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PAPER

Area laws and efficient descriptions of quantum many-body states

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Abstract

It is commonly believed that area laws for entanglement entropies imply that a quantum many-body state can be faithfully represented by efficient tensor network states—a conjecture frequently stated in the context of numerical simulations and analytical considerations. In this work, we show that this is in general not the case, except in one-dimension. We prove that the set of quantum many-body states that satisfy an area law for all Renyi entropies contains a subspace of exponential dimension. We then show that there are states satisfying area laws for all Renyi entropies but cannot be approximated by states with a classical description of small Kolmogorov complexity, including polynomial projected entangled pair states or states of multi-scale entanglement renormalisation. Not even a quantum computer with post-selection can efficiently prepare all quantum states fulfilling an area law, and we show that not all area law states can be eigenstates of local Hamiltonians. We also prove translationally and rotationally invariant instances of these results, and show a variation with decaying correlations using quantum error-correcting codes.

1. Introduction

Complex interacting quantum systems show a wealth of exciting phenomena, ranging from phase transitions of zero temperature to notions of topological order. A significant proportion of condensed matter physics is concerned with understanding the features emergent in quantum lattice systems with local interactions. However, naive numerical descriptions of such quantum systems require prohibitive resources, for the simple reason that the dimension of the underlying Hilbert space grows exponentially in the system size.

Yet, it has become clear in recent years that ground states—and a number of other natural states—usually occupy only a tiny fraction of this Hilbert space. This subset, which is sometimes referred to as the 'physical corner' of the Hilbert space (figure 3(a)), is commonly characterised by states having little entanglement. More precisely, they are characterised by the *area law* [1]: entanglement entropies grow only like the boundary area of any subset A of lattice sites

$$S(\rho_A) = O(|\partial A|) \tag{1}$$

and not extensively like its volume $|A|$ (figure 1). Such area laws have been proven for all gapped spin models in $D = 1$ [2–6]. In $D \geqslant 2$, area laws have only been proven in special cases, including free gapped bosonic and fermionic models [7–9], ground states in the same gapped phase as ones satisfying an area law [10, 11], models which have a suitable scaling for heat capacities [12], models whose Hamiltonian spectra satisfy related conditions [13, 14], frustration-free spin models [15], and models exhibiting local topological order [16]. The general expectation is that all gapped lattice models satisfy an area law. Proving a general area law for gapped lattice models in $D \geqslant 2$ has indeed become a milestone open problem in condensed-matter physics.

Area laws are at the core of powerful numerical algorithms, such as DMRG [17]. In $D = 1$, the situation is particularly clear: matrix-product states [18] essentially 'parameterise' those one-dimensional quantum states that satisfy an area law for some Renyi entropy S_α with $\alpha \in (0, 1)$. They approximate all such states provably well, which explains why essentially machine precision can be reached with such numerical tools [19, 20]. A

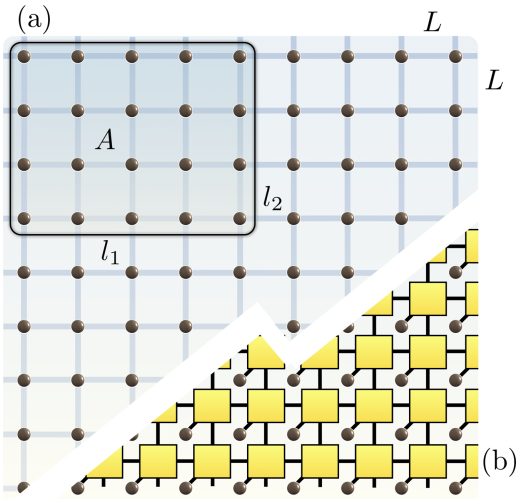

Figure 1. (a) There exist quantum states on D-dimensional cubic lattices in $D \geqslant 2$ such that $S_\alpha(\rho_A) = O(|\partial A|)$ for all $\alpha > 0$, but which cannot be approximated by efficient tensor network states, such as (b) polynomial projected entangled pair states.

common jargon is that similarly, projected entangled pair states (PEPS) [21], can approximate all states satisfying area laws in higher dimensions. In the same way, one expects those instances of tensor network states to capture the 'physical corner'.

In this work, we show that this jargon is not right: strictly speaking, area laws and the existence of efficient tensor network descriptions are unrelated. We show that there exist states that satisfy an area law for *every* Renyi entropy[4]

$$S_\alpha(\rho) = \frac{1}{1-\alpha}\log_2 \mathrm{tr}(\rho^\alpha), \quad \alpha \in [0, \infty), \tag{2}$$

but still, no efficient PEPS can be found. The same holds for multi-scale entanglement renormalisation (MERA) ansatzes [22], as well as all classes of states that have a short description (the precise meaning of this will be defined below). Not even a quantum computer with post-selection can efficiently prepare all states satisfying area laws. Moreover, not all states satisfying area laws are eigenstates of local Hamiltonians.

These conclusions follow from the main result of this work: in $D \geqslant 2$, the set of states satisfying area laws for all S_α contains a subspace whose dimension scales exponentially with the system size. By considering a very general notion of quantum state descriptions based on the theory of *quantum Kolmogorov complexity* [23], we then infer that this large subspace cannot be captured by efficient tensor network states.

However, our results should not be seen to indicate that area laws are not appropriate *intuitive* guidelines for approximations with tensor network states. We rather provide a significant step towards precisely delineating the boundary between those quantum many-body states that can be efficiently captured and those that cannot. We thus contribute to the discussion why PEPS and other tensor network states approximate natural states so well. Area laws without further qualifiers are, strictly speaking, inappropriate for this purpose as the 'corner' they parameterise is exponentially large. This work is hence a strong reminder that the programme of identifying that boundary is not finished yet.

2. Classically efficiently described states

We first review the concept of efficient classical descriptions of quantum states. The focus is on tensor network states, but the notion of efficient classical descriptions can be formulated in a much more general way. For our purposes, the following definition of efficiently describable quantum states will suffice (see also [23] for alternative definitions).

Definition 1 (Classical descriptions). A *classical description* of a pure quantum state $|\psi\rangle \in \mathcal{S}((\mathbb{C}^d)^{\otimes N})$ is a Turing machine that outputs the coefficients of $|\psi\rangle$ in the standard basis $\{|\mathbf{x}\rangle\colon \mathbf{x} \in [d]^N\}$ and halts. The *length* of

[4] Here, $S_1 = \lim_{\alpha\downarrow 1} S_\alpha = S$ is the familiar von-Neumann entropy and S_0 the binary logarithm of the Schmidt rank.

the classical description is the size of the Turing machine[5]. We say that the description is *polynomial* if its length is polynomial in N.

We emphasise that for a polynomial classical description we only require the *size* of the Turing machine to be polynomial, but not the *run-time* (which is necessarily exponential). Notice that the shortest length of a classical description for a given quantum state is a natural generalisation of the Kolmogorov complexity[6] to quantum states [23].

Example 2 (Tensor networks). States that can be written as polynomial tensor networks (i.e., they are defined on arbitrary graphs with bounded degree, have at most $O(\mathrm{poly}(N))$ bond-dimension and their tensor entries' Kolmogorov complexity is at most $O(\mathrm{poly}(N))$) are polynomially classically described states in the sense of definition 1. In particular, PEPS and MERA states with $O(\mathrm{poly}(N))$ bond-dimension and tensor entries of at most $O(\mathrm{poly}(N))$ Kolmogorov complexity are polynomially classically described states.

As a further interesting special case, we highlight that states that can be prepared by polynomial quantum circuits, even with post-selected measurement results, fall under our definition of classically described states.

Example 3 (Quantum circuits with post-selection). Suppose that $|\psi\rangle$ can be prepared by a quantum circuit of $O(\mathrm{poly}(N))$ gates from $|0\rangle^{\otimes O(\mathrm{poly}(N))}$, where we allow for post-selected measurement results in the computational basis. Then, a Turing machine that classically simulates the circuit constitutes a polynomial classical description in the sense of definition 1.

Example 4 (Eigenstates of local Hamiltonians). Suppose that $|\psi\rangle$ is an eigenvector of a local Hamiltonian with bounded interaction strength. Such Hamiltonians can be specified to arbitrary (but fixed) precision with polynomial Kolmogorov complexity. Thus, a Turing machine that starts from a polynomial description of the Hamiltonian and computes $|\psi\rangle$ by brute-force diagonalisation constitutes a polynomial classical description of $|\psi\rangle$ in the sense of definition 1.

3. Area laws and the exponential 'corner' of Hilbert space

Throughout the remainder of this work, we consider quantum lattice systems of local dimension d, arranged on a cubic lattice $[L]^D$ of fixed dimension $D > 1$, where $[L] := \{0, \ldots, L-1\}$. We show in this section that the set of states satisfying area laws for all S_α contains subspaces of exponential dimension. This result is then used in section 4 to conclude that such states in general do not have an efficient classical description. The case $D = 1$ is excluded since in this case, the question at hand has already been settled with the opposite conclusion [19, 20]. The local dimension is small and taken to be $d = 3$ for most of this work. There is no obvious fundamental reason, however, why such a construction should not also be possible for $d = 2$.

In the focus of attention are states that satisfy an area law for all α-Renyi entropies, in particular also for $\alpha < 1$.

Definition 5 (Strong area laws). A pure state $|\psi\rangle \in \mathcal{S}((\mathbb{C}^d)^{\otimes L^D})$ is said to satisfy a *strong area law* if there exists a universal constant c such that for all regions $A \subset [L]^D$, we have $S_0(\psi_A) \leqslant c|\partial A|$, where $\psi_A = \mathrm{tr}_{\bar{A}}\, |\psi\rangle\langle\psi|$.

Since $S_\alpha(\rho) \leqslant S_0(\rho)$ for all $\alpha > 0$, strong area law states in this sense also exhibit area laws for all Renyi entropies. Definition 5 is hence even stronger than the area laws usually quoted [1, 19, 20]. Here and later, we write $\psi = |\psi\rangle\langle\psi|$. For simplicity, we will for the remainder of this paper restrict our consideration to cubic regions only. It should be clear, however, that all arguments generalise to arbitrary regions $A \subset [L]^D$.

We now turn to showing that the 'physical corner' of states satisfying area laws in this strong sense is still very large: it contains subspaces of dimension $\exp(\Omega(L^{D-1}))$. We prove this by providing a specific class of quantum states that have that property. At the heart of the construction is an embedding of states defined on a $(D-1)$-dimensional qubit lattice into the D-dimensional qutrit one. Denote with $\mathcal{H}_L \subset \mathrm{span}\{|1\rangle, |2\rangle\}^{\otimes L^{D-1}}$ the subspace of translationally invariant states (with respect to periodic boundary conditions) on a $(D-1)$-

[5] For readers who are not familiar with Turing machines, a less formal but for our purposes equivalent definition is that a classical description of $|\psi\rangle$ is a (classical) computer program that computes the coefficients of $|\psi\rangle$ in the standard basis. The length of the description is then simply the length of the program.

[6] Recall that the Kolmogorov complexity of a classical string w is the size of the shortest Turing machine (or computer program) that outputs w and halts. It can be thought of as the shortest possible (classical) description of w. For an introduction to Turing machines and Kolmogorov complexity, see e.g. [24].

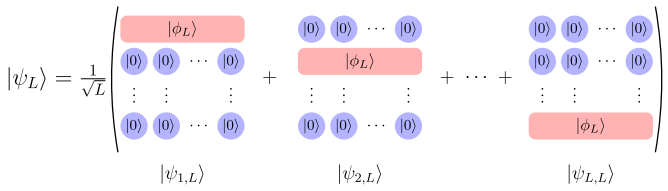

Figure 2. Schematic drawing of $|\psi_L\rangle$ in $D = 2$. $|\phi_L\rangle$ is an arbitrary translationally invariant state vector on L^{D-1} qubits with basis states $|1\rangle$, $|2\rangle$ in $(D-1)$-dimensions. Schmidt decompositions for $|\psi_L\rangle$ with respect to bi-partitions of the lattice can be readily obtained from the corresponding Schmidt decompositions of $|\phi_L\rangle$.

dimensional cubic lattice of L^{D-1} qubits. It is easy to show that $\dim(\mathcal{H}_L) \geqslant 2^{L^{D-1}}/L^{D-1}$. We start from the simplest translationally invariant construction on $\mathcal{H} := (\mathbb{C}^3)^{\otimes L^D}$ and discuss rotational invariance and decaying correlations below.

Theorem 6 (States satisfying strong area laws). *There exists an injective linear isometry $f\colon \mathcal{H}_L \to \mathcal{H}$ with the property that for all $|\phi_L\rangle \in \mathcal{H}_L$, $f(|\phi_L\rangle)$ satisfies a strong area law and is translationally invariant in all D directions.*

Proof. Given a state vector $|\phi_L\rangle \in \mathcal{H}_L$, define

$$|\psi_{k,L}\rangle := |0\rangle^{\otimes(k-1)L^{D-1}} \otimes |\phi_L\rangle \otimes |0\rangle^{\otimes(L-k)L^{D-1}} \in \mathcal{H}, \tag{3}$$

with $|\phi_L\rangle$ at the kth hyperplane of the lattice (figure 2). Define

$$|\psi_L\rangle := L^{-1/2}\sum_{k=1}^{L}|\psi_{k,L}\rangle, \tag{4}$$

which is translationally invariant. Any such state vector will satisfy a strong area law (in fact, a sub-area law): for any cubic subset $A = [l_1]\times\cdots\times[l_D]$, we have for the reduced state $(\psi_L)_A = \mathrm{tr}_{\bar{A}}\,|\psi_L\rangle\langle\psi_L|$ that

$$S_0((\psi_L)_A) \leqslant \log_2(2^{l_1\cdots l_{D-1}}l_D + 1) \leqslant 2\sum_{j=1}^{D}\prod_{k=1,k\neq j} l_k = |\partial A|, \tag{5}$$

where we used that the Schmidt rank with respect to the bi-partition A, $\bar{A}$ for each $|\psi_{k,L}\rangle$ with $k \in [l_D]$ is at most $2^{l_1\cdots l_{D-1}}$, and that since $|\phi_L\rangle$ is only supported on span$\{|1\rangle, |2\rangle\}$, the Schmidt vectors of $|\psi_{k,L}\rangle$ and $|\psi_{k',L}\rangle$ are orthogonal for $k \neq k' \in [l_D]$ such that in the distinguished Dth direction, the contribution to the Schmidt rank is additive and thus linear in l_D. Setting $f(|\phi_L\rangle) := |\psi_L\rangle$, we see that f has the desired properties.

4. Area laws and approximation by efficiently describable states

We now precisely state what we call an approximation of given pure states by polynomially classically described states.

Definition 7 (Approximation of quantum many-body states). A family of pure states $|\psi_L\rangle$ can be *approximated by polynomially classically described states* if for all $\varepsilon > 0$, there exist a polynomial p and pure states $|\omega_L\rangle$ with a classical description of length at most $p(L)$ such that for all L, $\||\psi_L\rangle\langle\psi_L| - |\omega_L\rangle\langle\omega_L|\|_1 \leqslant \varepsilon$.

Note that this is exactly the sense in which matrix-product states provide an efficient approximation of all one-dimensional states that satisfy an area law for some S_α with $\alpha \in (0, 1)$ [19]. We remark that definition 7 can be weakened without altering the results. We now turn to the main result:

Theorem 8 (Impossibility of approximating area law states). *Let $\tilde{\mathcal{H}}_L$ be a Hilbert space of dimension* $\exp(\Omega(\mathrm{poly}(L)))$. *Then there exist states in $\tilde{\mathcal{H}}_L$ that cannot be approximated by polynomially classically described states. In particular, not all translationally invariant strong area law states can be approximated by polynomially classically described states.*

Theorem 8 can be easily proven using a counting argument of ϵ-nets. Indeed, the number of states that can be parameterised by $O(\mathrm{poly}(L))$ many bits is at most $2^{O(\mathrm{poly}(L))}$. However, an ϵ-net covering the space of pure states in $\mathbb{C}^q$ requires at least $(1/\varepsilon)^{\Omega(q)}$ elements [25], which is much larger than $2^{O(\mathrm{poly}(L))}$ if $q = \exp(\Omega(\mathrm{poly}(L)))$ (see also [26, 27] on the topic of ε-nets for many-body states). Thus, the set of quantum states in $\tilde{\mathcal{H}}_L$ that have a polynomial classical description cannot form an ε-net for $\tilde{\mathcal{H}}_L$, which proves theorem 8.

We nevertheless also review the more involved proof from [23] using communication complexity in appendix A. This proof could, due to its more constructive nature, provide some insight into the structure of some strong area law states that cannot be approximated by polynomially classically described states.

4.1. Tensor network states

We saw that our definition of polynomial classical descriptions encompasses all efficient tensor network descriptions. Thus

Corollary 9 (**Tensor network states cannot approximate area law states**). *There exist translationally invariant strong area law states that cannot be approximated by polynomial tensor network states in the sense of example 2. In particular, not all translationally invariant strong area law states can be approximated by polynomial PEPS or MERA states.*

Notice the restriction to tensor networks whose tensor entries have a polynomial Kolmogorov complexity. This is required to ensure that the tensor network description is in fact polynomial. Indeed, a classical description depending on only polynomially many parameters $\lambda_1, \ldots, \lambda_{O(\mathrm{poly}(N))}$ (e.g., a PEPS with polynomial bond-dimension) is not necessarily already polynomial—for the latter, it is also necessary that each of the λ_i themselves can be stored efficiently. The notion of Kolmogorov complexity allows for the most general definition of tensor networks that can be stored with polynomial classical memory.

4.2. Quantum circuits

Example 3 shows that states prepared by a polynomial quantum circuit with post-selected measurement results have a polynomial classical description. Thus

Corollary 10 (**Post-selected quantum circuits cannot prepare area law states**). *There exist translationally invariant strong area law states that cannot be approximated by a polynomial quantum circuit with post-selection in the sense of example 3.*

In the light of the computational power of post-selected quantum computation [28], this may be remarkable.

4.3. Eigenstates of local Hamiltonians

Example 4 shows that eigenstates of local Hamiltonians with bounded interaction strengths also have a polynomial classical description. Thus

Corollary 11 (**Area law states without parent Hamiltonian**). *There exist translationally invariant strong area law states that cannot be approximated by eigenstates of local Hamiltonians.*

5. Rotationally invarant states and area laws

So far, the states in consideration were translationally but not rotationally invariant. However, by taking the superposition of appropriate rotations of (4), one can alter the above argument such that all involved states are also rotationally invariant, i.e. remain invariant under 90° rotations of the lattice. The details of this construction are given in appendix B.

Corollary 12 (**Approximation for translationally and rotationally invariant states**). *There exist translationally and rotationally invariant strong area law states that cannot be approximated by polynomially classically described states. In particular, corollaries 9–11 also hold for translationally and rotationally invariant states.*

6. Decaying correlations and area laws

One might wonder whether an exponentially dimensional subspace of strong area law states can be constructed while imposing decaying two-point correlations for distant observables, a property known to occur in ground states of local gapped Hamiltonians [29, 30]. It follows immediately from their definition that the states constructed in theorem 6 (and theorem 16 in appendix B) already satisfy an algebraic decay. Indeed, for all L and all local observables A, B on disjoint supports separated by an arbitrary distance ℓ,

$$\langle\psi_L|\ AB\ |\psi_L\rangle\ -\ \langle\psi_L|\ A\ |\psi_L\rangle\langle\psi_L|\ B\ |\psi_L\rangle = O(L^{-1}) = O(\ell^{-1}). \tag{6}$$

Using quantum error-correcting codes, it is however also possible to construct variations of the previous results such that for all L and all local observables A, B with disjoint supports,

$$\langle\psi_L|\ AB\ |\psi_L\rangle\ -\ \langle\psi_L|\ A\ |\psi_L\rangle\langle\psi_L|\ B\ |\psi_L\rangle\ = 0. \tag{7}$$

The details of this construction are given in appendix C.

Corollary 13 (Approximation for area law states with vanishing correlations of local observables). *There exist strong area law states with vanishing two-point correlations of all local observables on disjoint supports that cannot be approximated by polynomially classically described states. In particular, corollaries 9–11 also hold for states with vanishing correlations of local observables on disjoint supports.*

The translationally and rotationally invariant construction only gives algebraic decay (equation (6)). However, we conjecture that there also exist strong area law states which are translationally and rotationally invariant and simultaneously have exponentially small correlations for all local observables, but still cannot be approximated by polynomially classically described states.

7. Conclusion and outlook

We have shown that the set of states satisfying an area law in $D \geqslant 2$ comprises many states that do not have an efficient classical description: they cannot be described by efficient tensor networks, cannot be prepared by polynomial quantum circuits with post-selected measurements, and are also not eigenstates of local Hamiltonians. We have hence proven that the connection between entanglement properties and the existence of an efficient description is far more intricate than anticipated. These results are based on the simple observation that an arbitrary quantum state in $(D\ -\ 1)$ dimensions that is embedded into D dimensions satisfies a D-dimensional area law, thus implying that the set of area law states contains a subspace of exponential dimension. In other words, in $D \geqslant 2$, it is possible to 'dilute' the entanglement content and still arrive at a strong area law. We also demonstrated that the exponential scaling persists even if various physical properties, such as translational and rotational invariance, or decaying correlations of local observables, are imposed. We note however that while the latter can be extended to non-local observables of size $O(L^{D-1})$, our notion of decaying correlations is weaker than the exponential clustering property for ground states of gapped Hamiltonians, since this can involve all regions of unbounded size [5, 29]. It remains open whether our results are impeded if the stronger notion of exponential decay of correlations is imposed.

Area laws indeed suggest the expected entanglement behaviour of naturally occurring ground states. However, when put in precise contact with questions of numerical simulation, it turns out that satisfying an area law alone is not sufficient for efficient approximation. Picking up the metaphor of the introduction, the 'corner of states that can be efficiently described' is tiny compared to the 'physical corner' (figure 3).

A particularly exciting perspective arises from the observation that states with small entanglement content can go along with states having divergent bond dimensions in PEPS approximations. This may be taken as a suggestion that there may be states that are in the same phase if symmetries are imposed, but are being classified as being in different phases in a classification of phases of matter building upon tensor network descriptions [31–33]. It is the hope that the present work can be taken as a starting point of further endeavours towards understanding the complexity of quantum many-body states.

Acknowledgments

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Appendix A. Proof of theorem 8 using communication complexity

We now review the alternative proof of theorem 8 using communication complexity, which was given in [23]. Suppose two distant parties, Alice and Bob, each possess an n-bit string, $\mathbf{x}$ and $\mathbf{y}$, respectively. No communication between Alice and Bob is allowed, but they can communicate with a third party, Charlie, whose task is to guess whether or not $\mathbf{x} = \mathbf{y}$. We demand that Charlie may guess the wrong answer with a small (fixed) probability of at most $\delta > 0$. This is called the *equality problem*, which we denote by EQ(n). We now state some

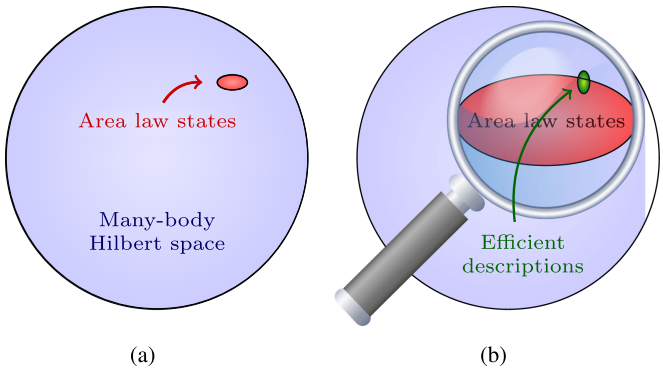

Figure 3. (a) The set of area law states is a tiny 'corner' of the many-body Hilbert space. (b) The set of states that can be efficiently described is tiny compared to the 'corner' of area law states.

known results [23, 34, 35] on the communication complexity, i.e. the minimum amount of communication required for solving the equality problem.

Lemma 14 (Equality problem for classical communication). *If Alice and Bob can only send classical information to Charlie, at least* $\Omega(\sqrt{n})$ *bits of communication are required to solve* $\mathrm{EQ}(n)$.

Lemma 15 (Quantum solution to equality problem).

(1) If Alice and Bob can send quantum information to Charlie, there exists a protocol for $\mathrm{EQ}(n)$ *using only* $O(\log n)$ *qubits of communication that is of the following form: Alice and Bob each prepare* $O(\log n)$ *qubit states* $|h(\mathbf{x})\rangle$ *and*[7] $|h(\mathbf{y})\rangle$, *respectively, which they send to Charlie. Charlie then applies a quantum circuit to* $|h(\mathbf{x})\rangle|h(\mathbf{y})\rangle|0\rangle$, *followed by a measurement of a single qubit whose outcome determines Charlie's guess.*

(2) There exists an $\varepsilon > 0$ *independent of n such that the protocol in (1) still works if instead, Alice and Bob send states to Charlie which are* ε*-close in trace distance*[8] *to* $|h(\mathbf{x})\rangle$ *and* $|h(\mathbf{y})\rangle$.

We now turn to the proof of theorem 8 .

Proof of theorem 8. We prove the claim by contradiction. Suppose that every state vector in $\tilde{\mathcal{H}}_L$ can be approximated by polynomially classically described states. Then in particular, all M-qubit states can be approximated by states with a classical description of length $O(\mathrm{poly}(N))$, where $M := \lfloor \log_2 \dim(\tilde{\mathcal{H}}_L) \rfloor$. Fix $\delta \in (0,\ 1)$ and let $\varepsilon > 0$ be as in lemma 15 (2). By lemma 15 (1), we can choose n with $\log n = \Theta(M)$ such that M qubits of communication suffice to solve $\mathrm{EQ}(n)$.

By assumption, $|h(\mathbf{x})\rangle$ and $|h(\mathbf{y})\rangle$ can be ε-approximated by states which have an $O(\mathrm{poly}(M))$ classical description. By lemma 15 (2), these states can be used instead of $|h(\mathbf{x})\rangle$ and $|h(\mathbf{y})\rangle$ in the quantum protocol to solve $\mathrm{EQ}(n)$. Now consider an alternative protocol using only classical communication to solve $\mathrm{EQ}(n)$ as follows: Alice and Bob send the classical description of their states to Charlie, who simulates the quantum circuit and the measurement from lemma 15 using the classical descriptions of the states. This protocol solves $\mathrm{EQ}(n)$ using only $O(\mathrm{poly}(M)) = O(\mathrm{poly}(\log n))$ bits of communication, contradicting lemma 14. Finally, by setting $\tilde{\mathcal{H}}_L := f(\mathcal{H}_L)$ with f and $\mathcal{H}_L$ as in theorem 6, the second part of theorem 8 follows. □

Appendix B. Translationally and rotationally invariant states

Corollary 12 follows directly from theorem 8 and the following theorem.

[7] The exact form of $|h(\mathbf{x})\rangle$ and $|h(\mathbf{y})\rangle$ is not important for our purpose—we will only need that they consist of $O(\log n)$ qubits. Interested readers are referred to [35].

[8] This was argued in [23] for the Euclidean vector distance but it is clear that the same holds for the trace distance.

Theorem 16 (Translationally and rotationally invariant area law states). *There exists an injective linear isometry* g: $\mathcal{G}_L \to \mathcal{H}$ *with* $\dim(\mathcal{G}_L) = \exp(\Omega(L^{D-1}))$ *such that for all* $|\phi_L\rangle \in \mathcal{G}_L$, $g(|\phi_L\rangle)$ *satisfies a strong area law and is translationally and rotationally invariant in all D directions.*

Theorem 16 can be proven with a minor modification of the proof of theorem 6. To start with, we replace $|\phi_L\rangle$ for each L by state vector on the translationally invariant subset $\mathcal{H}_L \subset (\mathbb{C}^2)^{\otimes L^{D-1}}$ which is also mirror symmetric, i.e. invariant under reflections, in all $(D-1)$ directions. Notice that the exact choice of the plane of symmetry in a given direction does not matter since we assume $|\phi_L\rangle$ to be translationally invariant. With $|\psi_L\rangle$ as in (4), we then consider, for the entire $[L]^D$ lattice, state vectors of the form

$$|\Psi_L\rangle := D^{-1/2}\sum_{j=1}^{D}\mathcal{R}_j\,|\psi_L\rangle, \tag{B1}$$

where $\mathbb{I} = \mathcal{R}_1, \ldots, \mathcal{R}_D$ rotate the entire lattice system such that $|\phi_L\rangle$ is arranged along each line of the cubic lattice in dimension D. Such a state is translationally and rotationally invariant, following from mirror symmetry. These states satisfy a strong area law: for any cubic subset $A \subset [L]^D$,

$$(\psi_L)_A = \mathrm{tr}_{\bar{A}}\,|\psi_L\rangle\langle\psi_L| = D^{-1}\sum_{j=1}^{D}\mathrm{tr}_{\bar{A}}(\mathcal{R}_j|\psi_L\rangle\langle\psi_L|\mathcal{R}_j^\dagger), \tag{B2}$$

since for $j \neq k$,

$$\mathrm{tr}_{\bar{A}}(\mathcal{R}_j|\psi_L\rangle\langle\psi_L|\mathcal{R}_k^\dagger) = 0. \tag{B3}$$

This can be seen by taking the partial trace with respect to a set C first. For simplicity of notation, for $D = 2$, consider w.l.o.g. distinguished subsets $A \subset [L]^D$ for which $A \cap C = \varnothing$ for $C := L \times [L]$. Then

$$\mathrm{tr}_{\bar{A}}(|\psi_L\rangle\langle\psi_L|\mathcal{R}_2^\dagger) = \mathrm{tr}_{\bar{A}\setminus C}\sum_{\mathbf{x}\in S}\langle\mathbf{x}|\psi_L\rangle\langle\psi_L|\mathcal{R}_2^\dagger|\mathbf{x}\rangle = 0, \tag{B4}$$

where $S = \{\mathbf{x}|\,\exists j\colon x_j \neq 0 \wedge x_k = 0\ \forall k \in [L]\setminus\{j\}\}$. An analogous argument holds for any dimension D. From these considerations, it follows that the area law is inherited by the area law valid for each individual $\mathcal{R}_j|\psi_L\rangle$. It is furthermore clear that the exponential scaling of the dimension is not affected by restricting to the subspace $\mathcal{G}_L \subset \mathcal{H}_L$ of mirror symmetric states. □

Appendix C. States with vanishing two point correlation functions for local observables

To prove corollary 13, consider a non-degenerate $[\![n, k, \Delta]\!]$-quantum error-correcting code C with $k/n = \Theta(1)$ and $\Delta/n = \Theta(1)$ [36]. Here n denotes the block size and k the number of encoded qubits. Δ is the so-called distance of the code. Since C is non-degenerate, the reduced density matrix of any $\Delta - 1$ qubits of any state in the code space of C is maximally mixed. By choosing $n = L^{D-1}$ and considering

$$|\psi_L\rangle := |C(\chi_L)\rangle \otimes |0\rangle^{(L-1)L^{D-1}}, \tag{C1}$$

where $|C(\chi_L)\rangle$ is an arbitrary state vector in the code space of C, we see that for all L and all observables A, B with disjoint support and whose joint support in the top hyperplane contains less than $\Delta = \Theta(L^{D-1})$ sites,

$$\langle\psi_L|\,AB\,|\psi_L\rangle - \langle\psi_L|\,A\,|\psi_L\rangle\langle\psi_L|\,B\,|\psi_L\rangle = 0. \tag{C2}$$

In particular, equation (C2) holds for local observables A, B. Clearly, states of the form (C1) obey a strong area law and since $k = \Theta(L^{D-1})$, we obtain a subspace of dimension $\exp(\Omega(L^{D-1}))$ of strong area law states with vanishing correlations of local observables. Corollary 13 now follows from theorem 8. □

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