# Chapter 2

# Second Fundamental CPS Problem of a Nonhomogeneous Body with a Doubly-Periodic Set of Cracks

## 2.1 Formulation, Solution of the Second Fundamental CPS Problem

The model of the elastic body of the second fundamental CPS problem is the same as Figure 1.1, the notations are the same as in Chapter 1. Denote $g_j^{\pm}(\tau) = u_j^{\pm}(\tau) + iv_j^{\pm}(\tau)$ as the displacements on the positive and negative sides of the point $\tau$ on the crack $\gamma_j$ $(j = 0, 1, \cdots, m-1)$, respectively. Assume that $g_j^{\pm}(\tau)$ are sufficiently smooth on $\gamma_j$ $(j = 0, 1, \cdots, m-1)$ and [49],

$$u_j^+(a_j) = u_j^-(a_j), \quad u_j^+(b_j) = u_j^-(b_j) \quad (j = 0, 1, \cdots, m-1),$$

$$v_j^+(a_j) = v_j^-(a_j), \quad v_j^+(b_j) = v_j^-(b_j),$$

or

$$g_j^+(a_j) = g_j^-(a_j), \quad g_j^+(b_j) = g_j^-(b_j).$$

By Lemma 1.1.1 we know the components of the displacement are doubly quasi-periodic due to the doubly-periodic stress distributions.

Now, consider the second fundamental CPS problem, where the displacements $g_j^{\pm}(t) = u_j^{\pm}(t) + iv_j^{\pm}(t)$, $t \in \gamma_j$ $(j = 0, 1, \cdots, m-1)$ and their cyclic increments $g_k$ $(k = 1, 2)$, and displacement $w(t), t \in \gamma$, with its cyclic increments $w_k$ $(k = 1, 2)$, are given. The displacement discontinuity $g(t) = [u^+(t) + iv^+(t)] - [u^-(t) + iv^-(t)]$ on $L$ is likewise given. The strain $e_3 = constant$. In this case, the external stress resultant principal vectors $X_{1j} + X_{2j}$ $(j = 0, 1, \cdots, m-1)$ are undetermined constants. Certainly, they must satisfy (1.28), to determine the elastic equilibrium.

Considering the displacement conditions, from formula (1.10), we have the boundary conditions on $\gamma_j$ and their congruent for the elastic system $(I)$

$$\kappa_j \phi^{\pm}(\tau) - \tau\overline{\phi'^{\pm}(\tau)} - \overline{\psi^{\pm}(\tau)} = 2\mu_j \left[ g_j^{\pm}(\tau) + \nu_j e_3 \tau \right], \tau \in \gamma_j, \ j = 0, 1, \cdots, m-1. \tag{2.1}$$

The external stresses applied to the two sides of $\mathcal{L}(m,n)$ must be in equilibrium, then, from formulae (1.19)-(1.20) we have

$$\phi^+(t) + t\overline{\phi'^+(t)} + \overline{\psi^+(t)} = \phi^-(t) + t\overline{\phi'^-(t)} + \overline{\psi^-(t)}, t \in L. \tag{2.2}$$

Moreover, by the displacement discontinuity conditions of the two sides of $\mathcal{L}(m,n)$, from formula (1.10) we have the boundary condition

$$\alpha^+\phi^+(t) - \beta^+[t\overline{\phi'^+(t)} + \overline{\psi^+(t)}] = \alpha^-\phi^-(t) - \beta^-[t\overline{\phi'^-(t)} + \overline{\psi^-(t)}] \\ -(\nu^+ - \nu^-)e_3 t + 2g(t), t \in L. \tag{2.3}$$

To ensure the double quasi-periodicity of the displacement, we have

$$\left[\kappa_z\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}\right]_z^{z+2\omega_k} = 2\mu_z g_k, k = 1, 2. \tag{2.4}$$

Similarly, for the elastic system ($I\!I$) we have

$$F^{\pm}(\tau) + \overline{F^{\pm}(\tau)} = w(t), \tau \in \gamma_j \ (j = 0, 1, \cdots, m-1), \tag{2.5}$$

$$F^{+}(t) + \overline{F^{+}(t)} = F^{-}(t) + \overline{F^{-}(t)}, t \in L, \tag{2.6}$$

$$\mu^{+}\left[F^{+}(t) - \overline{F^{+}(t)}\right] = \mu^{-}\left[F^{-}(t) - \overline{F^{-}(t)}\right], t \in L, \tag{2.7}$$

$$\left[F(z) + \overline{F(z)}\right]_z^{z+2\omega_k} = 2w_k, k = 1, 2, \tag{2.8}$$

where

$$\mu_j = \begin{cases} \mu^{+}, & \text{if } \gamma_j \subset S_0^{+}, \\ \mu^{-}, & \text{if } \gamma_j \subset S_0^{-}, \end{cases} \quad , \quad \kappa_j = \begin{cases} \kappa^{+}, & \text{if } \gamma_j \subset S_0^{+}, \\ \kappa^{-}, & \text{if } \gamma_j \subset S_0^{-}. \end{cases}$$

In order to solve the boundary value problem (2.1)-(2.4), the general representation of the solution will be constructed as follows,

$$\begin{aligned} \phi(z) = & \frac{1}{2\pi i}\int_{L\bigcup\gamma} \omega(t)\zeta(t-z)dt \\ & + \frac{1}{\kappa_z + 1}\sum_{j=0}^{m-1} A_j\left[\log\sigma(z-a_j)\sigma(z-b_j) - H_j(z)\right] \\ & + A_z z + C_j, \end{aligned} \tag{2.9}$$

$$\begin{aligned} \psi(z) = & -\sum_{j=0}^{m-1}\frac{\kappa_j}{2\pi i}\int_{\gamma_j}\overline{\omega(t)}\zeta(t-z)dt \\ & - \frac{\kappa_z}{\kappa_z + 1}\sum_{j=0}^{m-1}\overline{A_j}\left[\log\sigma(z-a_j)\sigma(z-b_j) - H_j(z)\right] \\ & - \frac{1}{2\pi i}\int_L \overline{\omega(t)}\zeta(t-z)dt - \frac{1}{2\pi i}\int_{L\bigcup\gamma}\overline{m(t)}\omega'(t)\zeta(t-z)dt \\ & + \frac{1}{2\pi i}\int_L \overline{H(t)}\zeta(t-z)dt - \frac{1}{2\pi i}\int_{\gamma}\left[\overline{h^{+}(t)} - \overline{h^{-}(t)}\right]\zeta(t-z)dt \\ & + D(z)\phi'(z) - D(z)\phi'(0) + B_z z, \end{aligned} \tag{2.10}$$

where $H(t)$ is an undetermined function, $C_j$ and

$$A_j = -\frac{X_{1j} + iX_{2j}}{4\pi},\ j = 0, 1, \cdots, m-1,$$

are undetermined constants,

$$h^{\pm}(t) = 2\mu_j g_j^{\pm}(t), t \in \gamma_j.$$

Substituting (2.9) and (2.10) into (2.2) and (2.4) we get

$$H(t_0) = Q(t_0) - \frac{2(\kappa^- - \kappa^+)}{(\kappa^+ - 1)(\kappa^- - 1)} Re\left[q_1 \int_l Q(t)d\overline{t}\right] t_0, \tag{2.11}$$

$$A^{\pm} = \frac{\kappa^{\pm} p_1^{\pm} - \overline{p_1^{\pm}}}{(\kappa^+)^2 - 1} + \frac{2\kappa^{\pm} Re[q_1 \int_l H(t)d\overline{t}]}{\kappa^- - 1}, \tag{2.12}$$

$$\overline{B^{\pm}} = p_2^{\pm} - q_2 \int_l H(t)d\overline{t}, \tag{2.13}$$

where

$$\begin{aligned} Q(t_0) = {} & \frac{(\kappa^+ - \kappa^-)t_0}{(\kappa^+ + 1)(\kappa^- + 1)} \left\{ \sum_{j=0}^{m-1} A_j \left[\log \sigma(t_0 - a_j)\sigma(t_0 - b_j) - H_j(t_0)\right] \right\} \\ & + \frac{(\kappa^+ - \kappa^-)t_0}{(\kappa^+ + 1)(\kappa^- + 1)} \left\{ \sum_{j=0}^{m-1} \overline{A_j} \left[\overline{\zeta(t_0 - a_j)} + \overline{\zeta(t_0 - b_j)} - H_j'(t_0)\right] \right\} \\ & - \left( \frac{p_1^+ - \overline{p_1^+}}{\kappa^+ - 1} - \frac{p_1^- - \overline{p_1^-}}{\kappa^- - 1} \right) t_0, \end{aligned}$$

$$\begin{aligned} p_1^{\pm} = {} & \frac{1}{8iS}\left[ -\kappa^{\pm}\delta_1 \int_{L\bigcup\gamma} \omega(t)dt + \sum_{j=0}^{m-1} \kappa_j \int_{\gamma_j} d\overline{t} + \int_l \omega(t)d\overline{t} + \int_{L\bigcup\gamma} m(t)\omega'(t)d\overline{t} \right] \\ & + \frac{\mu^{\pm}(\overline{\omega_1}h_2 - \overline{\omega_2}h_1)}{2iS} - \frac{1}{2\pi i}\int_{L\bigcup\gamma} \overline{\omega'(t)\zeta(t)}d\overline{t}, \end{aligned}$$

$$q_1 = -\frac{\delta_1}{8iS},$$

$$\begin{aligned} p_2^{\pm} = {} & \frac{1}{16iS}\kappa^{\pm} \int_{L\bigcup\gamma} \omega(t)dt \\ & - \frac{1}{8iS}\left[ \sum_{j=0}^{m-1} \kappa_j \int_{\gamma_j} \omega(t)d\overline{t} + \int_l \omega(t)d\overline{t} + \int_{L\bigcup\gamma} m(t)\omega'(t)d\overline{t} \right], \end{aligned}$$

$$q_2 = -\frac{1}{8iS}.$$

Substituting $H(t)$, $A_z$ and $B_z$ from equations (2.11), (2.12) and (2.13) into formulae (2.9) and (2.10), the boundary conditions (2.2) and (2.4) will be automatically satisfied.

Letting $z \to t_0 \in L$ and substituting equation (2.9) and (2.10) into equation (2.3), by employing the modified Plemelj formulae and taking $\kappa^{\pm}\beta^{\pm} = \alpha^{\pm}$ into account, we get the integral equation

$$\begin{aligned}
&(\alpha^+ + \alpha^- + \beta^+ + \beta^-)\omega(t_0) + \frac{\alpha^+ - \alpha^- + \beta^- - \beta^+}{\pi i}\int_L \omega(t)\overline{\zeta(t-t_0)}d\overline{t} \\
&+\frac{\beta^+ - \beta^-}{\pi i}\int_L \omega(t) d\left[\log\frac{\sigma(t-t_0)}{\overline{\sigma(t-t_0)}}\right] \\
&+\frac{\beta^+ - \beta^-}{\pi i}\int_L \overline{\omega(t)} d\left\{[m(t)-m(t_0)]\,\overline{\zeta(t-t_0)}\right\} \\
&-M_3[\omega(t),t_0] = N_3(t_0),
\end{aligned}\tag{2.14}$$

where

$$\begin{aligned}
M_3[\omega(t),t_0] &= \frac{\beta^+ - \beta^-}{\pi i}\sum_{j=0}^{m-1}\kappa_j\int_{\gamma_j}\omega(t)\overline{\zeta(t-t_0)}d\overline{t} \\
&+4\left(\frac{\beta^+}{\kappa^+ + 1} - \frac{\beta^-}{\kappa^- + 1}\right)\log|\sigma(t_0-a_j)\sigma(t_0-b_j) + H_j(t_0)| \\
&+2\left(\frac{\beta^+}{\kappa^+ + 1} - \frac{\beta^-}{\kappa^- + 1}\right)\sum_{j=0}^{m-1}\overline{A_j}\left[\overline{\zeta(t_0-a_j)} + \overline{\zeta(t_0-b_j)} - H_j'(t_0)\right],
\end{aligned}$$

$$N_3(t_0) = 4g(t_0) - \frac{\beta^+ - \beta^-}{\pi i}\int_{\gamma}\left[h^+(t) - h^-(t)\right]\overline{\zeta(t-t_0)}d\overline{t} + (\nu^+ - \nu^-)e_3 t_0.$$

Letting $z \to t_0 \in \gamma$ and substituting (2.9) and (2.10) into (2.1), we have

$$\begin{aligned}
&\kappa_j\omega(t_0) + \frac{\kappa_j}{2\pi i}\int_{\gamma_j}\omega(t) d\left[\log\frac{\sigma(t-t_0)}{\overline{\sigma(t-t_0)}}\right] \\
&+\sum_{k=0}^{m-1}{}'\frac{1}{2\pi i}\left\{\kappa_j\int_{\gamma_k}\omega(t)\zeta(t-t_0)dt - \kappa_k\int_{\gamma_k}\omega(t)\overline{\zeta(t-t_0)}d\overline{t}\right\} \\
&+M_4[\omega(t),t_0] = N_4(t_0),
\end{aligned}\tag{2.15}$$

where a prime at the summation sign indicates that the summation is carried out over all $k \neq j, k = 0, 1, \cdots, m-1$, and

$$
\begin{aligned}
M_4[\omega(t), t_0] = & \frac{1}{2\pi i}\left[\kappa_j \int_L \omega(t)\zeta(t-t_0)dt - \int_L \overline{\omega(t)\zeta(t-t_0)}d\bar{t}\right] \\
& + \frac{1}{2\pi i}\left\{\int_L \overline{\omega(t)} d\left\{[m(t)-m(t_0)]\,\overline{\zeta(t-t_0)}\right\}\right\} \\
& + \frac{2\kappa_j}{\kappa_j+1}\sum_{r=0}^{m-1} A_r \log|\sigma(t_0-a_r)\sigma(t_0-b_r) + H_r(t_0)| \\
& - \frac{m(t_0)}{\kappa_j+1}\sum_{r=0}^{m-1}\overline{A_r}\left[\overline{\zeta(t_0-a_r)} + \overline{\zeta(t_0-b_r)} - \overline{H_r'(t_0)}\right],
\end{aligned}
$$

$$
\begin{aligned}
N_4(t_0) = & -\frac{1}{2\kappa_j \pi i}\int_\gamma \left[h^+(t) - h^-(t)\right]\overline{\zeta(t-t_0)}d\bar{t} \\
& + \frac{1}{2\kappa_j}\left[h^+(t_0) + h^-(t_0)\right] + \nu_j e_3 t_0.
\end{aligned}
$$

Equations (2.14) and (2.15) as a whole constitute a singular integral equation of normal type along $L \bigcup \gamma$. We would find its solution in class $h_{2m}$, i.e., on account of (1.28), $X_{10} = -\sum_{j=1}^{m-1} X_{1j}$, $X_{20} = -\sum_{j=1}^{m-1} X_{2j}$, there are exactly $2m$ undetermined real constants: $X_{1j}$, $X_{2j}$ $(j = 1, \cdots, m-1)$, $ReC$ and $ImC$.

In order to solve the boundary value problems (2.5)-(2.8), we construct the solution in the form

$$
F(z) = \frac{1}{2\pi i}\int_{L\bigcup\gamma} i\Delta(t)\zeta(t-z)dt + Ez, \tag{2.16}
$$

where $\Delta(t)$ is an unknown real function, $E$ is an undetermined complex constant.

Substituting (2.16) into (2.8) we get a system of equations for $ReE$ and $ImE$

$$
\begin{cases}
2Re\,(\omega_1)\,ReE + 2Im\,(\omega_1)\,ImE = 2\left\{w_1 - \eta_1 Re\left[\delta^*(t)\right]\right\}, \\
2Re\,(\omega_2)\,ReE + 2Im\,(\omega_2)\,ImE = 2\left\{w_2 - \eta_2 Re\left[\delta^*(t)\right]\right\},
\end{cases} \tag{2.17}
$$

the determinant of which is

$$4\begin{vmatrix} Re\left(\omega_1\right) & Im\left(\omega_1\right) \\ Re\left(\omega_2\right) & Im\left(\omega_2\right) \end{vmatrix} = -S \neq 0.$$

Hence, we can obtain $ReE$, $ImE$ uniquely.

$$\begin{cases} ImE = \dfrac{1}{S}\left\{4Re\left(\omega_2\right)w_1 - Re\left(\omega_1\right)w_2 - \pi Im\left[\left(\delta_2+1\right)\Delta^*(t)\right]\right\}, \\ ReE = \dfrac{1}{iS}\left\{4Re\left(\omega_2\right)w_1 - Re\left(\omega_1\right)w_2 - \pi iRe\left[\left(\delta_2+1\right)\Delta^*(t)\right]\right\}. \end{cases} \tag{2.18}$$

Letting $z \to t_0 \in L$ and substituting (2.16) into (2.6), it is easy to see (2.6) will be identically satisfied.

Substituting (2.16) into (2.7) and (2.5) by the Plemelj formulae we get respectively

$$\Delta(t_0) + \frac{\mu^*}{2\pi i}\int_{L\bigcup\gamma}\Delta(t)d\left[\log\frac{\sigma(t-t_0)}{\overline{\sigma(t-t_0)}}\right] - 2\mu^* iRe(Et_0) = 0, \tag{2.19}$$

$$\frac{1}{\pi i}\int_{L\bigcup\gamma} i\Delta(t)d\log|\sigma(t-t_0)| + 2Re(Et_0) = w(t_0). \tag{2.20}$$

Equations (2.19) and (2.20) as a whole constitute a second kind Fredholm integral equation along $L\bigcup\gamma$.

## 2.2 Unique Solvability of the Second Fundamental CPS Problem

Now, we shall prove equations (2.14) and (2.15) to be uniquely solvable in $h_{2m}$. Analogously to Section 1.5, we should prove, $\omega_0(t) \equiv 0, t \in L\bigcup\gamma$ (hence $X_{1j}^0 = X_{2j}^0 = 0$, $ReC = ImC = 0, j = 0, 1, \cdots, m-1$) under homogeneous conditions of the second fundamental problem, that means $g_j^{\pm}(t) = 0, t \in \gamma$, $g(t) = 0, t \in L$, and $e_3 = 0$.

Let $\phi^0(z)$, $\psi^0(z)$, $H^0(t)$, $A_z^0$, $B_z^0$ be the corresponding values of $\phi(z)$, $\psi(z)$, $H(t)$, $A_z$, $B_z$ determined by (2.9)-(2.13) for $\omega(t) = \omega_0(t)$. By the uniqueness theorem [49], [40] of the second fundamental problem under homogeneous conditions, we have

$$\phi^0(z) = c_z, \quad \psi^0(z) = \kappa_z c_z. \tag{2.21}$$

Substituting (2.21) into (2.3) we obtain

$$(\kappa^+ + 1)c^+ = (\kappa^- + 1)c^-. \tag{2.22}$$

Due to $\phi^0(z)$ will be a single-valued function in this case, from (2.9) and (2.10) we arrive at the equalities

$$A_j^0 = X_{1j}^0 + iX_{2j}^0 = 0, \tag{2.23}$$

$$c_z = \frac{1}{2\pi i}\int_{L\bigcup\gamma} \omega_0(t)\zeta(t-z)dt + A_z^0 z, \tag{2.24}$$

$$\begin{aligned} \kappa_z c_z = & -\sum_{j=0}^{m-1}\frac{\kappa_j}{2\pi i}\int_{\gamma_j}\overline{\omega_0(t)}\zeta(t-z)dt \\ & -\frac{1}{2\pi i}\int_L \overline{\omega_0(t)}\zeta(t-z)dt - \frac{1}{2\pi i}\int_{L\bigcup\gamma}\overline{m(t)}\omega_0'(t)\zeta(t-z)dt \\ & +\frac{1}{2\pi i}\int_L \overline{H^0(t)}\zeta(t-z)dt + B_z^0 z. \end{aligned} \tag{2.25}$$

Comparing the quasi-periodic cyclic increments of the two sides of equations (2.24) and (2.25), respectively, we obtain

$$\frac{1}{2\pi i}\int_{L\bigcup\gamma}\omega_0(t)dt = 0, \tag{2.26}$$

$$A_z^0 = B_z^0 = 0, \tag{2.27}$$

$$\begin{aligned} & \sum_{j=0}^{m-1}\frac{\kappa_j}{2\pi i}\int_{\gamma_j}\overline{\omega_0(t)}dt - \frac{1}{2\pi i}\int_L\overline{\omega_0(t)}dt \\ & +\frac{1}{2\pi i}\int_{L\bigcup\gamma}\overline{m(t)}\omega_0'(t)dt - \frac{1}{2\pi i}\int_L\overline{H^0(t)}dt = 0. \end{aligned} \tag{2.28}$$

Substituting (2.26)-(2.28) into (2.24) and (2.25) by the Plemelj formulae we get

$$\omega_0(t) = c^+ - c^-, t \in L, \tag{2.29}$$

$$\omega_0(t) = 0, t \in \gamma. \tag{2.30}$$

Substituting (2.29) into (2.24), and let $z = 0 \in S_0^+$, by the Cauchy theorem we obtain

$$c^+ = 0.$$

Then, from (2.22) we find

$$c^- = 0.$$

Hence, it follows from (2.29) that

$$\omega_0(t) = 0, t \in L. \tag{2.31}$$

Referring to (2.30) and (2.31) we have proved

$$\omega_0(t) \equiv 0, t \in L \bigcup \gamma. \tag{2.32}$$

Using the method of Section 1.5, one may also prove the unique solvability of the equations (2.19) and (2.20).

Thus, the second fundamental CPS problem is solved uniquely.