Chapter 3

Mixed CPS Problem of a Nonhomogeneous Body with a Doubly-Periodic Set of Holes

3.1 Preliminaries and Kolosov Functions

The present investigation will be on the mixed CPS problem of a three-dimensional nonhomogeneous elastic body with a doubly-periodic set of holes of arbitrary shape on the x_1, x_2 transverse cross section (all holes being of course congruent on the x_1, x_2 plane). We assume that each periodic parallelogram is composed by two different isotropic materials, with modulus of elasticity $\kappa^{\pm}$ and Poisson ratio $\mu^{\pm}$, respectively, both of which contain a number of holes, m total in number. It is assumed that the interface of the two materials will be a simple, closed, smooth and non-intersecting contour $\mathcal{L} \equiv L$ $(mod\ 2\omega_1, 2\omega_2)$, oriented clockwise as its positive direction, e.g. in the periodic fundamental parallelogram P_{00}, L is the interface of the two materials (see Figure 3.1), denoted by γ_j $(j = 1, \cdots, m)$, and $\gamma = \bigcup\limits_{j=1}^{m} \gamma_j$ (i.e., by contours which do not intersect themselves and have no points in common), positive direction will be taken as indicated on Fig. 3.1. The inner region of γ_j $(j = 1, \cdots, m)$ will be denoted by S_0^j, the regions which is surrounded by L and $\mathit{\Gamma}$ in P_{00} except S_0^j, and the inner region of L except S_0^j will be denoted by D and D', respectively. Denote $S = D \bigcup D'$, for convenience, the origin will be chosen inside D.

The investigation will be restricted to doubly-periodic stress distributions in the elastic body. The external stress $X_{1n}(t) + iX_{2n}(t)$, on the hole boundaries, must now of course be equal in congrugent points. The so-called mixed CPS problem means the external stress $X_{1n}(t)+iX_{2n}(t)$ applied to some boundaries γ_j $(j \in I)$, and displacements $u_j(t)+iv_j(t) = h_j(t)$ with cyclic increments h_k $(k = 1, 2)$ for the remaining part of boundaries γ_j $(j \in I\!I)$, (of course the aggregate of the sets of subscripts $I \bigcup I\!I = \{1, 2, \cdots, m\}$), and displacement $w(t)$ on $\gamma_{I\!I}$ in x_3 direction with cyclic increments w_k (k=1,2), are given. The strain $e_3 = \mathit{constant}$. In addition, the displacement discontinuity for a passage through L is given by $g(t) = [u^+(t) + iv^+(t)] - [u^-(t) + iv^-(t)]\,, t \in L$, to find the state of elastic equilibrium.

For the double periodicity of the stress distribution and the equilibrium principle of forces, we have

$$\sum_{j \in I \cup I\!I} (X_{1j} + iX_{2j}) = 0, \tag{3.1}$$

where

$$X_{1j} + iX_{2j} = \int_{\gamma_j} [X_{1n}(t) + iX_{2n}(t)] ds,$$

$X_{1j} + iX_{2j}, j \in I$, being known because $X_{1n}(t) + iX_{2n}(t), t \in \gamma_j, j \in I$, are given, $X_{1j} + iX_{2j}$, $j \in I\!I$, will be undetermined constants.

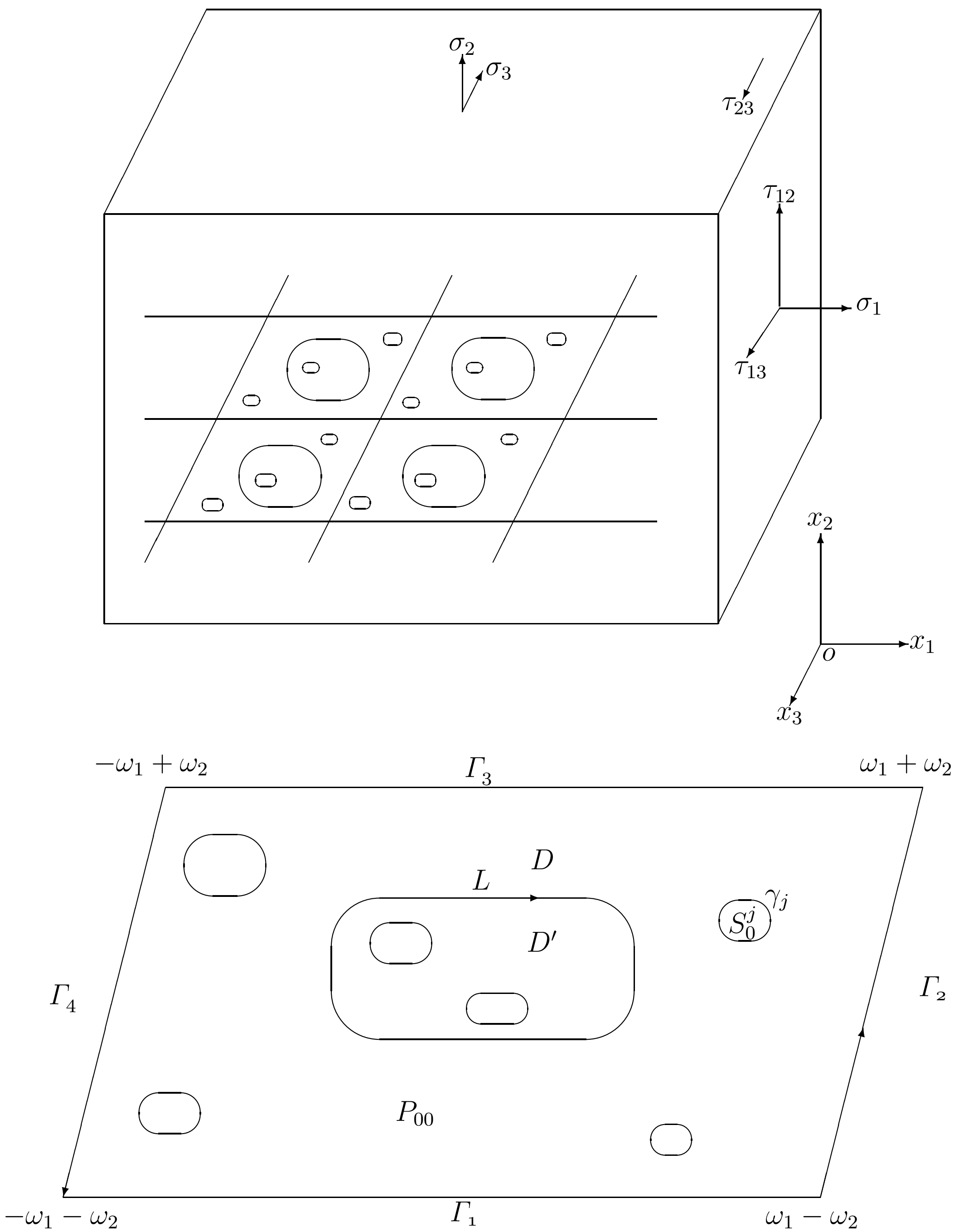

Figure 3.1: A nonhomogeneous body with a doubly-periodic set of holes

Denote

$$f(t) = i\int_{t_0}^{t} [X_{1n}(t) + iX_{2n}(t)]\, ds, t \in \gamma_I. \tag{3.2}$$

In spite of the components of stress and displacement are single-valued functions, the complex stress functions $\phi(z)$ and $\psi(z)$ may, in this case, be found to be multi-valued. However, we can separate the multi-valued parts of $\phi(z)$ and $\psi(z)$ by constructing Kolosov functions

$$\phi(z) = -\frac{1}{2\pi(\kappa_z + 1)} \sum_{j=0}^{m-1} [(X_{1j} + iX_{2j}) \log \sigma(z - z_j)] + \phi_0(z), \tag{3.3}$$

$$\psi(z) = \frac{\kappa_z}{2\pi(\kappa_z + 1)} \sum_{j=0}^{m-1} [(X_{1j} + iX_{2j}) \log \sigma(z - z_j)] + \psi_0(z). \tag{3.4}$$

where $\phi_0(z)$ and $\psi_0(z)$ are holomorphic functions and hence single-valued in S,

$$\kappa_z = \begin{cases} \kappa^+, & \text{if } z \in D, \\ \kappa^-, & \text{if } z \in D'. \end{cases}$$

Taking (3.1) into account, the aparted multi-valued parts , say, the first terms on the right-hand sides of (3.3) and (3.4) are doubly-periodic functions, hence, without loss generality, we can assume that the resultant vector of the tractions on γ_j $(j \in I)$ vanishes.

3.2 Formulation of the Mixed CPS Problem

By the external stress conditions on the boundaries of the holes, from formulae (1.19)-(1.20) and (3.2) we have

$$\phi(t) + t\overline{\phi'(t)} + \overline{\psi(t)} = f(t) + C_j(m,n), t \in \gamma_j \subset \gamma_I \bigcup \Omega_{mn}. \tag{3.5}$$

The external stresses applied to the two sides of $\mathcal{L}(m,n)$ must be in equilibrium, then, from formulae (1.19)-(1.20) we get

$$\phi^+(t) + t\overline{\phi'^+(t)} + \overline{\psi^+(t)} = \phi^-(t) + t\overline{\phi'^-(t)} + \overline{\psi^-(t)}, t \in L \bigcup \Omega_{mn}. \tag{3.6}$$

By the displacement discontinuity conditions of the two sides of $\mathcal{L}(m,n)$, from formula (1.10) we get the boundary condition

$$\alpha^+\phi^+(t) - \beta^+\left[t\overline{\phi'^+(t)} + \overline{\psi^+(t)}\right] = \alpha^-\phi^-(t) - \beta^-\left[t\overline{\phi'^-(t)} + \overline{\psi^-(t)}\right] \\ - (\nu^+ - \nu^-)e_3 t + 2g(t), t \in L\bigcup\Omega_{mn}. \qquad (3.7)$$

By the displacement conditions on the boundaries of the holes, from formula (1.10) we obtain

$$\kappa_j\phi(t) - t\overline{\phi'(t)} - \overline{\psi(t)} = 2\mu_j\left(h_j(t) + \nu_j e_3 t\right), t \in \gamma_j \subset \gamma_{I\!I}\bigcup\Omega_{mn}, \qquad (3.8)$$

$$\left[\kappa_z\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}\right]_z^{z+2\omega_k} = 2\mu_z h_k, k = 1,2, \qquad (3.9)$$

and

$$F^+(t) + \overline{F^+(t)} = F^-(t) + \overline{F^-(t)}, t \in L\bigcup\Omega(mn), \qquad (3.10)$$

$$\mu^+\left[F^+(t) - \overline{F^+(t)}\right] = \mu^-\left[F^-(t) - \overline{F^-(t)}\right], t \in L\bigcup\Omega_{mn}, \qquad (3.11)$$

$$F(t) - \overline{F(t)} = iC_j^*(m,n), t \in \gamma_I\bigcup\Omega_{mn}, \qquad (3.12)$$

$$F(t) + \overline{F(t)} = w(t), t \in \gamma_{I\!I}\bigcup\Omega_{mn}, \qquad (3.13)$$

$$\left[F(z) + \overline{F(z)}\right]_z^{z+2\omega_k} = 2w_k, k = 1,2, \qquad (3.14)$$

where

$$\mu_z = \begin{cases} \mu^+, z \in D, \\ \mu^-, z \in D,' \end{cases}$$

$$\kappa_k, \mu_k = \begin{cases} \kappa^+, \mu^+ & \text{when } \gamma_k \text{ is the boundary (part) of } D, \\ \kappa^-, \mu^- & \text{when } \gamma_k \text{ is the boundary (part) of } D', \end{cases}$$

$$\alpha^\pm = \frac{\kappa^\pm}{\mu^\pm}, \quad \beta^\pm = \frac{1}{\mu^\pm}, \quad \kappa^\pm = 3 - 4\nu^\pm,$$

$\mu^\pm$ are Lamé constants in D and D' and $\nu^\pm$ are Possion ratios in D and D', respectively.

3.3 Solution of the Mixed CPS Problem

In order to solve the boundary value problem (3.5)-(3.9), denoting $A_k = -(X_{1k} + iX_{2k})/2\pi,\ k \in I\!I$, we construct the general representation of the solution

$$
\begin{aligned}
\phi(z) = & \frac{1}{2\pi i}\int_{L\bigcup\gamma}\omega(t)\left[\zeta(t-z)-\zeta(t)\right]dt + \sum_{j\in I} b_j\zeta(z-z_j) \\
& + \frac{1}{\kappa_z+1}\sum_{k\in I\!I} A_k \log\ \sigma(z-z_k) + A_z z,
\end{aligned} \tag{3.15}
$$

$$
\begin{aligned}
\psi(z) = & \frac{1}{2\pi i}\int_{\gamma_I}\left[\overline{\omega(t)}dt + \omega(t)d\bar{t}\right]\left[\zeta(t-z)-\zeta(t)\right] \\
& - \frac{1}{2\pi i}\int_{L\bigcup\gamma}\omega(t)\left[\bar{t}\wp(t-z)-\rho_1(t-z)\right]dt \\
& - \frac{1}{2\pi i}\int_{L}\left[\overline{\omega(t)}dt - \omega(t)d\bar{t}\right]\left[\zeta(t-z)-\zeta(t)\right] \\
& - \sum_{k\in I\!I}\frac{\kappa_k}{2\pi i}\int_{\gamma_k}\overline{\omega(t)}\left[\zeta(t-z)-\zeta(t)\right]dt \\
& + \frac{1}{2\pi i}\int_{\gamma_{I\!I}}\omega(t)\left[\zeta(t-z)-\zeta(t)\right]d\bar{t} \\
& - \frac{1}{2\pi i}\int_{L}\overline{\omega(t)}\zeta(t-z)dt + \sum_{j\in I} b_j\left[\zeta(z-z_j)+\rho_1(z-z_j)\right] \\
& - \frac{\kappa_z}{\kappa_z+1}\sum_{k\in I\!I} A_k\log\ \sigma(z-z_k) \\
& + \frac{1}{2\pi i}\int_{L}\overline{H(t)}\left[\zeta(t-z)-\zeta(t)\right]dt + B_z z,
\end{aligned} \tag{3.16}
$$

where b_j, $A^{\pm}$, $B^{\pm}$ are undetermined constants,

$$
b_j = \frac{1}{2\pi i}\int_{\gamma_j}\left[\omega(t)d\bar{t} - \overline{\omega(t)}dt\right], j \in I, \tag{3.17}
$$

$$
A_z, B_z = \begin{cases} A^+, B^+, & z \in D, \\ A^-, B^-, & z \in D', \end{cases}
$$

$$
\rho_1(z) = \sum_{mn}{}'\left\{\frac{\overline{\Omega_{mn}}}{(z-\Omega_{mn})^2} - 2z\frac{\overline{\Omega_{mn}}}{(\Omega_{mn})^3} - \frac{\overline{\Omega_{mn}}}{(\Omega_{mn})^2}\right\}. \tag{3.18}
$$

$\rho_1(z)$ is a meromophic function with the properities [13], [14]

$$\rho_1(z+2\omega_k)-\rho_1(z)=2\overline{\omega_k}\wp(z)+2r_k, k=1,2. \tag{3.19}$$

$\wp(z)$ is the Weierstrass elliptic $\wp$ function [5]

$$\wp(z)=\frac{1}{z^2}+\sum_{m,n}{}'\left\{\frac{1}{(z-\Omega_{mn})^2}-\frac{1}{\Omega_{mn}^2}\right\},$$

which has properties

$$\wp(z)=\wp(-z),$$

$$\wp(z)=-\zeta'(z),$$

r_1 and r_2 are known constants satisfying

$$r_2\omega_1-r_1\omega_2=\eta_1\overline{\omega_2}-\eta_2\overline{\omega_1}=-\frac{\pi i}{2}\delta_2. \tag{3.20}$$

It is easy to verify that the function $\phi(z)$ and the expression $z\overline{\phi'(z)}+\overline{\psi(z)}$ obtained from (3.15) and (3.16) will both be indeed doubly quasi-periodic.

Substituting (3.15) and (3.16) into (3.9) we get a system of equations of unknown constants A_z and B_z, the determinant of which is

$$\begin{vmatrix}\omega_1 & -\overline{\omega_1}\\ \omega_2 & -\overline{\omega_2}\end{vmatrix}=-\frac{1}{2}iS\neq 0.$$

Hence we can obtain A_z, B_z uniquely as

$$A_z=\frac{\kappa_z R_z+\overline{R_z}}{\kappa_z^2-1}, \tag{3.21}$$

$$\begin{aligned}B_z=&\frac{\mu_z(\omega_1 h_2-\omega_2 h_1)}{2iS}+\frac{\kappa_z b(\omega_2-\omega_1)}{4iS}\\&+\frac{\kappa_z\pi}{4(\kappa_z+1)S}\sum_{k\in I\!I}A_k(z_k+\overline{z_k}\delta_2)+\frac{(\bar{b}-\bar{a})\pi\overline{\delta_2}}{8S},\end{aligned} \tag{3.22}$$

$$R_z = \frac{\mu_z(\overline{\omega_1}h_2 - \overline{\omega_2}h_1)}{2iS} + \frac{\kappa_z b(\overline{\omega_2} - \overline{\omega_1})}{4iS}$$
$$+\frac{\kappa_z \pi}{4(\kappa_z + 1)S} \sum_{k \in II} A_k(\overline{z_k} + \delta_2 z_k) + \frac{(\overline{a} + \overline{b})\pi\overline{\delta_2}}{8S},$$

$$a = \sum_{k \in II} \frac{\kappa_k}{\pi i} \int_{\gamma_k} \overline{\omega(t)}dt + \frac{1}{\pi i}\int_{\gamma_{II}} \omega(t)d\overline{t}$$
$$-\frac{2}{\pi i}\int_{\gamma_I} \omega(t)d\overline{t} - \frac{1}{\pi i}\int_L \overline{H(t)}dt, \tag{3.23}$$

$$b = \frac{1}{\pi i}\int_{\gamma_I} \left[\omega(t)d\overline{t} - \overline{\omega(t)}dt\right] - \frac{1}{\pi i}\int_{L \bigcup \gamma} \omega(t)dt. \tag{3.24}$$

Letting $z \to t_0 \in L$, on account of (3.21) and (3.22), and substituting (3.15) and (3.16) into (3.6) by the Plemelj formulae we get

$$H(t_0) = \frac{Im\int_l F^*(t)d\overline{t}}{4S}\left[\frac{\kappa^+ - \kappa^-}{(\kappa^- - 1)(\kappa^+ - 1)}\right] t_0 + F^*(t_0), \tag{3.25}$$

where

$$F^*(t_0) = \frac{2(\kappa^+ - \kappa^-)}{(\kappa^- - 1)(\kappa^+ - 1)} \sum_{k \in II} \left\{ Re\left[A_k \log\ \sigma(t_0 - z_k)\right] + t_0\overline{A_k\zeta(t_0 - z_k)}\right\}. \tag{3.26}$$

Letting $z \to t_0 \in \gamma_I$, on account of (3.21), (3.22) and (3.25), and substituting (3.15) and (3.16) into (3.5) by the Plemelj formulae after miscellaneous calculating we get a second kind Fredholm integral equation

$$\omega(t_0) + \frac{1}{2\pi i}\int_{\gamma_I} \omega(t)d\left[\log\frac{\sigma(t-t_0)\overline{\sigma(t)}}{\overline{\sigma(t-t_0)}\sigma(t)}\right] + \frac{1}{2\pi i}\int_{\gamma_{II}} \omega(t)\left[\zeta(t-t_0) - \zeta(t)\right]dt$$
$$+\frac{1}{\pi i}\int_L \omega(t)d\left[\log|\frac{\sigma(t-t_0)}{\sigma(t)}|\right] + \frac{1}{2\pi i}\int_L H(t)\left[\overline{\zeta(t-t_0)} - \overline{\zeta(t)}\right]d\overline{t}$$
$$+\sum_{k \in II}\frac{\kappa_k}{2\pi i}\int_{\gamma_k} \omega(t)\left[\overline{\zeta(t-t_0)} - \overline{\zeta(t)}\right]d\overline{t}$$
$$+\frac{1}{2\pi i}\int_{L\bigcup\gamma} \overline{\omega(t)}d\left\{\zeta_1(t-t_0) - (t-t_0)\left[\overline{\zeta(t-t_0)} - \overline{\zeta(t)}\right]\right\}$$
$$+\frac{t_0}{\kappa_j + 1}\sum_{k \in II}\overline{A_k\zeta(t_0 - z_k)} + \frac{1}{2\pi i}\int_{L\bigcup\gamma}\overline{\omega(t)\zeta(t)}dt$$

$$
\begin{aligned}
&+\frac{1}{\kappa_j+1}\sum_{k\in II} A_k\left[\log\ \sigma(t_0-z_k)-\kappa_j\log\overline{\sigma(t_0-z_k)}\right]\\
&+\sum_{j\in I} b_j\left[2Re\overline{\zeta(t_0-z_j)}+\overline{\rho_1(t_0-z_j)}-t_0\overline{\wp(t_0-z_j)}\right]\\
&+2(ReA_j)t_0+\overline{B_j t_0}=f(t_0)+C_j, t_0\in\gamma_j\subset\gamma_I. \qquad (3.27)
\end{aligned}
$$

Letting $z\to t_0\in\gamma_{II}$ and substituting (3.15) and (3.15) into (3.8) we have the following second kind Fredholm integral equation

$$
\begin{aligned}
&\kappa_j\omega(t_0)+\frac{\kappa_j}{2\pi i}\int_{\gamma_j}\omega(t)d\left[\log\frac{\sigma(t-t_0)\overline{\sigma(t)}}{\overline{\sigma(t-t_0)}\sigma(t)}\right]\\
&+\sum_{\substack{k\neq j\\ k\in II}}\left\{\kappa_j\int_{\gamma_k}\omega(t)\left[\zeta(t-t_0)-\zeta(t)\right]dt+\kappa_k\int_{\gamma_k}\omega(t)\left[\overline{\zeta(t-t_0)}-\overline{\zeta(t)}\right]d\overline{t}\right\}\\
&+\frac{1}{2\pi i}\int_{\gamma_I}\omega(t)d\left[\log\frac{\sigma(t-t_0)\overline{\sigma(t)}}{\overline{\sigma(t-t_0)}\sigma(t)}\right]-\frac{1}{\pi i}\int_L\omega(t)d\left[\log|\frac{\sigma(t-t_0)}{\sigma(t)}|\right]\\
&-\frac{1}{2\pi i}\int_{L\bigcup\gamma}\overline{\omega(t)}d\left\{\zeta_1(t-t_0)-(t-t_0)\overline{\zeta(t-t_0)}\right\}\\
&+\frac{\kappa_j+1}{2\pi i}\int_{L\bigcup\gamma_I}\omega(t)\left[\zeta(t-t_0)-\zeta(t)\right]dt+M_5[\omega(t),t_0]\\
&=N_5(t_0), t_0\in\gamma_j\subset\gamma_{II}, \qquad (3.28)
\end{aligned}
$$

where

$$
\begin{aligned}
M_5[\omega(t),t_0]=&\ \kappa_j\sum_{k\in I} b_k\zeta(t_0-z_k)+\frac{t_0}{\kappa_j+1}\sum_{k\in II}\overline{A_k\zeta(t_0-z_k)}\\
&-t_0\sum_{j\in I} b_j\overline{\wp(t_0-z_j)}+\frac{2\kappa_j}{\kappa_j+1}\sum_{k\in II} A_k\log|\sigma(t_0-z_k)|\\
&+(\kappa_jA_j+\overline{A_j})t_0+\overline{B_j t_0}-\frac{1}{2\pi i}\int_{L\bigcup\gamma}\overline{\omega(t)\zeta(t)}dt,
\end{aligned}
$$

$$
N_5(t_0)=2\mu_j\left(h_j(t)+\nu_j e_3 t\right)-\frac{1}{2\pi i}\int_L H(t)\left[\overline{\zeta(t-t_0)}-\overline{\zeta(t)}\right]d\overline{t}.
$$

Here we have defined the function $\zeta_1'(z)=\rho_1(z)$, $\zeta_1(0)=0$.

Letting $z\to t_0\in L$, substituting (3.15) and (3.16) into (3.7) and taking $\kappa^{\pm}\beta^{\pm}=\alpha^{\pm}$ into account, we obtain the following singular integral equation

$$
\begin{aligned}
&(\alpha^+ + \alpha^- + \beta^+ + \beta^-)\omega(t_0) \\
&+\frac{\alpha^+ - \alpha^- + \beta^- - \beta^+}{\pi i}\int_{L\bigcup\gamma}\omega(t)\left[\overline{\zeta(t-t_0)} - \overline{\zeta(t)}\right]d\bar{t} \\
&+\frac{\beta^+ - \beta^-}{\pi i}\int_{L\bigcup\gamma_I}\omega(t)d\left[\log\frac{\sigma(t-t_0)}{\overline{\sigma(t-t_0)}}\right] \\
&+\frac{\beta^+ - \beta^-}{\pi i}\int_{L\bigcup\gamma}\overline{\omega(t)}d\left[\overline{\zeta_1(t-t_0)} - (t-t_0)\overline{\zeta(t-t_0)}\right] \\
&-\frac{\beta^- - \beta^+}{\pi i}\left\{\sum_{k\in I\!I}\kappa_k\int_{\gamma_k}\omega(t)\left[\overline{\zeta(t-t_0)} - \overline{\zeta(t)}\right]d\bar{t} - \int_{\gamma_{I\!I}}\omega(t)\left[\zeta(t-t_0) - \zeta(t)\right]dt\right\} \\
&-M_6[\omega(t), t_0] = N_6(t_0), t_0 \in L, \qquad (3.29)
\end{aligned}
$$

where

$$
\begin{aligned}
M_6[\omega(t), t_0] = &\frac{\beta^+ - \beta^-}{\pi i}\int_{L\bigcup\gamma}\overline{\omega(t)\zeta(t)} \\
&+2\left(\frac{\beta^-}{\kappa^- + 1} - \frac{\beta^+}{\kappa^+ + 1}\right)t_0\sum_{k\in I\!I}\overline{A_k\zeta(t_0 - z_k)} \\
&-(\beta^+ - \beta^-)\sum_{j\in I}b_j\left[\overline{\zeta(t_0 - z_j)}t_0 - \overline{\wp(t_0 - z_j)}\right] - \overline{\rho_1(t_0 - z_j)} \\
&+4\left(\frac{\alpha^+}{\kappa^+ + 1} - \frac{\alpha^-}{\kappa^- + 1}\right)\sum_{k\in I\!I}A_k\log|\sigma(t_0 - z_k)| \\
&+2(\alpha^+ - \alpha^-)\sum_{j\in I}b_j\zeta(t_0 - z_j) - 2(\beta^+B^+ - \beta^-B^-)\overline{t_0} \\
&+2\left(\alpha^+A^+ - \alpha^-A^- - \beta^+\overline{A^+} + \beta^-\overline{A^-}\right)t_0,
\end{aligned}
$$

$$
N_6(t_0) = 4g(t_0) - \frac{\beta^+ - \beta^-}{\pi i}\int_L H(t)\overline{\zeta(t-t_0)}d\bar{t} + (\beta^+ - \beta^-)H(t_0) + (\nu^+ - \nu^-)e_3t_0.
$$

For the uniquenss of the solution, let [40], [49]

$$
C_j = -\int_{\gamma_j}\omega(t)ds, C_1 = 0, t \in \gamma_j \subset \gamma_I. \qquad (3.30)
$$

Thus, (3.27)-(3.29) as a whole constitute a normal type singular integral equation with doubly-periodic kernel on $L\bigcup\gamma$.

In order to solve the boundary value problems (3.10) - (3.14), we use the modified Sherman transform in this case,

$$F(z) = \frac{1}{2\pi i} \int_{L \bigcup \gamma} i\Delta(t)\zeta(t-z)dt + Ez. \tag{3.31}$$

Substituting (3.31) into (3.14) we can get ReE and ImE uniquely as functionals of $\Delta(t)$.

Substituting (3.31) into (3.10), it is obvious that (3.10) will be automatically satisfied.

Letting $z \to t_0 \in L$ and substituting (3.31) into (3.11) we get

$$\Delta(t_0) + \frac{\mu^*}{2\pi i} \int_{L \bigcup \gamma} \Delta(t) d \left[\log \frac{\sigma(t-t_0)}{\overline{\sigma(t-t_0)}} \right] - 2\mu^* i Re(Et_0) = 0, \tag{3.32}$$

where μ^* is given by (1.90).

Letting $z \to t_0 \in \gamma_j \subset \gamma_I$ and substituting (3.31) into (3.12) we have

$$\Delta(t_0) + \frac{1}{2\pi i} \int_{L \bigcup \gamma} \Delta(t) d \left[\log \frac{\sigma(t-t_0)}{\overline{\sigma(t-t_0)}} \right] - 2iRe(Et_0) - C_j^* = 0. \tag{3.33}$$

Letting at last $z \to t_0 \in \gamma_j \subset \gamma_{II}$ and substituting (3.31) into (3.13) we obtain

$$\frac{1}{2\pi i} \int_{L \bigcup \gamma} \Delta(t) d \left[\log |\sigma(t-t_0)| \right] - 2iRe(Et_0) = -iW(t_0). \tag{3.34}$$

Equations (3.32)-(3.34) combine into a second kind Fredholm integral equation.

3.4 Unique Solvability of the Mixed CPS Problem

At first, we prove the unique solvablity of equations (3.27)-(3.29). To do this, we must show that the homogeneous equation has no non-trival solutions, say, when $f(t) \equiv 0$, $h(t) \equiv 0$, $e_3 = 0$, after $C_j = C_j^0$ is taken, then $\omega_0(t) \equiv 0$ everywhere on $L \bigcup \gamma$ (and hence $C_j^0 = 0$ necessarily).

Let $\phi_0(z)$, $\psi_0(z)$, b_j^0, A_z^0, B_z^0, a_0, b_0, $H_0(t)$ and C_j^0 (for uniqueness e.g. $C_1^0 = 0$) be the corresponding values of $\phi(z)$, $\psi(z)$, b_j, A_z, B_z, a, b, $H(t)$ and C_j determined by equations (3.15)-(3.17), (3.21)-(3.25), and equation (3.30) for $\omega(t) = \omega_0(t)$. It is easy to verify that they satisfy the corresponding boundary conditions (3.5)-(3.9), which form the mixed fundamental problem under homogeneous conditions (and $C_1^0 = 0$). By the uniqueness theorem [49], [40], we have

$$\phi_0(z) = c_z, \quad \psi_0(z) = \kappa_z c_z, \tag{3.35}$$

Due to $C_1^0 = 0$, then

$$C_j^0 = 0, j = 2, \cdots, m, \tag{3.36}$$

and

$$(\kappa^+ + 1)c^+ = (\kappa^- + 1)c^-. \tag{3.37}$$

Now, as $\phi_0(z)$ is a single-valued function, it follows from (3.15) that we have

$$A_k^0 = 0, k \in I\!I. \tag{3.38}$$

Hence, we find $H_0(t) = 0$ from (3.25). Thus

$$c_z = \frac{1}{2\pi i}\int_{L\bigcup\gamma} \omega_0(t)\left[\zeta(t-z) - \zeta(t)\right]dt + \sum_{j\in I} b_j^0 \zeta(z - z_j) + A_z^0 z, \tag{3.39}$$

$$\begin{aligned}\kappa_z c_z = &\frac{1}{2\pi i}\int_{\gamma_I}\left[\overline{\omega_0(t)}dt + \omega_0(t)d\bar{t}\right]\left[\zeta(t-z) - \zeta(t)\right]\\ &-\frac{1}{2\pi i}\int_{L\bigcup\gamma}\omega_0(t)\left[\bar{t}\wp(t-z) - \rho_1(t-z)\right]dt\\ &-\frac{1}{2\pi i}\int_{L}\left[\overline{\omega_0(t)}dt - \omega_0(t)d\bar{t}\right]\left[\zeta(t-z) - \zeta(t)\right]\end{aligned}$$

$$
\begin{aligned}
&-\sum_{k\in I\!I}\frac{\kappa_k}{2\pi i}\int_{\gamma_k}\overline{\omega_0(t)}\,[\zeta(t-z)-\zeta(t)]\,dt\\
&+\frac{1}{2\pi i}\int_{\gamma_{I\!I}}\omega_0(t)\,[\zeta(t-z)-\zeta(t)]\,d\bar{t}\\
&+\sum_{j\in I}b_j^0\,[\zeta(z-z_j)+\rho_1(z-z_j)]+B_z^0 z. \qquad (3.40)
\end{aligned}
$$

Because the right-hand sides of equations (3.39) and (3.40) are doubly quasi-periodic, the cyclic increments of the two sides of (3.39) and (3.40) must be equal, respectively. Then, we get

$$
A_z^0=0, B_z^0=0, a_0=0, b_0=0. \qquad (3.41)
$$

By using the Plemelj formulae on L from (3.39), one obtains for $C_1^0=0$

$$
\omega_0(t)=c^+-c^-, t\in L. \qquad (3.42)
$$

Substituting (3.42) back into (3.39) and (3.40) after integrating by parts, we arrive at the equalities

$$
\begin{aligned}
c_z &= \frac{1}{2\pi i}\int_{\gamma}\omega_0(t)\,[\zeta(t-z)-\zeta(t)]\,dt\\
&+\frac{c^+-c^-}{2\pi i}\int_L[\zeta(t-z)-\zeta(t)]\,dt+\sum_{j\in I}b_j^0\zeta(z-z_j), \qquad (3.43)
\end{aligned}
$$

$$
\begin{aligned}
\kappa_z\overline{c_z} &= \frac{1}{2\pi i}\int_{\gamma_I}\overline{\omega_0(t)}dt\,[\zeta(t-z)-\zeta(t)]-\frac{1}{2\pi i}\int_{\gamma}\omega_0(t)\zeta(t)d\bar{t}\\
&-\frac{c^+-c^-}{2\pi i}\int_L\zeta(t)d\bar{t}+\frac{1}{2\pi i}\int_{\gamma}\omega_0\rho_1(t-z)dt\\
&-\frac{\overline{c^+}-\overline{c^-}}{2\pi i}\int_L[\zeta(t-z)-\zeta(t)]\\
&-\sum_{k\in I\!I}\frac{\kappa_k}{2\pi i}\int_{\gamma_k}\overline{\omega_0(t)}\,[\zeta(t-z)-\zeta(t)]\,dt. \qquad (3.44)
\end{aligned}
$$

The functions

$$
\chi_1(z)=c_z-\frac{c^+-c^-}{2\pi i}\int_L[\zeta(t-z)-\zeta(t)]\,dt, \qquad (3.45)
$$

$$\chi_2(z) = \kappa_z\overline{c_z} + \frac{c^+ - c^-}{2\pi i}\int_L \zeta(t)d\bar{t}$$
$$-\frac{\overline{c^+} - \overline{c^-}}{2\pi i}\int_L [\zeta(t-z) - \zeta(t)]\, dt, \tag{3.46}$$

are holomophic in S_0.

Putting $z = 0$ ($0 \in D \subset S_0$) and paying attention to the origin being located outside the region bounded by L, it follows from (3.45) and (3.46) by Cauchy theorem that

$$\chi_1(z) = c^+, \quad \chi_2(z) = \kappa^+\overline{c^+} + c^*, \tag{3.47}$$

where

$$c^* = \frac{c^+ - c^-}{2\pi i}\int_L \zeta(t)d\bar{t}. \tag{3.48}$$

We introduce the functions

$$\Phi_*(z) = \frac{1}{2\pi i}\int_\gamma \left[\omega_0(t) + \sum_{j\in I} b_j^0\zeta(t-z_j) - c^+\right][\zeta(t-z) - \zeta(t)]\, dt$$
$$= \begin{cases} 0, z \in S_0, \\ -i\phi_*(z), z \in S_0^j, j \in I \bigcup I\!I, \end{cases} \tag{3.49}$$

$$\Psi_*(z) = \frac{1}{2\pi i}\int_{\gamma_{I\!I}} \left[\omega_0(t) - \bar{t}\omega_0'(t) + \sum_{j\in I} b_j^0\zeta(t-z_j) + e - c^*\right][\zeta(t-z) - \zeta(t)]\, dt$$
$$-\sum_{k\in I\!I}\frac{\kappa_k}{2\pi i}\int_{\gamma_k}\overline{\omega_0(t)}\,[\zeta(t-z) - \zeta(t)]\, dt$$
$$-\frac{1}{2\pi i}\int_{\gamma_{I\!I}}[\bar{t}\omega_0'(t) + e^*][\zeta(t-z) - \zeta(t)]dt + Q(z)$$
$$= \begin{cases} 0, z \in S_0, \\ -i\psi_*(z), z \in S_0^j, j \in I \bigcup I\!I, \end{cases} \tag{3.50}$$

where

$$e = -\frac{1}{2\pi i}\int_{\gamma_I}\bar{t}\omega_0(t)\wp(t)dt - \kappa^+\overline{c^+},$$
$$e^* = \frac{1}{2\pi i}\int_{\gamma_{I\!I}}\bar{t}\omega_0(t)\wp(t)dt - \kappa^+\overline{c^+},$$

$$Q(z) = \frac{1}{2\pi i}\int_\gamma \omega_0(t)\rho_1(t-z)dt + \sum_{j\in I} b_j^0 \rho_1(z-z_j).$$

Then

$$\begin{aligned}\Phi_*^-(t) &= \frac{1}{i}\phi_*(t)\\ &= \begin{cases} \omega_0(t) + \sum\limits_{r\in I} b_r^0 \zeta(t-z_r) - c^+, t\in\gamma_j\subset\gamma_I, \\ \omega_0(t) - c^+, t\in\gamma_j\subset\gamma_{I\!I}, \end{cases}\end{aligned} \tag{3.51}$$

$$\begin{aligned}\Psi_*^-(t) &= \frac{1}{i}\psi_*(t)\\ &= \begin{cases} \overline{\omega_0(t)} - \bar{t}\omega_0'(t) + \sum\limits_{r\in I} b_r^0 \zeta(t-z_r) + e - c^*, \\ -\kappa_k\overline{\omega_0(t)} - \bar{t}\omega_0'(t) - e^*, t\in\gamma_j\subset\gamma_{I\!I}. \end{cases}\end{aligned} \tag{3.52}$$

Eliminating $\omega_0(t)$ from (3.51) and (3.52), we obtain

$$\begin{aligned}\phi_*(t) + t\overline{\phi_*'(t)} + \overline{\psi_*(t)} &= i\sum_{r\in I} b_r^0\left[\overline{\zeta(t-z_r)} - \zeta(t-z_r) - t\overline{\wp(t-z_r)}\right]\\ &\quad + i(\overline{e} - \overline{c^*} + c^+), t\in\gamma_j\subset\gamma_I,\end{aligned} \tag{3.53}$$

$$\kappa_k\phi_*(t) - t\overline{\phi_*'(t)} - \overline{\psi_*(t)} = i(\overline{e^*} + \kappa_k c^+ - \overline{c^*}), t\in\gamma_j\subset\gamma_{I\!I}. \tag{3.54}$$

Multiplying both sides of (3.53) by dt and integrating over $\gamma_j, j\in I$, we arrive at the equalities

$$\begin{aligned}\int_{\gamma_j}\left[\overline{\phi_*(t)}dt - \phi_*(t)d\bar{t}\right] &= i\sum_{r\in I} b_r^0 \int_{\gamma_j}\left[\overline{\zeta(t-z_r)}dt + \zeta(t-z_r)d\bar{t}\right]\\ &\quad -2\pi b_r^0, t\in\gamma_j\subset\gamma_I.\end{aligned}$$

Since the b_r^0 which are determined by (3.17) are real constants, hence

$$b_r^0 = 0, r\in I. \tag{3.55}$$

Therefore

$$\phi_*(t) + t\overline{\phi_*'(t)} + \overline{\psi_*(t)} = i(\overline{e} - \overline{c^*} + c^+), t\in\gamma_j\subset\gamma_I. \tag{3.56}$$

This is exactly the boundary value problem of the first fundamental problem in the absence of external forces. Applying the uniqueness theorem for the first fundamental problem [49], [40], we get

$$\phi_*(z) = i\epsilon_j z + c_j, \ \psi_*(z) = -\overline{d_j}, j \in I.$$

Then, from (3.51) and (3.55), it follows immediately that

$$\omega_0(t) = c^+ - \epsilon_j t + ic_j, t \in \gamma_j \subset \gamma_I. \tag{3.57}$$

Substituting (3.57) into (3.17), on account of (3.55) we have

$$\epsilon_j = 0, j \in I. \tag{3.58}$$

From (3.56) we obtain

$$c_j - d_j = i(c^+ + \overline{e} - \overline{c^*}). \tag{3.59}$$

Further, using successively (3.51)-(3.52), (3.55)-(3.59) we find

$$c_j = d_j = 0, j \in I; \ c^+ + \overline{e} - \overline{c^*} = 0. \tag{3.60}$$

It follows from equations (3.30), (3.36) and (3.37) that

$$c^+ = c^- = 0. \tag{3.61}$$

Referring to formulae (3.57)-(3.61) we arrive at the equality

$$\omega_0(t) = 0, t \in \gamma_j \subset \gamma_I. \tag{3.62}$$

From (3.61) and (3.42) we have

$$\omega_0(t) = 0, t \in L. \tag{3.63}$$

Similarly, taking (3.54) into consideration, we can calculate that

$$\omega_0(t) = 0, t \in \gamma_j \subset \gamma_{II}. \tag{3.64}$$

With the above, we have proved

$$\omega_0(t) \equiv 0, t \in L \bigcup \gamma. \tag{3.65}$$

In order to prove the unique solvability of the second kind Fredholm equations (3.32)-(3.34), consider the homogeneous equation, obtained from the homogeneous condition $w(t) = 0$.

Let $\Delta_0(t)$ be any solution of the homogeneous equation and $F_0(z)$ the corresponding expression for the function $F(z)$, analogously to [20], we have

$$F_0(z) = c_z^*, \tag{3.66}$$

where

$$c_z^* = \begin{cases} c^{*+}, z \in S_0, \\ c^{*-}, z \in S_0^j, j \in I \bigcup I\!I, \end{cases}$$

and $c^{*\pm}$ are complex constants.

From (3.10) and (3.11) we obtain

$$Re(c^{*+}) = Re(c^{*-}), \tag{3.67}$$

$$\mu^+ Im(c^{*+}) = \mu^- Im(c^{*-}). \tag{3.68}$$

It follows from (3.13) and (3.67) that

$$Re(c^{*+}) = Re(c^{*-}) = 0, \tag{3.69}$$

whence, by (3.66) and (3.31) we find

$$\Delta_0(t) = 0, t \in \gamma. \tag{3.70}$$

Referring to formulae (3.31), and (3.66) (3.70) we arrive at the equality

$$Im(c_z^*) = \frac{1}{2\pi i} \int_L i\Delta_0(t)\zeta(t-z)dt + E_0 z. \tag{3.71}$$

By the Plemelj formulae, we get

$$i\Delta_0(t) = i[Im(c^{*-}) - Im(c^{*+})], t \in L. \tag{3.72}$$

On account of the double quasi-periodicity of the two sides of (3.71), we have

$$\frac{1}{2\pi i}\int_L i\Delta_0(t)\zeta(t)dt = 0,\ E_0 = 0. \tag{3.73}$$

Putting $z = 0$ in (3.71) and taking (3.73) into consideration, we obtain

$$Im(c^{*+}) = \frac{1}{2\pi i}\int_L i\Delta_0(t)\zeta(t)dt = 0. \tag{3.74}$$

From (3.68) we get

$$Im(c^{*-}) = 0. \tag{3.75}$$

It follows from (3.72) that

$$\Delta_0(t) = 0, t \in L. \tag{3.76}$$

Finally, referring to (3.70) and (3.76) we find

$$\Delta_0(t) \equiv 0, t \in L\bigcup\gamma. \tag{3.77}$$