\mathbf{s})}$ with $1 \leq k < r$ there exists some $\mathbf{e}' \in \mathbf{I}_{n,1}^{(\mathbf{s})}$ such that $(\mathbf{e} + \mathbf{e}') \in \mathbf{I}_{n,k+1}^{(\mathbf{s})}$ and $|\mathbf{I}_{n,r}| = 1$.

Proof. $\mathbf{P}_n^{(\mathbf{s})}$ is Gorenstein if and only if $\mathbf{P}_n^{(\mathbf{s})}$ is level with exactly one canonical module generator. The canonical module of $\mathbb{k}[\mathbf{P}_n^{(\mathbf{s})}]$ for $\mathbf{P}_n^{(\mathbf{s})}$ level has $|\mathbf{I}_{n,r}^{(\mathbf{s})}|$ generators, as this is the leading coefficient of the h^* polynomial of $\mathbf{P}_n^{(\mathbf{s})}$. \square

We should note that in general Corollary 5.4.7 is less computationally useful than Theorem 5.1.2 or Theorem 5.1.1. However, it is unexpected, and indeed striking, that these conditions are equivalent when there exists an index i such that $\gcd(s_{i-1}, s_i) = 1$.

In the case of $\mathbf{s} \in \mathbb{Z}_{\geq 1}^2$, the conditions of Theorem 5.1.4 must always be satisfied. Therefore, we have the following result.

COROLLARY 5.4.8. The lecture hall polytope $\mathbf{P}_2^{(s_1, s_2)}$ is level for any $\mathbf{s} = (s_1, s_2)$.

Remark 5.4.9. By [HY18, Prop. 1.2], every lattice polygon is level. We state this result and its proof only to show how one can explicitly use Theorem 5.1.4 to determine levelness, especially in small dimensions.

Proof. Without loss of generality, suppose that $s_1 \leq s_2$. First note that if $\mathbf{I}_{2,2}^{(s_1, s_2)} = \emptyset$, then levelness is trivial. This trivial case consists of sequences $\mathbf{s} = (1, s_2)$ and $\mathbf{s} = (2, 2)$, which can be concretely shown to not have any inversion sequence with 2 ascents.

So, if $\mathbf{I}_{2,2}^{(s_1, s_2)} \neq \emptyset$, we have two cases either

- (i) $2 \leq s_1 < s_2$, or
- (ii) $3 \leq s_1 = s_2$.

In both cases, given $\mathbf{e} \in \mathbf{I}_{2,1}^{(s_1, s_2)}$ we will explicitly construct $\mathbf{f} \in \mathbf{I}_{2,1}^{(s_1, s_2)}$ such that $\mathbf{e} + \mathbf{f} \in \mathbf{I}_{2,2}^{(s_1, s_2)}$.

In case (i), take $\mathbf{e} = (e_1, e_2) \in \mathbf{I}_{2,1}^{(s_1, s_2)}$. We have three possible subcases:

- Suppose that $e_1 \geq 1$ and $e_2 = 0$. Let $\mathbf{f} = (f_1, f_2)$ where $f_1 = 0$ and $f_2 = s_2 - 1$. Then $\mathbf{e} + \mathbf{f} = (e_1, s_2 - 1) \in \mathbf{I}_{2,2}^{(s_1, s_2)}$ because

$$\frac{e_1}{s_1} \leq \frac{s_1 - 1}{s_1} < \frac{s_2 - 1}{s_2}.$$

- Suppose that $e_1 \geq 1$ and $e_2 \geq 1$. Note that $\mathbf{e} \in \mathbf{I}_{2,1}^{(s_1, s_2)}$ implies that $e_2 < s_2 - 1$ because we have

$$0 < \frac{e_1}{s_1} \leq \frac{s_1 - 1}{s_1} < \frac{s_2 - 1}{s_2}$$

So, if $e_2 = s_2 - 1$, then we will have that $\text{Asc}((e_1, e_2)) = \{0, 1\}$ which contradicts that $\mathbf{e} \in \mathbf{I}_{2,1}^{(s)}$. Let $\mathbf{f} = (0, s_2 - e_2 - 1)$. Then we have $\mathbf{e} + \mathbf{f} = (e_1, s_2 - 1) \in \mathbf{I}_{2,2}^{(s_1, s_2)}$ as shown above.

- Suppose that $e_1 = 0$ and $e_2 \geq 1$. Let $\mathbf{f} = (1, \min\{\lfloor \frac{s_2}{s_1} \rfloor, s_2 - e_2 - 1\}) \in \mathbf{I}_{2,1}^{(s_1, s_2)}$. Then, we have that

$$\mathbf{e} + \mathbf{f} = \begin{cases} (1, s_2 - 1) & \text{if } s_2 - e_2 - 1 \leq \lfloor \frac{s_2}{s_1} \rfloor \\ (1, e_2 + \lfloor \frac{s_2}{s_1} \rfloor) & \text{if } \lfloor \frac{s_2}{s_1} \rfloor < s_2 - e_2 - 1 \end{cases}$$

If the first case is true, then clearly $\mathbf{e} + \mathbf{f} \in \mathbf{I}_{2,2}^{(s_1, s_2)}$ by previous arguments. In the second case, notice that

$$\frac{1}{s_1} < \frac{\lfloor \frac{s_2}{s_1} \rfloor + 1}{s_2} \leq \frac{\lfloor \frac{s_2}{s_1} \rfloor + e_2}{s_2}$$

and hence $\mathbf{e} + \mathbf{f} \in \mathbf{I}_{2,2}^{(s_1, s_2)}$.

Now for case (ii), take $\mathbf{e} = (e_1, e_2) \in \mathbf{I}_{2,1}^{(s_1, s_2)}$. We have several possible subcases:

- Suppose that $e_1 = 0$ and $e_2 \geq 1$. If $e_2 > 1$, let $\mathbf{f} = (1, 0) \in \mathbf{I}_{2,1}^{(s_1, s_2)}$ and we have $\mathbf{e} + \mathbf{f} = (1, e_2) \in \mathbf{I}_{2,2}^{(s_1, s_2)}$. If $e_2 = 1$, then let $\mathbf{f} = (1, 1) \in \mathbf{I}_{2,1}^{(s_1, s_2)}$ and we have $\mathbf{e} + \mathbf{f} = (1, 2) \in \mathbf{I}_{2,2}^{(s_1, s_2)}$.
- Suppose that $e_1 \geq 1$ and $e_2 = 0$. If $e_1 < s_1 - 1$, let $\mathbf{f} = (0, s_2 - 1) \in \mathbf{I}_{2,1}^{(s_1, s_2)}$ and we have that $\mathbf{e} + \mathbf{f} = (e_1, s_2 - 1) \in \mathbf{I}_{2,2}^{(s_1, s_2)}$. If $e_1 = s_1 - 1$, then let $\mathbf{f} = (s_1 - 1, s_2 - 1) \in \mathbf{I}_{2,1}^{(s_1, s_2)}$ and we get that $\mathbf{e} + \mathbf{f} = (s_1 - 2, s_2 - 1) \in \mathbf{I}_{2,2}^{(s_1, s_2)}$.
- Suppose that $e_1 \geq 1$ and $e_2 \geq 1$. Note that $e_1 \geq e_2$. If $e_1 < s_1 - 1$, then let $\mathbf{f} = (0, s_2 - e_2 - 1) \in \mathbf{I}_{2,1}^{(s_1, s_2)}$ and we have that $\mathbf{e} + \mathbf{f} = (e_1, s_2 - 1) \in \mathbf{I}_{2,2}^{(s_1, s_2)}$. If $e_1 = s_1 - 1$, then let $\mathbf{f} = (s_1 - 1, s_2 - e_2 - 1) \in \mathbf{I}_{2,1}^{(s_1, s_2)}$ and we get that $\mathbf{e} + \mathbf{f} = (s_1 - 2, s_2 - 1) \in \mathbf{I}_{2,2}^{(s_1, s_2)}$.

Therefore, we satisfy the conditions of Theorem 5.1.4 in all cases. \square

The characterization allows for the construction of new level \mathbf{s} -lecture hall polytopes through the following corollaries.

COROLLARY 5.4.10. The lecture hall polytope $\mathbf{P}_n^{(s)}$ is level if and only if the lecture hall polytope $\mathbf{P}_{n+1}^{(1, s)}$ is level.

Proof. We can express any inversion sequence $e \in \mathbf{I}_{n+1}^{(1,s)}$ as

$$e = (0, e')$$

where $e' \in \mathbf{I}_n^{(s)}$. Thus, e satisfies the conditions of Theorem 5.1.4 exactly when e' satisfies the conditions. \square

Remark 5.4.11. One also has $\mathbf{P}_n^{(s)}$ level if and only if $\mathbf{P}_{n+1}^{(s,1)}$ level by applying an analogous argument.

COROLLARY 5.4.12. If both $\mathbf{P}_n^{(s)}$ and $\mathbf{P}_m^{(t)}$ are level, then $\mathbf{P}_{n+m+1}^{(s,1,t)}$ is level.

Proof. Any inversion sequence $e \in \mathbf{I}_{n+m+1}^{(s,1,t)}$ can be expressed as

$$e = (e_1, 0, e_2)$$

where $e_1 \in \mathbf{I}_n^{(s)}$ and $e_2 \in \mathbf{I}_m^{(t)}$. Subsequently, e satisfies the conditions of Theorem 5.1.4 when e_1 and e_2 both satisfy the conditions of Theorem 5.1.4. \square

Remark 5.4.13. It is worth noting that by combining Corollary 5.4.8 and Corollary 5.4.12 we can create an infinite family of level \mathbf{s} -lecture hall polytopes of arbitrary dimension. In particular, $\mathbf{P}_n^{(s)}$ is level when \mathbf{s} is any sequence satisfying $s_i = 1$ when $i = 0 \pmod{3}$.

5.5 CONCLUDING REMARKS AND FUTURE DIRECTIONS

One immediate avenue to continue this work would be using the levelness characterization to produce more tractable results in special cases. Furthermore, based on experimental evidence, we have the following conjecture for levelness in a large family of lecture hall polytopes:

CONJECTURE 5.5.1. Let $\mathbf{s} \in \mathbb{Z}_{\geq 1}^n$ be a sequence such that there exists some $\mathbf{c} \in \mathbb{Z}^n$ satisfying

$$c_j s_{j-1} = c_{j-1} s_j + \gcd(s_{j-1}, s_j)$$

for $j > 1$ with $c_1 = 1$. Then $\mathbf{P}_n^{(s)}$ is level.

This conjecture, if true, implies that $\mathcal{C}_n^{(s)}$ a Gorenstein cone is sufficient for $\mathbf{P}_n^{(s)}$ to be level. However, it should be noted that the characterization, though more efficient than explicitly computing the generators of the canonical module, can often be unwieldy for complicated computations. It may, in fact, be more effective to produce an alternative representation of the level property, perhaps in terms of local cohomology.

An additional future direction would be to consider levelness in \mathbf{s} -lecture hall cones. There is no canonical choice of grading for the \mathbf{s} -lecture hall cones as there is in the polytopes and the different gradings have different computational advantages (see [BBK⁺15, Ols17]). One must choose a grading before approaching this problem. Preliminary computations with respect to certain gradings suggests that (non-Gorenstein) level \mathbf{s} -lecture hall cones are quite rare.

“Few things are harder to put up with than a good example.” (Mark Twain)

This chapter is based on joint work with Yanxi Li, Johannes Rauh, and Ruriko Yoshida, see also our article [KLR18].

6.1 INTRODUCTION

Consider the system of linear equations and inequalities

$$A\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq 0, \quad (6.1)$$

where $A \in \mathbb{Z}^{m \times n}$ and $\mathbf{b} \in \mathbb{Z}^m$. Suppose that the solution set over the real numbers $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$ is not empty.

Problem 6.1.1. *Decide whether there exists an integral solution to the system (6.1) or not.*

Problem 6.1.1 is called the *integer feasibility problem*. To decide whether a system of equations is feasible is the first step in integer programming, where the goal is to find an “optimal” solution. Therefore, problem 6.1.1 can be solved computationally using a linear programming system that can handle integer constraints, such as `lp_solve` [BEN15]. However, this computational approach does not work if one wants to study a family of integer feasibility problems at the same time. Here, we consider the following problem:

Problem 6.1.2. *For fixed $A \in \mathbb{Z}^{m \times n}$, decide for which $\mathbf{b} \in \mathbb{Z}^m$ there exists an integral solution to the system (6.1).*

Problems 6.1.1 and 6.1.2 are of fundamental importance in many areas such as operations research, number theory, combinatorics, and statistics (see [TY08] and references within). For instance, the *Frobenius problem* is a simple-sounding yet wide-open integer feasibility problem, see e.g. [RA05] for an overview. For positive, coprime integers a_1, a_2, \dots, a_n , the Frobenius problem asks to find the biggest positive integer that cannot be expressed as a non-negative linear combination of the a_i ’s with integral coefficients. Even for $n = 4$, this is an active area of research.

The feasibility of (6.1) can be described in terms of the *semigroup*

$$\mathcal{M} := \mathcal{M}(A) := \{\mathbf{a}_1x_1 + \dots + \mathbf{a}_nx_n : x_1, \dots, x_n \in \mathbb{Z}_{\geq 0}\} \quad (6.2)$$

generated by the column vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$ of A . Here, $\mathbb{Z}_{\geq 0}$ denotes the set of non-negative integers, i.e., $\mathbb{Z}_{\geq 0} := \{0, 1, 2, \dots\}$. Moreover, we need the *cone*

$$K := \text{cone}(A) := \{\mathbf{a}_1x_1 + \dots + \mathbf{a}_nx_n : x_1, \dots, x_n \in \mathbb{R}_{\geq 0}\}$$

generated by the columns of A , where $\mathbb{R}_{\geq 0} := [0, \infty)$. Throughout this chapter, we assume that all cones are *pointed*, i.e., that they do not contain lines: if $\mathbf{v} \in K \setminus \{0\}$, then $-\mathbf{v} \notin K$. Finally, we need the *lattice*

$$\Lambda := \Lambda(A) := \{\mathbf{a}_1 x_1 + \cdots + \mathbf{a}_n x_n : x_1, \dots, x_n \in \mathbb{Z}\}$$

generated by the columns of A . In this chapter, we assume $\Lambda(A) = \mathbb{Z}^n$.

By definition, an integral solution to the system (6.1) exists if and only if $\mathbf{b} \in \mathcal{M}$. In general, it is difficult to check whether a given vector belongs to \mathcal{M} . However, it is much easier to check whether a given vector belongs to Λ or to K : To check whether $\mathbf{b} \in \Lambda$ is a problem of linear algebra (over the integers), and to check whether $\mathbf{b} \in K$ one can compute the inequality description of K and check whether \mathbf{b} satisfies all linear inequalities. Admittedly, computing the inequalities can be a difficult problem in itself, but usually it is still easier than the integer feasibility problem. Therefore, it makes sense to compare \mathcal{M} to the larger semigroup $\overline{\mathcal{M}} = K \cap \Lambda$, which is called the *saturation* of \mathcal{M} . Clearly, $\mathcal{M} \subset \overline{\mathcal{M}}$, and we call \mathcal{M} *saturated* (or *normal*) if $\mathcal{M} = \overline{\mathcal{M}}$. We define the set of *holes* H of the semigroup \mathcal{M} to be $H := \overline{\mathcal{M}} \setminus \mathcal{M}$.

If $\mathbf{b} \in H \subset \overline{\mathcal{M}}$, then the system

$$A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0,$$

has a solution $\mathbf{x} \in \mathbb{R}^n$ over the *reals*, but no solution $\mathbf{x} \in \mathbb{Z}_{\geq 0}^n$ over the integers.

In general, the set H may be infinite, but it is possible to write H as a finite union of finitely generated (affine) monoids. The first step is to compute the fundamental holes, where we say that a hole $\mathbf{h} \in H$ is *fundamental* if there is no other hole $\mathbf{h}' \in H$ such that $\mathbf{h} - \mathbf{h}' \in \mathcal{M}$. In contrast to H , the set $F \subseteq H$ of fundamental holes is always finite, as it is contained in the bounded set

$$P := \left\{ \sum_{i=1}^n \lambda_i \mathbf{a}_i : 0 \leq \lambda_1, \dots, \lambda_n < 1 \right\},$$

as shown in [TY08]. A finite algorithm to compute F is due to [HTY09b].

Once F is known, it is necessary to compute an explicit representation of the holes in $\mathbf{f} + \mathcal{M}$, where $\mathbf{f} \in F$. Hemmecke et. al. [HTY09b] showed how the set of holes in $\mathbf{f} + \mathcal{M}$ can be expressed as a finite union of finitely generated monoids using ideas from commutative algebra. Together with an algorithm to compute the fundamental holes, this gives a *finite* algorithm to compute an *explicit* representation of H , even for an infinite set H .

As shown by [AB03, TY08], computing the set of holes is polynomial in time in the input size of A if we fix the number of variables m and n (see the definition of input size in [Bar94]). Once we compute \mathcal{M} for a particular matrix A , we do not have to compute it again as it does not depend on \mathbf{b} .

In this chapter, we have implemented the algorithm introduced in [HTY09b] and we have applied our software to problems in combinatorics and statistics. We named the software HASE (**H**oles in **A**ffine **S**emigroups). It is available at <http://ehrhart.math>.

fu-berlin.de/People/fkohl/HASE/. The homepage also contains the input files that are needed to reproduce the examples that are discussed in this chapter.

This chapter is organized as follows: In Section 6.2, we outline the algorithm. The performance of the algorithm and possible ways to speed up the process are described in Section 6.3. In Section 6.4, we compute the set of holes for the *common diagonal effect model* [TTY08]. In Section 6.5, we show some computational experiments concerning the integer-decomposition property of polytopes and concerning a lifting algorithm for Markov bases, see [RS15]. We end with a discussion and open problems.

6.2 COMPUTING HOLES

In this section, we briefly describe our software and the implementation. The two main steps of the algorithm of [HTY09b] are:

1. Compute the set F of fundamental holes.
2. For each of the finitely many $\mathbf{f} \in F$, compute an explicit representation of the holes in $\mathbf{f} + \mathcal{M}$.

Our software outsources step 1 to **Normaliz**, see [BIR⁺]. **Normaliz** is a computer program that computes the saturation (or **Normalization**) of an affine semigroup. Usually, the saturation $\overline{\mathcal{M}}(A)$ is output in the form of a matrix A' such that $\overline{\mathcal{M}}(A) = \mathcal{M}(A')$, i.e., the saturation equals the semigroup generated by the columns of A' . Starting with version 3.0, **Normaliz** can also compute a second representation of $\overline{\mathcal{M}}(A)$ by giving a minimal set F' of “*generators of $\overline{\mathcal{M}}(A)$ as a $\mathcal{M}(A)$ -module.*” Formally, this says that

$$\overline{\mathcal{M}}(A) = \bigcup_{\mathbf{f} \in F'} (\mathbf{f} + \mathcal{M}(A)).$$

It is not difficult to see that $\mathbf{0} \in F'$ (since $\mathcal{M} \subseteq \overline{\mathcal{M}}$) and that $F := F' \setminus \{\mathbf{0}\}$ is the set of fundamental holes of $\mathcal{M}(A)$. For details how **Normaliz** computes the set F' , we refer to the documentation of **Normaliz**. As an illustration, Section 6.4 contains a description of the (fundamental and non-fundamental) holes of the common diagonal effect models.

It remains to determine the holes in $\mathbf{f} + \mathcal{M}$ for every fundamental hole $\mathbf{f} \in F$. Every **non-hole** belongs to $(\mathbf{f} + \mathcal{M}) \cap \mathcal{M}$ and if $\mathbf{z} \in (\mathbf{f} + \mathcal{M}) \cap \mathcal{M}$, then also $\mathbf{z} + A\boldsymbol{\lambda} \in (\mathbf{f} + \mathcal{M}) \cap \mathcal{M}$ for any $\boldsymbol{\lambda} \in \mathbb{Z}_{\geq 0}^n$. Consider the ideal

$$I_{A,\mathbf{f}} := \left\langle \mathbf{x}^\boldsymbol{\lambda} : \boldsymbol{\lambda} \in \mathbb{Z}_{\geq 0}^n, \mathbf{f} + A\boldsymbol{\lambda} \in (\mathbf{f} + \mathcal{M}) \cap \mathcal{M} \right\rangle, \quad (6.3)$$

where $\mathbf{x}^\boldsymbol{\lambda} := \prod_{i=1}^n x_i^{\lambda_i}$ is the monomial with exponent vector $\boldsymbol{\lambda}$. Then, $\mathbf{f} + A\boldsymbol{\lambda}$ is not a hole if and only if $\mathbf{x}^\boldsymbol{\lambda} \in I_{A,\mathbf{f}}$. So we need to find a description of the monomials *not* in $I_{A,\mathbf{f}}$. These monomials are called the *standard monomials*. There are algorithms for finding the standard monomials, once a generating set for the ideal $I_{A,\mathbf{f}}$ is known. A generating set of the ideal $I_{A,\mathbf{f}}$ is described by the following lemma:

LEMMA 6.2.1 ([HTY09b], Lemma 4.1). Let M be the set of \leq -minimal solutions $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathbb{Z}_{\geq 0}^{2n}$ to $\boldsymbol{f} + A\boldsymbol{\lambda} = A\boldsymbol{\mu}$, where the partial order \leq is given by coordinatewise comparison. Then

$$I_{A,\boldsymbol{f}} = \left\langle \boldsymbol{x}^\lambda : \exists \boldsymbol{\mu} \in \mathbb{Z}_{\geq 0}^n \text{ such that } (\boldsymbol{\lambda}, \boldsymbol{\mu}) \in M \right\rangle.$$

Therefore, we have to find minimal integral solutions to the above system of linear equations for every fundamental hole \boldsymbol{f} . For this task, we can again use **Normaliz**, or we can use the **zsolve** command of **4ti2**, see [tt08]. Usually, **zsolve** runs faster, so it is the default choice of HASE.

Once we have a generating set for $I_{A,\boldsymbol{f}}$, we can use a computer algebra software to find the standard monomials. In general, the set of standard monomials of a polynomial ideal can be infinite, but it has a finite representation in terms of *standard pairs*, which were established in [STV95]. HASE relies on **Macaulay2** [GS], which has the command **standardPairs**. A standard pair is a pair that consists of a monomial \boldsymbol{x}^λ and a set $\boldsymbol{x}^{\boldsymbol{\mu}_1}, \dots, \boldsymbol{x}^{\boldsymbol{\mu}_r}$ of monomials. Such a pair corresponds to the set of holes

$$\boldsymbol{f} + A(\boldsymbol{\lambda} + \sum_{i=1}^r c_i \boldsymbol{\mu}_i), \quad c_i \in \mathbb{Z}_{\geq 0},$$

and the set of all such standard pairs gives all holes in $\boldsymbol{f} + \mathcal{M}$.

6.3 PERFORMANCE OF THE ALGORITHM

As shown by [AB03, TY08], computing the set of holes H for the semigroup \mathcal{M} is polynomial in time in the input size of A if we fix the number of variables m and n (see the definition of the input size in [Bar94]). Still, computing H is a difficult problem, and our algorithm may fail to terminate due to limited memory or time even for reasonably-sized examples.

In the examples we computed, we experienced the following problems:

- **Normaliz** may fail to compute the set F of fundamental holes.
- For one of the fundamental holes $\boldsymbol{f} \in F$, **zsolve** or **Normaliz** may fail to find the \leq -minimal solutions to $\boldsymbol{f} + A\boldsymbol{\lambda} = A\boldsymbol{\mu}$.
- For one of the fundamental holes $\boldsymbol{f} \in F$, **Macaulay2** may fail to compute the standard pairs.

In this list, a failure means that either we ran out of memory or we ran out of time, i.e., we became impatient and aborted the computation.

If **Normaliz** fails, there is not much we can do. We really need the fundamental holes, and if computing the fundamental holes overstrains our computational resources at hand, it is very probable that the problem is just too difficult. The only thing we could do is to ask the developers of **Normaliz**, who are always up for a challenge, for advice.

If one of the later steps fails, there is much more that we could do. The translation of computing the holes of the form $\boldsymbol{f} + \mathcal{M}$ for a fundamental hole \boldsymbol{f} into a problem of

commutative algebra is not very direct, and there may be some room for improvements. We discuss one trick that we implemented in Section 6.3.1 below.

There may be a fourth problem: Namely, the set F may be extremely large. Thus, even if **Normaliz** computes F within reasonable time and if **zsolve** and **Macaulay2** find the hole monoids reasonably fast for each single hole, the total running time may be unacceptable. However, at least in this case it is relatively easy to obtain a good estimate for the total running time that would be needed, since in this case the cardinality of F is known, and the running times of **zsolve** and **Macaulay2** per fundamental hole can be estimated by their performance on the first few holes.

If F is very large, a natural remedy is to look for symmetries of the problem. However, currently symmetries are not implemented in HASE.

6.3.1 Speeding up **zsolve**

Let $\mathbf{f} \in F$ be a fundamental hole. As explained in Section 6.2, we want to solve the linear system $\mathbf{f} + A\boldsymbol{\lambda} = A\boldsymbol{\mu}$. This system can be simplified considerably if certain non-holes are known in advance. The simplest case is to look at the vectors $\mathbf{f} + \mathbf{a}_i$, where \mathbf{a}_i is a column of A .

Suppose that $\mathbf{f} + \mathbf{a}_i$ is not a hole. Then $\mathbf{f} + \mathbf{a}_i = A\boldsymbol{\mu}_0$ for some $\boldsymbol{\mu}_0$. Thus, $\mathbf{f} + \mathbf{a}_i + A\boldsymbol{\lambda} = A(\boldsymbol{\mu}_0 + \boldsymbol{\lambda})$. This shows that if $\mathbf{f} + \mathbf{a}_i$ is not a hole, then $\mathbf{f} + \mathbf{a}_i + \mathcal{M}$ contains no other holes. This implies that every minimal solution to $\mathbf{f} + A\boldsymbol{\lambda} = A\boldsymbol{\mu}$ has $\lambda_i = 0$. Let A' be the matrix A with the i^{th} column \mathbf{a}_i dropped. Then, instead of solving $\mathbf{f} + A\boldsymbol{\lambda} = A\boldsymbol{\mu}$, we may just as well solve $\mathbf{f} + A'\boldsymbol{\lambda}' = A\boldsymbol{\mu}$. Observe that this leads to a linear system with one variable fewer. If we can identify many columns \mathbf{a}_i that we can drop, we can speed up the computation of the holes in $\mathbf{f} + \mathcal{M}$.

This idea is implemented in HASE and can be activated using the option `--trick`. With this option, HASE does the following instead of solving $\mathbf{f} + A\boldsymbol{\lambda} = A\boldsymbol{\mu}$:

1. For each column \mathbf{a}_i of A , check whether $\mathbf{f} + \mathbf{a}_i$ is a hole.
2. Let A' be the matrix with columns those \mathbf{a}_i for which $\mathbf{f} + \mathbf{a}_i$ is a hole.
3. Compute the minimal solutions to $\mathbf{f} + A'\boldsymbol{\lambda}' = A\boldsymbol{\mu}$ (using either **zsolve** or **Normaliz**).
4. Use **Macaulay2** to compute the standard pairs of the ideal $I_{A',\mathbf{f}}$.

Step 1 is an integer feasibility problem. HASE uses the open source (mixed-integer) linear programming system **lp.solve** [BEN15] to solve this problem. Usually, this is a relatively quick step (and if it is not, it is again an indication that our original problem is too difficult).

In the last step, observe that the trick also leads to a smaller ideal $I_{A',\mathbf{f}}$ or, more precisely, an ideal in a smaller ambient ring ($I_{A,\mathbf{f}}$ and $I_{A',\mathbf{f}}$ will in fact have the same generators). This, however, should not lead to a big speed-up, since the command **standardPairs** in **Macaulay2** will usually realize when variables do not appear in the generating set of an ideal.

6.4 COMMON DIAGONAL EFFECT MODELS

In this section, we consider the common diagonal effect models (CDEM) introduced by [HTY09a]. Under this model, we consider a $d \times d$ contingency table with row sums, column sums, and the diagonal sum fixed. The CDEM and its generalizations have also been studied in [OT12]. The results in this section were obtained by computing small examples using HASE to build a conjecture.

Let $A \in \mathbb{Z}^{(2d+1) \times d^2}$ be the matrix that computes the row sums, column sums, and also the diagonal sum of a $d \times d$ table. For instance, if $d = 3$, we have

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The cone K generated by the columns of A lies in the hyperplane

$$\sum_{i=1}^d z_i = \sum_{i=d+1}^{2d} z_i, \quad (6.4)$$

since this linear equality is satisfied by all columns of A . Our goal is to describe the Hilbert basis of the saturation $\overline{\mathcal{M}}$ of the semigroup \mathcal{M} generated by the columns of A . First, we define a set \mathcal{F} that will later turn out to be the set of fundamental holes of \mathcal{M} .

Definition 6.4.1. Let \mathbf{a}_{ij} be the $((i-1)d+j)^{\text{th}}$ column of A , and let

$$\mathbf{h}_{kl} := \frac{1}{2}(\mathbf{a}_{ll} + \mathbf{a}_{lk} + \mathbf{a}_{kl} + \mathbf{a}_{kk}).$$

Finally, let $\mathcal{F} := \{\mathbf{h}_{kl} : k, l \in [d], k < l\}$, where $[d] := \{1, 2, \dots, d\}$.

There are $\binom{d}{2}$ choices for l and k . Since every choice yields a different vector, we get $\#\mathcal{F} = \binom{d}{2}$. The next lemma shows that \mathcal{F} consists of holes.

LEMMA 6.4.2. $\mathcal{F} \subset K \cap \mathbb{Z}^{2d+1}$, and $\mathcal{F} \subseteq \overline{\mathcal{M}} \setminus \mathcal{M}$.

Proof. \mathbf{a}_{ij} has a 1 in the j^{th} coordinate and in the $(d+i)^{\text{th}}$ coordinate. Moreover, if $i = j$, then there is a 1 in the $(2d+1)^{\text{th}}$ coordinate. So every vector of the form $2\mathbf{h}_{kl} = \mathbf{a}_{ll} + \mathbf{a}_{lk} + \mathbf{a}_{kl} + \mathbf{a}_{kk}$ has a 2 in the l^{th} , k^{th} , $(d+l)^{\text{th}}$, $(d+k)^{\text{th}}$ and in the $(2d+1)^{\text{st}}$ coordinate, and all other coordinates are 0. Thus, $\mathcal{F} \subset K \cap \mathbb{Z}^{2d+1} = \overline{\mathcal{M}}$.

It remains to show that $\mathcal{F} \cap \mathcal{M} = \emptyset$. So suppose we have a non-negative integral linear combination of the \mathbf{a}_{ij} 's that lies in \mathcal{F} . To get a 1 in the l^{th} coordinate, we need a

generator \mathbf{a}_{il} and to get a 1 in the k^{th} coordinate we need a generator $\mathbf{a}_{i'k}$. Since there has to be a 1 in the $(d+l)^{\text{th}}$ and $(d+k)^{\text{th}}$ coordinate, we see that $i, i' \in \{l, k\}$. Note that we cannot use a different generator to obtain a 1 in these coordinates, since otherwise we would get another 1 in the first d coordinates. If there are more than 2 generators, then either there is an entry bigger than 1 or there are at least five 1's in the first $2d$ coordinates.

If $i = i'$, then without loss of generality our linear combination is $\mathbf{a}_{ll} + \mathbf{a}_{lk}$, which has a 2 in the $(d+l)^{\text{th}}$ coordinate. If $i \neq i'$, then either we have $\mathbf{a}_{kl} + \mathbf{a}_{lk}$, which has a 0 in the last entry, or we have $\mathbf{a}_{ll} + \mathbf{a}_{kk}$ which has a 2 in the last entry. Hence, we have $\mathcal{F} \cap \mathcal{M} = \emptyset$. To see that all elements in \mathcal{F} are indeed fundamental holes, one checks that each vector of the form $\mathbf{h}_{kl} - \mathbf{a}_{ij}$ is not in $\overline{\mathcal{M}}$. \square

We have now identified a set of fundamental holes. In [TY08, proof of Proposition 3.1], Takemura and Yoshida have shown that the set of fundamental holes is contained in

$$P := \left\{ \sum_{i,j \in [d]} \lambda_{ij} \mathbf{a}_{ij} : 0 \leq \lambda_{ij} < 1 \text{ for } i, j \in [d] \right\}.$$

To identify the fundamental holes, we can focus on P . Moreover, this proposition also implies that the (minimal) Hilbert basis is contained in the closure of P . The next theorem describes the (minimal) Hilbert basis for $\overline{\mathcal{M}}$.

THEOREM 6.4.3. The minimal Hilbert basis for $\overline{\mathcal{M}}$ is given by

$$\mathcal{H} := \{\mathbf{a}_{ij}\}_{i,j \in [d]} \cup \mathcal{F}.$$

Proof. Let $\mathbf{z} = (z_1, z_2, \dots, z_{2d+1}) \in P \cap \mathbb{Z}^{2d+1} \setminus \{0\}$. Since $\mathbf{z} \in P$, we have that $\mathbf{z} = \sum_{ij} \lambda_{ij} \mathbf{a}_{ij}$ with $\lambda_{ij} < 1$. Let $S := z_1 + z_2 + \dots + z_d = z_{d+1} + z_{d+2} + \dots + z_{2d}$. For every $i, j \in [d]$, we have $z_i = \sum_{j'=1}^d \lambda_{ij'}$ and $z_{d+j} \geq \lambda_{ij}$. This implies

$$S - z_{d+i} = \sum_{j \in [d] \setminus \{i\}} z_{d+j} \geq \left[\sum_{j \in [d] \setminus \{i\}} \lambda_{ij} \right] \stackrel{z \text{ lattice point}}{=} z_i,$$

and hence

$$z_i + z_{d+i} \leq S. \tag{6.5}$$

We show that every non-negative integer vector \mathbf{z} that satisfies (6.5) is a non-negative integer combination of \mathcal{H} . To do this, we show that if $\mathbf{z} \neq 0$, then there is an element $\mathbf{a} \in \mathcal{H}$ such that $\mathbf{z} - \mathbf{a}$ is non-negative and satisfies (6.5). Observe that subtracting \mathbf{a} from \mathbf{z} decreases the right hand side S , so we need to make sure that the left hand side also decreases for those $i \in [d]$ for which (6.5) holds with equality.

First, suppose that $z_{2d+1} > 0$.

If there are two indices l and k where equality holds in (6.5), then

$$z_k + z_{d+k} = S \geq z_k + z_l \quad \text{and} \quad z_l + z_{d+l} = S \geq z_{d+k} + z_{d+l}.$$

It follows that $z_{d+k} = z_l$ and $z_{d+l} = z_k$. Thus, z_l and z_k are the only nonzero entries among the first d coordinates. It is easy to check that in this case, $\mathbf{z} - \frac{1}{2}(\mathbf{a}_{ll} + \mathbf{a}_{lk} + \mathbf{a}_{kl} + \mathbf{a}_{kk})$ is non-negative and satisfies (6.5).

If there is only one index i for which equality holds in (6.5), we can check that $\mathbf{z} - \mathbf{a}_{ii}$ again is non-negative and satisfies (6.5). If there is no i for which equality holds, we pick i such that the pair of indices $(i, d+i)$ with $z_i, z_{d+i} \neq 0$ contains the biggest entry and subtract \mathbf{a}_{ii} . Then $\mathbf{z} - \mathbf{a}_{ii}$ is non-negative and satisfies (6.5).

It remains to discuss the case $z_{2d+1} = 0$. To express \mathbf{z} as a non-negative linear integral combination of \mathbf{a}'_{ij} s where $i \neq j$, we translate the problem to a matching problem. We have two labeled multi-sets $\bigcup_{i:z_i>0} \bigcup_{r=1}^{z_i} \{i\}$ and $\bigcup_{j:z_{d+j}>0} \bigcup_{s=1}^{z_{d+j}} \{j\}$, both of cardinality S . Writing \mathbf{z} as a non-negative integer combination of the \mathbf{a}_{ij} corresponds to a matching between the two sets. The matching has to be *proper* in the sense that we only match elements i, j with $i \neq j$. For example, if $\mathbf{z} = (2, 0, 2, 2, 0, 2, 0)$, the two multi-sets are both equal to $\{1, 1, 3, 3\}$. In this example, there is (up to symmetry) only one proper matching that matches 1 to 3 and 3 to 1, corresponding to the identity $\mathbf{z} = \mathbf{a}_{1,3} + \mathbf{a}_{3,1} + \mathbf{a}_{1,3} + \mathbf{a}_{3,1}$.

It remains to show that there always exist such a proper matching. This can either be seen directly by induction or by appealing to Hall's marriage theorem, noting that (6.5) always ensures that the marriage condition is satisfied.

This finishes the proof \mathcal{H} is a Hilbert basis. It is straightforward to check that \mathcal{H} is indeed minimal. \square

Knowing the Hilbert basis for $\overline{\mathcal{M}}$, we can completely describe the set of fundamental holes.

COROLLARY 6.4.4. \mathcal{F} is the set of fundamental holes of \mathcal{M} .

Proof. We have already seen that every element in \mathcal{F} is a fundamental hole. It only remains to show that there are no other fundamental holes. Any fundamental hole is a non-negative integer combination of the Hilbert basis \mathcal{H} . Clearly, this combination cannot involve the columns \mathbf{a}_{ij} (otherwise the combination would not be fundamental). Thus, it suffices to show that the sum of two holes in \mathcal{H} is not a hole. This follows from the identity $\mathbf{h}_{ij} + \mathbf{h}_{kl} = \mathbf{a}_{kk} + \mathbf{a}_{ll} + \mathbf{a}_{ij} + \mathbf{a}_{ji}$. \square

We have now seen that there are exactly $\binom{d}{2}$ *fundamental* holes. However, this semigroup has *infinitely* many holes:

THEOREM 6.4.5. The set of holes in the set $\mathbf{h}_{kl} + \mathcal{M}$ is the union of the two monoids

$$\mathbf{h}_{kl} + \mathbb{Z}_{\geq 0}\mathbf{a}_{kk} + \mathbb{Z}_{\geq 0}\mathbf{a}_{kl} + \mathbb{Z}_{\geq 0}\mathbf{a}_{lk} + \mathbb{Z}_{\geq 0}\mathbf{a}_{ll} \quad (6.6a)$$

and

$$\mathbf{h}_{kl} + \sum_{i=1}^d \mathbb{Z}_{\geq 0}\mathbf{a}_{ii}. \quad (6.6b)$$

Proof. Fix $k, l \in [d]$, $k < l$. If $i \neq j$ and $i \notin \{k, l\}$, then

$$\mathbf{h}_{kl} + \mathbf{a}_{ij} = \mathbf{a}_{kk} + \mathbf{a}_{il} + \mathbf{a}_{lj}$$

assuming that $j \neq l$. If $j = l$, then we get

$$\mathbf{h}_{kl} + \mathbf{a}_{ij} = \mathbf{a}_{ll} + \mathbf{a}_{kl} + \mathbf{a}_{ik}.$$

Thus, if $i \neq j$ and $i \notin \{k, l\}$, then $\mathbf{h}_{kl} + \mathbf{a}_{ij}$ is not a hole. Similarly, if $j \notin \{k, l\}$, then $\mathbf{h}_{kl} + \mathbf{a}_{ij}$ is not a hole. Thus, if $\mathbf{h}_{kl} + \sum_{r=1}^s \mathbf{a}_{i_r j_r}$ is a hole, then either $i_r = j_r$ or $\{i_r, j_r\} = \{k, l\}$ for each r . We claim that either $i_r = j_r$ for all r , or $\{i_r, j_r\} = \{k, l\}$ for all r . This implies that each hole is as in the statement of the theorem. The claim follows from the computation

$$\mathbf{h}_{kl} + \mathbf{a}_{ii} + \mathbf{a}_{kl} = \mathbf{a}_{kk} + \mathbf{a}_{ll} + \mathbf{a}_{il} + \mathbf{a}_{ki},$$

which is valid whenever $i \notin \{k, l\}$ and $k \neq l$.

It remains to see that every integer vector in (6.6) is indeed a hole. Let \mathbf{h} be of the form (6.6a), and suppose that $\mathbf{h} = \sum_{ij} \lambda_{ij} \mathbf{a}_{ij}$ with $\lambda_{ij} \geq 0$. Then $\lambda_{ij} \neq 0$ only for $\{i, j\} \subseteq \{k, l\}$, because $h_i = 0$ or $h_j = 0$ for $i, j \notin \{k, l\}$. The matrix A restricted to the columns \mathbf{a}_{ij} with $\{i, j\} \subseteq \{k, l\}$ equals

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

up to rows with only zeros. Since this matrix has rank four, the representation of \mathbf{h} as a linear combination is unique. However, by assumption, \mathbf{h} has a representation of the form $\mathbf{h} = \mathbf{h}_{kl} + \dots$ in which the coefficients are not integers (but half integers). Thus, \mathbf{h} is a hole.

Finally, let \mathbf{h} be of the form (6.6b). Suppose that $\mathbf{h} = \sum_{ij} \lambda_{ij} \mathbf{a}_{ij}$ with $\lambda_{ij} \in \mathbb{Z}_{\geq 0}$, and let $S = \sum_{i=1}^d h_i = \sum_{ij} \lambda_{ij}$. Note that $h_{2d+1} = S - 1$. Therefore, \mathbf{h} is the sum of $S - 1$ “diagonal” columns \mathbf{a}_{ii} and one “off-diagonal” column \mathbf{a}_{ij} . This is not possible, since, by assumption, the column sums and the row sums are the same. \square

Remark 6.4.6. Note the following two properties of the hole monoids:

1. The two monoids corresponding to a single hole \mathbf{h}_{kl} are not disjoint.
2. The hole monoids corresponding to two different fundamental holes $\mathbf{h}_{kl}, \mathbf{h}_{k'l'}$ are not disjoint: For example,

$$\mathbf{h}_{12} + \mathbf{a}_{33} = \mathbf{h}_{23} + \mathbf{a}_{11}.$$

6.5 COMPUTATIONAL EXPERIMENTS

Semigroups play an important role in combinatorics, in discrete geometry, and in combinatorial commutative algebra. As we have seen in Chapter 2, the interplay between these areas is nicely exemplified by the theory of lattice polytopes. One can associate a semigroup to every lattice polytope. The Hilbert function of the corresponding graded semigroup ring turns out to be the Ehrhart function of the lattice polytope, see also [MS05, Section 12.1].

It is of particular interest to determine whether this semigroup has holes. For example, if the semigroup has holes, then there is no unimodular triangulation. Therefore, the algebraic structure is closely related to geometric properties. In Section 6.5.1, we present a computational result regarding the linear ordering polytope.

Holes of semigroups also play a role when computing Markov bases, as was recently shown by [RS15]. We give a brief example in Section 6.5.2.

6.5.1 The Integer-Decomposition Property and Linear Order Polytopes

This section is dedicated to examine whether the n^{th} linear order polytope P_n has the integer-decomposition property. Sturmfels and Welker already showed that for $n \leq 6$, the n^{th} linear ordering polytope satisfies the IDP, see [SW12, Theorem 6.1].

For any permutation π of n elements, we define

$$v_{ij}(\pi) = \begin{cases} 1 & \text{if } \pi(i) > \pi(j) \\ 0 & \text{otherwise,} \end{cases}$$

where $1 \leq i < j \leq n$. We follow the definition of [Kat13, Section 3.3] and define the n^{th} linear ordering polytope P_n as the convex hull of the $n!$ vectors $\mathbf{v}(\pi) := (v_{ij}(\pi))_{1 \leq i < j \leq n} \in \mathbb{R}^{\binom{n}{2}}$. Note that the vertices are the only integer points of P_n . A python program that generates the matrix can be downloaded from the HASE homepage. After a bit more than a month of computation time on a linux machine with 16 Intel processors (Intel(R) Xeon(R) CPU E5-2687W v2 at 3.40GHz), the program confirmed that P_7 also has the IDP.

THEOREM 6.5.1. The n^{th} linear ordering polytope P_n has the integer-decomposition property for $n \leq 7$.

Recent improvements made by the Normaliz team make it *much faster* to validate this result. Unfortunately — at least with our computational tools — checking whether P_8 has the IDP is still out of reach. The question whether or not P_n satisfies the IDP for all $n \in \mathbb{Z}_{\geq 1}$ is still open.

6.5.2 Lifting Markov bases and Gröbner bases

Recently, Rauh and Sullivant [RS15] have proposed a new iterative algorithm to compute Markov bases and Gröbner bases of toric ideals in which a key step is to understand the

holes of an associated semigroup. We do not explain this theory here, but we summarize two examples that arose in this context and that can now be reproduced using HASE.

The first example is from the computation of the Markov basis of the binary complete bipartite graph $K_{3,N}$, as computed by [RS14]. The associated semigroup has two fundamental holes. Each fundamental hole has one associated monoid, generated by eight generators. The input file `K31codz.mat` for HASE can be downloaded from the HASE homepage. Within a few seconds, HASE produces the following output:

```
Normaliz found 2 fundamental holes.
Standard pairs of [1 1 1 1 1 1 1 1 1 1 1 0 1 1 0 1 0 0 1]:
1: {x1, x2, x4, x7, x9, x10, x12, x15}
Standard pairs of [1 1 1 1 1 1 1 1 1 1 1 1 0 0 1 0 1 1 0]:
1: {x0, x3, x5, x6, x8, x11, x13, x14}
```

The same method can be used to compute a Markov basis for the binary 3×3 -grid. In this case, one needs to understand the holes of a larger semigroup. Again, the input file `3x3codz.mat` can be downloaded from the HASE homepage.

This problem turns out to be much more difficult for HASE, and in fact, after waiting 24 hours for HASE to finish we became impatient and aborted the program, even when the option `--trick` was activated.

Surprisingly, it turns out that the set of holes itself can be computed with some extra information: While the semigroup has 32 fundamental holes, there are only three symmetry classes. The time that HASE spends on a single hole varies greatly, even within a symmetry class. So all that is needed to finish the computation is to find representatives of the three symmetry classes such that the hole monoid computations run through relatively quickly. The current version of HASE cannot be used to run the algorithm on a subset of the fundamental holes (but it is not difficult to do this manually by looking at HASE's source code). This shows once again how important it is to take symmetry into account.

6.6 DISCUSSION AND OPEN PROBLEMS

There are many open problems concerning semigroups and holes of semigroups. In this section, we just want to briefly mention a non-representative selection of open problems.

As mentioned in the beginning, the Frobenius problem is still open if there are more than two generators. There are several computational results, see e.g. [BEZ03]. It might be possible to use the structure of the holes, i.e. which hole is based on which fundamental hole, to say something about the Frobenius number. Alternatively, one could use a slightly modified version of HASE to compute the Frobenius number explicitly.

As briefly discussed in Section 6.5.1, holes in semigroups coming from lattice polytopes are of particular interest as they reflect geometric properties. Therefore, another application of HASE is to describe the semigroup coming from a user specified lattice polytope.

“Mathematics is a collection of cheap tricks and dirty jokes.” (Lipman Bers)

This chapter is based on the HASE manual which can be found at <http://ehrhart.math.fu-berlin.de/People/fkohl/HASE/Manual/>, see [KLR16].

HASE is a program that computes holes of pointed, affine semigroups. It is an implementation of the algorithm described by Hemmecke, Yoshida, and Takemura, see [HTY09b]. The program itself is written in Python 3, but it internally uses (and hence is dependent on) Normaliz [BIR⁺], Macaulay2 [GS], and optionally on 4ti2 [tt08]. The dependence on 4ti2 however is not strict, as there is the `-Nsolve` option avoiding 4ti2. The source code can be found in the Appendix, see Chapter 8. In this chapter, we want to explicitly demonstrate how one can use HASE to compute the holes in certain semigroups.

7.1 THE BASICS

7.1.1 Notation

In this section, we want to briefly recall the notation that we have used in Chapter 6. Let $A \in \mathbb{Z}^{m \times n}$ be a matrix with integral entries. Moreover, let $\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \dots\}$ be the set of nonnegative integers and let $\mathbb{R}_{\geq 0} = [0, \infty)$. Let $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ be the columns of a matrix A . Then we define

1. the *semigroup generated by the columns of A* as the set

$$\mathcal{M} := \mathcal{M}(A) := \{\mathbf{a}_1 x_1 + \dots + \mathbf{a}_n x_n : x_1, \dots, x_n \in \mathbb{Z}_{\geq 0}\},$$

2. the *cone generated by the columns of A* as

$$K := \text{cone}(A) := \{\mathbf{a}_1 x_1 + \dots + \mathbf{a}_n x_n : x_1, \dots, x_n \in \mathbb{R}_{\geq 0}\},$$

and

3. the *lattice $\Lambda = \Lambda(A)$ generated by the columns A* as

$$\Lambda := \Lambda(A) := \{\mathbf{a}_1 x_1 + \dots + \mathbf{a}_n x_n : x_1, \dots, x_n \in \mathbb{Z}\}.$$

The following definitions have already been introduced in Chapter 2 and Chapter 6, but we will briefly recall them here. The semigroup $\overline{\mathcal{M}} = K \cap \Lambda$ is called the *saturation* of the semigroup \mathcal{M} with respect to the lattice $\Lambda(A)$. It follows that $\mathcal{M} \subset \overline{\mathcal{M}}$ and we

call \mathcal{M} *saturated* if $\mathcal{M} = \overline{\mathcal{M}}$ (this is also called *normal*). We call $H = \overline{\mathcal{M}} \setminus \mathcal{M}$ the set of *holes* of the semigroup \mathcal{M} . A hole $h \in H$ is *fundamental* if there is no other hole $h' \in H$ such that $h - h' \in \mathcal{M}$. Note that in contrast to H , F is always finite as it is contained in the the set

$$P := \left\{ \sum_{i=1}^n \lambda_i \mathbf{a}_i : 0 \leq \lambda_1, \dots, \lambda_n < 1 \right\},$$

as shown in [TY08]. Let us illustrate these definitions in a small example.

Example 7.1.1 (The Frobenius Problem). Let p, q be two positive, coprime integers. In the notation above, this means $A = (p, q) \in \mathbb{R}^{1 \times 2}$. The semigroup generated by the columns of A is

$$\mathcal{M}(A) = \{n \in \mathbb{Z}_{\geq 0} : n = ap + bq, \text{ where } a, b \in \mathbb{Z}_{\geq 0}\}.$$

It is a direct consequence of Bezout's theorem that the lattice generated by p and q is $\Lambda(A) = \mathbb{Z}$. However, if p and q are **not** coprime, then $\Lambda(A) \neq \mathbb{Z}$. Furthermore, we have that $\text{cone}(A) = \mathbb{R}_{\geq 0}$. The saturation $\overline{\mathcal{M}} = \text{cone}(A) \cap \Lambda(A) = \mathbb{R}_{\geq 0} \cap \mathbb{Z} = \mathbb{Z}_{\geq 0}$. Thus, the set of holes $H = \overline{\mathcal{M}} \setminus \mathcal{M}$ is the of nonnegative integers that cannot be written as a positive, linear, integral combination of p and q . It is a theorem that there are only *finitely* many holes in $\mathcal{M}(A)$. The Frobenius problem asks to find the largest number that cannot be written as such a combination of p and q . This number is called the **Frobenius number** and it is denoted $g(p, q)$. Moreover, it can be shown that

$$g(p, q) = pq - p - q.$$

For a proof this theorem, we refer the reader to [BR07, Theorem 1.2, p. 6].

If we now choose $p = 3$ and $q = 7$, we can see that

$$H = \{1, 2, 4, 5, 8, 11\}.$$

and

$$F = \{1, 2\}.$$

One can generalize this problem to more than two generators. In this case — except for some smaller special cases — the Frobenius problem remains wide open.

7.1.2 A First Example

This example is taken from [HTY09b, Ex. 2.2]. Let us now turn to an explicit computation of holes in an affine semigroup. Let $\mathcal{M}(A)$ be the semigroup generated by $(1, 0)^t$, $(1, 2)^t$, $(1, 3)^t$, and $(1, 4)^t$. This means that

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 \end{pmatrix}.$$

To compute the holes with HASE, we need to create an input-file *example.mat* which looks like


```

2 4
1 1 1 1
0 2 3 4

```

The first line specifies the dimensions of the matrix and the other lines describe the matrix A . We have to make sure that this file is in the same folder as Normaliz and ZSolve, which is an application from 4ti2. Now we can run the program by entering

```
python3 hase.py --normaliz "./normaliz" -v example
```

into the terminal. The output looks like

```

Using temporary directory /tmp/tmp080qluh7/
Read example.mat. 4 generators in dimension 2.
>>>> Running normaliz to compute the fundamental holes.
Normaliz found 1 fundamental holes.
Looking at hole [1 1]
>>>> Running zsolve to find the minimal solutions to f = Am - A1.
>>>> Running M2 to compute the standard pairs. Output:
1: {x0}

```

This output means that there is one fundamental hole $(1, 1)^t$ and H can be described as

$$H = \{(1, 1)^t + m(1, 0)^t : m \in \mathbb{Z}_{\geq 0}\}.$$

As mentioned in the preface, there is the option `-Nsolve`, which uses Normaliz to determine the integer solutions instead of using 4ti2. The command now looks like

```
python3 hase.py --normaliz "./normaliz" --Nsolve "./normaliz" -v beispiel
```

The output now is

```

Read beispiel.mat. 4 generators in dimension 2.
>>>> Running normaliz to compute the fundamental holes.
Normaliz found 1 fundamental holes.
Looking at hole [1 1]
>>>> Running Normaliz to find the minimal solutions to f = Am - A1.
>>>> Running M2 to compute the standard pairs. Output:
1: {x0}

```

7.1.3 Frobenius Revisited

Let's now turn to the Frobenius problem. We will start with a very small example to nicely interpret the output. With the notation of Example 7.1.1, we set $p = 3$ and $q = 5$. For technical reasons, we embed the problem into \mathbb{R}^2 . Our input file *frobenius.mat* then looks like:

```

2 2
3 5
0 0

```

HASE gives the following output:

```

Using temporary directory /tmp/tmp6qedfn44/
Read frobenius.mat. 2 generators in dimension 2.
>>>> Running normaliz to compute the fundamental holes.
Normaliz found 2 fundamental holes.
Looking at hole [1 0]
>>>> Running zsolve to find the minimal solutions to f = Am - A1.
>>>> Running M2 to compute the standard pairs. Output:
  1: {}
  x0: {}
  2
  x0 : {}
Looking at hole [2 0]
>>>> Running zsolve to find the minimal solutions to f = Am - A1.
>>>> Running M2 to compute the standard pairs. Output:
  1: {}
  x1: {}

```

There are only two fundamental holes, but there are several standard pairs. The x'_i 's in the output refer to the $(i - 1)^{\text{st}}$ generator. For the hole $(1, 0)$ this simply means, we have the holes $(1, 0)$, then the hole $(1, 0) + (3, 0)$ and $(1, 0) + 2 \cdot (3, 0)$. Moreover, we have the holes with base point $(2, 0)$, namely $(2, 0)$ and $(2, 0) + (5, 0)$. Note that the list of holes is redundant, as the hole $(7, 0)$ appears with two different base points. In higher dimensions we cannot expect a finite set of holes, so based on every hole there will be a monoid. This monoid is trivial in our case, which explains the empty brackets after the variables.

7.1.4 A Bigger Example

Now let us turn to a more complex problem, namely to the Common Diagonal Effect Model (CDEM) which we have already encountered in Section 6.4. The CDEM examines (square) marginal tables, where the row sums, the column sums, and the diagonal sum are fixed. Let

$$M = \begin{pmatrix} m_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \vdots & \vdots & \dots & \vdots \\ m_{n1} & m_{n2} & \dots & m_{nn} \end{pmatrix}$$

be an $n \times n$ marginal table. We embed M into \mathbb{R}^{n^2} using the isomorphism

$$M \mapsto (m_{11}, m_{12}, \dots, m_{1n}, m_{21}, m_{22}, \dots, m_{2n}, \dots, m_{n1}, m_{n2}, \dots, m_{nn}) =: \tilde{M}.$$

Now we can describe the conditions that the row sums, columns sums, and diagonal sum is fixed by a linear equation of the form

$$A\tilde{M} = \mathbf{b}$$

for some $A \in \mathbb{R}^{n^2 \times n^2}$ and $\mathbf{b} \in \mathbb{R}^{n^2}$. In particular, if $n = 4$ then $A \in \mathbb{Z}^{9 \times 16}$ is the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let us determine the holes of the semigroup generated by the columns of A . We save A in the file 44.mat:

```
9 16
1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0
0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0
0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0
0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1
1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1
1 0 0 0 0 1 0 0 0 0 1 0 0 0 0 1
```

Now we can run our program with the following command:

```
python3 hase.py --normaliz "./normaliz" -v 44
```

The output will be:

```
Normaliz found 6 fundamental holes.
Standard pairs of [0 0 1 1 0 0 1 1 1]:
  1: {x10, x11, x14, x15}
  1: {x0, x5, x10, x15}
Standard pairs of [0 1 0 1 0 1 0 1 1]:
  1: {x5, x7, x13, x15}
  1: {x0, x5, x10, x15}
Standard pairs of [0 1 1 0 0 1 1 0 1]:
  1: {x5, x6, x9, x10}
  1: {x0, x5, x10, x15}
Standard pairs of [1 0 0 1 1 0 0 1 1]:
  1: {x0, x5, x10, x15}
  1: {x0, x3, x12, x15}
Standard pairs of [1 0 1 0 1 0 1 0 1]:
  1: {x0, x5, x10, x15}
  1: {x0, x2, x8, x10}
Standard pairs of [1 1 0 0 1 1 0 0 1]:
  1: {x0, x5, x10, x15}
  1: {x0, x1, x4, x5}
```

An explicit description of the Hilbert basis and the structure of the holes of $\mathcal{M}(A)$ is described in Chapter 6, see also [KLR18].

7.2 MORE OPTIONS

In this section, we want to give a very brief overview over the different options. Enter

```
python3 hase.py --normaliz "./normaliz" --help
```

in your terminal to see all possible flags.

```
usage: hase.py [-h] [-k] [--M2 M2] [--normaliz NORMALIZ]
              [--zsolve ZSOLVE] [-t] [--time]
              [--lp_solve LP_SOLVE] [-v]
              filename
```

Compute the hole monoids of an affine semigroup.

positional arguments:

filename the name of the file containing the generators,
without the ending ".mat" (as columns of a matrix, in
4ti2 .mat format)

optional arguments:

-h, --help show this help message and exit

-k, --keep-temporary-files

with this option, temporary files are created in the
same directory (using FILENAME-him as a basename) and
are not deleted when the program finishes. Note that
the temporary files for different fundamental holes
get the same name, so only the temporary files used
when computing the monoid of the last fundamental hole
are preserved.

--M2 M2 the command to call Macaulay2 (including the path)

--normaliz NORMALIZ the command to call normaliz (including the path)

--zsolve ZSOLVE the command to call zsolve (including the path)

--Nsolve the command to use Nsolve instead of zsolve (including path)

-t, --trick for each fundamental hole f and generator g, check if
f+g is a hole. Only compute the hole monoid among the
remaining generators.

--time time each zsolve command

--lp_solve LP_SOLVE the command to call lp_solve (including the path)

-v, --verbose increase output verbosity. Use several times ("-vv")

to further increase output verbosity

“If you had done something twice, you are likely to do it again.” (Brian Kernighan and Bob Pike (The Unix Programming Environment, p. 97)

We want to end this thesis by appending the source code of HASE. The implementation of the flag `--trick` is due to Johannes Rauh.

Listing 8.1: Source Code of HASE

```
#!/usr/bin/env python3

import sys
import os
import numpy as np

import argparse # command line processing
import warnings # to avoid warning when reading empty matrix in .mat format
import tempfile # to create temporary files in dedicated directory
import subprocess # to call other commands safely

##### parse arguments
parser = argparse.ArgumentParser(description="Compute the hole monoids of an affine
semigroup.")
parser.add_argument("filename", help="the name of the file containing the generators,
without the ending \".mat\" (as columns of a matrix, in 4ti2 .mat format)")
parser.add_argument("-k", "--keep-temporary-files", help="with this option, temporary
files are created in the same directory (using FILENAME-him as a basename) and
are not deleted when the program finishes. Note that the temporary files for
different fundamental holes get the same name, so only the temporary files used
when computing the monoid of the last fundamental hole are preserved.", action="
store_true")
parser.add_argument("--M2", help="the command to call Macaulay2 (including the path
)", default='M2')
parser.add_argument("--normaliz", help="the command to call normaliz (including the
path)", default='normaliz')
parser.add_argument("--zsolve", help="the command to call zsolve (including the path)",
default='zsolve')
parser.add_argument("-t", "--trick", help="for each fundamental hole f and generator g,
check if f+g is a hole. Only compute the hole monoid among the remaining
generators.", action="store_true")
parser.add_argument("--time", help="time each zsolve command", action="store_true")
parser.add_argument("--lp_solve", help="the command to call lp_solve (including the
path)", default='lp_solve')
parser.add_argument("-v", "--verbose", action="count", help="increase output verbosity.
Use several times (\\"-vv\\") to further increase output verbosity", default=0)
parser.add_argument("-n", "--Nsolve", help="use normaliz instead of zsolve to solve
linear integer equations.", action="store_true")
```

```

args = parser.parse_args()

##### input/output of files
def getMatMatrix(filename):
    with warnings.catch_warnings(): # numpy complains if the file is empty (or contains
        only the header line)
        warnings.simplefilter("ignore")
        matrix = np.loadtxt(filename, dtype=int, skiprows=1) # , ndmin=2 : does
            not work for matrices with one size equal to zero
    return matrix

# motivated by: http://stackoverflow.com/questions/5914627/prepend-line-to-beginning-
of-a-file
def apppend(filename,header='',footer=''):
    with open(filename, 'r+') as f:
        content = f.read()
        f.seek(0, 0)
        f.write(header.rstrip('\r\n') + '\n' + content.rstrip('\r\n') + '\n' + footer)

def putMatMatrix(filename,A):
    # np.savetxt(filename, A, fmt='%i', header=str(A.shape[0])+" "+str(A.shape[1]), comments
    =") ## for numpy > 1.7.0
    np.savetxt(filename, A, fmt='%i')
    apppend(filename, header=str(A.shape[0])+" "+str(A.shape[1]))

def putMatConstVector(filename,size,entry):
    "Writes a vector of identical entries to a file in 4ti2 format"
    with open(filename,'w') as f:
        f.write('1_' + str(size) + "\n")
        for i in range(size):
            f.write(entry + '_')
        f.write("\n")

def putNormalizMatrix(filename,A,add=''):
    "Writes the matrix A to file filename in Normaliz format. Last line is add."
    # np.savetxt(filename, A.T, fmt='%i', header=str(A.shape[1])+"\n"+str(A.shape[0]), footer
    =add, comments='') ## for numpy > 1.7.0
    np.savetxt(filename, A.T, fmt='%i')
    apppend(filename, header=str(A.shape[1])+"\n"+str(A.shape[0]), footer=add)

def readNormalizFundHoles(filename):
    FundHoles = set()
    with open(filename,'r') as outfile:
        line=""
        while not "embedding_dimension=" in line:
            line = outfile.readline()
        dimension_cone = int(line.strip().split()[-1]) # the last part of the line is
            the embedding dimension
        FundHoles = np.empty((0,dimension_cone),dtype=np.int )
        # line that contains "module generators" and ends with ':'
        while not "module_generators_over_original_monoid:" in line:
            line = outfile.readline()
        line = outfile.readline()

```



```

        while line != "\n":
            hole=np.array(line.strip().split(), dtype=int)
            if np.any(hole):
                FundHoles = np.append(FundHoles, [hole], axis=0)
            line = outfile.readline()
    return FundHoles # - {''} # remove the empty generator

def readNormalizInhom(filename):
    SolInhom=set()
    with open(filename,'r') as outfile:
        line = ""
        while not "embedding" in line:
            line = outfile.readline()
        dimension_matrix = int(line.strip().split()[-1])-1 # the -1 is due to the
            fact that they have the inhomogeneous part as the last column
        SolInhom=np.empty((0,dimension_matrix),dtype=np.int)

        while not "module_generators:" in line:
            line=outfile.readline()
        line=outfile.readline()
        while line != "\n":
            solution=np.array(line.strip().rsplit('\t', 1)[0].split(), dtype=int) #
                rsplit removes the last entry (inhomogeneous part)
            SolInhom = np.append(SolInhom, [solution], axis=0)
            line=outfile.readline()

    return SolInhom

### set up temporary files
if args.keep_temporary_files:
    dirname = ""
else:
    dir = tempfile.TemporaryDirectory()
    dirname = dir.name + "/"
    if args.verbose:
        print ("Using temporary directory" + dirname)
filename = dirname + os.path.basename(args.filename) + "-him" # use this as the base for
    all temporary files

### read the matrix

try:
    matrix = getMatMatrix(args.filename + ".mat")
except FileNotFoundError:
    print("ERROR:_Could_not_read_matrix_from_file_" + args.filename + ".mat.", file
        =sys.stderr)
    if args.verbose:
        print("Maybe_" + args.filename + ".mat_does_not_exist?", file=sys.stderr)
    sys.exit(1)
except (StopIteration,ValueError,UserWarning):
    print("ERROR:_Could_not_read_matrix_from_file_" + args.filename + ".mat.", file
        =sys.stderr)

```

```

    if args.verbose:
        print("Maybe a syntax error in " + args.filename + ".mat?", file=sys.
              stderr)
    sys.exit(1)

if len(matrix.shape) <= 1: # matrix is not a matrix, but a vector or so...
    if (matrix.shape[0] == 0):
        print("ERROR: Input matrix (read from file " + args.filename + ".mat)
              seems to be empty.", file=sys.stderr)
        sys.exit(1)
    else:
        matrix.shape=(1,matrix.shape[0])

dimension_cone = matrix.shape[0] # ambient dimension of the cone
number_generators = matrix.shape[1] # number of generators of the semigroup

if args.verbose:
    print("Read " + args.filename + ".mat." + str(number_generators) + "
          generators in dimension " + str(dimension_cone) + ".");

if args.verbose:
    print(">>>>> Running normaliz to compute the fundamental holes.")

putNormalizMatrix(filename + ".in",matrix,add="cone")
if args.verbose >= 2: # normaliz is quite verbose, so make it quiet, unless verbosity level is
    at least 2:
    normalizopts = ["-c"]
else:
    normalizopts = []

try:
    subprocess.check_call([args.normaliz,"-M"] + normalizopts + [filename])
except subprocess.CalledProcessError as err:
    print("ERROR: Normaliz failed with exit code:" + str(err.returncode), file=sys.
          stderr)
    sys.exit(1)

try:
    FundHoles = readNormalizFundHoles(filename + ".out")
except:
    print("ERROR: Could not read Normaliz output.", file=sys.stderr)
    sys.exit(1)

# if args.verbose:
print("Normaliz found " + str(len(FundHoles)) + " fundamental holes.");

### Find set of minimal solutions to f = Am - A1

if not args.Nsolve: # When using zsolve: prepare auxilliary files and verbosity options
    if args.verbose >= 3:
        zsolveopts = ["-v"]
    elif args.verbose == 2:
        zsolveopts = []

```

```

else: # zsolve is too verbose, so make it quiet, unless verbosity level is at least 2:
    zsolveopts = ["-q"]

putMatConstVector(filename + ".rel", dimension_cone, "=")
if not args.trick: # prepare auxilliary files for zsolve only once
    concmatrix = np.concatenate((matrix, -matrix), axis=1)
    putMatMatrix(filename + ".mat", concmatrix)
    putMatConstVector(filename + ".sign", 2*number_generators, "1")

# for run_var in range(number_fundholes):
for hole in FundHoles:
    if args.verbose:
        print("Looking at hole", end="")
        print(hole)
    ##### implement Johannes' trick: check each generator whether it is necessary:
    if args.trick:
        necessary_generators = []
        for k in range(number_generators):
            with open(filename + '.lp', 'w') as lpfile:
                lpfile.write('min:;\n')
                for j in range(number_generators): # positivity constraints
                    lpfile.write('x' + str(j) + '>=0;\n')
                for i in range(dimension_cone): # margin constraints
                    iszero = True
                    for j in range(number_generators):
                        Aij = matrix.item((i,j))
                        if Aij != 0:
                            if iszero:
                                iszero = False
                            else:
                                lpfile.write("_+_\n")
                            if Aij != 1:
                                lpfile.write(str(Aij) + "_\n")
                            if (number_generators > 10 and j < 10):
                                lpfile.write("_\n")
                                lpfile.write("x" + str(j))
                    if not iszero: # a zero line in the matrix A would
                        lead to the equation 0 = 0, which leads to an
                        error of lp_solve
                        lpfile.write("_=_\n" + str(hole[i] + matrix.
                            item((i,k))) + ";\n")

                # integer constraint:
                lpfile.write("int_x0")
                for i in range(1, number_generators):
                    lpfile.write(",_x" + str(i))
                lpfile.write(";\n")
            if args.verbose:
                print(">>>>_Trick:_Running_lp_solve_to_exclude_certain
                    _generators.")

        if args.verbose >= 1:
            lpret = subprocess.call([args.lp_solve, "-S1", filename + ".lp

```

```

        ])
    else:
        with open(os.devnull,"w") as defnull:
            lppret = subprocess.call([args.lp_solve, "-S1",
                                     filename + ".lp"], stdout=defnull)
        if lppret==2: # lp_solve returns exit code 2 if problem is not feasible
            necessary_generators.append(k)
        elif lppret != 0: # any exit code except 0 or 2 indicates a problem
            # raise subprocess.CalledProcessError(lpret,args.lp_solve + "
            -S1 " + filename + ".lp")
            print("ERROR:lp_solve_failed_with_exit_code:" + str(
                lppret), file=sys.stderr)
            sys.exit(1)

    if not args.Nsolve: # write auxilliary files if using zsolve
        concmatrix = np.concatenate((matrix,-np.take(matrix,
            necessary_generators,axis=1)),axis=1)
        putMatMatrix(filename + ".mat", concmatrix)
        putMatConstVector(filename + ".sign", number_generators + len(
            necessary_generators), "1")

    if args.Nsolve: # run normaliz
        # prepare the matrix
        ### there should be a cleaner way to construct concmatrix! hole is of type
        np.array.
        f = np.zeros((1,int(dimension_cone)),dtype=np.int)

    if args.trick:
        concmatrix = np.concatenate((matrix,-np.take(matrix,
            necessary_generators,axis=1),-np.array([hole]).T),axis=1)
    else:
        concmatrix = np.concatenate((matrix,-matrix,-np.array([hole]).T),
            axis=1)

    putNormalizMatrix(filename + "nsolve.in", concmatrix.T, add="
        inhom.equations")

    if args.verbose:
        print(">>>>>Running Normaliz to find the minimal solutions
            to f=Am-Al.")
    try:
        subprocess.check_call([args.normaliz]+[filename + "nsolve"])
    except subprocess.CalledProcessError as err:
        print("ERROR:Normaliz_failed_with_exit_code:" + str(err.
            returncode), file=sys.stderr)
        sys.exit(1)

    try:
        zinhom=readNormalizInhom(filename + "nsolve.out")
    except:
        print("ERROR:Could_not_read Normaliz_output", file=sys.
            stderr)
        sys.exit(1)

```

```

numzinhom = len(zinhom)

else: # run zsolve
    putMatMatrix(filename + ".rhs", np.array([hole]))
    if args.verbose:
        print(">>>>> Running zsolve to find the minimal solutions to f
            _=Am_ _Al.")
    try:
        if args.time:
            subprocess.check_call(["time", args.zsolve] + zsolveopts + [
                filename])
        else:
            subprocess.check_call([args.zsolve] + zsolveopts + [filename])
    except subprocess.CalledProcessError as err:
        print("ERROR: zsolve failed with exit code:" + str(err.
            returncode), file=sys.stderr)
        sys.exit(1)

#####IMPORTANT#####
#In contrast to the paper, we have to take the SECOND HALF of each solution (m,l),
#since we entered the linear system differently!
    try:
        zinhom = getMatMatrix(filename + ".zinhom")
    except:
        print("ERROR: Could not read zsolve output.", file=sys.stderr)
        sys.exit(1)
    numzinhom = len(zinhom)

if numzinhom == 0:
    print('Standard pairs of ' + str(hole) + ':')
    ##### implement Johannes' trick: need to change the numbering
    according to necessary_generators
    if args.trick:
        # output the necessary generators.
        # replace is needed to have the same output format as M2
        print('l:l: ' + str(necessary_generators).replace("[", "{x").replace("
            ,", ",x").replace("]", "}"))
    else:
        print('l:l: ' + str(list(range(number_generators))).replace("[", "{
            x").replace(",", ",x").replace("]", "}"))

else:

    if len(zinhom.shape) == 1:
        zinhom.shape = (1, zinhom.shape[0])
        seconhalf = np.take(zinhom, range(number_generators, zinhom.shape[1]), axis
            =1)
    #create input file for macaulay2
    with open(filename + ".m2", 'w') as m2file:
        m2file.write('toMonomial = V -> product_for_i_from_0_to_#_
            generators_R_1 list_(R.i)^(V#i); -- turn a vector into a
            monomial\n')
        m2file.write('toMonomialIdeal = M -> monomialIdeal_for_i_from_0

```

```

        to_numRows(M)-1).list_toMonomial_flatten_entries_(M^{i});
        --_turn_a_matrix_into_a_monomial_ideal_(each_row_is_a_
        monomial)\n')

m2file.write('M=_matrix_') #write matrix M whose rows are the
    lambdas
notfirst = False
for line in secondhalf:
    if notfirst:
        m2file.write(';')
    else:
        notfirst = True
        line.tofile(m2file,sep=" ")
m2file.write(");\n") #matrix M is now written
m2file.write('n=%d;_--_number_of_variables_\n' %int(secondhalf.
    shape[1]))
m2file.write('f=_map_ZZ^\n') #create n by n identity matrix
#### implement Johannes' trick: need to change the numbering
    according to necessary_generators
if args.trick:
    m2file.write('R=QQ[')
    for i in necessary_generators:
        m2file.write('x' + str(i) + ',')
    m2file.write('Degrees=>_entries_f,_MonomialOrder=>_
        Lex;\n') #Degrees makes ring multigraded
else:
    m2file.write('R=QQ[vars(52..51+n),_Degrees=>_entries_f,_
        MonomialOrder=>_Lex;\n') #Degrees makes ring
        multigraded
# variables starting at 52 are x0, x1, ...
m2file.write('I=toMonomialIdeal(M);\n')
m2file.write('SPL=_standardPairs_I;\n') # compute the standard
    pairs
m2file.write('for_entry_in_SPL_do_\n')
m2file.write('_<<<"_<<"_<<_toString(entry#0)_<<":_<<_
    rsort(entry#1)_<<_endl;\n')

m2file.write('quit();\n')

if args.verbose:
    print(">>>>>_Running_M2_to_compute_the_standard_pairs_...
        Output:")
else:
    print('Standard_pairs_of_' + str(hole) + ':')
try:
    subprocess.check_call([args.M2,"--script",filename + ".m2"])
except subprocess.CalledProcessError as err:
    print("ERROR:_Macaulay2_failed_with_exit_code:_ + str(err.
        returncode), file=sys.stderr)
    sys.exit(1)

```

BIBLIOGRAPHY

- [AB03] K. Woods A. Barvinok, *Short rational generating functions for lattice point problems*, J. Amer. Math. Soc. **16** (2003), 957–979.
- [AHK77] K. Appel, W. Haken, and J. Koch, *Every planar map is four colorable. II. Reducibility*, Illinois J. Math. **21** (1977), no. 3, 491–567. MR 0543793 (58 #27598b)
- [Alm98] Gert Almkvist, *Asymptotic formulas and generalized Dedekind sums*, Experiment. Math. **7** (1998), no. 4, 343–359. MR 2000d:11126
- [Bar94] A. Barvinok, *Polynomial time algorithm for counting integral points in polyhedra when the dimension is fixed*, Math of Operations Research **19** (1994), 769–779.
- [Bat94] Victor V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, J. Algebraic Geom. **3** (1994), no. 3, 493–535. MR 1269718
- [BBK⁺15] Matthias Beck, Benjamin Braun, Matthias Köppe, Carla D. Savage, and Zafeirakis Zafeirakopoulos, *s-lecture hall partitions, self-reciprocal polynomials, and Gorenstein cones*, Ramanujan J. **36** (2015), no. 1-2, 123–147. MR 3296715
- [BBK⁺16] ———, *Generating functions and triangulations for lecture hall cones*, SIAM J. Discrete Math. **30** (2016), no. 3, 1470–1479. MR 3531728
- [Bel58] Richard Bellman, *On a routing problem*, Quart. Appl. Math. **16** (1958), 87–90. MR 0102435
- [BEMPS10] Laura Beaudin, Joanna Ellis-Monaghan, Greta Pangborn, and Robert Shrock, *A little statistical mechanics for the graph theorist*, Discrete Math. **310** (2010), no. 13-14, 2037–2053. MR 2629922
- [BEN15] Michel Berkelaar, Kjell Eikland, and Peter Notebaert, *lp_solve : Open source (Mixed-Integer) Linear Programming system*, 2015.
- [BEZ03] Matthias Beck, David Einstein, and Shelemyahu Zacks, *Some experimental results on the Frobenius problem*, Experiment. Math. **12** (2003), no. 3, 263–269. MR 2034391
- [BG09] Winfried Bruns and Joseph Gubeladze, *Polytopes, rings, and K-theory*, Springer Monographs in Mathematics, Springer, Dordrecht, 2009. MR 2508056
- [BGT02] Winfried Bruns, Joseph Gubeladze, and Ngô Viêt Trung, *Problems and algorithms for affine semigroups*, Semigroup Forum **64** (2002), no. 2, 180–212. MR 1876854
- [BH93] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR MR1251956 (95h:13020)
- [BH17] Gabriele Balletti and Akihiro Higashitani, *Universal inequalities in ehrhart theory*, arxiv (2017).

- [BHM08] Matthias Beck, Christian Haase, and Asia R. Matthews, *Dedekind-Carlitz polynomials as lattice-point enumerators in rational polyhedra*, Math. Ann. **341** (2008), no. 4, 945–961, arXiv:0710.1323. MR MR2407333 (2009k:11161)
- [BIR⁺] W. Bruns, B. Ichim, T. Roemer, R. Sieg, and C. Soeger, *Normaliz. algorithms for rational cones and affine monoids*, Available at <https://www.normaliz.uni-osnabrueck.de>.
- [BK14] Matthias Beck and Florian Kohl, *Rademacher-Carlitz polynomials*, Acta Arith. **163** (2014), no. 4, 379–393. MR 3217673
- [BL16] Petter Brändén and Madeleine Leander, *Lecture hall p -partitions*, ArXiv e-prints (2016), 1609.02790.
- [BME97a] Mireille Bousquet-Mélou and Kimmo Eriksson, *Lecture hall partitions*, Ramanujan J. **1** (1997), no. 1, 101–111. MR 1607531
- [BME97b] ———, *Lecture hall partitions. II*, Ramanujan J. **1** (1997), no. 2, 165–185. MR 1606188
- [BP03] Matthias Beck and Dennis Pixton, *The Ehrhart polynomial of the Birkhoff polytope*, Discrete Comput. Geom. **30** (2003), no. 4, 623–637, arXiv:math.CO/0202267. MR 2 013 976
- [BR07] Matthias Beck and Sinai Robins, *Computing the continuous discretely*, Undergraduate Texts in Mathematics, Springer, New York, 2007, Integer-point enumeration in polyhedra. MR 2271992
- [BR15] ———, *Computing the continuous discretely*, second ed., Undergraduate Texts in Mathematics, Springer, New York, 2015, Integer-point enumeration in polyhedra, With illustrations by David Austin. MR 3410115
- [Bri88] M. Brion, *Points entiers dans les polyèdres convexes*, Ann. Sci. École Norm. Sup. **4** (1988), no. 21, 653–663.
- [BS] Petter Brändén and Liam Solus, (*personal communication*).
- [BZ06] Matthias Beck and Thomas Zaslavsky, *Inside-out polytopes*, Adv. Math. **205** (2006), no. 1, 134–162. MR 2254310
- [Ciu98] Mihai Ciucu, *An improved upper bound for the 3-dimensional dimer problem*, Duke Math. J. **94** (1998), no. 1, 1–11. MR 1635888
- [CLS11] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. MR 2810322
- [Cox15] David A. Cox, *Mirror symmetry and polar duality of polytopes*, Symmetry **7** (2015), no. 3, 1633–1645. MR 3407829
- [CR90] D. Cvetković and P. Rowlinson, *The largest eigenvalue of a graph: a survey*, Linear and Multilinear Algebra **28** (1990), no. 1-2, 3–33. MR 1077731
- [Ded53] Richard Dedekind, *Erläuterungen zu den Fragmenten xxviii*, Collected Works of Bernhard Riemann, Dover Publ., New York, 1953, pp. 466–478.
- [Die59] Ulrich Dieter, *Das Verhalten der Kleinschen Funktionen $\log \sigma_{g,h}(\omega_1, \omega_2)$ gegenüber Modultransformationen und verallgemeinerte Dedekindsche Summen*, J. Reine Angew. Math. **201** (1959), 37–70. MR 21 #3397

- [DNH97] Emanuela De Negri and Takayuki Hibi, *Gorenstein algebras of Veronese type*, J. Algebra **193** (1997), no. 2, 629–639. MR 1458806
- [EHHSM15] Viviana Ene, Jürgen Herzog, Takayuki Hibi, and Sara Saeedi Madani, *Pseudo-Gorenstein and level Hibi rings*, J. Algebra **431** (2015), 138–161. MR 3327545
- [EK18] Alexander Engström and Florian Kohl, *Transfer-matrix methods meet Ehrhart theory*, Advances in Mathematics **330** (2018), 1 – 37.
- [FLS07] Michael Freedman, László Lovász, and Alexander Schrijver, *Reflection positivity, rank connectivity, and homomorphism of graphs*, J. Amer. Math. Soc. **20** (2007), no. 1, 37–51. MR 2257396
- [FP05] Shmuel Friedland and Uri N. Peled, *Theory of computation of multidimensional entropy with an application to the monomer-dimer problem*, Adv. in Appl. Math. **34** (2005), no. 3, 486–522. MR 2123547
- [GS] Daniel R. Grayson and Michael E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at <http://www.math.uiuc.edu/Macaulay2/>.
- [HH16] Takayuki Hibi and Akihiro Higashitani, *Integer decomposition property of free sums of convex polytopes*, Ann. Comb. **20** (2016), no. 3, 601–607. MR 3537921
- [Hib87] Takayuki Hibi, *Distributive lattices, affine semigroup rings and algebras with straightening laws*, Commutative algebra and combinatorics (Kyoto, 1985), Adv. Stud. Pure Math., vol. 11, North-Holland, Amsterdam, 1987, pp. 93–109. MR 951198
- [Hib88] ———, *Level rings and algebras with straightening laws*, J. Algebra **117** (1988), no. 2, 343–362. MR 957445
- [Hib90] ———, *Some results on Ehrhart polynomials of convex polytopes*, Discrete Math. **83** (1990), no. 1, 119–121. MR 1065691
- [Hib92] ———, *Algebraic combinatorics on convex polytopes*, Carlsaw Publications, Glebe, 1992. MR 3183743
- [HKN16] Johannes Hofscheier, Lukas Katthän, and Benjamin Nill, *Ehrhart theory of spanning lattice polytopes*, arxiv (2016).
- [HKT] Christian Haase, Florian Kohl, and Akiyoshi Tsuchiya, *Levelness of order polytopes*, (in preparation).
- [HM07] Agnes M. Herzberg and M. Ram Murty, *Sudoku squares and chromatic polynomials*, Notices Amer. Math. Soc. **54** (2007), no. 6, 708–717. MR 2327972
- [HM16] Takayuki Hibi and Kazunori Matsuda, *Quadratic Gröbner bases of twinned order polytopes*, European J. Combin. **54** (2016), 187–192. MR 3459062
- [Hoc72] M. Hochster, *Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes*, Ann. of Math. (2) **96** (1972), 318–337. MR 0304376
- [HOT16] Takayuki Hibi, McCabe Olsen, and Akiyoshi Tsuchiya, *Gorenstein properties and integer decomposition properties of lecture hall polytopes*, 2016, arXiv:1608.03934.
- [HOT17] ———, *Self dual reflexive simplices with Eulerian polynomials*, Graphs Combin. (2017), To Appear; DOI 10.1007/s00373-017-1781-8.

- [HPPS14] Christian Haase, Andreas Paffenholz, Lindsay C. Piechnik, and Francisco Santos, *Existence of unimodular triangulations — positive results*, arXiv:1405.1687 (2014).
- [HS16] Takayuki Hibi and Liam Solus, *Facets of the r -stable (n, k) -hypersimplex*, Ann. Comb. **20** (2016), no. 4, 815–829. MR 3572388
- [HTY09a] H. Hara, A. Takemura, and R. Yoshida, *A markov basis for conditional test of common diagonal effect in quasi-independence model for two-way contingency tables*, J of Computational Statistics and Data Analysis **53** (2009), 1006–1014.
- [HTY09b] R. Hemmecke, A. Takemura, and R. Yoshida, *Computing holes in semi-groups and its applications to transportation problems*, Contributions to Discrete Mathematics **4** (2009), no. 1, 81 – 91.
- [Huh12] June Huh, *Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs*, J. Amer. Math. Soc. **25** (2012), no. 3, 907–927.
- [HY18] Akihiro Higashitani and Kohji Yanagawa, *Non-level semi-standard graded Cohen-Macaulay domain with h -vector (h_0, h_1, h_2)* , J. Pure Appl. Algebra **222** (2018), no. 1, 191–201. MR 3681002
- [HZ74] Friedrich Hirzebruch and Don Zagier, *The Atiyah-Singer Theorem and Elementary Number Theory*, Publish or Perish Inc., Boston, Mass., 1974. MR 58 #31291
- [ILL⁺07] Srikanth B. Iyengar, Graham J. Leuschke, Anton Leykin, Claudia Miller, Ezra Miller, Anurag K. Singh, and Uli Walther, *Twenty-four hours of local cohomology*, Graduate Studies in Mathematics, vol. 87, American Mathematical Society, Providence, RI, 2007. MR 2355715
- [JS14] Katharina Jochemko and Raman Sanyal, *Arithmetic of marked order polytopes, monotone triangle reciprocity, and partial colorings*, SIAM J. Discrete Math. **28** (2014), no. 3, 1540–1558. MR 3262594
- [JVV90] F. Jaeger, D. L. Vertigan, and D. J. A. Welsh, *On the computational complexity of the Jones and Tutte polynomials*, Math. Proc. Cambridge Philos. Soc. **108** (1990), no. 1, 35–53. MR 1049758
- [Kat13] Lukas Katthän, *Decomposing sets of inversions*, Electron. J. Combin. **20** (2013), Paper 40, 16. MR 3035050
- [KKMSD73] G. Kempf, Finn Faye Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal embeddings. I*, Lecture Notes in Mathematics, Vol. 339, Springer-Verlag, Berlin-New York, 1973. MR 0335518
- [KLRY16] Florian Kohl, Yanxi Li, Johannes Rauh, and Ruriko Yoshida, *An introduction to HASE*, 2016, available at <http://ehrhart.math.fu-berlin.de/People/fkohl/HASE/Manual/>.
- [KLRY18] Florian Kohl, Yanxi Li, Johannes Rauh, and Ruriko Yoshida, *Semigroups — a computational approach*, Advanced Studies in Pure Mathematics **77** (2018), 155–170.
- [Knu81] Donald E. Knuth, *The Art of Computer Programming. Vol. 2*, second ed., Addison-Wesley Publishing Co., Reading, Mass., 1981. MR 83i:68003
- [KO] Florian Kohl and McCabe Olsen, *Level algebras and s -lecture hall polytopes*, <https://arxiv.org/abs/1710.10892>.

- [LM08] Per Håkan Lundow and Klas Markström, *Exact and approximate compression of transfer matrices for graph homomorphisms*, LMS J. Comput. Math. **11** (2008), 1–14. MR 2379936
- [Mey57] Curt Meyer, *Über einige Anwendungen Dedekindscher Summen*, J. Reine Angew. Math. **198** (1957), 143–203. MR 21 #3396
- [Mey00] Carl Meyer, *Matrix analysis and applied linear algebra*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000, With 1 CD-ROM (Windows, Macintosh and UNIX) and a solutions manual (iv+171 pp.). MR 1777382
- [Miy17] Mitsuhiro Miyazaki, *On the generators of the canonical module of a Hibi ring: a criterion of level property and the degrees of generators*, J. Algebra **480** (2017), 215–236. MR 3633306
- [MS79] Werner Meyer and Robert Sczech, *Über eine topologische und zahlentheoretische Anwendung von Hirzebruchs Spitzenauflösung*, Math. Ann. **240** (1979), no. 1, 69–96. MR 82c:10035
- [MS05] Ezra Miller and Bernd Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005. MR 2110098
- [Ols17] McCabe Olsen, *Hilbert bases and lecture hall partitions*, Ramanujan J. (2017), To Appear; DOI 10.1007/s11139-017-9933-2.
- [OT12] Mitsunori Ogawa and Akimichi Takemura, *Markov bases for typical block effect models of two-way contingency tables*, J. Multivariate Anal. **112** (2012), 219–229. MR 2957298
- [Pom93] James E. Pommersheim, *Toric varieties, lattice points and Dedekind sums*, Math. Ann. **295** (1993), no. 1, 1–24. MR 1198839
- [PS13] Thomas W. Pensyl and Carla D. Savage, *Lecture hall partitions and the wreath products $C_k \wr S_n$* , Integers **12B** (2012/13), no. Proceedings of the Integers Conference 2011, Paper No. A10, 18. MR 3055684
- [PS13] ———, *Rational lecture hall polytopes and inflated Eulerian polynomials*, Ramanujan J. **31** (2013), no. 1-2, 97–114. MR 3048657
- [RA05] J. L. Ramírez Alfonsín, *The Diophantine Frobenius problem*, Oxford Lecture Series in Mathematics and its Applications, vol. 30, Oxford University Press, Oxford, 2005. MR 2260521
- [RS14] Johannes Rauh and Seth Sullivant, *The Markov basis of $K_{3,N}$* , Preprint, available at <https://arxiv.org/abs/1406.5936> (2014).
- [RS15] ———, *Lifting Markov bases and higher codimension toric fiber products*, Journal of Symbolic Computation **74** (2015), 276–307.
- [Sav16] Carla D. Savage, *The mathematics of lecture hall partitions*, J. Combin. Theory Ser. A **144** (2016), 443–475. MR 3534075
- [Sch86] A. Schrijver, *Theory of linear and integer programming*, Wiley-Interscience Series in Discrete Mathematics, John Wiley & Sons Ltd., Chichester, 1986, A Wiley-Interscience Publication. MR MR874114 (88m:90090)

- [Sch03] Alexander Schrijver, *Combinatorial optimization. Polyhedra and efficiency. Vol. A*, Algorithms and Combinatorics, vol. 24, Springer-Verlag, Berlin, 2003, Paths, flows, matchings, Chapters 1–38. MR 1956924
- [Sol98] David Solomon, *Algebraic properties of Shintani’s generating functions: Dedekind sums and cocycles on $\mathrm{PGL}_2(\mathbf{Q})$* , Compositio Math. **112** (1998), no. 3, 333–362. MR MR1631700 (99g:11137)
- [SS12] Carla D. Savage and Michael J. Schuster, *Ehrhart series of lecture hall polytopes and Eulerian polynomials for inversion sequences*, J. Combin. Theory Ser. A **119** (2012), no. 4, 850–870. MR 2881231
- [Sta73] Richard P. Stanley, *Acyclic orientations of graphs*, Discrete Math. **5** (1973), 171–178. MR 0317988
- [Sta77] ———, *Cohen-Macaulay complexes*, Higher combinatorics (Proc. NATO Advanced Study Inst., Berlin, 1976), NATO Adv. Study Inst. Ser., Ser. C: Math. and Phys. Sci., 31, Reidel, Dordrecht, 1977, pp. 51–62.
- [Sta78] ———, *Hilbert functions of graded algebras*, Advances in Math. **28** (1978), no. 1, 57–83. MR 0485835
- [Sta86] ———, *Two poset polytopes*, Discrete Comput. Geom. **1** (1986), no. 1, 9–23. MR 824105
- [Sta91] ———, *On the Hilbert function of a graded Cohen-Macaulay domain*, J. Pure Appl. Algebra **73** (1991), no. 3, 307–314. MR 1124790
- [Sta96] ———, *Combinatorics and commutative algebra*, second ed., Progress in Mathematics, vol. 41, Birkhäuser Boston, Inc., Boston, MA, 1996. MR 1453579
- [Sta09] Alan Stapledon, *Inequalities and Ehrhart δ -vectors*, Trans. Amer. Math. Soc. **361** (2009), no. 10, 5615–5626. MR 2515826
- [Sta12] Richard P. Stanley, *Enumerative combinatorics. Volume 1*, second ed., Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012. MR 2868112
- [Sta16] Alan Stapledon, *Additive number theory and inequalities in Ehrhart theory*, Int. Math. Res. Not. IMRN (2016), no. 5, 1497–1540. MR 3509934
- [Stu96] Bernd Sturmfels, *Gröbner bases and convex polytopes*, University Lecture Series, vol. 8, American Mathematical Society, Providence, RI, 1996. MR MR1363949 (97b:13034)
- [STV95] Bernd Sturmfels, Ngô Việt Trung, and Wolfgang Vogel, *Bounds on degrees of projective schemes*, Math. Ann. **302** (1995), no. 3, 417–432. MR 1339920
- [SV12] Carla D. Savage and Gopal Viswanathan, *The $1/k$ -Eulerian polynomials*, Electron. J. Combin. **19** (2012), no. 1, Paper 9, 21. MR 2880640
- [SV15] Carla D. Savage and Mirkó Visontai, *The s -Eulerian polynomials have only real roots*, Trans. Amer. Math. Soc. **367** (2015), no. 2, 1441–1466. MR 3280050
- [SW12] Bernd Sturmfels and Volkmar Welker, *Commutative algebra of statistical ranking*, J. Algebra **361** (2012), 264–286. MR 2921622

- [Tei07] Vladimir B Teif, *General transfer matrix formalism to calculate dna-protein-drug binding in gene regulation: Application to or operator of phage λ .*, Nucleic Acids Research **35.11** (2007).
- [tt08] 4ti2 team, *4ti2—a software package for algebraic, geometric and combinatorial problems on linear spaces*, Available at www.4ti2.de, 2008.
- [TTY08] A. Takemura, P. Thomas, and R. Yoshida, *Holes in semigroups and their applications to the two-way common diagonal effect model*, In: Proceedings of the 2008 International Conference on Information Theory and Statistical Learning, ITSL (2008).
- [TY08] A. Takemura and R. Yoshida, *A generalization of the integer linear infeasibility problem*, Discrete Optimization **5** (2008), no. 1, 36–52.
- [Whi32] Hassler Whitney, *A logical expansion in mathematics*, Bull. Amer. Math. Soc. **38** (1932), no. 8, 572–579. MR 1562461
- [Zie95] Günter M. Ziegler, *Lectures on Polytopes*, Springer-Verlag, New York, 1995, Revised edition, 1998; “Updates, corrections, and more” at www.math.tu-berlin.de/~ziegler. MR 96a:52011

LIST OF FIGURES

| | | |
|-------------|--|----|
| Figure 2.1 | The unit square P and its dilates $2P$ and $3P$ (green), the cone over P (gray) and the (conical hull of the) canonical module (blue). | 11 |
| Figure 2.2 | The box P and its dilate $2P$, the cone over P (gray) and the (conical hull of the) canonical module (blue). | 12 |
| Figure 2.3 | Deletion-contraction along e | 14 |
| Figure 2.4 | G , an acyclic orientation of G , and a cyclic orientation of G | 14 |
| Figure 2.5 | $G = C_3$ and the two walks of length 2 from v_1 to v_1 | 16 |
| Figure 3.1 | The Cartesian graph product illustrated. | 23 |
| Figure 3.2 | M_{P_3} | 25 |
| Figure 3.3 | Adjacency matrix of M_{P_3} for 4 colors. | 26 |
| Figure 3.4 | Integer points in dashed triangles correspond to proper k -colorings of P_2 | 29 |
| Figure 3.5 | P_3 with vertex x colored by 3, the corresponding inside-out polytope (P, \mathcal{H}) , and the <i>induced</i> inside-out polytope $(\overline{P}, \overline{\mathcal{H}})$ | 34 |
| Figure 3.6 | L matrix where we only quotient out by permutations of colors. | 37 |
| Figure 3.7 | An illustration of the proper 3-colorings counted by the first row of L | 38 |
| Figure 3.8 | In (a), we illustrate the general form of a coloring counted by $(L^5)_{11}$. In (b), we give an explicit example of a 3-coloring counted by $(L)_{11}^5$ | 38 |
| Figure 3.9 | $P_3 \times P_6$ | 39 |
| Figure 3.10 | The connected graphs on at most five vertices with a hidden symmetry. | 41 |
| Figure 3.11 | All $(1, 2, 3)$ -restricted, compatible pairs of acyclic orientations and (not necessary proper) colorings of $P_3 \times P_2$ | 43 |
| Figure 4.1 | The Hasse diagram of the poset $\Pi = \{i \lessdot j \gtrdot k\}$, $\Pi \cup \{\infty\}$, and $\Pi \cup \{-\infty\}$ | 50 |
| Figure 4.2 | The order and chain polytope of the poset described in Figure 4.1. | 51 |
| Figure 4.3 | Edges belonging to longest chain are colored red. | 52 |
| Figure 4.4 | Fink's poset and the minimal element \mathbf{y} illustrated. | 54 |
| Figure 4.5 | Ordinal sum of a chain of length 3 and an antichain of length 2. | 59 |
| Figure 4.6 | Original poset (on the right) and the two quotient posets (on the left and in the middle). | 60 |
| Figure 5.1 | Two three-dimensional lecture hall polytopes (not drawn to scale) | 71 |

LIST OF TABLES

| | | |
|-----------|---|----|
| Table 5.1 | Palindromic s -Eulerian Polynomials | 78 |
|-----------|---|----|

ZUSAMMENFASSUNG

Diese Arbeit beschäftigt sich mit den Eigenschaften und Anwendungen von Gitterpolytopen, sowie Eigenschaften von Halbgruppen. Die nötigen Grundlagen werden in Kapitel 2 beschrieben.

In Kapitel 3 werden zulässige Färbungen von einer speziellen Familie von Graphen untersucht. Diese Familie ist durch das kartesische Produkt $G \times P_n$ bzw. $G \times C_n$ eines beliebigen Graphs G mit einem Pfad- (P_n) beziehungsweise einem Kreisgraph C_n mit n Ecken gegeben. Um diese Problemstellung zu untersuchen, werden Transfermatrixmethoden mit umgestülpten Polytopen (*inside-out polytopes*) kombiniert. Um die Grösse der Transfermatrix zu beschränken, werden Gruppenwirkungen, die durch die zugrundeliegende Symmetrie gegeben sind, ausgenutzt. Dies führt zu einer expliziten Formel für das chromatische Polynom für Graphen der Form $G \times P_n$. Desweiteren wird das asymptotische Verhalten der Anzahl der zulässigen Färbungen von $G \times C_n$ beschrieben.

In Kapitel 4 werden Halbordnungen und die assoziierten Ordnungspolytope (*order polytopes*) auf die algebraische Level-Eigenschaft untersucht. Diese Eigenschaft verallgemeinert die Gorenstein-Eigenschaft und beschreibt die Struktur der minimalen Elemente des kanonischen Moduls. Während es einfach ist, die Gorenstein-Eigenschaft für Halbordnungen zu klassifizieren, verhält sich die Level-Eigenschaft subtiler. Wir benutzen gewichtete, gerichtete Graphen, um eine vollständige Klassifizierung zu erreichen. Mit dieser Klassifizierung beschreiben wir eine neue, unendliche Familie von Halbordnungen, die die Level-Eigenschaft besitzen. Desweiteren wird eine Klassifizierung der Level-Eigenschaft für Alkopolytope (*alcoved polytopes*) gegeben.

Die Gorenstein- und die Level-Eigenschaft von \mathfrak{s} -Hörsaal-Polytopen (*s-lecture hall polytopes*) wird in Kapitel 5 untersucht. In der Literatur wurde bisher nur die Gorenstein-Eigenschaft für Hörsaal-Kegel untersucht, siehe [BBK⁺15]. Aufbauend auf diesen Resultaten klassifizieren wir die Gorenstein-Eigenschaft der Hörsaal-Polytope für alle \mathfrak{s} -Folgen. Die Gorenstein-Eigenschaft hat direkte Implikationen für die Struktur einer Erzeugendenfunktion. Weiterhin geben wir auch eine Charakterisierung der Level-Eigenschaft in Form von Inversionsfolgen. Wir illustrieren, wie man diese Charakterisierung benutzen kann, um zu zeigen, dass jedes 2-dimensionale Hörsaal-Polytop die Level-Eigenschaft hat. Der Hauptpunkt ist hier, explizit zu zeigen, wie man diese Klassifizierung in niedrigen Dimensionen anwenden kann.

Die Level- und die Gorenstein-Eigenschaften können durch die Struktur eines Halbgruppenrings beschrieben werden. Kapitel 6 widmet sich der Frage, wann solche Halbgruppen(ringe) Löcher haben können. Diese Arbeit baut auf Resultaten aus [HTY09b] auf, wo ein Algorithmus zum Finden und Beschreiben solcher Löcher beschrieben wird. Dieser Algorithmus wurde von uns implementiert. Die dazugehörige Software kann unter <http://ehrhart.math.fu-berlin.de/People/fkohl/HASE/> gefunden werden. Mithilfe dieser Software konnten wir die Löcher des gewöhnlichen Diagonaleffekt-Modells (*common diagonal effect model*) beschreiben, sowie zeigen, dass die assoziierte Halbgruppe des linearen Ordnungspolytops (*linear ordering polytope*) keine Löcher hat.

Kapitel 7 kann als Bedienungsanleitung der dazugehörigen Software angesehen werden. Anhand von expliziten Beispielen werden die Bedienung der Software beschrieben, sowie verschiedene Optionen illustriert. Desweiteren zeigen wir, wie man das Programm benutzen kann, um das Frobenius-Problem zu untersuchen. Im Anhang ist der Quellcode von HASE angegeben.

SELBSTSTÄNDIGKEITSERKLÄRUNG

Hiermit versichere ich, dass ich alle Hilfsmittel und Hilfen angegeben und auf dieser Grundlage die Arbeit selbstständig verfasst habe. Zudem wurde die Arbeit auch noch nicht in einem früheren Promotionsverfahren eingereicht.

Berlin, July 7, 2018

Florian Kohl