



Relations between positivity, localization and degrees of freedom: The Weinberg–Witten theorem and the van Dam–Veltman–Zakharov discontinuity



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ABSTRACT

The problem of accounting for the quantum degrees of freedom in passing from massive higher-spin potentials to massless ones, and the inverse problem of “fattening” massless tensor potentials of helicity $\pm h$ to their massive $s = |h|$ counterparts, are solved – in a perfectly ghost-free approach – using “string-localized fields”.

This approach allows to overcome the Weinberg–Witten impediment against the existence of massless $|h| \geq 2$ energy–momentum tensors, and to qualitatively and quantitatively resolve the van Dam–Veltman–Zakharov discontinuity concerning, e.g., very light gravitons, in the limit $m \rightarrow 0$.

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1. Introduction

In relativistic quantum field theory, the quantization of interacting massless or massive classical potentials of higher spin ($s \geq 1$) either violates Hilbert space positivity which is an indispensable attribute of the probability interpretation of quantum theory, or leads to a violation of the power counting bound of renormalizability whose maintenance requires again a violation of positivity.

In order to save positivity for those quantum fields which correspond to classically gauge invariant observables, one usually formally extends the theory by adding degrees of freedom in the form of negative metric Stückelberg fields and “ghosts” without a counterpart in classical gauge theories. The justification for this quantum gauge setting is that one can extract from the indefinite metric Krein space a Hilbert space that the gauge invariant operators generate from the vacuum.

This situation is satisfactory as far as the vacuum sector and the perturbative construction of a unitary gauge-invariant S-matrix are concerned. However the theory remains incomplete in that it provides no physical interpolating fields that mediate between the

causal localization of the field theory and the analytic structure of the S-matrix in terms of fields that connect charged states with the vacuum. Expressed differently, gauge theory allows to compute the perturbative S-matrix, but cannot construct its off-shell extension on a Hilbert space.

There are two famous results about the higher-spin massless case. The first is the Weinberg–Witten theorem [24] which states that for $s \geq 2$, no point-localized stress-energy tensor exists such that the Poincaré generators are moments of its zero-components. This result also obstructs the semiclassical coupling of massless higher-spin matter to gravity.

The second is the DVZ observation due to van Dam and Veltman [25] and to Zakharov [28], that in interacting models with $s \geq 2$, scattering amplitudes are discontinuous in the mass at $m = 0$, i.e., the scattering of matter through exchange of massless gravitons (say) is significantly different from the scattering via gravitons of a very small mass.

Both problems can be addressed, without being plagued by the positivity troubles of gauge theories, with the help of “string-localized quantum fields” defined in the physical Hilbert space. The latter may be regarded as a fresh start to Mandelstam’s attempts [12] to reformulate gauge theories as full-fledged theories in which all fields live on the Hilbert space of the field strength. The new point of view was triggered by a new approach to Wigner’s infinite-spin representation [17], that proved to be useful also

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for finite spin. String-localized potentials for finite spin s are integrals over their field strengths along a “string” $x + \mathbb{R}_0^+ e$, see Eq. (1.1), Eq. (2.2), or Eq. (3.3), where e is a (spacelike) direction. This evidently does not change the particle content. The main benefit of these potentials is their improved UV dimension $d_{UV} = 1$ rather than $d_{UV} = s + 1$, admitting renormalizable interactions that are otherwise excluded by power-counting.

A point-localized massive spin s potential can be split up into a string-localized potential that has a massless limit, and derivatives of one or more so-called “escort fields”. The role of the latter is to separate off derivative terms from the interaction Lagrangean or from conserved currents, that do not contribute to the S -matrix or to charges and Poincaré generators, respectively. They thus “carry away” all non-renormalizable UV fluctuations and singularities in the limit $m \rightarrow 0$.

How this works, may be illustrated in the case of QED [21, 22,13]: The coupling to the indefinite Maxwell potential A^F (“F” stands for “Feynman gauge”) is replaced by a coupling $j^\mu A_\mu^P$ to the massive Proca potential A^P . This avoids negative-norm states, but the interaction is non-renormalizable because of the UV dimension 2 of the Proca potential. Now, the decomposition (see Sect. 2) $A_\mu^P(x) = A_\mu(x, e) - m^{-1} \partial_\mu a(x, e)$ into a string-localized potential and its escort is brought to bear: $A_\mu(x, e)$ has UV dimension 1 and is regular at $m = 0$. The UV-divergent part of the interaction is carried away by the escort field: $-m^{-1} j^\mu \partial_\mu a(e) = -\partial_\mu (m^{-1} j^\mu a(e))$ is a total derivative and may be discarded from the interaction Lagrangean. The remaining string-localized (but equivalent to the point-localized) interaction $j^\mu A_\mu(e)$ has UV dimension 4, and remains renormalizable at $m = 0$.

The ongoing analysis of perturbation theory with string-localized interactions [8,13,16] gives strong evidence that the resulting theory is order-by-order renormalizable, and equivalent to the “usual” QED. The scattering matrix can be made independent of the string direction e , provided a suitable renormalization condition is satisfied. In that case, interacting observable fields are string-independent and hence local. These conditions can be seen as an analogue of Ward identities imposed in order to ensure BRST invariance in point-localized but indefinite approaches [20,5], see also footnote 1 in Sect. 6. Indeed, the conditions can also be formulated in a cohomological manner. Yet, the precise relation between gauge invariance and string-independence remains to be explored.

We give more details, especially on the preservation of causality, for the (much easier) case of the coupling of a massive vector field to an external source in Sect. 2.1.

Whereas string-localized perturbation theory is still in its infancy, the problems of massless currents and energy-momentum tensors as well as the continuous passage from free massive fields to their massless helicity counterparts can be completely solved. The presentation of this solution is the principal aim of this letter, including also the opposite direction, sometimes (in connection with the Higgs mechanism) referred to as “fattening”.

1.1. Overview of results

We outline the general picture for arbitrary integer spin s , referring to [15] for further details. As the case $s = 2$ already exhibits all the features of the general case, we focus on $s = 1$ and $s = 2$ in Sect. 2 and Sect. 3.

The 2-point functions of covariant massless potentials are indefinite polynomials in the metric tensor $\eta_{\mu\nu}$, while their field strengths (curl in all indices) are positive. (By “positive”, it is understood “positive-semidefinite”, accounting for null states due to equations of motion like $\partial^\mu F_{\mu\nu} = 0$.) Alternatively, the field strengths can be constructed, without reference to a potential, directly on the Fock space over the unitary massless helicity $h = \pm s$

Wigner representations of the Poincaré group. This is exposed in standard textbooks, e.g., [23]. One can construct potentials in the Coulomb gauge on the same Hilbert space, but one gets into conflict with Poincaré covariance: Lorentz transformations result in an operator-valued gauge transformation due to the affine nature of the Wigner phase. When the potentials are required for interactions, and one has to compromise between positivity or Lorentz invariance, preference is usually given to covariance.

For some early treatments of massive free tensor fields of higher spin, see [4,7]. We freely adopt the name “Proca” for all spins $s \geq 1$. The Proca potentials are symmetric traceless and conserved tensors $A_{\mu_1 \dots \mu_s}^P(x)$ of rank s . Their 2-point functions obtained from the (m, s) Wigner representation [23] are polynomials in the positive projection orthogonal to the momentum (sign convention $\eta_{00} = +1$)

$$-\pi_{\mu\nu}(p) = -\eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}$$

with coefficients dictated by symmetry and tracelessness. The momenta in the numerator cause the UV dimension $d_{UV} = s + 1$ and, by power counting, jeopardize the renormalizability of minimal couplings to currents.

The potentials evidently admit no massless limit. Only their field strengths $F_{[\mu_1 \nu_1] \dots [\mu_s \nu_s]}$ exist at $m = 0$ because the curls kill the terms with momentum factors.

We define symmetric free tensor fields $A_{\mu_1 \dots \mu_r}^{(r)}(x, e)$ of rank $0 \leq r \leq s$ on the Fock space of the massive field strengths such that

- All $A^{(r)}$ have UV dimension $d_{UV} = 1$ and are regular in the massless limit.
- The potential A^P can be decomposed in a way that (i) all contributions of UV dimension > 1 are isolated as derivatives of the “escort” fields $A^{(r)}$ of lower rank $r < s$, and (ii) the singular behaviour at $m \rightarrow 0$ is manifest in the expansion coefficients (inverse powers of m).
- The massive fields $A^{(r)}$ are coupled among each other through their traces and divergences. In the massless limit, they become traceless and conserved, and their field equations and 2-point functions decouple.
- At $m = 0$, the escort $A^{(0)}$ is the canonical massless scalar φ . The tensors $A^{(r>0)}$ are potentials for the field strengths of helicity $h = \pm r$ [23]. They were previously constructed [18] without an approximation from $m > 0$.
- Conversely, the given massless potential $A^{(s)}$ of any helicity $h = \pm s$ can be made massive (“fattening”) in the same way as for scalar and Dirac fields, namely by simply changing the dispersion relation $p^0 = \omega_m(\vec{p})$. The fattened field brings along with it all lower rank fields $A^{(r)}$ by virtue of the coupling through the divergence. We give a surprisingly simple formula involving only derivatives, to restore the exact Proca potential A^P from the fattened $A^{(s)}$.
- The massless limit shows the way to construct a stress-energy tensor for the massless fields that decouples into a direct sum of mutually commuting stress-energy tensors $T^{(r)}$ for the helicity potentials $A^{(r)}$.
- None of these constructions refers to a classical action principle. The quantization is manifest and without ghosts from the outset.

The massless limit describes the exact splitting of the (m, s) Wigner representation into massless helicity representations with $h = \pm r$ ($r = 1, \dots, s$) and $h = 0$.

In particular, the number $2s + 1$ of one-particle states at fixed momentum is preserved. In contrast, the “fattening” of the massless potential of helicity s increases the number of one-particle states, because its 2-point function is a semi-definite quadratic

form of rank 2 that becomes rank $2s + 1$ under the deformation of the dispersion relation.

These facts yield the obvious explanation of the DVZ discontinuity [25,28] in linearized gravity coupled to external sources: The spin 2 Proca potential A^P (or its analog in the indefinite Feynman gauge) is not continuously connected with a massless helicity $h = \pm 2$ potential. At each positive mass, the former has contributions from all $r \leq 2$. Rejecting at $m = 0$ the helicities $|h| < 2$ causes the discontinuity. We shall exhibit in Sect. 3 that in the ghost-free setting, at $m = 0$ only the helicities $h = \pm 2$ (the linearized massless gravity) and $h = 0$ survive in the coupling to the external source. The $h = \pm 2$ part is the linearized massless gravity. The additional scalar $\varphi(x)$ is the massless limit of the scalar escort field $A^{(0)}(x, e)$ and couples to the trace of the stress-energy tensor.

These results state the preservation of degrees of freedom (of free fields coupled to external sources) in a ghost-free language. The DVZ discontinuity arises by dropping the scalar “by hand” at $m = 0$. In contrast, there exist several ideas to explain how the scalar, and hence the discontinuity, could be instead dynamically suppressed in the presence of suitable interactions. Notably, Vainshtein [26] has presented a model with a non-perturbative screening mechanism that is effective only at small distances. It would be extremely rewarding to see how this screening emerges in a ghost-free approach. Other authors appeal to curvature effects or extra dimensions, cf. the review [9]. We hope that our physical approach to free fields will also contribute to a better understanding of the interacting models.

The stated properties of the massless potentials and stress-energy tensors are clearly at variance with many No-Go theorems, including the Weinberg–Witten theorem. This is possible because they are string-localized. Their 2-point functions involve, instead of the singular (as $m \rightarrow 0$) tensor $\pi_{\mu\nu}(p)$ or indefinite tensor $\eta_{\mu\nu}$, a suitable tensor $E_{\mu\nu}(p)$ whose substitution into the 2-point functions (i) preserves positivity, (ii) does not affect the field strengths, and (iii) has a regular limit $m \rightarrow 0$.

The No-Go theorems may be attributed to the fact that such a tensor $E_{\mu\nu}(p)$ does not exist, if it is allowed to be a function of the momentum only. Instead,

$$E(e, e')_{\mu\nu}(p) := \eta_{\mu\nu} - \frac{p_\mu e_\nu}{(pe)_+} - \frac{e'_\mu p_\nu}{(pe')_+} + \frac{(ee')p_\mu p_\nu}{(pe)_+(pe')_+}$$

(where $i/(k)_+ = i/(k+i0)$ is the Fourier transform of the Heaviside function) are distributions in p and two four-vectors e, e' . If $E_{\mu\nu}$ is substituted for $\pi_{\mu\nu}$ or $\eta_{\mu\nu}$, the potentials depend on e , but the field strengths will not.

In momentum space, the integration

$$X(x, e) \equiv (I_e X)(x) := \int_0^\infty d\lambda X(x + \lambda e) \tag{1.1}$$

produces the denominators $i((pe) + i0)^{-1}$ in the creation part and $-i((pe) - i0)^{-1}$ in the annihilation part. Thus, fields whose 2-point functions are polynomials in $E_{\mu\nu}$ are necessarily localized along the “string” $x + \mathbb{R}_0^+ e$.

String-localization requires some comments. First, it is not a feature of the associated particles, but of the fields that may be used to couple them to other particles. (The only exception are particles in the infinite-spin representations [17,10], that are beyond the scope of this letter.)

Eq. (1.1) (and its generalizations involving several integral operations I_e) imply the Poincaré transformations of string-localized fields

$$U_{a,\Lambda} A_{\mu_1 \dots \mu_r}(x, e) U_{a,\Lambda}^* = \left(\prod_i \Lambda^{v_i} \mu_i \right) A_{\nu_1 \dots \nu_r}(a + \Lambda x, \Lambda e), \tag{1.2}$$

i.e., the direction of the string is transformed along with its apex x and the tensor components of the field tensor.

There is no conflict with the principle of causality, which is as imperative in relativistic quantum field theory as Hilbert space positivity. String-localized fields satisfy causal commutation relations according to their localization: two fields commute whenever their strings are pointwise spacelike separated. There are sufficiently many spacelike separated pairs of spacelike or lightlike strings to construct scattering states by asymptotic cluster properties (Haag–Ruelle theory). For this reason, scattering theory requires $e^2 \leq 0$.

String-localized interactions admit couplings of physically massive tensor potentials without spontaneous symmetry breaking (cf. Sect. 6). Instead, when coupling self-interacting massive vector bosons (like W and Z bosons) via their string-localized potentials, the string-independence can only be achieved with the help of a boson with properties like the Higgs, including a quartic self-interaction [22]. Its role is, however, not the generation of the mass, but the preservation of the renormalizability and locality.

Examples of new renormalizable interactions in the string-localized setting could be the coupling of matter to gravitons through the string-localized potentials $A^{(2)}$, and perhaps the self-coupling of gravitons.

In the sequel, we give more details for spin 1 and 2. All displayed linear relations between fields follow from their definitions by integrals and derivatives of point-localized fields, e.g., by inspection of their integral representations in terms of creation and annihilation operators.

We write 2-point functions throughout as

$$(\Omega, X(x)Y(y)\Omega) = \int d\mu_m(p) \cdot e^{-ip(x-y)} \cdot {}_m M^{X,Y}(p),$$

$$\text{where } d\mu_m(p) = \frac{d^4 p}{(2\pi)^3} \delta(p^2 - m^2) \theta(p^0).$$

2. Spin one

The 2-point function of the massless Feynman gauge potential

$${}_0 M^{A_\mu^F, A_\nu^F} = -\eta_{\mu\nu}$$

is indefinite. Its curl $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Maxwell field with positive 2-point function

$${}_0 M^{F_{\mu\nu}, F_{\kappa\lambda}} = -p_\mu p_\kappa \eta_{\nu\lambda} + p_\mu p_\lambda \eta_{\nu\kappa} + p_\nu p_\kappa \eta_{\mu\lambda} - p_\nu p_\lambda \eta_{\mu\kappa}.$$

The massive Proca potential satisfies $\partial^\mu A_\mu^P = 0$. Its positive 2-point function is

$${}_m M^{A_\mu^P, A_\nu^P} = -\pi_{\mu\nu}(p). \tag{2.1}$$

The curl kills the term $p_\mu p_\nu / m^2$, so the field strength is regular at $m = 0$. The field equation $\partial^\mu F_{\mu\nu} = -m^2 A_\nu^P$ gives back the potential in terms of its field strength.

Only for $s = 1$, the massless limit can be achieved with point-localized fields: by inspection of their 2-point functions, ${}_m A_\mu^P$ is regular at $m = 0$, where it decouples from $F_{\mu\nu}$ and becomes the derivative of the canonical scalar free field φ with ${}_0 M^{\partial_\mu \varphi, \partial_\nu \varphi} = p_\mu p_\nu$.

In the string-localized setting, the massless scalar emerges without derivative. We define

$$A_\mu^{(1)}(x, e) \equiv A_\mu(x, e) := (I_e F_{\mu\nu})(x) e^\nu \equiv \int_0^\infty d\lambda F_{\mu\nu}(x + \lambda e) e^\nu, \tag{2.2}$$

$$A^{(0)}(x, e) \equiv a(x, e) := -m^{-1} \partial^\mu A_\mu(x, e). \tag{2.3}$$

$A_\mu(e)$ is regular in the massless limit because $F_{\mu\nu}$ is. That $a(e)$ is also regular can be seen from

$${}_m M^{A_\mu(-e), A_\nu(e')} = -E(e, e')_{\mu\nu}(p), \quad (2.4)$$

which implies by the definition of $a(e)$

$${}_m M^{a(-e), A_\nu(e')} = O(m), \quad {}_m M^{a(-e), a(e')} = 1 + O(m^2).$$

(As the fields are distributions also in e [14], we have to admit independent string directions e, e' . The choice “ $-e$ ” is a convenience paying off for higher spin [15].)

At $m = 0$, the fields a and A_μ decouple, and converge to the massless scalar and (as the terms $O(p/(pe))$ in Eq. (2.4) do not contribute to $F_{\mu\nu}$) to a string-localized massless potential for the Maxwell field strength.

In addition, one gets the decomposition underlying the QED example in Sect. 1

$$A_\mu^P(x) = A_\mu(x, e) - m^{-1} \partial_\mu a(x, e). \quad (2.5)$$

The taming of the UV behaviour is seen from Eq. (2.4): the momentum factors in the denominators of $E(e, e')$ balance those in the numerators [14].

The string-localized field $A_\mu(x, e)$ for $e = (1, \vec{0})$ coincides with the Coulomb gauge field A_μ^C . The well-known non-locality of the Coulomb gauge potential reflects the fact that two timelike strings are never spacelike separated. Its failure of covariance (when e_0 is fixed) is due to Eq. (1.2) which requires an additional gauge transformation to bring Λe_0 back to e_0 .

It may also be interesting to notice that one can average the potential $A_\mu(x, e)$ in e over the spacelike sphere with $e^0 = 0$. The resulting field is again the Coulomb gauge potential.

Similarly, for fixed spacelike e , $A_\mu(e)$ coincides with an axial gauge potential satisfying $e^\mu A_\mu = 0$. However, this relation is not used as a gauge condition to reduce the degrees of freedom before quantization, but instead the potentials for all e coexist simultaneously on the Fock space of the field strength, and they covariantly transform into each other according to Eq. (1.2). By specifying the 2-point function for spacelike e as a distribution rather than a function with a singularity, the manifestly string-localized representation Eq. (2.2) of the axial gauges is revealed, and the mutual commutativity of axial gauge fields for different directions is discovered.

2.1. Preservation of causality

One might worry that a string-localized interaction Lagrangean could spoil the causality of the resulting perturbation theory. We sketch here why this does not happen. We choose the easiest and most transparent example: the interaction of a massive vector field with a conserved external (classical) current $j^\mu(x)$. It is essential that the string-dependence of the interaction Lagrangean is a total derivative:

$$L_{\text{int}}(x, e) = A_\mu(x, e) j^\mu(x) = A_\mu^P(x) j^\mu(x) + \partial_\mu(\phi(x, e) j^\mu(x)) \quad (2.6)$$

so that the classical action, and hence the lowest order of the S-matrix, is independent of e . (In the massless case, neither A^P nor $\phi(x, e) = m^{-1} a(x, e)$ exist, but the variation of $L_{\text{int}}(e)$ w.r.t. e^κ is still a total derivative, because $\partial_{e^\kappa} A_\mu(x, e) = \partial_\mu(I_e A_\kappa)(x)$, where I_e is the integral along the string as in Eq. (2.2). This crucial feature is shared by many other interactions of interest [21,22,8], cf. examples in Sect. 6, where the renormalizability conditions in higher orders are less trivial than in the present external source problem.)

In order that the causal S-matrix

$$S_e[gj] := T \exp i \int d^4x g(x) A_\mu(x, e) j^\mu(x)$$

is independent of e in the limit $g \rightarrow \text{const}$, the decomposition Eq. (2.6) must continue to hold “under the time-ordering”. This can be formulated as a condition on the string-localized Feynman propagator $i(\Omega, T A_\mu(x, e) A_{\mu'}(x', e') \Omega)$, which is quite non-trivial because the time-ordering must be taken along the strings $x + \mathbb{R}_0^+ e, x' + \mathbb{R}_0^+ e'$, and the singularities at intersections of strings have to be carefully analyzed and renormalized [14,13]. The condition can be fulfilled and gives the unique answer

$$i(\Omega, T A_\mu(x, e) A_{\mu'}(x', e') \Omega) = (-\eta_{\mu\mu'} - \partial_\mu e_{\mu'} I_e + \partial_{\mu'} e'_\mu I_{-e'} + (ee') \partial_\mu \partial_{\mu'} I_e I_{-e'}) G_F(x - x'),$$

where $G_F(x - x')$ is the scalar Feynman propagator.

We proceed by computing the interacting potential in the setting of causal perturbation theory based on Bogoliubov's formula [2]

$$A_\mu^{\text{int}}(x, e) := S_{e_0}[j]^{-1} \frac{-i\delta}{\delta f^\mu(x, e)} S_{e_0}[j, f] \Big|_{f=0}$$

where $S_{e_0}[j, f] = T \exp i \int d^4x (A_\mu(x, e_0) j^\mu(x) + \int d\sigma(e) A_\mu(x, e) \times f^\mu(x, e))$. In this approach, renormalization amounts to the proper definition (as distributions) of propagators and their products. As it can be done in position space, it is best suited to control causality. In the external source problem, no further renormalization is necessary. Namely, by Wick's Theorem, we get

$$A_\mu^{\text{int}}(x, e) = A_\mu(x, e) + \int d^4x' g(x') G_{\mu,\mu'}^{\text{ret}}(x, e; x', e_0) j^{\mu'}(x')$$

where the string-localized retarded Green function $G^{\text{ret}} = i(\Omega, (T[AA'] - A'A)\Omega)$ equals

$$G_{\mu,\mu'}^{\text{ret}}(x, e; x', e_0) = (-\eta_{\mu\mu'} - \partial_\mu e_{\mu'} I_e + \partial_{\mu'} e_0 I_{-e_0} + (ee_0) \partial_\mu \partial_{\mu'} I_e I_{-e_0}) G^{\text{ret}}(x - x').$$

The contributions depending on e_0 vanish in the limit $g \rightarrow \text{const}$, because j^μ is conserved, hence $A_\mu^{\text{int}}(x, e)$ is independent of e_0 . The remaining contributions can be written as

$$A_\mu^{\text{int}}(x, e) = A_\mu(x, e) + (A_\mu^{\text{cl}}(x) + \partial_\mu \phi^{\text{cl}}(x, e)) \cdot \mathbf{1}$$

where

$$A_\mu^{\text{cl}}(x) = -g \int d^4x' G^{\text{ret}}(x - x') j_\mu(x'),$$

$$\phi^{\text{cl}}(x, e) = -g \int d^4x' G^{\text{ret}}(x - x') j(x', e)$$

are classical fields with sources $j^\mu(x)$ and $j(x, e) = e_\mu \int_0^\infty d\lambda j^\mu(x + \lambda e)$, respectively. The field strength is then manifestly independent of e , and coincides with the solution in the point-localized setting – except that the latter has a δ -function ambiguity for the Feynman propagator of the Proca field due to its bad UV behaviour. In the string-localized setting, the ambiguity is fixed ($= 0$). For more details, and for the QED case with a quantum source, see [13].

3. Spin two

The case $s = 2$ is largely analogous, but the decoupling at $m = 0$ requires a second step.

The positive 2-point function of the massless field strength $F_{[\mu\kappa][\nu\lambda]}$ can be represented as the curl of the (auxiliary) indefinite 2-point function of the Feynman gauge potential

$${}_0M^{A_{\mu\nu}^F, A_{\kappa\lambda}^F} = \frac{1}{2}[\eta_{\mu\kappa}\eta_{\nu\lambda} + \eta_{\mu\nu}\eta_{\kappa\lambda}] - \frac{1}{2}\eta_{\mu\nu}\eta_{\kappa\lambda}. \quad (3.1)$$

The coefficient $-\frac{1}{2}$ of the last term ensures that there are precisely two helicity states.

The symmetric, traceless and conserved massive Proca 2-point function is

$${}_mM^{A_{\mu\nu}^P, A_{\kappa\lambda}^P} = \frac{1}{2}[\pi_{\mu\kappa}\pi_{\nu\lambda} + \pi_{\mu\lambda}\pi_{\kappa\nu}] - \frac{1}{3}\pi_{\mu\nu}\pi_{\kappa\lambda}. \quad (3.2)$$

The coefficient $-\frac{1}{3}$ of the last term ensures the vanishing of the trace. The formulae for the massive and massless field strengths differ *only* by this coefficient. In particular, the massless field strength is not the limit of the massive field strength as $m \rightarrow 0$.

In the string-localized setting, we define the massive potential

$$A_{\mu\nu}(x, e) := (I_e^2 F_{[\mu\kappa][\nu\lambda]})(x)e^\kappa e^\lambda \quad (3.3)$$

with I_e as in Eq. (2.2) iterated twice, and its escort fields

$$\begin{aligned} a_\mu^{(1)}(x, e) &:= -m^{-1} \partial^\nu A_{\mu\nu}(x, e), \\ a^{(0)}(x, e) &:= -m^{-1} \partial^\mu a_\mu^{(1)}(x, e). \end{aligned} \quad (3.4)$$

Eq. (3.2) implies

$$\begin{aligned} {}_mM^{A_{\mu\nu}(-e), A_{\kappa\lambda}(e')} &= \frac{1}{2}[E(e, e')_{\mu\kappa}E(e, e')_{\nu\lambda} + (\kappa \leftrightarrow \lambda)] \\ &\quad - \frac{1}{3}E(e, e)_{\mu\nu}E(e', e')_{\kappa\lambda}, \end{aligned} \quad (3.5)$$

and one obtains the escort correlations with Eq. (3.4). The correlations between even and odd rank fields are $O(m)$ and decouple in the massless limit. The odd-odd and even-even correlations become

$$\begin{aligned} {}_0M^{a_\mu^{(1)}(-e), a_\nu^{(1)}(e')} &= -\frac{1}{2}E(e, e')_{\mu\nu}(p), \\ {}_0M^{A_{\mu\nu}(-e), a^{(0)}(e')} &= -\frac{1}{3}E(e, e)_{\mu\nu}(p), \\ {}_0M^{a^{(0)}(-e), a^{(0)}(e')} &= \frac{2}{3} \end{aligned} \quad (3.6)$$

up to $O(m^2)$. $A_{\mu\nu}(e)$ and $a(e)$ do not decouple at $m = 0$, in fact one has $\eta^{\mu\nu}A_{\mu\nu}(e) = -a(e)$. In order to decouple the fields, notice that the operator

$$E_{\mu\nu}(e, e) = \eta_{\mu\nu} + (e_\nu \partial_\mu + e_\mu \partial_\nu)I_e + e^2 \partial_\mu \partial_\nu I_e^2$$

acts in momentum space on the creation and annihilation parts by multiplication with $E(e, e)_{\mu\nu}(p)$ and with $E(e, e)_{\mu\nu}(-p) = E(-e, -e)_{\mu\nu}(p)$, respectively. Thus,

$$A_{\mu\nu}^{(2)}(e) := A_{\mu\nu}(e) + \frac{1}{2}E_{\mu\nu}(e, e)a^{(0)}(e) \quad (3.7)$$

decouples from $a^{(0)}$, and its 2-point function is the same as Eq. (3.5) but with the proper coefficient $-\frac{1}{2}$ rather than $-\frac{1}{3}$ for the last term. Thus, taken at $m = 0$, $A^{(2)}$ is a string-localized potential for the massless field strength to which it is related by the same formula as Eq. (3.3). It is, unlike other potentials, positive, traceless and conserved.

In the massless limit, $A^{(0)}(e) = \sqrt{3/2}a^{(0)}(e)$ becomes the e -independent massless scalar field by Eq. (3.6). $A_\mu^{(1)} := \sqrt{2}a_\mu^{(1)}$ is the same string-localized Maxwell potential as obtained from $s = 1$.

The generalization of Eq. (2.5),

$$\begin{aligned} A_{\mu\nu}^P(x) &= A_{\mu\nu}^{(2)}(x, e) - \sqrt{1/6}E_{\mu\nu}(e, e)A^{(0)}(x, e) - \\ &\quad - \frac{\sqrt{1/2}}{m}(\partial_\mu A_\nu^{(1)} + \partial_\nu A_\mu^{(1)})(x, e) \\ &\quad + \frac{\sqrt{2/3}}{m^2}\partial_\mu \partial_\nu A^{(0)}(x, e), \end{aligned} \quad (3.8)$$

quantifies the singular lower helicity contributions to A^P .

Now, turning to the DVZ problem, we may couple linearized massive gravity in a Minkowski background to a conserved stress-energy source by

$$S_{\text{int}}(e) = \int d^4x A_{\mu\nu}(x, e)T^{\mu\nu}(x). \quad (3.9)$$

Because by Eq. (3.8), $A_{\mu\nu}(e)$ differs from $A_{\mu\nu}^P$ only by derivatives, the action is independent of e . At $m > 0$, all five states of the graviton couple to the source. In the limit $m \rightarrow 0$, we have by Eq. (3.7)

$$A_{\mu\nu}(x, e) = A_{\mu\nu}^{(2)}(x, e) - \sqrt{1/6}\eta_{\mu\nu}\varphi(x) + \text{derivatives},$$

where $\varphi(x) = \sqrt{3/2}\lim_{m \rightarrow 0} a^{(0)}(x, e)$ is the massless scalar field decoupled from the helicity-2 potential $A^{(2)}(x, e)$. Thus,

$$\lim_{m \rightarrow 0} S_{\text{int}}(e) = \int d^4x A_{\mu\nu}^{(2)}(x, e)T^{\mu\nu}(x) - \sqrt{1/6} \int d^4x \varphi(x) T_\mu^\mu(x). \quad (3.10)$$

The first term satisfies the L - Q condition (see Sect. 6) and therefore equals the e -independent action for linearized pure massless gravity. With a classical source, it can be treated exactly as in Sect. 2.1.

We have thus (along with the known decoupling of the helicity ± 1 degrees of freedom) explicitly identified the scalar field that is responsible for the DVZ discontinuity, as the limit of the escort field on the massive Hilbert space. Our result is formally similar to Zakharov's who writes (in an indefinite gauge) instead the massless coupling $A_{\mu\nu}^F(x, e)T^{\mu\nu}(x)$ as the limit of the massive coupling plus a compensating scalar ghost [28]. We want to emphasize the change of perspective when one avoids unphysical ghost degrees of freedom.

As at spin 1, also the spin-2 potentials $A_{\mu\nu}(e)$ and $A_{\mu\nu}^{(2)}(e)$ at fixed e can be regarded as axial gauges. The averaging over the string directions with $e^0 = 0$ is only possible for $A_{\mu\nu}^{(2)}(e)$ at $m = 0$, and yields again the radiation gauge potential $A_{0\mu}^C(x) = 0$.

The case of general integer spin [15] is very similar to $s = 2$, except for the more involved combinatorics.

4. String-localized stress-energy tensor

The stress-energy tensor is by no means unique. It must be conserved and symmetric so that the generators

$$P_\sigma = \int_{x_0=t} d^3x T_{0\sigma}, \quad M_{\sigma\tau} = \int_{x_0=t} d^3x (x_\sigma T_{0\tau} - x_\tau T_{0\sigma})$$

are independent of the time t ; and the commutators with the generators must implement the infinitesimal Poincaré transformations given by the Wigner representation. (The commutators are fixed by the 2-point functions.) But one may add “irrelevant” local terms as long as they do not change the generators.

One choice of a stress-energy tensor that produces the correct generators is the “reduced stress-energy tensor” ($\times = \mu_2 \dots \mu_s$ is a multi-index)

$$T_{\rho\sigma}^{\text{red}} := (-1)^s \left[-\frac{1}{4} :A_{\mu\times}^P \overset{\leftrightarrow}{\partial}_\rho \overset{\leftrightarrow}{\partial}_\sigma A^{P\mu\times} : \right. \\ \left. - \frac{s}{2} \partial^\mu \left(:A_{\rho\times}^P \overset{\leftrightarrow}{\partial}_\sigma A_{\mu\times}^P : + (\rho \leftrightarrow \sigma) \right) \right]. \quad (4.1)$$

It differs by “irrelevant terms” from the Hilbert stress-energy tensor, defined as the variation of a suitable generally covariant action w.r.t. the metric. The first term in Eq. (4.1) also appears in [4]. The second term does not contribute to the momenta, but is needed to ensure the correct Lorentz transformations [15].

Expanding A^P into $A^{(r)}(e)$ resp. $A^{(r)}(e')$, and discarding irrelevant terms (involving derivatives of escort fields) that “carry away” all singularities when $m \rightarrow 0$, one gets a string-localized stress-energy tensor that admits a massless limit. Discarding more terms that are irrelevant at $m = 0$, one decouples it as the sum over $r \leq s$ of

$$T_{\rho\sigma}^{(r)}(e, e') = (-1)^r \left[-\frac{1}{4} :A_{\mu\times}^{(r)}(e) \overset{\leftrightarrow}{\partial}_\rho \overset{\leftrightarrow}{\partial}_\sigma A^{(r)\mu\times}(e') : \right. \\ \left. - \frac{r}{4} \partial^\mu \left(:A_{\rho\times}^{(r)}(e) \overset{\leftrightarrow}{\partial}_\sigma A_{\mu\times}^{(r)}(e') : + (e \leftrightarrow e') \right) \right] \quad (4.2)$$

understood as distributions in two independent directions e, e' . As in Eq. (3.3),

$$A_{\mu_1 \dots \mu_r}^{(r)}(x, e) = (I_e^r F_{[\mu_1 \nu_1] \dots [\mu_r \nu_r]}^{(r)})(x) e^{\nu_1} \dots e^{\nu_r}$$

can be expressed in terms of the massless field strengths.

As the massless potentials $A^{(r)}$ mutually commute, the generators defined by $T^{(r)}$ separately implement the Poincaré transformations of $A^{(r)}$. Massless higher-spin currents of charged potentials are constructed similarly. For details see [15].

That the Weinberg–Witten theorem can be evaded with non-local densities, was pointed out earlier in [11], where examples with unpaired helicities were given. Eq. (4.2) involving string integrals over field strengths is perhaps the most conservative alternative, also in comparison with other proposals to couple massless higher-spin matter to gravity [6,27,1,9].

5. “Fattening”

The 2-point functions of the massless and massive string-localized potentials $A^{(s)}$ (for any spin) are the same polynomials in the tensor $E_{\mu\nu}(p)$, except that the argument p of the functions $E_{\mu\nu}$ is taken on the respective mass-shell. Thus, one obtains the massive field $A^{(s)}$ from the massless field $A^{(s)}$ just by changing the dispersion relation $p^0 = \omega_m(\vec{p})$. As the massive 2-point function was constructed on the Hilbert space of the Proca potential, this deformation preserves positivity. Through the coupling to the lower escort fields, it brings back all spin components of the Proca field. Indeed, the latter is restored from the massive potential $A^{(s)}$ by

$$A_{\mu_1 \dots \mu_s}^P(x) = (-1)^s m M A_{\mu_1 \dots \mu_s}^{P, A^P} A_{\nu_1 \dots \nu_s}^{(s)} \Big|_m(x, e),$$

where in this formula $m M A_{\mu_1 \dots \mu_s}^{P, A^P}$ is understood as a differential operator (a polynomial in $\pi_{\mu\nu} = \eta_{\mu\nu} + m^{-2} \partial_\mu \partial_\nu$).

6. Outlook: interactions

A crucial question is which physical interactions that otherwise are non-renormalizable or cannot be formulated on a Hilbert space, are accessible by couplings to string-localized fields. Although the actual perturbation theory is not the subject of this letter, we give an overview of possible interactions.

One class of interactions (called “ L - V -pairs”) are of the form

$$L_{\text{int}}(e) = L_{\text{int}} + \partial_\mu V^\mu(e)$$

where L_{int} is a possibly non-renormalizable string-independent interaction Lagrangean, and the string-localized $L_{\text{int}}(e)$ is renormalizable. The string-dependent derivative term ∂V disposes of the strong short-distance fluctuations of L_{int} , typically by means of escort fields.

Further constraints may arise in order to secure the e -independence of the perturbative S -matrix in higher orders in the coupling constant, and higher order interactions may be needed. We refer to these as “induced” interactions.

All couplings $A_{\mu_1 \dots \mu_s}(x, e) j^{\mu_1 \dots \mu_s}(x)$ of massive potentials to conserved currents are of L - V type, but also cubic self-couplings of massive vector bosons

$$f_{abc} F_{\mu\nu}^a(x) A^{b\mu}(x, e) A^{c\nu}(x, e) \\ + m^2 f_{abc} A^{av}(x, e) \partial_\nu \phi^b(x, e) \phi^c(x, e)$$

where $\phi^a(e)$ are the escort fields as in Sect. 2.1, and f_{abc} is totally anti-symmetric; or more general expressions admitting vector bosons (W and Z) of different masses. A second order constraint is that f_{abc} must satisfy the Jacobi identity, so they are the structure constants of some Lie algebra (without a gauge principle having been imposed). Induced interactions in this case are the quartic Yang–Mills terms, as well as an additional coupling to a Higgs boson of arbitrary mass and with a potential such that $\frac{m_H^2}{2} H^2 + V(H)$ is the usual Higgs potential with one of its minima at $H = 0$.¹

While in the non-abelian case the Higgs coupling is induced from the cubic self-coupling of the vector bosons, an abelian Higgs coupling $A_\mu(x, e) A^\mu(x, e) H(x)$ may be chosen directly provided it is completed to a power-counting renormalizable L - V -pair

$$A_\mu(e) A^\mu(e) H + A_\mu(e) [\phi(e) \overset{\leftrightarrow}{\partial}^\mu H] - \frac{m_H^2}{2} \phi(e)^2 H \\ = A_\mu^P A^{P\mu} H + \partial_\mu V^\mu$$

with $V^\mu = A^\mu(e) \phi(e) H + \frac{1}{2} \phi(e)^2 \overset{\leftrightarrow}{\partial}^\mu H$. A Higgs potential is induced in higher orders.

A more general class (called “ L - Q -pairs”) are interaction Lagrangeans $L_{\text{int}}(e)$, for which

$$\partial_{e^\kappa} L_{\text{int}}(e) = \partial_\mu Q_\kappa^\mu(e)$$

holds, which is sufficient to secure the e -independence of $S_{\text{int}}(e) = \int d^4x L_{\text{int}}(x, e)$. These include couplings $A_{\mu_1 \dots \mu_s}(x, e) j^{\mu_1 \dots \mu_s}(x)$ of massless potentials to conserved (classical or quantum) sources, for which a point-localized potential on the Hilbert space does not exist. Namely, $\partial_{e^\kappa} A_{\mu_1 \dots \mu_s}(x, e)$ is a sum of gradients. A general understanding of L - Q pairs, and which terms they induce in higher orders in order to maintain renormalizability, is presently under investigation.

Perturbative correlation functions of observables (composite fields that are e -independent in the free theory) remain e -independent when the interaction is switched on. This has to be secured by Ward identities, that, e.g., allow to pull derivatives out of time-ordered products.

¹ Some of these claims have not yet been conclusively established, but work is in progress [16]. They are true in the analogous BRST approach pursued by Scharf et al. [20,5], and up to now the same patterns are always repeated in the string-localized approach, where e -independence replaces BRST invariance.

7. Conclusion

We have identified string-localized potentials for massive particles of integer spin s on the Hilbert space of their field strengths, that admit a smooth massless limit to decoupled potentials with helicities $h = \pm r$, $r \leq s$. We have presented an inverse “fattening” prescription via a manifestly positive deformation of the 2-point function. The approach provides a way around the Weinberg–Witten theorem, and explicitly and quantitatively exhibits the origin of the DVZ discontinuity.

Our results also allow to approximate string-localized fields in the massless infinite-spin Wigner representations [17] by the massive scalar escort fields $A^{(0)}$ of spin $s \rightarrow \infty$, $m^2 s(s+1) = \kappa^2 = \text{const.}$ (Work in progress [19].)

String-localized fields are a device to formulate quantum interactions in terms of a given particle content, that allow to take into full account the well-known conflicts between point-localization and positivity. With their use, positivity is manifest, while localization is controlled by renormalized causal perturbation theory, as presently investigated in [8,16,13]. It bears formal analogies with BRST renormalization, but is more economic (avoiding unphysical degrees of freedom), and much closer to the fundamental principles of relativistic quantum field theory.

It was shown in the framework of algebraic quantum field theory, that to connect scattering states with the vacuum, may in certain theories require operations localized in narrow spacelike cones; and in the presence of a mass gap it cannot be worse than that [3]. The emerging perturbation theory using string-localized fields is the practical realization of this insight.

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