Chapter 1

First Fundamental CPS Problem of a Nonhomogeneous Body with a Doubly-Periodic Set of Cracks

1.1 Preliminaries, Definition and Lemmas

We consider a three-dimensional piecewise homogeneous isotropic elastic body with a doubly-periodic set of cracks on the x_1, x_2 transverse cross section. The primitive periods will be $2\omega_1$, $2\omega_2$ with $Im\left(\frac{\omega_2}{\omega_1}\right) > 0$, $Im(\omega_1) = 0$. The doubly-periodic fundamental parallelogram on the x_1, x_2 plane of the elastic body will be denoted by P_{00}. Its vertices are $\pm\omega_1 \pm \omega_2$. The boundary of P_{00} will be denoted by Γ with the positive direction taken to be anticlockwise, and $\Gamma = \bigcup_{i=1}^{4} \Gamma_i$. It will be assumed that inside each doubly-periodic parallelogram $P_{mn}(m, n = 0, \pm 1, \cdots)$ there exists a hole, and let another different solid isotropic material be inserted into this hole. The interface of the two isotropic materials is composed by simply, closed, smooth and non-intersecting contours denoted by $\mathcal{L} \equiv L$ $(mod\ 2\omega_1, 2\omega_2)$, oriented clockwise as its positive direction. L is the interface of the two materials in P_{00} (see Fig. 1.1). We will assume that there exists a system of m cracks distributed in the two materials in P_{00}, denoted by $\gamma_j = \widehat{a_j b_j}$ $(j = 0, 1, \cdots, m-1)$, which are smooth and non-intersecting curve segements, and non-intersecting with L. The positive directions of γ_j $(j = 0, 1, \cdots, m-1)$ are taken to be from a_j to b_j, and we denote $\gamma = \bigcup_{j=0}^{m-1} \gamma_j$. The system of cracks inside $P_{mn}(m, n = 0, \pm 1, \pm \cdots)$ is congruent to γ in P_{00}. The elastic regions located on the left-hand and right-hand sides of L in P_{00} except cracks are denoted by S_0^+ and S_0^-, respectively (see Figure 1.1), the aggregate of $S_0^\pm$ and all its congruent regions will be denoted by $S^\pm$, with modulus of elasticity $\kappa^\pm$ and Poisson ratio $\mu^\pm$, respectively, hence the elastic region on the transverse cross section of the elastic body will be $S = S^+ \bigcup S^-$. The origin will be chosen inside S_0^+ and will not be on the cracks.

The investigation on the doubly-periodic problem will be restricted to doubly-periodic stress distributions in the elastic body

$$\begin{cases} \sigma_1(z + 2\omega_k) = \sigma_1(z) = \sigma_1(x_1, x_2), \\ \sigma_2(z + 2\omega_k) = \sigma_2(z) = \sigma_2(x_1, x_2), \\ \tau_{12}(z + 2\omega_k) = \tau_{12}(z) = \tau_{12}(x_1, x_2), \\ \tau_{13}(z + 2\omega_k) = \tau_{13}(z) = \tau_{13}(x_1, x_2), \\ \tau_{23}(z + 2\omega_k) = \tau_{23}(z) = \tau_{23}(x_1, x_2), \end{cases} \tag{1.1}$$

where $z = x_1 + ix_2$.

By the generalized Hook's Law, the strain tensors e_1, e_2, e_3, e_{12}, e_{13}, and e_{23} can be expressed in terms of the stress components by the following formulae

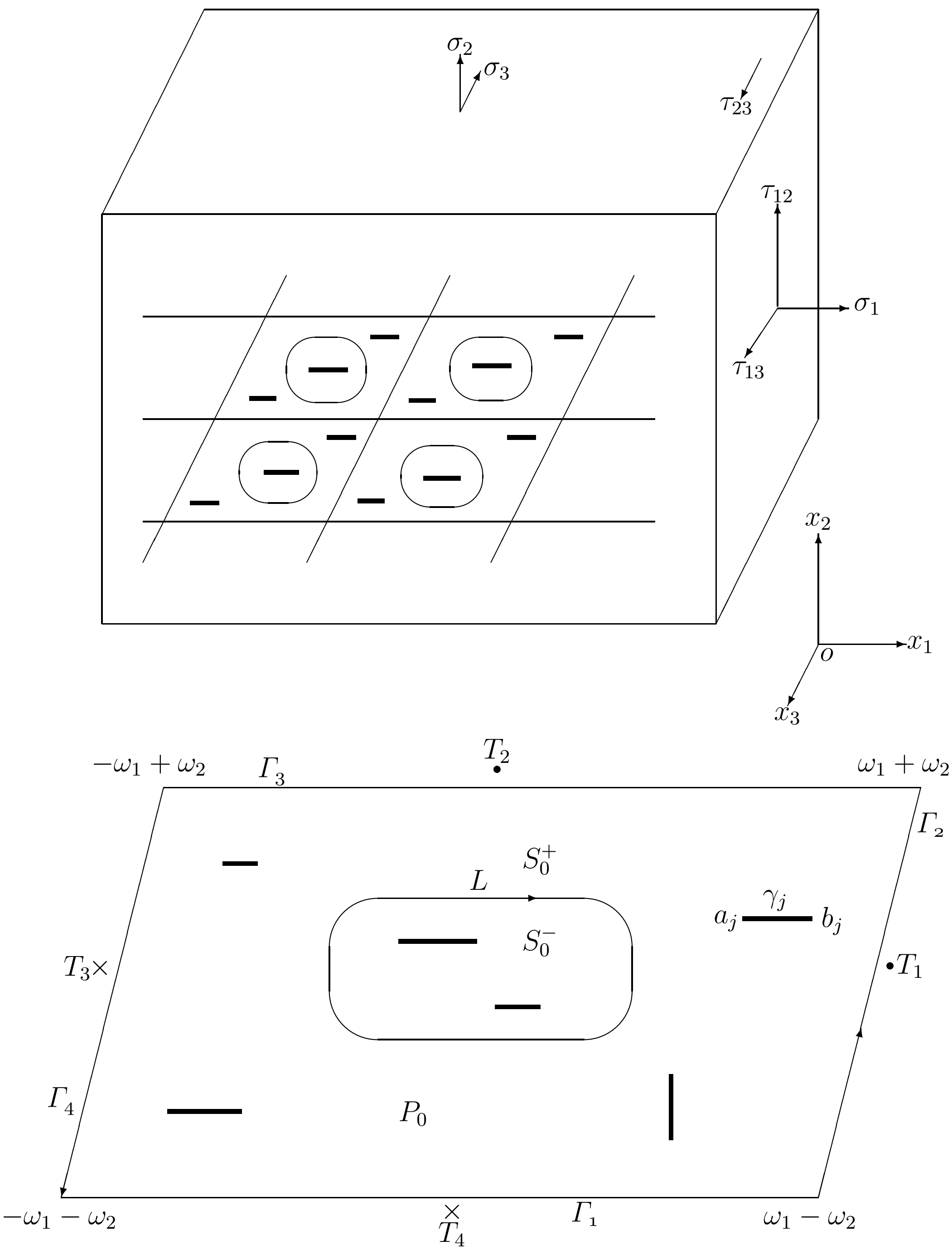

Figure 1.1: A nonhomogeneous body with a doubly-periodic set of cracks

(see [3], [49])

$$\begin{cases} e_1 = \frac{1}{E}[\sigma_1 - \nu(\sigma_2 + \sigma_3)], \\ e_2 = \frac{1}{E}[\sigma_2 - \nu(\sigma_1 + \sigma_3)], \\ e_3 = \frac{1}{E}[\sigma_3 - \nu(\sigma_1 + \sigma_2)], \\ e_{12} = \frac{1+\nu}{E}\tau_{12}, e_{23} = \frac{1+\nu}{E}\tau_{23}, e_{13} = \frac{1+\nu}{E}\tau_{13}, \end{cases} \tag{1.2}$$

where ν is the Poisson's ratio and E the Young's modulus.

And the strain components in terms of the displacements are determined by the formulae

$$\begin{cases} e_1 = \frac{\partial u}{\partial x_1}, e_2 = \frac{\partial v}{\partial x_2}, e_3 = \frac{\partial w}{\partial x_3}, \\ e_{13} = \frac{1}{2}\left(\frac{\partial u}{\partial x_3} + \frac{\partial w}{\partial x_1}\right), e_{12} = \frac{1}{2}\left(\frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2}\right), \\ e_{23} = \frac{1}{2}\left(\frac{\partial w}{\partial x_2} + \frac{\partial v}{\partial x_3}\right). \end{cases} \tag{1.3}$$

The so-called *complete plane strain* (**CPS**) state is

$$\begin{cases} \sigma_1 = \sigma_1(x_1, x_2), \sigma_2 = \sigma_2(x_1, x_2), \\ \tau_{12} = \tau_{12}(x_1, x_2), \tau_{13} = \tau_{13}(x_1, x_2), \\ \tau_{23} = \tau_{23}(x_1, x_2), e_3 = constant. \end{cases} \tag{III}$$

We can resolve the special three-dimensional elastic system (*III*) into two linearly independent plane elastic systems by the superposition principle of forces [20], one is the *generalized plane strain* state

$$\begin{cases} \sigma_1 = \sigma_1(x_1, x_2), \sigma_2 = \sigma_2(x_1, x_2), \\ \tau_{12} = \tau_{12}(x_1, x_2), \tau_{13} = \tau_{23} = 0, \\ e_3 = constant, \end{cases} \tag{I}$$

and another is the *longitudinal displacement* state

$$\begin{cases} \sigma_1 = \sigma_2 = \tau_{12} = 0, \\ \tau_{13} = \tau_{13}(x_1, x_2), \tau_{23} = \tau_{23}(x_1, x_2), \\ e_3 = 0. \end{cases} \tag{II}$$

The stress equilibrium equations for the plane elastic system (I)

$$\begin{cases} \dfrac{\partial \sigma_1}{\partial x_1} + \dfrac{\partial \tau_{12}}{\partial x_2} = 0, \\ \dfrac{\partial \sigma_2}{\partial x_2} + \dfrac{\partial \tau_{12}}{\partial x_1} = 0, \end{cases} \tag{1.4}$$

and the compatibility equation

$$\Delta(\sigma_1 + \sigma_2) = 0 \tag{1.5}$$

together can be simplified to one single equation with one single unknown function $U(x_1, x_2)$ which is called a real stress function or Airy function [49],

$$\Delta^2 U \equiv \frac{\partial^4 U}{\partial x_1^4} + 2\frac{\partial^4 U}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 U}{\partial x_2^4} \tag{1.6}$$

where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

Equation (1.6) is called *biharmonic* equation.

Furthermore, according to Goursat's formula, the biharmonic function $U(x_1, x_2)$ may be represented in terms of two analytic functions [49]

$$u(x_1, x_2) = Re[\overline{z}\phi(z) + \chi(z)], z = x_1 + ix_2,$$

where $\phi(z)$ and $\chi(z)$ are analytic functions in S.

The derivatives of $U(x_1, x_2)$ can be expressed in a very simple manner by the help of two so-called complex stress functions $\phi(z)$ and $\psi(z)$,

$$\frac{\partial U}{\partial x_1} + i\frac{\partial U}{\partial x_2} = \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)}, \tag{1.7}$$

where $\psi(z) = \chi'(z)$. The functions $\phi(z)$ and $\psi(z)$ are referred to as Goursat functions in some references, e.g. [46], etc.

Then, the stress and displacement components for the elastic system (I) will be expressed by the complex stress functions $\phi(z)$, $\psi(z)$, $\Phi(z) = \phi'(z)$ and $\Psi(z) = \psi'(z)$ as follows [49],[20]

$$\sigma_1 + \sigma_2 = 4Re\{\Phi(z)\}, \tag{1.8}$$

$$\sigma_2 - \sigma_1 + 2i\tau_{12} = 2[\overline{z}\Phi'(z) + \Psi(z)], \tag{1.9}$$

$$2\mu_z(u + iv) = \kappa_z\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} - 2\mu_z e_3 \nu_z z, \tag{1.10}$$

$$\sigma_3 = 2\mu_z(1 + \nu_z)e_3 + \nu_z(\sigma_1 + \sigma_2), \tag{1.11}$$

$$w = e_3 x_3, \tag{1.12}$$

where

$$\mu_z = \begin{cases} \mu^+, z \in S^+, \\ \mu^-, z \in S^-, \end{cases}, \quad \nu_z = \begin{cases} \nu^+, z \in S^+, \\ \nu^-, z \in S^-, \end{cases}$$

$$\mu^{\pm} = \frac{E^{\pm}}{2(1 + \nu^{\pm})},$$

u, v, and w are displacement components in x_1, x_2, x_3 directions. $\kappa_z = 3 - 4\nu_z$. $E^{\pm}$ (> 0), $\nu^{\pm}$ $(0 < \nu^{\pm} < \frac{1}{2})$ are elastic Young's modulus and Poisson ratio of the materials in $S^{\pm}$, respectively.

The longitudinal stress and displacement components for the elastic system (II), taking double periodicity into account, will be expressed by the third complex stress function $F(z)$ which is called the complex *torsion* function [49] as follows [20]

$$\tau_{13} - i\tau_{23} = 2\mu_z F'(z), \tag{1.13}$$

$$w = F(z) + \overline{F(z)}. \tag{1.14}$$

Now, we consider the properties of the complex stress functions and displacements when the stress distributions are doubly-periodic. At first, we introduce

Definition 1 *Let $2\omega_1$, $2\omega_2$ be a pair of primitive periods such that $Im(\omega_1/\omega_2) > 0$, a function which is analytic (except isolated singular points) and satisfies the equations*

$$\begin{cases} f(z + 2\omega_1) = f(z) + f_1, \\ f(z + 2\omega_2) = f(z) + f_2, \end{cases} \tag{1.15}$$

for all values of z for which $f(z)$ exists, is called a doubly quasi-periodic function. Here the constants f_1 and f_2 are called cyclic increments of $f(z)$.

Then we have

Lemma 1.1.1 *When the stress distributions are doubly-periodic in the elastic body, then the displacements are doubly quasi-periodic.*

Proof From the equations (1.1) and (1.2), it follows immediately that all strain components e_1, e_2, e_3, e_{12}, e_{23} and e_{13} are doubly-periodic. Hence, from equation (1.3) the displacement components u, v and w are doubly quasi-periodic.

Lemma 1.1.2 *When the stress distributions are doubly-periodic in the elastic body, then the complex stress function $\phi(z)$, the expression $z\overline{\phi'(z)} + \overline{\psi(z)}$ and the complex torsion function $F(z)$ are all doubly quasi-periodic.*

Proof Due to the doubly-periodic stress distributions in the elastic body, from equations (1.1) and (1.8), $\Phi(z)$ must have a doubly-periodic real part, and hence it must be a doubly-periodic function, i.e.

$$\Phi(z + 2\omega_k) = \Phi(z), \ (k = 1, 2). \tag{1.16}$$

Integration of equation (1.16) results in

$$\phi(z + 2\omega_k) = \phi(z) + 2\alpha_k, \ (k = 1, 2). \tag{1.17}$$

This result expresses the fact that $\phi(z)$ is a quasi-periodic function with cyclic increments $2\alpha_k$.

From equation (1.10) we have

$$z\overline{\phi'(z)} + \overline{\psi(z)} = \kappa_z \phi(z) - 2\mu_z e_3 \nu_z z - 2\mu_z(u + iv). \tag{1.18}$$

Taking Lemma 1.1.1 into account, we know immediately the right hand of equation (1.18) is doubly quasi-periodic, so that the expression $z\overline{\phi'(z)} + \overline{\psi(z)}$ is doubly quasi-periodic. Due to equations (1.1), (1.13) and (1.14), one obtains $F'(z)$ is doubly-periodic, then, $F(z)$ is doubly quasi-periodic.

Let $X_1 + iX_2$ be the external resultant principal vector of the tractions exerted on an arc AB, then [49],

$$X_1 + iX_2 = \int_{AB} (X_{1n} + iX_{2n})\, ds, \tag{1.19}$$

where $X_{1n} + iX_{2n}$ is the external stress on the arc AB. Because [49]

$$X_{1n} = \frac{d}{ds}\left(\frac{\partial U}{\partial x_2}\right), X_{2n} = -\frac{d}{ds}\left(\frac{\partial U}{\partial x_1}\right).$$

or in complex form

$$(X_{1n} + iX_{2n})\, ds = -id\left(\frac{\partial U}{\partial x_1} + i\frac{\partial U}{\partial x_2}\right).$$

Hence, on account of equation (1.7),

$$\begin{aligned} X_1 + iX_2 &= -i\left[\frac{\partial U}{\partial x_1} + i\frac{\partial U}{\partial x_2}\right]_A^B \\ &= -i\left[\phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)}\right]_A^B. \end{aligned} \tag{1.20}$$

where $[\ \ ]_A^B$ denotes the increment of the expression in the brackets as the point z passes along the arc from A to B.

Now, in our case, let $F_k = X_{1\Gamma_k} + iX_{2\Gamma_k}$ $(k = 1, 2)$, be the external resultant principal vectors on the boundary Γ_k $(k = 1, 2)$ of the fundamental periodic parallelogram P_{00}. Due to the equilibrium of forces, that of $\mathit{\Gamma}_3$ and $\mathit{\Gamma}_4$ will be $-F_1$ and $-F_2$, repectively. In fact,

$$F_k = \int_{\mathit{\Gamma}_k} [X_{1n}(\tau) + iX_{2n}(\tau)]d\sigma, \tag{1.21}$$

where $X_{1n}(\tau) + iX_{2n}(\tau), \tau \in \mathit{\Gamma}_k$, be the external stress functions on $\mathit{\Gamma}_k$, σ is the arc-length parameter of $\mathit{\Gamma}$.

Then, from equation (1.19)-(1.21)

$$\left[\phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)}\right]_{\mathit{\Gamma}_k} = iF_k, k = 1, 2. \tag{1.22}$$

Let

$$f(\tau) = i\int_0^\tau [X_{1n}(\tau) + iX_{2n}(\tau)]d\sigma, \tau \in \mathit{\Gamma}. \tag{1.23}$$

Because

$$\int_{\mathit{\Gamma}} [X_{1n}(\tau) + iX_{2n}(\tau)]d\sigma = F_1 + F_2 - F_1 - F_2 = 0, \tag{1.24}$$

hence there is a single-valued function $f(\tau)$ on $\mathit{\Gamma}$ such that

$$F_k = -i[f(\tau)]_{\mathit{\Gamma}_k}, k = 1, 2, \tag{1.25}$$

where $[\ \ ]_{\mathit{\Gamma}_k}$ denotes the increment of the expression in brackets along $\mathit{\Gamma}_k$ in positive direction.

Let $X_{1n}^{\pm}(\tau)+iX_{2n}^{\pm}(\tau), \tau \in \gamma_j$, be the external stress function on the positive and negative sides of γ_j $(j = 0, 1, \cdots, m-1)$ satisfying a Hoelder condition. $X_{1j}^{\pm} + iX_{2j}^{\pm} = \int_{\gamma_k}[X_{1n}(\tau) + iX_{2n}(\tau)]ds$ $(j = 0, 1, \cdots, m-1)$ will be the external resultant principal vectors on the positive and negative sides of γ_j, where s is the arc-length parameter of γ_j. Hence,

$$X_{1j} + iX_{2j} = (X_{1j}^{+} + iX_{2j}^{+}) + (X_{1j}^{-} + iX_{2j}^{-}).$$

We set

$$f_j^{+}(\tau) = i\int_{a_j}^{\tau} [X_{1n}^{+}(\tau) + iX_{2n}^{+}(\tau)]ds, \tag{1.26}$$

$$f_j^-(\tau) = i(X_{1j} + iX_{2j}) - i\int_{a_j}^{\tau} [X_{1n}^-(\tau) + iX_{2n}^-(\tau)]ds. \tag{1.27}$$

According to the equilibrium principle of forces, taking (1.24) into account, one has

$$\sum_{j=0}^{m-1} (X_{1j} + iX_{2j}) = 0. \tag{1.28}$$

When each $X_{1j} + iX_{2j} = 0$ on γ_j $(j = 0, 1, \cdots, m-1)$, then formulae (1.26) and (1.27) become

$$f_j^{\pm}(\tau) = \pm i\int_{a_j}^{\tau} [X_{1n}^{\pm}(\tau) + iX_{2n}^{\pm}(\tau)]ds, \tag{1.29}$$

and thereby

$$f_j^{\pm}(a_j) = 0, \quad f_j^+(b_j) = f_j^-(b_j). \tag{1.30}$$

Let

$$F(\tau) = f_j^+(\tau) - f_j^-(\tau), \quad G(\tau) = f_j^+(\tau) + f_j^-(\tau), \tau \in \gamma_j, \tag{1.31}$$

so that

$$F(a_j) = F(b_j) = 0. \tag{1.32}$$

Denote

$$f_j^{\pm}(\tau) = f_{j1}^{\pm}(\tau) + f_{j2}^{\pm}(\tau),$$

then, from (1.29)

$$df_{j1}^{\pm}(\tau) = \mp X_{2n}^{\pm}(\tau)ds, \quad df_{j2}^{\pm}(\tau) = \mp X_{1n}^{\pm}(\tau)ds.$$

Thus, the resultant moments on the two sides of γ_j about the origin of the coordinate system will be

$$\begin{aligned} M_j^{\pm} &= \int_{\gamma_j} [t_1 X_{2n}^{\pm}(\tau) - t_2 X_{1n}^{\pm}(\tau)]ds \\ &= \mp \int_{a_j}^{b_j} [t_1 df_{j1}^{\pm}(\tau) - t_2 df_{j2}^{\pm}(\tau)] \\ &= \mp Re \int_{a_j}^{b_j} \overline{\tau} df_j^{\pm}(\tau) \\ &= \mp Re[\overline{\tau} f_j^{\pm}(\tau)]_{a_j}^{b_j} \pm Re \int_{a_j}^{b_j} f_j^{\pm}(\tau) d\overline{\tau}, \end{aligned} \tag{1.33}$$

where $\tau = t_1 + it_2$.

Remembering (1.30), one gets

$$M_j = M_j^+ + M_j^- = Re \int_{a_j}^{b_j} [f_j^+(\tau) - f_j^-(\tau)] d\overline{\tau}. \tag{1.34}$$

The resultant moment on Γ about the origin of the coordinate system will be

$$\begin{aligned} M_\Gamma &= \int_\Gamma [\xi X_{2n}(t) - \eta X_{1n}(t)] d\sigma \\ &= Im \int_\Gamma \overline{t}[X_{1n}(t) + iX_{2n}(t)] d\sigma, \end{aligned} \tag{1.35}$$

where we have put $t = \xi + i\eta$.

On the other hand, considering the double periodicity of the stress components,

$$X_{1n}(t + 2\omega_k) = -X_{1n}(t), \quad X_{2n}(t + 2\omega_k) = -X_{2n}(t), \tag{1.36}$$

from (1.35), taking (1.21) into account, we obtain

$$M_\Gamma = 2Im[\overline{\omega_1}F_2 - \overline{\omega_2}F_1]. \tag{1.37}$$

According to the equilibrium principle of resultant moments,

$$M_\Gamma + \sum_{j=0}^{m-1} M_j = 0. \tag{1.38}$$

Then from (1.34) and (1.37), one obtain

$$2Im[\overline{\omega_1}F_2 - \overline{\omega_2}F_1] + \sum_{j=0}^{m-1} \left\{ \int_{a_j}^{b_j} [f_j^+(\tau) - f_j^-(\tau)] d\overline{\tau} \right\} = 0. \tag{1.39}$$

Therefore

$$Re \left\{ \int_\gamma [f^+(\tau) - f^-(\tau)] d\overline{\tau} - 2i[\overline{\omega_1}F_2 - \overline{\omega_2}F_1] \right\} = 0. \tag{1.40}$$

The complex stress functions $\phi(z)$ and $\psi(z)$ need not always be single-valued in a multiply connected region, although of course the components of stress and displacement are single-valued. In order to separate the multi-valued parts of $\phi(z)$ and $\psi(z)$, we need to construct the Kolosov functions.

1.2 Kolosov Functions

Here we need not only to separate the multi-valued parts but must keep the double peridicity of them. For doing this, we prove

Lemma 1.2.1 *A doubly quasi-periodic function $f(z)$ with cyclic increments f_1 and f_2, which is analytic (except at the simple pole $z = z_0$), can be expressed as*

$$f(z) = \delta_1^0 z + \delta_2^0 \zeta(z - z_0) + C, \tag{1.41}$$

where

$$\delta_1^0 = \frac{1}{\pi i}\left(f_2\eta_1 - f_1\eta_2\right), \delta_2^0 = \frac{1}{\pi i}\left(f_1\omega_2 - f_2\omega_1\right),$$

C is an arbitrary constant, $\zeta(z)$ is the Weierstrass zeta function, namely

$$\zeta(z) = \frac{1}{z} + \sum_{m,n}{}' \left\{\frac{1}{z - \Omega_{mn}} + \frac{1}{\Omega_{mn}} + \frac{z}{\Omega_{mn}^2}\right\},$$

where $\Omega_{mn} = 2m\omega_1 + 2n\omega_2$, the prime of $\sum$ above indicates that m and n will not be zero simultaneously. In fact, $\zeta(z)$ is a doubly quasi-periodic function with cyclic increments $2\eta_k$, $k = 1, 2$, such that

$$\zeta(z + 2\omega_k) = \zeta(z) + 2\eta_k, \tag{1.42}$$

$$\eta_k = \zeta(\omega_k), k = 1, 2,$$

$$\omega_2\eta_1 - \omega_1\eta_2 = \frac{\pi}{2}i. \tag{1.43}$$

Proof Let $f(z)$ be any doubly quasi-periodic function with cyclic increments f_1 and f_2, which is analytic (except at the simple pole $z = z_0$). Because $\delta_1^0 z + \delta_2^0\zeta(z - z_0)$ is a doubly quasi-periodic function with cyclic increments f_1 and f_2, which has only a simple pole at the point $z = z_0$ due to (1.42) and (1.43), then, $f(z) - [\delta_1^0 z + \delta_2^0\zeta(z - z_0)]$ is therefore a doubly-periodic analytic function without any pole in the complex plane. By Liouville's theorem it should be merely a constant, namely

$$f(z) - \left[\delta_1^0 z + \delta_2^0 \zeta(z - z_0)\right] = C,$$

and so (1.41) is established.

Corollary 1.2.1 *A doubly quasi-periodic elliptic function $f(z)$ with cyclic increments f_1 and f_2, which has no poles, has the representation*

$$f(z) = \delta_1 z + C, \tag{1.44}$$

where, C is an arbitrary constant,

$$\delta_1 = \frac{f_1}{2\omega_1} = -\frac{f_2}{2\omega_2}.$$

Now, we construct the Kolosov functions in our case as follows.

$$\begin{aligned}\phi(z) = &\frac{-1}{4\pi(\kappa_z + 1)} \sum_{j=0}^{m-1} \{(X_{1j} + iX_{2j}) \left[\log \sigma(z - a_j)\sigma(z - b_j) - H_j(z)\right]\} \\ &+\phi_0(z),\end{aligned} \tag{1.45}$$

$$\begin{aligned}\psi(z) = &\frac{\kappa_z}{4\pi(\kappa_z + 1)} \sum_{j=0}^{m-1} \{(X_{1j} + iX_{2j}) \left[\log\ \sigma(z - a_j)\sigma(z - b_j) - H_j(z)\right]\} \\ &+D(z)\phi'(z) - D(z)\phi'(0) + \psi_0(z),\end{aligned} \tag{1.46}$$

where

$$H_j(z) = \int_{\gamma_j} h_j(\tau)\zeta(\tau - z)d\tau,$$

$$h_j(\tau) = \frac{2\tau - a_j - b_j}{b_j - a_j}.$$

It is easy to prove that $H_j(z)$ will be a doubly-periodic function with the same singularities as $\log(z - a_j)$ and $\log(z - b_j)$ at a_j, b_j, respectively. $\sigma(z)$ is the Weierstrass *sigma* function

$$\sigma(z) = z \prod_{m,n}{}' \left(1 - \frac{z}{\Omega_{mn}}\right) exp(\frac{z}{\Omega_{mn}} + \frac{z^2}{2\Omega_{mn}}),$$

which has a relationship with $\zeta(z)$, namely

$$\zeta(z) = \frac{\sigma'(z)}{\sigma(z)}. \tag{1.47}$$

Here we have constructed by Lemma 1.2.1

$$D(z) = \delta_1 \zeta(z) + \delta_2 z, \tag{1.48}$$

$$\delta_1 = \frac{2}{\pi i}(\overline{\omega_2}\omega_1 - \overline{\omega_1}\omega_2), \ \delta_2 = \frac{2}{\pi i}(\overline{\omega_1}\eta_2 - \overline{\omega_2}\eta_1).$$

$D(z)$ will be a doubly quasi-periodic function with cyclic increments $-2\overline{\omega_k}$ $(k = 1, 2)$,

$$D(z + 2\omega_k) = D(z) - 2\overline{\omega}_k, k = 1, 2. \tag{1.49}$$

We note that the first parts (multi-valued parts) of the right-hand sides of equations (1.45) and (1.46) are doubly-periodic functions, respectively, as a result of (1.28) and the property of $\sigma(z)$, namely

$$\sigma(z + 2\omega_k) = -e^{2\eta_k(z+\omega_k)}\sigma(z), k = 1, 2.$$

Then, $\phi_0(z)$ and $\psi_0(z)$ are single-valued, sectionally holomophic, doubly quasi-periodic functions.

1.3 Formulation of the First Fundamental CPS Problem

Let us consider the first fundamental CPS problem: The external stress functions $X_{1n}^{\pm} + iX_{2n}^{\pm}$ on the positive and the negative sides of γ_j are given, respectively. This implies that $f_j^{\pm}$ are known functions from (1.26) and (1.27), and the discontinuties in the displacements for a passage through L are given, too. This means if we denote $u^{\pm}(t)+iv^{\pm}(t)$ as the displacements on the positive and negative sides of the point $t\in L$, respectively, then the displacement discontinuity function

$$g(t)=[u^{+}(t)+iv^{+}(t)]-[u^{-}(t)+iv^{-}(t)] \tag{1.50}$$

will be given. Assume $X^{\pm}_{1n}(\tau)+iX^{\pm}_{2n}(\tau)\in H(\gamma_j),\tau\in\gamma_j,\ g(t)\in H(L),t\in L$. In addition, the stress resultant principal vectors F_k, on $\mathit{\Gamma}_k$, k=1,2, in the x_1,x_2 plane, and the shear stress $T_k, k=1,2$, in the x_3 direction are given, too. The strain $e_3=constant$. Then, by the external stress conditions, from formulae (1.29) and (1.19)-(1.20) we have the boundary conditions on γ_j and their congruents for the elastic system (I)

$$\phi^{\pm}(\tau)+\tau\overline{\phi'^{\pm}(\tau)}+\overline{\psi^{\pm}(\tau)}]=f^{\pm}_j(\tau)+C_j(m,n),\tau\in\gamma_j(m,n)=\gamma_j\bigcup\Omega_{mn}, \tag{1.51}$$

where the $C_j(m,n)$ may be arbitrarily fixed constants.

The external stresses applied to the two sides of $\mathcal{L}(m,n)$ must be in equilibrium. Then, from formulae (1.19)-(1.20) we have

$$\begin{aligned}\phi^{+}(t)\ +\ & t\overline{\phi'^{+}(t)}+\overline{\psi^{+}(t)}]\\ &=\phi^{-}(t)+t\overline{\phi'^{-}(t)}+\overline{\psi^{-}(t)}],t\in\mathcal{L}(m,n)=L\bigcup\Omega_{mn},\end{aligned} \tag{1.52}$$

Moreover, by the displacement discontinuity conditions of the two sides of $\mathcal{L}(m,n)$, from formula (1.10) we get the boundary condition

$$\begin{aligned}\alpha^{+}\phi^{+}(t)\ -\ & \beta^{+}[t\overline{\phi'^{+}(t)}+\overline{\psi^{+}(t)}]-\nu^{+}e_3t\\ &=\alpha^{-}\phi^{-}(t)-\beta^{-}[t\overline{\phi'^{-}(t)}+\overline{\psi^{-}(t)}]\\ &\quad -\nu^{-}e_3t+2g(t),t\in\mathcal{L}(m,n),\end{aligned} \tag{1.53}$$

where we have put

$$\alpha^{\pm}=\frac{\kappa^{\pm}}{\mu^{\pm}},\quad \beta^{\pm}=\frac{1}{\mu^{\pm}}.$$

On account of the double quasi-periodicity of the complex stress function $\phi(z)$ and the expression $z\overline{\phi'(z)}+\overline{\psi(z)}$, we have

$$\left[\phi(z)+z\overline{\phi'(z)}+\overline{\psi(z)}\right]_{\Gamma_k}=iF_k, k=1,2. \tag{1.54}$$

Similarly, we have the following boundary conditions for the plane elastic system (II)

$$F^{\pm}(\tau)-\overline{F^{\pm}(\tau)}=iC_j^*(m,n), \tau\in\gamma_j(m,n), \tag{1.55}$$

$$F^{+}(t)+\overline{F^{+}(t)}=F^{-}(t)+\overline{F^{-}(t)}, t\in\mathcal{L}(m,n), \tag{1.56}$$

$$\mu^{+}\left[F^{+}(t)-\overline{F^{+}(t)}\right]=\mu^{-}\left[F^{-}(t)-\overline{F^{-}(t)}\right], t\in\mathcal{L}(m,n), \tag{1.57}$$

$$\mu_z\left[F(z)-\overline{F(z)}\right]_{\Gamma_k}=|\omega_k|T_k, k=1,2, \tag{1.58}$$

where $C_j^*(m,n)$'s may be arbitrarily fixed constants.

In fact, paying attention to the Kolosov functions (1.45) and (1.46), without loss of generality, we may assume that each $X_{1j}+iX_{2j}=0$, then

$$\phi(z)=\phi_0(z),\quad \psi(z)=D(z)\phi'(z)-D(z)\phi'(0)+\psi_0(z). \tag{1.59}$$

For the double periodicity, we only need to consider the solution in the fundamental periodic-parallelogram [41]. After substituting (1.59) into the boundary conditions (1.51)-(1.54), we get

$$\phi_0^{\pm}(\tau)+m(\tau)\overline{\phi_0'^{\pm}(\tau)}+\overline{\psi_0^{\pm}(\tau)}=f_{j0}^{\pm}(\tau)+C_j, \tau\in\gamma_j, \tag{1.60}$$

$$\phi_0^{+}(t)+m(t)\overline{\phi_0'^{+}(t)}+\overline{\psi_0^{+}(t)}=\phi_0^{-}(t)+m(t)\overline{\phi_0'^{-}(t)}+\overline{\psi_0^{-}(t)}, t\in L, \tag{1.61}$$

$$\begin{aligned}\alpha^{+}\phi_0^{+}(t)-\beta^{+}[m(t)\overline{\phi_0'^{+}(t)}+\overline{\psi_0^{+}(t)}]&=\alpha^{-}\phi_0^{-}(t)-\beta^{-}[m(t)\overline{\phi_0'^{-}(t)}+\overline{\psi_0^{-}(t)}]\\&\quad+g_0(t), t\in L,\end{aligned} \tag{1.62}$$

$$\left[\phi_0(z)+m(z)\overline{\phi_0'(z)}-\overline{D(z)\phi_0'(z)}+\overline{\psi_0(z)}\right]_{\Gamma_k}=iF_k, k=1,2, \tag{1.63}$$

where

$$m(z) = z + \overline{D(z)} \tag{1.64}$$

will be a non-analytic doubly-periodic function [27],

$$f_{j0}^{\pm}(\tau) = f_j^{\pm}(\tau) + \overline{\phi_0'(0)D(\tau)},$$

$$g_0(t) = 2g(t) - (\beta^+ - \beta^-)\overline{\phi_0'(0)D(t)} + (\nu^+ - \nu^-)e_3 t.$$

1.4 Solution, Reduction to Integral Equations

In order to solve the boundary value problems (1.60)-(1.63), a new unknown function $\omega(t) \in H, t \in L\bigcup\gamma$, will be introduced, and we construct the general representation for the solution in the form

$$\phi_0(z) = \frac{1}{2\pi i}\int_{L\bigcup\gamma} \omega(t)\zeta(t-z)dt + A_z z, \tag{1.65}$$

$$\begin{aligned} \psi_0(z) &= -\frac{1}{2\pi i}\int_{L\bigcup\gamma} [\overline{\omega(t)} + \overline{m(t)}\omega'(t)]\zeta(t-z)dt \\ &\quad + \frac{1}{2\pi i}\int_{\gamma} \overline{F(\tau)}\zeta(\tau - z)d\tau + B_z z, \end{aligned} \tag{1.66}$$

where

$$A_z = \begin{cases} A^+, & \text{if } z \in S_0^+, \\ A^-, & \text{if } z \in S_0^-, \end{cases} \qquad B_z = \begin{cases} B^+, & \text{if } z \in S_0^+, \\ B^-, & \text{if } z \in S_0^-. \end{cases}$$

After integrating by parts, we have

$$\phi_0'(z) = \frac{1}{2\pi i}\int_{L\bigcup\gamma} \omega'(t)\zeta(t-z)dt + A_z. \tag{1.67}$$

We assume temporarily [39]

$$\omega(a_j) = \omega(b_j) = 0, j = 0, 1, \cdots, m-1, \tag{1.68}$$

the validity of it will be proved later.

Substituting equations (1.65)-(1.67) into the left-hand side of the condition (1.63), we get

$$\begin{aligned}
&[\phi_0(z) \ + \ z\overline{\phi_0'(z)} + \overline{D(z)\phi_0'(z)} - \overline{\psi_0(z)}\Big]_{\Gamma_k} \\
&\quad = \frac{i\eta_k}{\pi}\int_{L\bigcup\gamma}\omega(t)dt + \frac{\overline{\eta_k}}{\pi i}\int_{\gamma}F(\tau)d\overline{\tau} \\
&\quad + \frac{i\overline{\eta_k}}{\pi}\int_{L\bigcup\gamma}[\omega(t) + m(t)\overline{\omega'(t)}]d\overline{t} \\
&\quad - \frac{\omega_k}{\pi i}\int_{L\bigcup\gamma}\overline{\omega'(t)\eta(t)dt} \\
&\quad + 4(ReA^+)\omega_k + 2\overline{B^+}\overline{\omega_k}.
\end{aligned} \tag{1.69}$$

If one replaces $\phi_0(z)$ by $\phi_0(z) + i\epsilon z + c$ (ϵ is a real and c is a complex constant), the left-hand side of equation (1.63) is not altered, then the state of stress is neither changed. So A^+, B^+ can not be obtained directly from the system of algebraic equations obtained from equation (1.63) and equation (1.69). It will be expedient to modify the system of algebraic equations by adding ImA^+ to ReA^+ in (1.69) [34]. As will be proved later

$$ImA^+ = 0, \tag{1.70}$$

if the solution exists. Thus, we get the following modified system of algebraic equations

$$\begin{aligned}
&4A^+\omega_k + 2\overline{B^+}\overline{\omega_k}\frac{i\eta_k}{\pi}\int_{L\bigcup\gamma}\omega(t)dt - \frac{\omega_k}{\pi i}\int_{L\bigcup\gamma}\overline{\omega'(t)\eta(t)dt} \\
&+\frac{i\overline{\eta_k}}{\pi}\int_{L\bigcup\gamma}[\omega(t) + m(t)\overline{\omega'(t)}]d\overline{t} + \frac{\overline{\eta_k}}{\pi i}\int_{\gamma}F(\tau)d\overline{\tau} = iF_k, k = 1, 2.
\end{aligned} \tag{1.71}$$

It can be rewritten as

$$\begin{aligned}
4A^+\omega_1 + 2\overline{B^+}\overline{\omega_1} = &\frac{\eta_1}{\pi i}\int_{L\bigcup\gamma}\omega(t)dt + \frac{\omega_1}{\pi i}\int_{L\bigcup\gamma}\overline{\omega'(t)\eta(t)dt} \\
&+\frac{\overline{\eta_1}}{\pi i}\int_{L\bigcup\gamma}[\omega(t) + m(t)\overline{\omega'(t)}]d\overline{t} \\
&+\frac{i\overline{\eta_1}}{\pi}\int_{\gamma}F(\tau)d\overline{\tau} + iF_1,
\end{aligned}$$

$$
\begin{aligned}
4A^+\omega_2 + 2\overline{B^+}\overline{\omega_2} = &\frac{\eta_2}{\pi i}\int_{L\bigcup\gamma}\omega(t)dt + \frac{\omega_2}{\pi i}\int_{L\bigcup\gamma}\overline{\omega'(t)\eta(t)}d\bar{t} \\
&+\frac{\overline{\eta_2}}{\pi i}\int_{L\bigcup\gamma}[\omega(t)+m(t)\overline{\omega'(t)}]d\bar{t} \\
&+\frac{i\overline{\eta_2}}{\pi}\int_{\gamma}F(\tau)d\overline{\tau} + iF_2.
\end{aligned}
\tag{1.72}
$$

The determinant of the system of equations (1.72) is

$$
\begin{vmatrix} 4\omega_1 & 2\overline{\omega_1} \\ 4\omega_2 & 2\overline{\omega_2} \end{vmatrix} = 8(\omega_1\overline{\omega_2} - \omega_2\overline{\omega_1}) = -4iS \neq 0, \tag{1.73}
$$

where

$$
S = 2i(\omega_1\overline{\omega_2} - \omega_2\overline{\omega_1}) \tag{1.74}
$$

is exactly the area of the fundamental periodic-parallelogram P_{00} (on account of $Im(\omega_1) = 0$). Hence, we can obtain A^+, B^+ uniquely as

$$
\begin{aligned}
A^+ = &\frac{1}{4S}\left\{2\left[\overline{\omega_1}F_2 - \overline{\omega_2}F_1\right] + i\int_{\gamma}F(\tau)d\overline{\tau} - i\delta_2\int_{L\bigcup\gamma}\omega(t)dt\right\} \\
&-\frac{i}{4S}\int_{L\bigcup\gamma}\left[\omega(t)+m(t)\overline{\omega'(t)}\right]d\bar{t} + \frac{1}{4\pi i}\int_{L\bigcup\gamma}\overline{\omega'(t)\eta(t)}d\bar{t}, \\
\overline{B^+} = &\frac{1}{2S}\left\{2\left[\omega_2F_1 - \omega_1F_2\right] + i\overline{\delta_2}\int_{\gamma}F(\tau)d\overline{\tau} - i\int_{L\bigcup\gamma}\omega(t)dt\right\} \\
&-\frac{i\overline{\delta_2}}{2S}\int_{L\bigcup\gamma}\left[\omega(t)+m(t)\overline{\omega'(t)}\right]d\bar{t}.
\end{aligned}
\tag{1.75}
$$

By use of the modified Plemelj formulae [27]

$$
\widetilde{\Phi}^{\pm}(t) = \pm\frac{1}{2}\widetilde{\phi}(t) + \frac{1}{2\pi i}\int_L\widetilde{\phi}(\tau)\zeta(\tau - t)d\tau,
$$

where

$$
\widetilde{\Phi}(z) = \frac{1}{2\pi i}\int_L\widetilde{\phi}(\tau)\zeta(\tau - z)dt.
$$

Letting $z \to t \in L$ and substituting equations (1.65), (1.66) into equation (1.61) we have

$$
A^+t + m(t)\overline{A^+} + \overline{B^+}\bar{t} = A^-t + m(t)\overline{A^-} + \overline{B^-}\bar{t}, t \in L. \tag{1.76}
$$

Obviously the two sides of equation (1.76) are doubly quasi-periodic, then, the cyclic increments of them must be equal, respectively,

$$\begin{cases} A^+\omega_1 + \overline{B^+}\overline{\omega_1} = A^-\omega_1 + \overline{B^-}\overline{\omega_1}, \\ A^+\omega_2 + \overline{B^+}\overline{\omega_2} = A^-\omega_2 + \overline{B^-}\overline{\omega_2}. \end{cases} \tag{1.77}$$

Hence, on account of $Im\,(\omega_2/\omega_1) > 0$,

$$A^+ = A^-, \quad B^+ = B^-,$$

follows immediately. Then equation (1.61) will be identically satisfied.

For convenience, we denote

$$A^+ = A^- = A, \quad B^+ = B^- = B. \tag{1.78}$$

Letting $z \to t_0 \in \gamma$ and substituting equations (1.65) and (1.66) into equation (1.60) , either for the positive or negative boundary value, by employing the modified Plemelj formulae and taking equations (1.68) and (1.78) into account, we get the same equation

$$\frac{1}{\pi i}\int_{L\bigcup\gamma}\omega(t)\zeta(t-t_0)dt - \frac{1}{2\pi i}\int_\gamma \omega(t)d\left[\log\frac{\sigma(t-t_0)}{\overline{\sigma(t-t_0)}}\right]$$
$$-\frac{1}{2\pi i}\int_\gamma \overline{\omega(t)}d\left\{[m(t)-m(t_0)]\,\overline{\zeta(t-t_0)}\right\} + M_1[\omega(t),t_0] = N_1(t_0), \tag{1.79}$$

where

$$M_1[\omega(t),t_0] = 2(ReA)t_0 + \overline{B}\overline{t_0} + \overline{D(t_0)}\left[\overline{A} - \frac{1}{2\pi i}\int_{L\bigcup\gamma}\overline{\omega'(t)\zeta(t)d\bar{t}}\right] - C_j,$$

$$N_1(t_0) = \frac{1}{2\pi i}\int_\gamma F(t)\overline{\zeta(t-t_0)}d\overline{\tau} + \frac{1}{2}G(t_0).$$

Substituting equations (1.65) and (1.66) into equation (1.62) we have

$$\frac{1}{2}(\alpha^+ + \alpha^- + \beta^+ + \beta^-)\omega(t_0) + \frac{\alpha^+ - \alpha^-}{2\pi i}\int_{L\bigcup\gamma}\omega(t)\zeta(t-t_0)dt$$
$$+\frac{\beta^- + \beta^+}{2\pi i}\int_{L\bigcup\gamma}\overline{\omega(t)}d\left\{[m(t)-m(t_0)]\,\overline{\zeta(t-t_0)}\right\}$$
$$+\frac{\beta^- + \beta^+}{2\pi i}\int_{L\bigcup\gamma}\omega(t)\overline{\zeta(t-t_0)}d\bar{t} + M_2[\omega(t),t_0] = N_2(t_0), \tag{1.80}$$

where

$$M_2[\omega(t), t_0] = (\alpha^+ - \alpha^-)At_0 + (\beta^- - \beta^+)\overline{Bt_0} - A(\beta^+ - \beta^-)m(t_0)$$
$$+(\beta^- - \beta^+)\left[\overline{A} - \frac{1}{2\pi i}\int_{L\bigcup\gamma}\overline{\omega'(t)\zeta(t)d\overline{t}}\right]\overline{D(t_0)},$$

$$N_2(t_0) = g(t_0) - \frac{\beta^+ - \beta^-}{2\pi i}\int_\gamma F(\tau)\overline{\zeta(\tau - t_0)}d\overline{\tau}.$$

Then, equations (1.79) and (1.80) as a whole constitute a singular integral equation with the Weierstrass zeta kernel along $L\bigcup\gamma$, the dominant part of the singular operator corresponding to it will be

$$A(t_0)\omega(t_0) + \frac{B(t_0)}{\pi i}\int_{L\bigcup\gamma}\omega(t)\zeta(t - t_0)dt, \tag{1.81}$$

where

$$A(t_0) = \begin{cases} 0, & \text{if } t_0 \in \gamma, \\ \alpha^+ + \alpha^- + \beta^+ + \beta^-, & \text{if } t_0 \in L, \end{cases}$$

$$B(t_0) = \begin{cases} 1, & \text{if } t_0 \in \gamma, \\ \alpha^+ - \alpha^- - \beta^+ + \beta^-, & \text{if } t_0 \in L. \end{cases}$$

Since $A(t_0) \pm B(t_0) \neq 0$ on $L\bigcup\gamma$, so it is a normal type singular integral equation. We need to find its solution in class h_{2m}, i.e., $\omega(a_j)$, $\omega(b_j)$ have to be finite. Now, we verify that, if this equation has a solution in h_{2m}, then equation (1.68) is really valid. In fact, due to $F(a_j) = F(b_j) = 0$, hence the expression

$$\frac{1}{2\pi i}\int_\gamma F(t)\overline{\zeta(t - t_0)}d\overline{t}$$

will be bounded at the end points $t_0 = a_j, b_j$ of γ_j. Then the left-hand side of equation (1.79) is bounded at the points $t_0 = a_j, b_j$, too. However, the integrals of the right-hand side of equation (1.79) are regular integrals except the term

$$\frac{1}{\pi i}\int_{L\bigcup\gamma}\omega(t)\zeta(t - t_0)dt. \tag{1.82}$$

Comparing the two sides of equation (1.79), then, the integral term (1.82) must be bounded at the points $t_0 = a_j, b_j$. Hence (1.68) is really valid, otherwise, (1.82) has at least a logarithmic singularity.

From (1.56) we get

$$F^{\pm}(z) = I(z), z \in S_0^{\pm}. \tag{1.83}$$

In order to solve the boundary value problem (1.56)-(1.58), we construct the modified Sherman transform

$$I(z) = \frac{1}{2\pi i}\int_L i\Delta(t)\zeta(t-z)dt + \frac{1}{2\pi i}\int_\gamma \Delta(t)\zeta(t-z)dt + Ez, \tag{1.84}$$

where $\Delta(t)$ is a new unknown real function, E is an undetermined complex constant.

Substituting equation (1.84) into equation (1.58) by separating the real and imaginary parts of E, we get a system of algebraic equations for the unknown real constants ReE and ImE

$$(\omega_k - \overline{\omega_k})ReE + (\omega_k + \overline{\omega_k})ImE = \frac{1}{4\mu^+}|\omega_k|T_k - \frac{1}{\pi i}Re[\eta_k\Delta(t)], k = 1, 2, \tag{1.85}$$

the determinant of coefficients of which will be

$$\begin{vmatrix} \omega_1 - \overline{\omega_1} & i(\omega_1 + \overline{\omega_1}) \\ \omega_2 - \overline{\omega_2} & i(\omega_2 + \overline{\omega_2}) \end{vmatrix} = S \neq 0. \tag{1.86}$$

Thus, taking equation (1.43) into account, we can obtain ReE and ImE uniquely.

$$\begin{cases} ReE = \dfrac{Im\Delta(t)}{2S} + \dfrac{Re[i\delta_2\Delta(t)]}{2\pi S} + \dfrac{Re(\omega_2)|\omega_1|T_1 - (Re(\omega_1)|\omega_2|T_2}{\mu^+ S}i, \\ ImE = -\dfrac{Im\Delta(t)}{2S} - \dfrac{Re[i\delta_2\Delta(t)]}{2\pi S} + \dfrac{Im(\omega_2)|\omega_1|T_1 - (Im(\omega_1)|\omega_2|T_2}{\mu^+ S}i. \end{cases} \tag{1.87}$$

Substituting equation (1.84) into equation (1.56), it is observed that equation (1.56) is identically satisfied. Then, substituting equation (1.84) into equations (1.57), (1.55) we obtain

$$\Delta(t) + \frac{\mu^*}{2\pi i}\int_L \Delta(t) d\left[\log\frac{\sigma(t-t_0)}{\overline{\sigma(t-t_0)}}\right] \; -\frac{\mu^*}{\pi}\int_\gamma \Delta(t) d \log|\sigma(t-t_0)| \\ -2i\mu^* Re(Et) = 0, \tag{1.88}$$

and

$$\frac{1}{2\pi i}\int_L i\Delta(t) d\left[\log\frac{\sigma(t-t_0)}{\overline{\sigma(t-t_0)}}\right] \; +\frac{\mu^*}{\pi}\int_\gamma \Delta(t) d\; \log|\sigma(t-t_0)| \\ -2iIm(Et) - ic_j^* = 0, \tag{1.89}$$

respectively, where

$$\mu^* = \frac{\mu^+ - \mu^-}{\mu^+ + \mu^-}. \tag{1.90}$$

Equations (1.88) and (1.89) as a whole constitute a Fredholm integral equations of the second kind.

1.5 Unique Solvability of the First Fundamental CPS Problem

At first, it will now be shown that, if equations (1.79) and (1.80) have a solution $\omega(t)$, then, by A, B obtained from equation (1.75), and $\phi_0(z)$, $\psi_0(z)$ from equations (1.65) and (1.66), $\phi(z)$, $\psi(z)$ from equations (1.45) and (1.46), then (1.70) will be satisfied. In fact, taking the modified system of equation (1.71) into account,

$$\left[\phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)}\right]_{\Gamma_k} = i\left[F_k - 4(ImA)\omega_k\right], k = 1, 2. \tag{1.91}$$

This means the external resultant principal vectors on the boundary Γ_k ($k = 1, 2$) of the fundamental periodic-parallelogram P_{00} are $F_k - 4(ImA)\omega_k$

($k = 1, 2$) as a result of (1.20). According to the equilibrium principle of resultant moments, we have

$$Re\left\{\int_{\gamma}[f^{+}(\tau) - f^{-}(\tau)]d\overline{\tau} - 2i[\overline{\omega_1}F_2 - \overline{\omega_2}F_1] + 8i(ImA)(\omega_2\overline{\omega_1} - \omega_1\overline{\omega_2})\right\} = 0. \tag{1.92}$$

Considering the condition (1.40), we get

$$8(ImA)Im(\omega_2\overline{\omega_1} - \omega_1\overline{\omega_2}) = 0. \tag{1.93}$$

Because

$$8Im(\omega_2\overline{\omega_1} - \omega_1\overline{\omega_2}) = 4S \neq 0,$$

then,

$$ImA = 0. \tag{1.94}$$

Therefore, (1.70) is indeed satisfied.

It will now be proved that if C_j ($j = 0, 1, \cdots, m-1$) will be suitably chosen (also uniquely), then equations (1.79) and (1.80) are solvable in h_{2m}. Similarly to [49], for this purpose, the homogeneous equation, obtained from equation (1.79) and (1.80) for $f_j^{\pm}(\tau) = 0, g(t) = 0, e_3 = 0$, will be considered and it will be shown that it has no non-trival solutions. This means if $\omega_0(t)$ be any solution of this equation, after $C_j = C_j^0$ ($j = 0, 1, \cdots, m-1$) be taken, then $\omega_0(t) = 0$ everywhere on $L \bigcup \gamma$ (and hence $C_j^0 = 0$ necessarily).

It is obvious that (1.40) is satisfied in this case. Let $\phi_0^0(z)$, $\psi_0^0(z)$, $\phi^0(z)$, $\psi^0(z)$, A^0, B^0 be the corresponding values of $\phi_0(z)$, $\psi_0(z)$, $\phi(z)$, $\psi(z)$, A, B determined by equations (1.65), (1.66), (1.45), (1.46), and equation (1.75) for $\omega(t) = \omega_0(t)$. It is easy to verify that they satisfy the corresponding boundary conditions (1.51)-(1.54), which form the first fundamental problem under homogeneous conditions. By the uniqueness theorem [39], [49] , the elastic region may only have a rigid motion.

$$\phi^0(z) = i\epsilon_z z + c_z,\ \psi^0(z) = -\overline{c_z}, \tag{1.95}$$

with a sectionally real constant

$$\epsilon_z = \begin{cases} \epsilon^+, & \text{if } z \in S_0^+, \\ \epsilon^-, & \text{if } z \in S_0^-, \end{cases}$$

and a sectionally complex constant

$$c_z = \begin{cases} c^+, & \text{if } z \in S_0^+, \\ c^-, & \text{if } z \in S_0^-. \end{cases}$$

From equations (1.45), (1.46) and (1.95), it follows that

$$\phi_0^0(z) = \phi^0(z) = i\epsilon_z z + c_z, \;\; \psi_0^0(z) = \psi^0(z) = -\overline{c_z}. \tag{1.96}$$

Substituting equation (1.96) into equation (1.61) we arrive at the equality

$$\epsilon^+ = \epsilon^-. \tag{1.97}$$

Substituting equation (1.96) into equation (1.62) we get

$$\begin{cases} (\alpha^+ + \beta^+)\epsilon^+ = (\alpha^- + \beta^-)\epsilon^-, \\ (\alpha^+ + \beta^+)c^+ = (\alpha^+ + \beta^+)c^-. \end{cases} \tag{1.98}$$

Referring to formulae (1.65) and (1.66), and noting (1.96), we have

$$\phi_0^0(z) = i\epsilon_z z + c_z = \frac{1}{2\pi i}\int_{L\bigcup\gamma} \omega_0(t)\zeta(t-z)dt + A^0 z, \tag{1.99}$$

$$\psi_0^0(z) = -\overline{c_z} = -\frac{1}{2\pi i}\int_{L\bigcup\gamma} [\overline{\omega_0(t)} + \overline{m(t)}\omega_0'(t)]\zeta(t-z)dt + B^0 z. \tag{1.100}$$

When $z \in S_0^+$, considering quasi-periodicity of the two sides of equations (1.99) and (1.100), we obtain

$$A^0 = i\epsilon^+, \;\; B^0 = 0, \tag{1.101}$$

$$\frac{1}{2\pi i}\int_{L\bigcup\gamma} \omega_0(t)dt = 0, \;\;\; \frac{1}{2\pi i}\int_{L\bigcup\gamma} [\overline{\omega_0(t)} + \overline{m(t)}\omega_0'(t)]dt = 0. \tag{1.102}$$

Taking equation (1.101) into account, from equation (1.75) we can get

$$A^0 = 0. \tag{1.103}$$

Hence, from equations (1.101) and (1.97) we have

$$\epsilon^+ = \epsilon^- = 0. \tag{1.104}$$

By the Plemelj formulae from equation (1.99), we get immediately

$$\omega_0(t) = \phi_0^{0+}(t) - \phi_0^{0-}(t) = 0, t \in \gamma, \tag{1.105}$$

and

$$\omega_0(t) = c^+ - c^-, t \in L. \tag{1.106}$$

Substituting equations (1.105), (1.106) back into equations (1.99), (1.100) we obtain

$$c_z = \frac{c^+ - c^-}{2\pi i} \int_L \zeta(t - z)dt, \quad -\overline{c_z} = \frac{\overline{c^+} - \overline{c^-}}{2\pi i} \int_L \zeta(t - z)dt. \tag{1.107}$$

Setting $z = 0$ in equation (1.107), we get

$$c^+ = 0. \tag{1.108}$$

From equation (1.98) we obtain

$$c^- = 0. \tag{1.109}$$

At last, from equations (1.105), (1.108), (1.109) and (1.106) we have

$$\omega_0(t) \equiv 0, t \in L \bigcup \gamma. \tag{1.110}$$

If only the distribution of stress are required (not necessarily for displacements), we may take derivatives of both sides of equations (1.79), (1.80) and so obtain a singular integral equation with density function $\Omega(\tau) = \omega'(\tau)$, which has to be solved in class h_0 with the additional conditions

$$\int_{\gamma_j} \Omega(\tau) d\tau = 0, j = 0, 1, \cdots, m-1. \tag{1.111}$$

The obtained equations do not contain any undetermined constants, which simplifies the process of solving.

We now turn to prove the equations (1.88) and (1.89) to be uniquely solvable. To do this, we must prove $\Delta_0(t) \equiv 0, t \in L \bigcup \gamma$ under homogeneous conditions of the first fundamental problem. Analogously to [20], in this case we have

$$I(z) = I_z, \quad I_z = \begin{cases} I^+, & \text{if } z \in S_0^+, \\ I^-, & \text{if } z \in S_0^-, \end{cases} \tag{1.112}$$

where I^+, I^- are constants.

Substituting equation (1.112) into equations (1.56)-(1.57), we get

$$ReI^+ = ReI^-, \tag{1.113}$$

$$\mu^+ ImI^+ = \mu^- ImI^-. \tag{1.114}$$

Then, referring to equation (1.84) we arrive at the equality

$$I_z = \frac{1}{2\pi i} \int_L i\Delta_0(t)\zeta(t-z)dt + \frac{1}{2\pi i} \int_\gamma \Delta_0(t)\zeta(t-z)dt + E^0 z. \tag{1.115}$$

On account of the doubly-periodic properties of the two sides of equation (1.115) we obtain

$$\frac{1}{2\pi i} \int_L i\Delta_0(t)dt + \frac{1}{2\pi i} \int_\gamma \Delta_0(t)dt = 0. \tag{1.116}$$

$$E_0 = 0. \tag{1.117}$$

By the Plemelj formulae on γ, from equations (1.112) and (1.115) we get

$$\Delta_0(t) = 0, t \in \gamma. \tag{1.118}$$

By the Plemelj formulae on L, from equations (1.112) and (1.115) we have

$$\Delta_0(t) = I^+ - I^-,\ t \in L. \qquad (1.119)$$

Substituting equations (1.116)-(1.118), into equation (1.115) we obtain

$$I_z = \frac{I^+ - I^-}{2\pi} \int_L \zeta(t-z)dt. \qquad (1.120)$$

Putting $z = 0$ in equation (1.120), it follows immediately that

$$I^+ = 0. \qquad (1.121)$$

Substituting equation (1.121) into equations (1.113) and (1.114), we find $I^- = 0$. Then, from equation (1.119) we get

$$\Delta_0(t) = 0, t \in L. \qquad (1.122)$$

Hence, referring to equations (1.118) and (1.122) we have proved

$$\Delta_0(t) \equiv 0, t \in L \bigcup \gamma. \qquad (1.123)$$

Thus, the first fundamental CPS problem is solved.