Semicompressible Ocean Thermodynamics and Boussinesq Energy Conservation

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Abstract: Equations more accurate than the Boussinesq set that still filter out sound were recently introduced. While these equations were shown to have a consistent potential energy, their thermodynamical behavior and associated implications were not fully analyzed. These shortcomings are remedied in the present note that argues both sets are fully consistent from a thermodynamic perspective. It is further argued that both sets remain computationally competitive with the Boussinesq set.

1. Introduction

Dewar *et al.* [1] recently developed two sets of ocean dynamical equations that include some degree of compressibility and exclude sound waves. They were argued to be simultaneously more accurate than the Boussinesq equations and computationally competitive with them. These were referred to as the Type I and Type II Semicompressible equations.

In subsequent communications with Prof. R. Klein, he correctly pointed out that the thermodynamic consistency of these equations had not yet been demonstrated. Thermodynamic consistency in this context means that total energy is conserved and that an entropy variable exists for the system that obeys the second thermodynamic law. This remark was made in comparison to pseudoincompressible equations derived in [2] for a general equation of state in which a much more thorough thermodynamic analysis had been undertaken. The purpose of this short communication is to analyze similar characteristics of both types of the semicompressible equations.

2. The Semicompressible Equations

As discussed in [1], the Type I equations are

$$\rho^*(\tfrac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + 2\Omega\times\boldsymbol{u}) = -\nabla p - [\rho^* - \rho_r(z) + \tfrac{p\rho_r}{c_s^2\rho^*}]g\boldsymbol{k} + \nabla\cdot\rho^*\nu^{visc}\nabla\boldsymbol{u}$$
$$\tfrac{\partial}{\partial t}\rho^* + \nabla\cdot(\rho^*\boldsymbol{u}) = 0 \tag{1a}$$
$$\rho^* = \rho(S_A, \Theta, P^*)$$

$$\tfrac{d}{dt}\Theta = \Theta^o$$
$$\tfrac{d}{dt}S_A = S_A^o \tag{1b}$$

where density, ρ^*, is evaluated using Conservative Temperature, Θ, Absolute Salinity, S_A and the hydrostatic pressure, P^*, determined from the reference density profile, ρ_r (at most a function of z) according to

$$\frac{\partial}{\partial z}P^* = -\rho_r g \tag{2}$$

The quantity ν^{visc} is viscosity and $\nu^* = 1/\rho^*$.

De Szoeke and Samelson [3] have shown that the hydrostatic Boussinesq equations are equivalent to the compressible hydrostatic Navier-Stokes equations written in pressure coordinates. Equation (1a) improves upon the classic hydrostatic balance in the hydrostatic limit through the presence of the sound speed. The density approximation it represents is a more accurate approximation of true density than ρ^*, density evaluated at the hydrostatic pressure. Equation (1) is also more flexible in that they can be used in non-hydrostatic settings.

The notation X^o in Equation (1) denotes the non-advective contributions to X. Absolute Salinity, being a purely conserved quantity (ignoring chemistry, see [4]), the form is clear

$$S_A^o = -\nu\nabla \cdot \boldsymbol{F_S} \tag{3}$$

where $\boldsymbol{F_S}$ is the diffusive flux of Absolute Salinity. We purposely leave the form for Θ^o unspecified, as part of the present exercise is to determine it. Other notation is standard.

The equations in Equation (1) satisfy the energy equation

$$\frac{\partial}{\partial t}(\rho^*(K + h_I^+)) + \nabla \cdot (\boldsymbol{u}(\rho^*(K + h_I^+)) + p\boldsymbol{u}) = pR + \nabla \cdot \rho^*\nu^{visc}\nabla K - \rho^*\epsilon \tag{4}$$

with potential energy played by the quantity

$$h_I^+ = -\int_{P_o}^{P^*}\frac{\rho - \rho_r}{\rho\rho_r}dP = \int_{P_o}^{P^*}\frac{b_I}{g\rho_r}dP \tag{5}$$

The quantity $\epsilon = \nu^{visc}|\nabla\boldsymbol{u}|^2$ is the viscous dissipation of kinetic energy, b_I is the Type I buoyancy, the quantity R

$$R = -\frac{\partial}{\partial\Theta}\nu^*\frac{d}{dt}\Theta - \frac{\partial}{\partial S_A}\nu^*\frac{d}{dt}S_A \tag{6}$$

denotes diffusive terms and Equation (1b) demonstrates the presence of compressibility in these equations.

The Type II equations are

$$\rho_{II}(\frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + 2\Omega\times\boldsymbol{u}) = -\nabla p - (\rho^* - \rho_{II})g\boldsymbol{k} - \frac{p}{c_{s_{II}}^2}g\boldsymbol{k} + \nabla \cdot \rho^{II}\nu^{visc}\nabla\boldsymbol{u}$$
$$\frac{\partial}{\partial t}\rho_{II} + \nabla \cdot (\rho_{II}\boldsymbol{u}) = \nabla \cdot (\rho_{II}\boldsymbol{u}) = 0 \tag{7a}$$
$$\rho^* = \rho(S_A, \Theta, P^*)$$

$$\frac{d}{dt}\Theta = \Theta^o$$
$$\frac{d}{dt}S_A = S_A^o \tag{7b}$$

where

$$\rho_{II}(z) = \rho_{II}(0)e^{-\int_o^z\frac{g}{c_{s_{II}}^2}dz} \tag{8}$$

is either a chosen reference density profile yielding the sound speed profile $c_{s_{II}}^2$ or a profile constructed from a chosen sound speed profile, $c_{s_{II}}^2$. In either case, density, ρ_{II}, is a function of depth only and the hydrostatic pressure is computed from it

$$\frac{\partial}{\partial z}P^* = -\rho_{II}g \tag{9}$$

Compressibility appears in these equations in Equation (7b) where the time independence of ρ_{II} has been recognized. The Type II equations satisfy the energy constraint

$$\frac{\partial}{\partial t}(\rho_{II}(K + h_{II}^+)) + \nabla \cdot (\boldsymbol{u}\rho_{II}(K + h_{II}^+) + p\boldsymbol{u}) = \nabla \cdot \rho^{II} \nu^{visc}\nabla K - \rho_{II}\epsilon \tag{10}$$

where the role of potential energy is now played by

$$h_{II}^+ = -\int_{P_o}^{P^*} \frac{\rho - \rho_{II}}{\rho_{II}^2} dP = \int_{P_o}^{P^*} \frac{b_{II}}{g\rho_{II}} dP \tag{11}$$

The quantity b_{II} is the Type II buoyancy. The advantages and disadvantages of these equations are discussed in [1]. Extracting Equations (1) and (7) from the full Navier–Stokes equations is akin to substituting the Boussinesq density for the full density in most places, and retaining a reference density profile where needed for potential energy consistency.

3. Semicompressible Thermodynamics

We now consider how to augment (1) and (7) with a consistent thermodynamic behavior. The First Law of Thermodynamics is

$$\frac{d}{dt}h - \nu\frac{d}{dt}P = T\frac{d}{dt}\eta + \mu\frac{d}{dt}S_A = -\frac{1}{\rho}\nabla \cdot \boldsymbol{F_Q} + \epsilon \tag{12}$$

where T is temperature, $h(\eta, S_A, P)$ is specific enthalpy, η is specific entropy, specific volume is denoted by $\nu = 1/\rho$, P is pressure and μ is the relative chemical potential of salt in seawater. The quantity ϵ is the heat of viscous energy dissipation, and $\boldsymbol{F_Q}$ is a generalized heat flux. Enthalpy is defined by

$$h = e + P\nu \tag{13}$$

where e is specific internal energy. Enthalpy from Equation (12) is seen to depend naturally on the variables η, S_A, and P, and in this form plays the role of a thermodynamic potential from which all thermodynamic variables can be obtained by differentiation. For example, derivatives of enthalpy with respect to its natural variables are T, μ, and ν, respectively. The derivation of Equation (12) is discussed extensively in the TEOS-10 manual ([4]; see also [5,6]).

Equation (12) is exact. The semicompressible equations consider modifications to the thermodynamic variables caused mainly by the difference between static and full pressure, and we analyze Equation (12) from that perspective. In what follows, we will adopt entropy, salinity, and pressure as state variables, and will often cast pressure dependency in terms of static pressure. For example,

$$h = h^*(S_A, \eta, P^*) + h_P^*(S_A, \eta, P^*)p + O(p^2) \tag{14}$$

where P^* is static pressure and p the dynamic pressure, given by $p = P - P^*$. The superscript (*) will denote quantities evaluated at the static pressure. Specific volume, density, *etc.* will be written in a manner similar to Equation (14). Note, according to Equation (12), Equation (14) is equivalent to

$$h = h^*(S_A, \eta, P^*) + \nu^* p \tag{15}$$

3.1. Type I Thermodynamics

We first develop what is a consistent statement of Type I thermodynamics. We begin by considering the form of Equation (12) to $O(p^2)$; *i.e.*,

$$\frac{d}{dt}(h^* + v^*p) = (v^* + p\frac{\partial}{\partial P}v)\frac{d}{dt}P^* + v^*\frac{d}{dt}p + T^+\frac{d}{dt}\eta + \mu^+\frac{d}{dt}S_A + O(p^2) \tag{16}$$

where

$$T^+ = T^* + p\frac{\partial}{\partial P}T \tag{17}$$

and similarly for μ^+. Evaluating the derivative of the static pressure, Equation (16) becomes

$$\frac{d}{dt}(h^* + v^*p) = (v^* + p\frac{\partial}{\partial P}v)(-wg\rho_r) + v^*\frac{d}{dt}p + T^+\frac{d}{dt}\eta + \mu^+\frac{d}{dt}S_A \tag{18}$$

While Equation (18) is literally correct to $O(p^2)$, it will turn out that a less accurate approximation, to $O(p)$, is required on the right hand side of Equation (12)

$$T^+\frac{d}{dt}\eta + \mu^+\frac{d}{dt}S_A = -v^*\nabla \boldsymbol{F_Q} + \epsilon \tag{19}$$

The reason for this will become clear later, here we stress only that it assures thermodynamic consistency of the Type I equations.

3.2. Type I Mechanical Energy Equation

The mechanical energy equation derived from Equation (1a) is

$$\rho^*\frac{d}{dt}K = -\boldsymbol{u} \cdot \nabla p - wg(\rho^* - \rho_r + \frac{\rho_r p}{\rho^* c_s^2}) + \nabla \cdot \rho^* v^{visc}\nabla K - \rho^*\epsilon \tag{20}$$

where $K = (\boldsymbol{u} \cdot \boldsymbol{u})/2$ is the kinetic energy density. Introducing the gravitational potential $\Phi = gz$ and using Equation (18)

$$\rho^*(\frac{d}{dt}K + \Phi + h^* + pv^*) = -\boldsymbol{u} \cdot \nabla p + \frac{d}{dt}p + \nabla \cdot \rho^* v^{visc}\nabla K - \rho^*\epsilon + \rho^*(T^+\frac{d}{dt}\eta + \mu^+\frac{d}{dt}S_A) \tag{21}$$

Using Equation (12)

$$\frac{\partial}{\partial P}h^* = v^* \tag{22}$$

so

$$h^* = h_o + \int_{P_o}^{P^*} v dP = h_o + \int_{P_o}^{P^*} \frac{1}{\rho_r}dP + \int_{P_o}^{P^*} \frac{b}{\rho_r g}dP \tag{23}$$

where

$$b = -g\frac{(\rho^* - \rho_r)}{\rho^*} \tag{24}$$

The quantity h_o is recognized as the potential enthalpy defined by [6], and the last integral in Equation (23) is the so-called dynamic enthalpy of [7]. The quantity P_o is a constant reference surface pressure. The middle integral in Equation (23) reduces upon inspection to $-\Phi$.

Thus, Equation (21) becomes

$$\rho^*\frac{d}{dt}(K + h_o + h_I^+) = -\nabla \cdot p\boldsymbol{u} + \nabla \cdot \rho^* v^{visc}\nabla K - \rho^*\epsilon + \rho^*(T^+\frac{d}{dt}\eta + \mu^+\frac{d}{dt}S_A) \tag{25}$$

Lastly, we exploit Equation (19) to obtain

$$\rho^* \frac{d}{dt}(K + h_o + h_I^+) = -\nabla \cdot p\boldsymbol{u} + \nabla \cdot \rho^* \nu^{visc} \nabla K - \nabla \cdot \boldsymbol{F_Q} \tag{26}$$

which is in conservative form. This implies the Type I equations have a well-formed energy principle, provided they are augmented by Equations (18) and (19) as their thermodynamic equations. Note that the entropy in Equation (19) can be rewritten as

$$\rho^* \frac{d}{dt}\eta = -\nabla \cdot \frac{\boldsymbol{F_Q} - \mu^+ \boldsymbol{F_S}}{T^+} + (\boldsymbol{F_Q} - \mu^+ \boldsymbol{F_S}) \cdot \nabla(\frac{1}{T^+}) - \boldsymbol{F_S} \cdot \nabla(\frac{\mu^+}{T^+}) + \frac{\rho^* \epsilon}{T^+} \tag{27}$$

which is entropy "non-conservation" in its familiar form. Hence, the Type I equations also have a consistent entropy variable.

At this point, a practical difficulty is that entropy is not a typical ocean modelling variable. Instead, McDougall's Conservative Temperature

$$\Theta = \frac{h_o}{c_P^o} \tag{28}$$

McDougall ([6]) is preferable and so it is necessary to work out a Conservative Temperature equation to complement (27). From Equation (18), one finds

$$\frac{d}{dt}h^* - \nu^* \frac{d}{dt}P^* = -\frac{1}{\rho^*}\nabla \cdot \boldsymbol{F_Q} - p\frac{\partial}{\partial P}T\frac{d}{dt}\eta - p\frac{\partial}{\partial P}\mu\frac{d}{dt}S_A + \epsilon \tag{29}$$

By definition

$$h_o = h(\eta, S_A, P_o) \tag{30}$$

so

$$\frac{d}{dt}h_o = \theta\frac{d}{dt}\eta + \mu_o\frac{d}{dt}S_A \tag{31}$$

where θ is potential temperature and $\mu_o = \mu(\eta, S_A, P_o)$. Potential enthalpy is thus related to entropy via

$$\frac{d}{dt}\eta = \frac{\frac{d}{dt}h_o - \mu_o\frac{d}{dt}S_A}{\theta} \tag{32}$$

From Equation (23)

$$\frac{d}{dt}h_o = c_p^o\frac{d}{dt}\Theta = -\frac{1}{\rho^*}\nabla \cdot \boldsymbol{F_Q} - (T^+ - \theta)\frac{d}{dt}\eta - (\mu^+ - \mu_o)\frac{d}{dt}S_A + \epsilon \tag{33}$$

so using Equation (32)

$$\frac{d}{dt}\Theta = \frac{-\nu^*\nabla \cdot \boldsymbol{F_Q} + \epsilon}{c_P^o(1 + \underline{\frac{(T^+ - \theta)}{\theta}})} - \frac{((\frac{T^+}{\theta})\mu_o - \mu^+)\frac{d}{dt}S_A}{c_P^o(1 + \underline{\frac{(T^+ - \theta)}{\theta}})} \tag{34}$$

The underlined terms in Equation (34) are quite small, according to [8]. Neglecting them leads to the statement that Conservative Temperature is very accurately portrayed as a conserved quantity; however, all of those terms must be included in order to ensure full energy conservation.

It is seen, therefore, that the Type I equations yield energy conservation, but the cost is the insertion of a large quantity ($c_P^o\Theta$) into the energy equation. A similar result was labelled in [7] as a "crippling" disadvantage of the Boussinesq-Bernoulli equation. It might thus be more useful to work with a partial equation when considering energy; *i.e.*, one that doesn't explicitly include Conservative Temperature, but in any case, the full potential enthalpy Equation (34) is required. We have also found the Type I Semicompressible equations have a proper entropy variable and conserve total energy, and so are

fully thermodynamically consistent. These equations are very similar to the pseudo-incompressible equation set for general equations of state in [2].

3.3. Type II Equations: Potential, Kinetic, and Thermal Energies

The Type II equations are quite similar to the Type I equations except for the appearance of the reference density profile, rather than the Boussinesq density, in the mass conservation equation and in front of the momentum acceleration. Most of the analysis carries over in a manner similar to the Type I analysis, with the one important distinction that the velocity divergence diagnosed from mass conservation has no diabatic contributions

$$\nabla \cdot \boldsymbol{u}_{II} = \frac{gw}{c_{sII}^2} \tag{35}$$

Therefore, the mechanical energy equation for the system analogous to Equation (21) is

$$\rho_{II} \frac{d}{dt} K = -\nabla \cdot p\boldsymbol{u} + p\underline{\frac{wg}{c_{sII}^2}} - wg(\rho^* - \rho_{II} + \frac{p}{c_{sII}^2}) + \nabla \cdot \rho^{II} \nu^{visc} \nabla K - \rho_{II}\epsilon \tag{36}$$

with the underlined terms cancelling. Taking the material derivative of Equation (11) and substituting in Equation (36) yields

$$
\begin{aligned}
\rho_{II} \frac{d}{dt} K &= -\nabla \cdot p\boldsymbol{u} + \rho_{II} \int_{P_o}^{P^*} \frac{\partial}{\partial S_A} \left[\frac{b_{II}}{g\rho_r} \right] dP \frac{d}{dt} S_A \\
&+ \rho_{II} \int_{P_o}^{P^*} \frac{\partial}{\partial \eta} \left[\frac{b_{II}}{g\rho_r} \right] dP \frac{d}{dt}\eta - \rho_{II} \frac{d}{dt} h_{II}^+ + \nabla \cdot \rho^{II} \nu^{visc} \nabla K - \rho_{II}\epsilon
\end{aligned}
\tag{37}
$$

To put Equation (37) in conservative form requires the potential enthalpy equation

$$\frac{d}{dt} h^* = \frac{d}{dt} h_o + \nu^* \frac{d}{dt} P^* + \int_{P_o}^{P^*} \frac{\partial \nu}{\partial S_A} dP \frac{d}{dt} S_A + \int_{P_o}^{P^*} \frac{\partial \nu}{\partial \eta} dP \frac{d}{dt}\eta \tag{38}$$

which is equivalent to Equation (33). This leads to

$$\rho_{II} \frac{d}{dt} (K + h_{II}^+ + c_P^o \Theta) = -\nabla \cdot p\boldsymbol{u} - \nabla \cdot \boldsymbol{F_Q} + \nabla \cdot \rho^{II} \nu^{visc} \nabla K \tag{39}$$

The Type II Conservative Temperature equation becomes

$$\frac{d}{dt} \Theta = \frac{-\nu_{II} \nabla \cdot \boldsymbol{F_Q} + \epsilon - ((\frac{T^+}{\theta})\mu_o - \mu^+)\frac{d}{dt} S_A}{c_P^o \frac{T^+}{\theta}} \tag{40}$$

The most significant difference relative to the Type I set is in Equation (35), where the diabatic contributions to expansion are absent. Like for Type I, it is still necessary to retain all terms for Conservative Temperature, rather than the approximate form recommended in [6]. We conclude that the Type II equations also have a consistent mechanical and thermal energy structure. Our Type II equations are very similar to the thermodynamically consistent anelastic set derived in [9].

3.4. A Comparison to the Boussinesq Set

To ensure full energetic consistency for the semicompressible equations, it has been necessary to carry along some additional non-conservative effects in the calculations. We now analyze the full energetics behavior of the Boussinesq set to gauge any additional computational burden of the semicompressible set relative to them. If we adopt Conservative Temperature, salinity, and pressure as

the thermodynamic coordinates and follow the procedure outlined in [7], we arrive at the Boussinesq version of the Bernoulli equation

$$\frac{d}{dt}(K + p + h_B) = \frac{\partial}{\partial t}p + \int_{P_o}^{P^*}\left[\frac{\partial}{\partial S_A}\frac{b}{g\rho_o}\right]dP\,\frac{d}{dt}S_A + \int_{P_o}^{P^*}\left[\frac{\partial}{\partial \Theta}\frac{b}{g\rho_o}\right]dP\,\frac{d}{dt}\Theta - \epsilon \tag{41}$$

(see Equation (48) in [7]) where $b = -g(\rho^* - \rho_o)/\rho^*$ is the Boussinesq buoyancy and ρ_o is the Boussinesq reference density. Dynamic enthalpy as defined in [7] is

$$h_B = \frac{1}{g\rho_o}\int_{P_o}^{P^*} b\,dP \tag{42}$$

Returning to a form of the potential enthalpy consistent with the Boussinesq approximation

$$h^*(S_A, \Theta, P^*) = h_o + \frac{1}{\rho_o}\int_{P_o}^{P^*}(1 + \frac{b}{g})dP \tag{43}$$

where h_o is potential enthalphy, and taking its material derivative yields

$$\frac{d}{dt}h^* - v^*\frac{d}{dt}P^* = c_P^o\frac{d}{dt}\Theta \;\;+\;\; \int_{P_o}^{P^*}\left[\frac{\partial}{\partial S_A}\frac{b}{g\rho_o}\right]dP\,\frac{d}{dt}S_A$$
$$+\;\; \int_{P_o}^{P^*}\left[\frac{\partial}{\partial \Theta}\frac{b}{g\rho_o}\right]dP\,\frac{d}{dt}\Theta = -\frac{1}{\rho_o}\nabla\cdot\boldsymbol{F_Q} + \epsilon \tag{44}$$

Using Equation (44) to eliminate the pressure integrals in Equation (41) turns it into the conservative Boussinesq energy equation

$$\frac{d}{dt}(K + p + h_B + c_P^o\Theta) = \frac{\partial}{\partial t}p + \nabla\cdot\rho^{II}v^{visc}\nabla K - \frac{1}{\rho_o}\nabla\cdot\boldsymbol{F_Q} \tag{45}$$

The Boussinesq Conservative Temperature equation is given by

$$\frac{d}{dt}\Theta = \frac{-v_o\nabla\cdot\boldsymbol{F_Q} + \epsilon - ((\frac{T^*}{\theta})\mu_o - \mu^+)\frac{d}{dt}S_A}{c_P^o\frac{T^*}{\theta}} \tag{46}$$

The point of this section is that to obtain a conservative energy equation out of the Boussinesq set, it is necessary to carry along the same additional diffusive terms as for the semi-compressible equations. A similar result has recently been derived in [10,11]. This is not done in any Boussinesq ocean circulation model of which we are aware.

4. Summary

The Type I and II semicompressible equations discussed in [1], which are analogues of the pseudo-incompressible and anelastic equations known in meteorology, have undergone an examination of their thermodynamic structure. To summarize, the Types I and II equations are fully consistent from an energetic and thermodynamic perspective. To obtain this, it is necessary to include a more complete equation for Conservative Temperature as a part of the equation sets than is normally done. Interestingly, it turns out the Boussinesq equations must also carry analogous quantities to be in purely conservative energetic form. We thus see the semicompressible equations remain computationally competitive with the Boussinesq equations while admitting some degree of compressibility. Other results here include the associated definitions of the dynamic enthalpy variables belonging to the sets.

From a simple computer throughput perspective, this analysis recommends Type II relative to Type I because they are slightly less computationally intensive. However, the Type II equations

represent the oceanic density structure by a single vertical profile. Any such profile cannot perform well globally, so the "cost" of the simplicity is a less accurate density representation. This promises to be the most troubling when computing sea level changes via the continuity equation. The additional accuracy provided by the Boussinesq density in the Type I set might well compensate for the (slight) additional complexity.

Any further exploration with these equation sets should bear in mind the above points.

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