



# Helicity decoupling in the massless limit of massive tensor fields <sup>☆</sup>

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## Abstract

Massive and massless potentials play an essential role in the perturbative formulation of particle interactions. Many difficulties arise due to the indefinite metric in gauge theoretic approaches, or the increase with the spin of the UV dimension of massive potentials. All these problems can be evaded in one stroke: modify the potentials by suitable terms that leave unchanged the field strengths, but are not polynomial in the momenta. This feature implies a weaker localization property: the potentials are “string-localized”. In this setting, several old issues can be solved directly in the physical Hilbert space of the respective particles: We can control the separation of helicities in the massless limit of higher spin fields and conversely we recover massive potentials with  $2s + 1$  degrees of freedom by a smooth deformation of the massless potentials (“fattening”). We construct stress–energy tensors for massless fields of any helicity (thus evading the Weinberg–Witten theorem). We arrive at a simple understanding of the van Dam–Veltman–Zakharov discontinuity concerning, e.g., the distinction between a massless or a very light graviton. Finally, the use of string-localized fields opens new perspectives for interacting quantum field theories with, e.g., vector bosons or gravitons.

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<sup>☆</sup> An abridged version of this paper, focusing on spin  $s = 1$  and  $s = 2$ , is [25].

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## 1. Overview

The purpose of this contribution is to formulate and investigate a unified setting for potentials describing both massless and massive vector and tensor bosons, that live in Hilbert space (i.e., without negative-norm states even at intermediate steps). The Hilbert space is that of the field strengths, which have no positivity problems. Our focus is here on the free fields, and in particular on the limit  $m \rightarrow 0$  that is smooth in this setting. We comment in appropriate places on the issues concerning renormalizable interactions, in particular the improved UV behaviour, and refer for more details to the literature [37,38,24,17,26,23].

Hilbert space positivity is quantum theory's most basic attribute which is indispensable for its probability interpretation. In the standard formulation, it enters through the identification of quantum states with unit rays in a Hilbert space. The von Neumann uniqueness theorem concerning Heisenberg's commutation relations in terms of operators in a Hilbert space secures the positivity of quantum mechanics. With the more general notion of states as positive linear functionals of  $*$ -algebras, positivity is secured via the GNS construction leading again to representations in Hilbert space.

In contrast to Born's quantum mechanical localization in terms of probability (the argument of the wave function directly refers to the position of a particle), the positivity issue in QFT is more demanding and detaches the causal localization of fields from the localization of particles: it is impossible to assign a probability such that  $\phi(x)$  creates with certainty a particle at the point  $x$  [28]. Classical field theory has no structural feature whose quantization guarantees that the corresponding quantum fields act in a Hilbert space, and canonical quantization of massless fields generically introduces negative-norm states. For low spin  $s < 1$  this problem is absent but, as Gupta and Bleuler first pointed out, starting from  $s = 1$ , only by using additional negative-metric degrees of freedom can one maintain a formal analogy to the classical covariant gauge potentials at a point.

There has been extensive work on higher spin fields by Fronsdal [15], Rindani et al. [32], Francia et al. [14], Bekaert et al. [2], Sagnotti [34], and in particular by Vasiliev [13,40,41,6,30] (to name only a few).<sup>1</sup> While Fronsdal proves the positivity of the 2-point function contracted with constrained sources, most of the more recent work concentrates on Lagrangians and field equations without even addressing the crucial issues of quantization: positivity (semi-definite two-point functions, actions of fields in a ghost-free Hilbert space, ...) and causal localization (commutation relations at spacelike distance).

An alternative to canonical quantization is to start from Wigner's classification of unitary positive energy representations of the Poincaré group. This approach takes care of positivity from the start, and one can fully concentrate on the interplay of covariance with causal localizability. In order to construct covariant fields on the Wigner Fock space, one uses intertwiner functions  $u_\alpha(p)$  which mediate between unitary representations of the stabilizer group and matrix representations for covariant field multiplets; for  $m > 0$  this was done by Weinberg [42]. Since the intertwiners determine the two-point functions, they also determine the commutators, and therefore control causal commutativity. In the case of half-integer spin, causality is incompatible with positivity and has to be replaced by anti-commutation relations instead. For spin 1, as expected,

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<sup>1</sup> The strings in the titles of some of these papers have nothing in common with the strings in our paper. The prevalent idea to assign a "quantum mechanical" notion of localization to (super)strings is at variance with the causal localization of quantum field theory, as discussed in [36, Sect. 2].

massless vector potentials  $A_\mu(x)$  localized at points  $x$  (which are precisely what is needed for QED) do not exist on the Hilbert space.

During the last decade we have learned that the Wigner representation theory contains much more information concerning causal localization and associated intertwiners [27,29]. Modifying the intertwiners does not change the particle content, but will modify properties of the fields. It may improve the short-distance dimension at the price of a weaker localization. With the new flexibility, one can construct massive vector potentials that admit renormalizable interactions and massless vector potentials directly on the Hilbert space, and even causal fields that transform in Wigner’s infinite-spin representation [27]. All these fields are localized on “strings” = rays extending from a point to spacelike infinity.

This kind of localization was actually proven long ago to be *necessary* to connect the vacuum with charged one-particle states. Buchholz [4] gave a nonperturbative proof that electrical charge-carrying quantum fields cannot be compactly localized and that the tightest localized covariant fields cannot be better localized than on spacelike half-lines. Buchholz and Fredenhagen [5], by an analysis of how the corresponding state functionals on the observables differ, showed that string-localization may be necessary in massive theories; and at the same time *sufficient* to construct many-particle scattering states in the asymptotic time limits.

Taking advantage of this new flexibility, one can reformulate all perturbative interactions of the Standard Model directly on the Hilbert space of the physical asymptotic particles, without recourse to ghosts and BRST methods. The emerging program of causal perturbation theory with string-localized fields [37,38,24,17,26,23] suggests that it allows to compute the same scattering matrix as the BRST gauge theory approach, but unlike the latter, it also allows to construct (non-observable) interpolating fields between the vacuum and the charged states, that live on the Hilbert space and are just local enough to allow for scattering theory.

The reformulation is classically equivalent to the usual one in that the Lagrangian differs only by a total derivative.

**Example 1.1.** [38] We want to illustrate this in the case of massive or massless vector bosons coupled to a conserved current  $j^\mu(x)$ . The interaction part of the action is

$$S_{\text{int}} = \int d^4x A_\mu(x) j^\mu(x). \tag{1.1}$$

Unlike the integrand, the integral does not depend on the choice of the potential when the field strength is given. Namely, by Poincaré’s Lemma, any two potentials differ by a gradient  $\partial_\mu \alpha$ , and  $\int d^4x \partial_\mu \alpha j^\mu = \int d^4x \partial_\mu (\alpha j^\mu) = 0$ . Indeed, the classical equation of motion derived by Hamilton’s Principle contains only the field strength.

We are going to use potentials that depend on a direction  $e$  in Minkowski spacetime and are manifestly localized along the string  $x + \mathbb{R}_0^+ e$ :

$$A_\mu(x, e) = \int_0^\infty ds F_{\mu\nu}(x + se) e^\nu. \tag{1.2}$$

One easily sees that  $\partial_\mu A_\nu(x, e) - \partial_\nu A_\mu(x, e) = F_{\mu\nu}(x)$ . From these facts, the  $e$ -independence of

$$S_{\text{int}}(e) = \int d^4x A_\mu(x, e) j^\mu(x) \tag{1.3}$$

is manifest.

The preservation of the equivalence of  $S_{\text{int}}(e)$  and  $S_{\text{int}}$  at the quantum level, i.e., the  $e$ -independence of the S matrix  $T \exp i \int d^4x L_{\text{int}}(e)$ , is an issue of renormalization conditions, whose satisfiability beyond the lowest orders is presently under investigation. Interacting fields can then be constructed with the Bogoliubov S matrix method [3]. The interacting potential and Dirac field will depend on  $e$ , but their field strength and current are expected to be  $e$ -independent.

The merit of the string-localized approach is twofold: String-localized interacting fields that connect the vacuum with scattering states are constructed in the Hilbert space, and in such a way that observables are string-independent, and hence causality is secured. In the massless case,  $A_\mu(x, e)$  is defined on the physical Hilbert space of the field strength which has only two photon states; and in the massive case,  $A_\mu(x, e)$  has UV dimension  $d_{UV} = 1$  instead of  $d_{UV} = 2$  for the Proca potential  $A_\mu^P$ , see Eq. (1.7) vs. Eq. (1.11). Thus, the coupling  $A_\mu(x, e) j^\mu(x)$  of a massive vector boson to the Dirac current of dimension  $d_{UV} = 3$  is renormalizable. The strong short-distance fluctuations of the Proca potentials have been “carried away” by the discarded derivative term.

We have carried out this program in the easiest case: the coupling of a string-localized vector potential to a conserved classical current [25]. The interacting field is found to be

$$A_\mu^{\text{int}}(x, e) = A_\mu(x, e) + A_\mu^{\text{cl}}(x) + \partial_\mu \phi^{\text{cl}}(x, e),$$

where  $A_\mu(x, e)$  is the free quantum field, and  $A_\mu^{\text{cl}}(x)$  and  $\phi^{\text{cl}}(x, e)$  are the classical retarded solutions associated with the current  $j_\mu(x)$  and with the source  $j_e(x) := \int_0^\infty ds j_\mu(x + se)e^\mu$ , respectively. Clearly, the interacting field strength is  $e$ -independent.

See Sect. 1.2 for the analogous case of massive and massless gravity.

The same strategy to secure causality of the perturbation theory applies whenever the string-dependence of an interaction term  $L_{\text{int}}(e)$  is a total derivative, so that the classical action  $S_{\text{int}}(e)$  is  $e$ -independent. In the case of massive tensor fields, one may have the form  $L_{\text{int}}(x, e) = L_{\text{int}}^P(x) + \partial_\mu V^\mu(x, e)$ , where  $L_{\text{int}}^P$  is point-localized but non-renormalizable; its UV-divergences are absorbed by the derivative term such that  $L_{\text{int}}(e)$  is string-localized and renormalizable. Interactions of massless particles do not possess an equivalent point-local Lagrangian in the Hilbert space, but for the  $e$ -independence of the action it is sufficient that  $\partial_{e^\kappa} L_{\text{int}}(e) = \partial_\mu Q_\kappa^\mu$ .

However, the  $e$ -independence of the causal S matrix requires at the quantum level that the time-ordering can be defined in such a way that total derivatives are preserved. These conditions impose already in lowest orders certain constraints on the possible interactions, that are all realized in the Standard Model: the Lie algebra structure of cubic couplings of several species of vector bosons [38]; the presence of a Higgs field when there are non-Abelian massive vector bosons [38,26]; and the chirality of their coupling to fermions [17]. In scalar massive QED, the cubic part of the string-local minimal coupling induces also the quartic part [38], so that the full fibre-bundle-like structure of the quantum theory turns out to be a consequence of imposing  $e$ -independence of the unitary S matrix, hence of positivity and causality, rather than a classical local gauge symmetry. We wonder whether this remarkable feature extends also to higher spin and gravitational couplings, possibly demanding additional couplings to lower spin fields.

These observations for  $s = 1$  resp. speculations for  $s = 2$  are very analogous to the analysis by Scharf et al. [8,9,35,7] (pursued in the gauge-theoretic indefinite-metric and point-localized setting) where the fibre bundle structure and the presence of a Higgs boson are consequences of BRST invariance of causal perturbation theory with self-interacting massive vector bosons. For spin 2, BRST invariance requires to supplement the cubic self-interaction of perturbative gravity in a unique way by higher-order terms that eventually sum up to the full Einstein gravity [35].

Indeed, like BRST invariance, also the condition of  $e$ -independence can be formulated in a cohomological manner. Yet, the precise relation between gauge invariance and string-independence remains to be explored. But beyond this analogy, it becomes clear that the role of the Higgs boson is not the generation of the mass, but the preservation of the renormalizability and locality under the constraints imposed by positivity [38,26].

*1.1. Properties of string-localized fields*

Let us turn back to Eq. (1.2), that we shall henceforth write in a short-hand notation:

$$A_\mu(x, e) = (I_e F_{\mu\nu})(x)e^\nu \tag{1.4}$$

where  $I_e$  stands for the integration over the string in the direction  $e$ . This potential is certainly not a fundamental field variable; it is some “useful function” of the field strength. Exploiting the freedom to define different fields in terms of the same creation and annihilation operators, we understand string-localized potentials mainly as a device to set up renormalizable interaction terms that are equivalent to but better behaved than their non-renormalizable point-localized counterparts.

It is also clear that string-localization is not a feature of the associated particles, which are always the same massive or massless particles specified by the Wigner representation. (The only exception would be particles in the infinite-spin representations [27,21,31], whose fields are “intrinsically string-localized”, i.e., not representable as integrals like Eq. (1.2). This case is beyond the scope of the present paper.)

Working with  $A_\mu(e)$  is not in conflict with the principle of causality, which is as imperative in relativistic quantum field theory as Hilbert space positivity. While their field strengths are point-localized, the string-localized potentials satisfy causal commutation relations according to their localization: two such operators commute whenever every point on the string  $x + \mathbb{R}_0^+ e$  is spacelike to every point on the other string  $x' + \mathbb{R}_0^+ e'$ . If the strings are chosen spacelike, such pairs of spacelike separated strings are abundant. In this work, we assume  $e^2 = -1$  without loss of generality.

It follows from Eq. (1.2) (and later generalizations involving tensor fields and/or iterations of the integral operation  $I_e$ ) that the Poincaré transformations of string-localized fields are

$$U(a, \Lambda)A_{\mu_1 \dots \mu_r}(x, e)U(a, \Lambda)^* = \Lambda^{\nu_1}_{\mu_1} \dots \Lambda^{\nu_r}_{\mu_r} A_{\nu_1 \dots \nu_r}(a + \Lambda x, \Lambda e), \tag{1.5}$$

i.e., the direction of the string is transformed along with its apex  $x$  and the tensor components of the field tensor. Unlike a fixed string direction, a transforming direction does not violate covariance. See Remark 1.2 for further comments, comparing with “axial gauges” with fixed directions.

We shall write 2-point functions throughout as

$$(\Omega, X(x)Y(y)\Omega) = \int d\mu_m(p) \cdot e^{-ip(x-y)} \cdot {}_m M^{X,Y}(p), \tag{1.6}$$

where  $d\mu_m(p) = \frac{d^4 p}{(2\pi)^3} \delta(p^2 - m^2)\theta(p^0)$ . Our sign convention is  $\eta_{00} = +1$ .

From the 2-point function of the field strengths, one can compute the 2-point function of their string-localized potentials of any mass  $m > 0$  or  $m = 0$ :<sup>2</sup>

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<sup>2</sup> Our choice to consider correlations between fields with strings  $-e$  and  $e'$  is a convention to simplify notations, that will pay off when it comes to higher spin.

$${}_m M^{A_\mu(-e), A_\nu(e')} = -E(e, e')_{\mu\nu}(p), \tag{1.7}$$

where  $E(e, e')_{\mu\nu}(p)$  is the distribution in  $p, e,$  and  $e'$

$$E(e, e')_{\mu\nu}(p) := \eta_{\mu\nu} - \frac{p_\mu e_\nu}{(pe)_+} - \frac{e'_\mu p_\nu}{(pe')_+} + \frac{(ee')p_\mu p_\nu}{(pe)_+(pe')_+}. \tag{1.8}$$

The denominators  $1/(pe)_+ = 1/((pe + i0))$  arise from the integrations  $\int_0^\infty du e^{ip(x+ue)} = \frac{i}{p \cdot e + i0} e^{ipx}$ . Eq. (1.7) should be compared with the massless 2-point function in the Gupta–Bleuler (Feynman gauge) approach:

$${}_0 M^{A_\mu^F, A_\nu^F} = -\eta_{\mu\nu}, \tag{1.9}$$

the massless 2-point function in the Coulomb gauge  $A_0^C = 0, \vec{\nabla} \cdot \vec{A}^C = 0$ :

$${}_0 M^{A_i^C, A_j^C} = \delta_{ij} - \frac{p_i p_j}{|\vec{p}|^2}, \tag{1.10}$$

and the 2-point function of the Proca potential related to the massive field strength by the Proca field equation  $m^2 A_\mu^P(x) = \partial^\nu F_{\mu\nu}(x)$ :

$${}_m M^{A_\mu^P, A_\nu^P} = -\eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2} =: -\pi_{\mu\nu}(p). \tag{1.11}$$

Eq. (1.9) is point-localized but obviously indefinite, Eq. (1.10) is positive<sup>3</sup> but non-covariant and completely non-local [42], Eq. (1.11) is point-localized and positive but has short-distance dimension  $d_{UV} = 2$  due to the momenta in the numerator. Moreover, it obviously does not admit a massless limit.

In contrast, the string-localized 2-point function Eq. (1.7) is positive and covariant (in the sense of Eq. (1.5)) and it has  $d_{UV} = 1$  for all  $m \geq 0$ , and the massless case is smoothly connected to the massive case.

All 2-point functions produce the same 2-point function for the field strengths  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ :

$${}_m M^{F_{\mu\nu}, F_{\kappa\lambda}} = -p_\mu p_\kappa \eta_{\nu\lambda} + p_\nu p_\kappa \eta_{\mu\lambda} + p_\mu p_\lambda \eta_{\nu\kappa} - p_\nu p_\lambda \eta_{\mu\kappa}, \tag{1.12}$$

because Eq. (1.9)–Eq. (1.7) all differ only by terms proportional to  $p$  that are “killed” by the curl.

Indeed, the string-localized potential  $A_\mu(x, e)$  for  $e = (1, 0, 0, 0)$  coincides with the Coulomb gauge potential  $A_\mu^C$ . The well-known non-locality of the Coulomb gauge potential [42] reflects the fact that two timelike strings are never spacelike separated. It may be interesting to notice that one can average the potential  $A_\mu(x, e)$  in  $e$  over the spacelike sphere with  $e^0 = 0$ . The resulting potential is again the Coulomb gauge potential Eq. (1.10). (Similar statements also hold for  $s > 1, m = 0$ , but the averaging must be very carefully performed.)

**Remark 1.2.** By definition, or by inspection of the 2-point function,  $A_\mu(x, e)$  satisfies the relation  $e^\mu A_\mu(x, e) = 0$ . This is the axial gauge condition if  $e$  is spacelike; indeed, Eq. (1.7) coincides with the respective gauged 2-point functions. We emphasize that these gauge conditions are not used to reduce the degrees of freedom before quantization, but that instead the potentials for all

<sup>3</sup> By “positive”, it is actually understood “positive-semidefinite”, accounting for the null states due to equations of motion.

$e$  coexist simultaneously on the Fock space of the field strength, and they covariantly transform into each other according to Eq. (1.5).

By specifying the 2-point function for spacelike  $e$  as a distribution rather than a function with a singularity, we reveal the manifestly string-localized representation Eq. (1.2) of the axial gauges, and we discover the mutual commutativity of axial gauge potentials for different directions.

Unlike the Proca potential, the string-localized potential is not conserved. Let

$$a(x, e) := -m^{-1} \cdot \partial^\mu A_\mu(x, e). \tag{1.13}$$

One sees from the 2-point function Eq. (1.7) that in spite of the factor  $m^{-1}$ ,  $a(x, e)$  is regular in the massless limit:

$${}_m M^{a(-e), A_\nu(e')} = im \left( \frac{e_\nu}{(pe)_+} - \frac{(ee')p_\nu}{(pe)_+(pe')_+} \right), \tag{1.14}$$

$${}_m M^{a(-e), a(e')} = 1 - m^2 \frac{(ee')}{(pe)_+(pe')_+}. \tag{1.15}$$

At  $m = 0$ ,  $A_\mu(e)$  and  $a(e)$  decouple, and Eq. (1.15) is independent of  $e$  and  $e'$ , hence  $\varphi(x) = a(x, e)|_{m=0}$  is independent of  $e$ . Its one-particle state is the remnant of the massive particle state with longitudinal angular momentum, decoupled from the massless string-localized Maxwell potential.

Finally, we have the identity (underlying Example 1.1)

$$A_\mu^P(x) = A_\mu(x, e) - m^{-1} \partial_\mu a(x, e), \tag{1.16}$$

that can also be seen from the definitions, using the Proca field equations, or from the 2-point functions.

### 1.2. Spin $\geq 2$ : DVZ discontinuity and Weinberg–Witten theorem

The case of spin 2 (and higher) exhibits several new features as compared to spin 1, apart from the analogous problems with positivity, covariance, short-distance dimension and massless limit. We here only sketch some pertinent results for spin 2, that are proven for general integer spin in Sect. 3.

The first new feature is the DVZ observation due to van Dam and Veltman [39] and Zakharov [44], that in interacting models with  $s \geq 2$ , scattering amplitudes are discontinuous in the mass at  $m = 0$ , i.e., the scattering on massless gravitons (say) is significantly different from the scattering on gravitons of a very small mass. The DVZ discontinuity has been used to argue that, by measuring the deflection of light in a gravitational field, gravitons must be exactly massless.

The second new feature is the Weinberg–Witten theorem [43,22,20] about the higher-spin massless case. It states that for  $s \geq 2$ , no Lorentz covariant point-localized stress–energy tensor exists such that the Poincaré generators are moments of its zero-components:

$$P_\sigma \stackrel{\dagger}{=} \int_{x^0=t} d^3\vec{x} T_{0\sigma}, \quad M_{\sigma\tau} \stackrel{\dagger}{=} \int_{x^0=t} d^3\vec{x} (x_\sigma T_{0\tau} - x_\tau T_{0\sigma}). \tag{1.17}$$

The absence of a Lorentz covariant stress–energy tensor also obstructs the semiclassical coupling of massless higher spin matter to gravity. More precisely, because there exists a stress–energy ten-

sor in terms of potentials whose Lorentz transforms involve gauge transformations, the problem is shifted to the challenge of finding gauge invariant couplings to the gravitational field.<sup>4</sup>

The DVZ discontinuity can be traced back to the fact that for spin 2 (or higher), the massless limit of the massive field strength

$$F_{[\mu\nu][\kappa\lambda]} = \partial_\mu \partial_\kappa A_{\nu\lambda} - \partial_\nu \partial_\kappa A_{\mu\lambda} - \partial_\mu \partial_\lambda A_{\nu\kappa} + \partial_\nu \partial_\lambda A_{\mu\kappa} \tag{1.18}$$

exists but differs from the massless field strength. This is seen explicitly by inspection of the 2-point functions, which in the massive case is the curl (taken in all indices) of the positive 2-point function of the spin-2 Proca field<sup>5</sup>

$${}_m M^{A_{\mu\nu}^P, A_{\kappa\lambda}^P} = \frac{1}{2} \left[ \pi_{\mu\kappa} \pi_{\nu\lambda} + \pi_{\mu\lambda} \pi_{\nu\kappa} \right] - \frac{1}{3} \pi_{\mu\nu} \pi_{\kappa\lambda}, \tag{1.19}$$

and in the massless case is the curl of

$${}_0 M^{A_{\mu\nu}^F, A_{\kappa\lambda}^F} = \frac{1}{2} \left[ \eta_{\mu\kappa} \eta_{\nu\lambda} + \eta_{\mu\lambda} \eta_{\nu\kappa} \right] - \frac{1}{2} \eta_{\mu\nu} \eta_{\kappa\lambda}, \tag{1.20}$$

where  $A^F$  is the Feynman gauge potential. We emphasize, however, that the massless field strength is autonomously defined on the positive Fock space over the helicity  $h = \pm 2$  Wigner representations, and the indefinite Feynman gauge potential does not exist on this Hilbert space.

Applying the curls to both Eq. (1.19) and Eq. (1.20), the difference between the tensors  $\pi_{\mu\nu}$  (Eq. (1.11)) and  $\eta_{\mu\nu}$  disappears; but the different coefficient  $-\frac{1}{3}$  vs.  $-\frac{1}{2}$  of the last term survive. These coefficients are intrinsic features of the underlying massive and massive representations: they fix the number of linearly independent states in the one-particle spaces whose scalar product is given by the 2-point functions of the field strengths ( $2s + 1 = 5$  in the massive case, 2 in the massless case).

In order to analyze the DVZ discontinuity with the help of string-localized fields, we have to properly decouple in the massless limit the lower helicities  $h = 0, \pm 1$ , that all contribute to the massive field, from the helicities  $h = \pm 2$  of the graviton. Let us first study the decoupling.

The massive string-localized potential is

$$A_{\mu\nu}(x, e) := (I_e^2 F_{[\mu\kappa][\nu\lambda]})(x) e^\kappa e^\lambda \tag{1.21}$$

such that its double curl gives back the massive field strength. Its 2-point function is

$$\begin{aligned} {}_m M^{A_{\mu\nu}(-e), A_{\kappa\lambda}(e')} &= \\ &= \frac{1}{2} \left[ E(e, e')_{\mu\kappa} E(e, e')_{\nu\lambda} + (\kappa \leftrightarrow \lambda) \right] - \frac{1}{3} E(e, e)_{\mu\nu} E(e', e')_{\kappa\lambda}. \end{aligned} \tag{1.22}$$

Unlike the spin-2 Proca potential, the string-localized potential is not conserved. We define the escort fields

$$\begin{aligned} a_\mu^{(1)}(x, e) &:= -m^{-1} \partial^\nu A_{\mu\nu}(x, e), \\ a^{(0)}(x, e) &:= -m^{-1} \partial^\mu a_\mu^{(1)}(x, e). \end{aligned} \tag{1.23}$$

They are also regular at  $m = 0$  because  $\partial^\nu F_{[\mu\kappa][\nu\lambda]} = -m^2 F_{[\mu\kappa]\lambda}^P$  (the partial field strength [10]), and  $\partial^\mu F_{[\mu\kappa]\lambda}^P = -m^2 A_{\kappa\lambda}^P$ . Moreover, the identity

<sup>4</sup> We thank the referee for this more precise formulation of the issue.

<sup>5</sup> Albeit historically incorrect, we adopt the name ‘‘Proca field’’ also for higher spin.



$$a^{(0)}(x, e) = -\eta^{\mu\nu} A_{\mu\nu}(x, e) \tag{1.24}$$

holds, as well as the decomposition

$$A_{\mu\nu}^P(x) = A_{\mu\nu}(x, e) - m^{-1}(\partial_\mu a_\nu^{(1)}(x, e) + \partial_\nu a_\mu^{(1)}(x, e)) + m^{-2}\partial_\mu\partial_\nu a^{(0)}(x, e). \tag{1.25}$$

From Eq. (1.22) all other 2-point functions can be computed by descending with Eq. (1.23). One finds that  $a_\mu^{(1)}(e)$  decouples from  $A_{\mu\nu}(e)$  and from  $a^{(0)}(e)$  in the limit  $m \rightarrow 0$ , but the latter two do not decouple from each other:

$${}_0M^{a^{(0)}(-e), A_{\mu\nu}(e')} = -\frac{1}{3}E(e', e')_{\mu\nu}(p) \tag{1.26}$$

$${}_0M^{a^{(0)}(-e), a^{(0)}(e')} = \frac{2}{3}. \tag{1.27}$$

In order to decouple them, let

$$A_{\mu\nu}^{(2)}(x, e) := A_{\mu\nu}(x, e) + \frac{1}{2}E_{\mu\nu}(e, e)a^{(0)}(x, e), \tag{1.28}$$

where the integro-differential operator

$$E(e, e)_{\mu\nu} = \eta_{\mu\nu} + (e_\nu\partial_\mu + e_\mu\partial_\nu)I_e + e^2\partial_\mu\partial_\nu I_e^2 \tag{1.29}$$

acts by multiplication with  $E(e, e)_{\mu\nu}(p)$  and  $E(e, e)_{\mu\nu}(-p) = E(-e, -e)_{\mu\nu}(p)$  on the creation and annihilation parts, respectively. With this redefinition, the decoupling is exact at  $m = 0$ , and at all  $m \geq 0$

$$\begin{aligned} {}_mM^{A_{\mu\nu}^{(2)}(-e), A_{\kappa\lambda}^{(2)}(e')} &= \\ &= \frac{1}{2}\left[E(e, e')_{\mu\kappa}E(e, e')_{\nu\lambda} + (\kappa \leftrightarrow \lambda)\right] - \frac{1}{2}E(e, e)_{\mu\nu}E(e', e')_{\kappa\lambda}, \end{aligned} \tag{1.30}$$

which is exactly Eq. (1.22) except for the coefficient  $-\frac{1}{2}$  of the last term. Thus, at  $m = 0$ ,  $A^{(2)}$  coincides with the string-localized potential

$$A_{\mu\nu}^{(2)}(x, e) = (I_e^2 F_{[\mu\kappa][\nu\lambda]}^{(m=0)})(x)e^\kappa e^\lambda \tag{1.31}$$

associated with the massless field strength. The appropriately normalized field  $A_\mu^{(1)}(e) := \sqrt{2} \cdot a_\mu^{(1)}(e)$  converges at  $m = 0$  to the string-localized Maxwell potential, and  $A^{(0)}(e) := \sqrt{3/2} \cdot a^{(0)}(e)$  converges to the  $e$ -independent, canonically normalized, scalar field  $\varphi(x)$ . Thus, the fields  $A^{(r)}(e)$  with  $r = 0, 1, 2$  parametrize the exact decoupling of helicities  $h = \pm r$  at  $m = 0$ .

The result generalizes to arbitrary integer spin: Linear combinations of  $A_{\mu_1\dots\mu_s}(e)$  and its lower rank escort fields acted on by the operator  $E(e, e)_{\mu\nu}$  yield fields  $A_{\mu_1\dots\mu_r}^{(r)}(e)$  for every  $0 \leq r \leq s$  which in the massless limit are decoupled potentials of helicity  $h = \pm r$  field strengths.  $A^{(0)}(e)$  becomes independent of  $e$  and is the massless scalar field. Of course, the total number  $2s + 1$  of one-particle states is preserved. All these potentials have short-distance dimension  $d_{UV} = 1$  at  $m \geq 0$  and are suited for setting up renormalizable perturbation theory.

Now, returning to the DVZ problem, we may couple perturbative massive gravity in a Minkowski background to a conserved stress–energy source by

$$S_{\text{int}}(e) = \int d^4x A_{\mu\nu}(x, e)T^{\mu\nu}(x). \tag{1.32}$$

Because by Eq. (1.25),  $A_{\mu\nu}(e)$  differs from  $A_{\mu\nu}^P$  only by derivatives, the action is independent of  $e$ . At  $m > 0$ , all five states of the graviton couple to the source. In the limit  $m \rightarrow 0$ , we have by Eq. (1.28)

$$A_{\mu\nu}(x, e) = A_{\mu\nu}^{(2)}(x, e) - \sqrt{1/6} \eta_{\mu\nu} \varphi(x) + \text{derivatives},$$

where  $\varphi(x) = \sqrt{3/2} \lim_{m \rightarrow 0} a^{(0)}(x, e)$  is the string-independent massless scalar field decoupled from the helicity-2 potential  $A^{(2)}(x, e)$ . Thus,

$$\lim_{m \rightarrow 0} S_{\text{int}}(e) = \int d^4x A_{\mu\nu}^{(2)}(x, e) T^{\mu\nu}(x) - \sqrt{1/6} \int d^4x \varphi(x) T_{\mu}^{\mu}(x). \tag{1.33}$$

The first (pure massless gravity) contribution is independent of  $e$  by virtue of [Corollary 3.12](#).

We have thus explicitly identified the scalar field that is responsible for the DVZ discontinuity, as the limit of the escort field on the massive Hilbert space. This is formally equivalent with Zakharov’s reading who writes instead the massless coupling  $A_{\mu\nu}^F(x, e) T^{\mu\nu}(x)$  as the limit of the massive coupling plus a compensating scalar ghost [44]; but we emphasize that our reading does not involve unphysical ghost degrees of freedom.

The same decoupling of helicities in terms of  $A^{(r)}(e)$  with  $0 \leq r \leq s$  also allows to construct a massless stress–energy tensor for arbitrary helicity. It is quadratic in  $A^{(r)}(e)$ , hence also string-localized. There is no conflict with the Weinberg–Witten theorem, that assumes point-localized fields.

It was already discussed in [22] that the Weinberg–Witten theorem does not exclude non-local densities. The string-localized stress–energy tensors realize this possibility. As compared to other proposals [13,40,41,20,2] evoking an interplay of infinitely many spins, M-theory, and non-commutative geometry, the string-localized stress–energy tensors of [Proposition 4.6](#) for every pair of helicities  $h = \pm s$  are perhaps the most conservative way around the Weinberg–Witten theorem. They are even “less non-local” than the examples with unpaired helicities proposed in [22].

We are presently investigating how they may be used to (semiclassically) couple massless higher spin matter to gravity. Similar as in [Example 1.1](#) or Eq. (1.32), this requires to identify string-independent actions involving a string-dependent stress–energy tensor. A straightforward ansatz  $h_{\mu\nu} T^{\mu\nu}(e)$  with  $h$  a point-localized massive or string-localized massless tensor may require additional terms to ensure string-independence. Cf. also Sect. 1.

The construction of the string-localized massless stress–energy tensor proceeds along the following lines. For more details, see Sect. 3.

For  $s = 2$  one may start at  $m > 0$  from the Pauli–Fierz Lagrangian [11] whose field equations are the spin-2 Proca (or rather Fierz) equations [10]. The Hilbert stress–energy tensor is defined by the variation w.r.t. the metric of its generally covariant version. One may as well start from a simpler “reduced” massive stress–energy tensor whose densities differ by spatial derivatives; it therefore yields the same Poincaré generators Eq. (1.17), and is as good for the purpose. The reduced stress–energy tensor easily generalizes to arbitrary spin, in fact without the need for a free higher spin Lagrangian. Since the 2-point function determines the commutator, one can (and must) verify that the generators implement the correct infinitesimal Poincaré transformations.

(To our surprise, the generators associated with the stress–energy tensor given by Fierz [10] implement the correct translations, but wrong Lorentz transformations.)

By inserting the decomposition

$$A_{\mu\nu}^P(x) = A_{\mu\nu}(x, e) - m^{-1} (\partial_{\mu} a_{\nu}^{(1)} + \partial_{\nu} a_{\mu}^{(1)})(x, e) + m^{-2} \partial_{\mu} \partial_{\nu} a^{(0)}(x, e), \tag{1.34}$$

respectively its generalization Eq. (3.10) to arbitrary spin, into the reduced stress–energy tensor, it turns out that all contributions with negative powers of the mass multiplied by derivatives of escort fields can be combined into spatial derivative terms that do not contribute to the generators, see Appendix B. Discarding these terms, one arrives at a “regular” stress–energy tensor that admits a massless limit. It is quadratic in escort fields  $a_{\mu_1 \dots \mu_r}^{(r)}$  for all  $0 \leq r \leq s$ ; but not yet of much use because these fields are still coupled through their traces at  $m = 0$ .

By expressing the massless escort fields  $a_{\mu_1 \dots \mu_r}^{(r)}$  in terms of the decoupling massless potentials  $A_{\mu_1 \dots \mu_r}^{(r)}$ , one may again discard contributions from the regular stress–energy tensor that do not contribute to the generators at  $m = 0$ . The resulting stress–energy tensor is quadratic in the massless potentials  $A_{\mu_1 \dots \mu_r}^{(r)}$ . Because the latter mutually commute, this stress–energy tensor is a direct sum of massless stress–energy tensors for all  $0 \leq r \leq s$ . These separately yield by Eq. (1.17) the generators of the helicity  $h = \pm r$  subrepresentations of the massless limit of the massive spin  $s$  representation.

## 2. General $s$ : preliminaries on point-localized fields

### 2.1. Massive case

The massive Proca field  $A_{\mu_1 \dots \mu_s}^P$  of spin  $s$  is a completely symmetric traceless and conserved tensor field satisfying the Klein–Gordon equation:

$$\eta^{\mu_i \mu_j} A_{\mu_1 \dots \mu_s}^P = 0, \quad \partial^{\mu_j} A_{\mu_1 \dots \mu_s}^P = 0, \quad (\square + m^2) A_{\mu_1 \dots \mu_s}^P = 0, \tag{2.1}$$

with 2-point function

$$m M^{A_{\mu_1 \dots \mu_s}^P, A_{\nu_1 \dots \nu_s}^P} = (-1)^s \sum_{2n \leq s} \tilde{\beta}_n^s (\pi_{\mu\mu})^n (\pi_{\nu\nu})^n (\pi_{\mu\nu})^{s-2n}. \tag{2.2}$$

The sum extends over all inequivalent attributions of the available indices of the given form, namely either  $\pi_{\mu_i \mu_j}$ ,  $\pi_{\nu_i \nu_j}$ , or  $\pi_{\mu_i \nu_j}$ , to the schematically displayed factors. Due to the factors of  $p$  in the numerator, the short-distance dimension of  $A^P$  is  $d_{UV} = s + 1$ .

The coefficients  $\tilde{\beta}_n^s = \frac{n!}{s!(\frac{s}{2}-n)_n}$  ensure the vanishing of the traces, while the fact that  $\pi_{\mu\nu}(p)p^\nu = 0$  ensures the vanishing of the divergences. (The alternating overall sign is due to our sign convention of the metric.)

The same formula can also be derived from the  $(m, s)$  Wigner representation:

$$A_{\mu_1 \dots \mu_s}^P = \int d\mu_m(p) \left[ e^{ipx} \sum_{a=1}^{2s+1} u_{\mu_1 \dots \mu_s}^a(p) a_a^*(p) + \text{h.c.} \right] \tag{2.3}$$

where the tensors  $u_{\mu_1 \dots \mu_s}^a(p)$  intertwine the  $(m, s)$  Wigner representation of the Lorentz group with the symmetric traceless tensor representation [42], and the spin indices  $a$  are summed over. The coefficients  $\tilde{\beta}_n^s$  in Eq. (2.2) arise due to the projection operator (involved in the intertwiners  $u_{\mu_1 \dots \mu_s}^a$ ) onto the spin  $s$  representation in the  $s$ -fold tensor product of vector representations of the little group  $SO(3)$  (= traceless symmetric tensors in  $(\mathbb{C}^3)^{\otimes s}$  [18, Eq. (1.13)]).

To keep track of the combinatorics for general  $s$ , it will be advantageous to trade the indices for an “orientation vector”  $f \in \mathbb{R}^4$  and write

$$X(f) \equiv X_{\mu_1 \dots \mu_r} f^{\mu_1} \dots f^{\mu_r},$$

when  $X$  is a symmetric rank  $r$  tensor. Then the divergence  $(\partial X)_{\mu_2 \dots \mu_r} := \partial^{\mu_1} X_{\mu_1 \dots \mu_r}$  and the trace  $(\text{Tr } X)_{\mu_3 \dots \mu_r} := \eta^{\mu_1 \mu_2} X_{\mu_1 \dots \mu_r}$  are given by

$$r \cdot (\partial X)(f) = (\partial_x \cdot \partial_f X)(f), \quad r(r - 1) \cdot (\text{Tr } X)(f) = \square_f X(f).$$

In this notation, the Proca 2-point function Eq. (2.2) is written

$${}_m M^{A^P(f), A^P(f')} = (-1)^s \sum_{2n \leq s} \beta_n^s (f^t \pi f)^n (f'^t \pi f')^n (f^t \pi f')^{s-2n} \tag{2.4}$$

whose coefficients differ from  $\tilde{\beta}_n^s$  by a counting factor of equivalent terms:

$$\beta_n^s = \left[ \binom{s}{2n} (2n - 1)!! \right]^2 (s - 2n)! \cdot \tilde{\beta}_n^s = \frac{1}{4^n n!} \frac{s!}{(s - 2n)!} \frac{1}{(\frac{1}{2} - s)_n}. \tag{2.5}$$

In  $D$  dimensions (where  $\text{Tr}(\pi) = D - 1$ , little group  $\text{SO}(D - 1)$ ), the Pochhammer symbol  $(\frac{1}{2} - s)_n$ , arising from the projection onto traceless symmetric tensors in  $(\mathbb{C}^{D-3})^{\otimes s}$ , would be replaced by  $(\frac{s-D}{2} - s)_n$ .

### 2.2. Massless case

For the massless case, point-localized covariant potentials with a positive 2-point function do not exist. From the pair of Wigner representations ( $m = 0, h = \pm s$ ) one can construct point-localized covariant field strengths  $F_{[\mu_1 \nu_1] \dots [\mu_s \nu_s]}^{(s)}$  whose 2-point function are the curls of the indefinite 2-point function

$${}_0 M^{A^F_{\mu_1 \dots \mu_s}, A^F_{\nu_1 \dots \nu_s}} = (-1)^s \sum_{2n \leq s} \tilde{\gamma}_n^s (\eta_{\mu\mu})^n (\eta_{\nu\nu})^n (\eta_{\mu\nu})^{s-2n}, \tag{2.6}$$

(notation as in Eq. (2.2)) or equivalently,

$${}_0 M^{A^F(f), A^F(f')} = (-1)^s \sum_{2n \leq s} \gamma_n^s (f^t \eta f)^n (f'^t \eta f')^n (f^t \eta f')^{s-2n}. \tag{2.7}$$

The coefficients are

$$\gamma_n^s = \left[ \binom{s}{2n} (2n - 1)!! \right]^2 (s - 2n)! \cdot \tilde{\gamma}_n^s = \frac{1}{4^n n!} \frac{s!}{(s - 2n)!} \frac{1}{(1 - s)_n}. \tag{2.8}$$

In  $D$  dimensions (little group  $\text{E}(D - 2) = \text{SO}(D - 2) \times \mathbb{R}^{D-2}$  with  $\mathbb{R}^{D-2}$  represented trivially), the Pochhammer symbol  $(1 - s)_n$ , arising from the projection onto traceless symmetric tensors in  $(\mathbb{C}^{D-2})^{\otimes s}$ , would be  $(\frac{6-D}{2} - s)_n$ .

### 3. String-localized fields: general integer spin $s$

Throughout this section the spin  $s$  is fixed, and does not always appear explicitly in the notation; i.e., fields like  $a^{(r)}$  or numbers like  $\beta_{nn}^{rr'}$  will depend also on  $s$ .

Let  $A^P_{\mu_1 \dots \mu_s}$  and  $F_{[\mu_1 \nu_1] \dots [\mu_s \nu_s]}$  be the Proca potential and its field strength. Let  $e$  be a (space-like) unit vector. We introduce the symmetric string-localized potential

$$a_{\mu_1 \dots \mu_s}^{(s)}(x, e) := (I_e^s F_{[\mu_1 \nu_1] \dots [\mu_s \nu_s]})(x) e^{\nu_1} \dots e^{\nu_s} \tag{3.1}$$

defined on the Wigner Fock space over the  $(m, s)$  Wigner representation. It is another potential for  $F_{[\mu_1 \nu_1] \dots [\mu_s \nu_s]}$ , but unlike  $A^P_{\mu_1 \dots \mu_s}$ , it is neither traceless nor conserved.

We display the 2-point function  ${}_m M^{a^{(s)}(-e), a^{(s)}(e)}$ . Every factor  $\pi$  in Eq. (2.2) is hit by two of the matrices  $J(p, e')$  or  $\bar{J}(p, -e) = J(p, e)$ . We therefore define

$$E_{\mu\nu}(e_1, e_2)(p) := (J(p, e_1)\pi(p)J(p, e_2)^t)_{\mu\nu} = (J(p, e_1)\eta J(p, e_2)^t)_{\mu\nu} \tag{3.2}$$

which is precisely the distribution defined in Eq. (1.8), and abbreviate (for  $f, f' \in \mathbb{R}^4$ )

$$E_{ff} \equiv f^t E(e, e) f, \quad E_{ff'} \equiv f^t E(e, e') f', \quad E_{f'f'} \equiv f'^t E(e', e') f'.$$

Then we have from Eq. (2.2):

$$m M^{a^{(s)}(-e)(f), a^{(s)}(e')(f')} = (-1)^s \sum_n \beta_n^s (E_{ff})^n (E_{f'f'})^n (E_{ff'})^{s-2n}. \tag{3.3}$$

Because  $E(e, e')(p)$  is a homogeneous function of  $p$ , the short-distance dimension of  $a^{(s)}$  is  $d_{UV} = 1$ .

### 3.1. Escort fields

In order to establish the relation between  $A^P$  and  $a^{(s)}$ , and to control the massless limit, we introduce the escort fields for  $0 \leq r < s$

$$a_{\mu_1 \dots \mu_r}^{(r)}(x, e) := -m^{-1} \cdot \partial^\mu a_{\mu_1 \dots \mu_r \mu}^{(r+1)}(x, e). \tag{3.4}$$

**Remark 3.1.**  $a_{\mu_1 \dots \mu_s}^{(s)}$  coincides with the string-localized field denoted  $A_{\mu_1 \dots \mu_s}$  in [37,38,24].  $a^{(r)}$  ( $r < s$ ) are related to the escort fields  $\phi^{(r)}$  introduced there by derivatives of lower  $\phi^{(q)}$  ( $q < r$ ) and an overall power of the mass:

$$a_{\mu_1 \dots \mu_s}^{(r)}(x, e) = m^{s-r} \sum_{q \leq r} \partial_{\mu \dots \mu} \phi_{\mu_1 \dots \mu}^{(q)}(x, e)$$

where for each  $q \leq r$  the sum extends over all  $\binom{r}{q}$  inequivalent permutations of the indices. This can be seen from [24, Eq. (4)] by taking divergences and using the Klein–Gordon equation. Lemma 3.4 below justifies our departure from the previous definition.

The definition Eq. (3.1) involves the operations curl, contraction with  $e$  and string integration on each Lorentz index of  $A_{\mu_1 \dots \mu_s}^P$ . This means that  $a_{\mu_1 \dots \mu_s}^{(s)}$  arises from Eq. (2.3) by multiplication of the intertwiner  $u_{v_1 \dots v_s}^\alpha$  with the matrix

$$J_\mu{}^\nu(p, e) = \delta_\mu^\nu - \frac{p_\mu e^\nu}{(pe)_+}, \tag{3.5}$$

in each Lorentz index. It is obviously

$$J_\mu{}^\nu(p, e) p_\nu = 0, \quad e^\mu J_\mu{}^\nu(p, e) = 0. \tag{3.6}$$

**Corollary 3.2.** The “axial gauge” condition (cf. Remark 1.2)  $e^\mu a_{\mu\mu_2 \dots \mu_r}^{(r)}(e) = 0$  holds.

**Proof.** Evident from the second of Eq. (3.6) and the definition Eq. (3.4).  $\square$

The following property secures the string-independence of actions  $S_{\text{int}}(e)$  like Eq. (1.3) and Eq. (1.32).

**Corollary 3.3.** The string-dependence of the string-localized potential  $a_{\mu_1 \dots \mu_r}^{(s)}(e)$  is a sum of derivatives:

$$\partial_{e^\kappa} a_{\mu_1 \dots \mu_s}^{(s)}(e) = m^{-1} \sum_i \partial_{\mu_i} (I e a_{\dots \mu_{i-1} \kappa \mu_{i+1} \dots}^{(s)}(e)). \tag{3.7}$$

**Proof.** Evident by computing the derivative  $\partial_{e^\kappa} J_\mu^\nu = -\frac{p_\mu}{(pe)_+} J_\kappa^\nu$ .  $\square$

This formula together with Eq. (1.5) also explains why Lorentz transformations of axial gauge potentials at fixed  $e$  formally involve an “operator-valued gauge transformation”.

The conservation of  $A^P$  means that  $p^{\mu_i} u_{\mu_1 \dots \mu_s}^a(p) = 0$  in Eq. (2.3). Because  $ip^\mu J_\mu^\nu = ip^\nu - im^2 \frac{e^\nu}{(pe)_+}$ , it follows for  $r \leq s$

$$a_{\mu_1 \dots \mu_r}^{(r)}(x, e) = \int d\mu_m(p) \left[ e^{ipx} \prod_{k=1}^r J_{\mu_k}^{\nu_k}(p, e) \prod_{k=r+1}^s \frac{ime^{\nu_k}}{(pe)_+} \sum_a u_{\nu_1 \dots \nu_s}^a(p) a_a^*(p) + \text{h.c.} \right] \tag{3.8}$$

**Lemma 3.4.** *The fields  $a_{\mu_1 \dots \mu_r}^{(r)}(x, e)$  ( $0 \leq r \leq s$ ) are regular in the limit  $m \rightarrow 0$ .*

**Proof.** The 2-point functions of  $a_{\mu_1 \dots \mu_r}^{(r)}$  arise from Eq. (2.2) by multiplying with the matrices  $J$  and contracting with  $im \frac{e}{(pe)_+}$  according to Eq. (3.8). By the first of Eq. (3.6), every matrix  $J$  kills one singular factor  $p_\mu p_\nu / m^2$  of Eq. (2.2), and the powers of  $m$  coming with the contractions balance the remaining singularity.  $\square$

We have the (preliminary) decomposition of the point-localized field  $A_{\mu_1 \dots \mu_s}^P$  into  $a^{(s)}$  and its escort fields:

**Proposition 3.5.** *The massive point-localized potential of spin  $s$  can be written as*

$$A_{\mu_1 \dots \mu_s}^P(x) = \prod_{k=1}^s (\delta_{\mu_k}^{\nu_k} + m^{-2} \partial_{\mu_k} \partial^{\nu_k}) a_{\nu_1 \dots \nu_s}^{(s)}(x, e). \tag{3.9}$$

*It decomposes into regular string-localized escort fields with inverse mass coefficients*

$$A_{\mu_1 \dots \mu_s}^P(x) = a_{\mu_1 \dots \mu_s}^{(s)}(x, e) + \sum_{r < s} (-m^{-1})^{s-r} \partial_{\mu_1} \dots \partial_{\mu_r} a_{\mu_{r+1} \dots \mu_s}^{(r)}(x, e) \tag{3.10}$$

where for each  $r < s$  the sum extends over all  $\binom{s}{r}$  inequivalent permutations of the indices. From this it is manifest that  $A^P$  and  $a^{(s)}$  have the same field strength.

**Proof.** In momentum space, the differential operator in Eq. (3.9) is  $\pi^{\otimes s}$ . The identity follows from Eq. (3.8) (with  $r = s$ ), because  $\pi J = \pi$  and  $\pi^{\otimes s} u^a = u^a$  since  $A^P$  is conserved. The derivatives  $m^{-2} \partial_\mu \partial^\nu$  involved in Eq. (3.9) turn  $a^{(r)}$  into  $-m^{-1} \partial_\mu a^{(r-1)}$  by Eq. (3.4). This gives Eq. (3.10).  $\square$

The string-localized fields  $a^{(r)}$  are dynamically coupled among each other. We have

**Proposition 3.6.** *The regular escort fields  $a_{\mu_1 \dots \mu_r}^{(r)}$  are coupled through the field equations*

$$\partial^{\mu_1} a_{\mu_1 \dots \mu_r}^{(r)} = -m a_{\mu_2 \dots \mu_r}^{(r-1)}, \quad \eta^{\mu_1 \mu_2} a_{\mu_1 \dots \mu_r}^{(r)} = -a_{\mu_3 \dots \mu_r}^{(r-2)}. \tag{3.11}$$

By the first equation, every escort  $a^{(r)}$  still “contains” all the lower escorts  $a^{(r')}$  ( $r' < r$ ). The divergence will decouple in the massless limit from the lower escorts, while the trace doesn’t. Subtracting the traces would instead bring back the coupling through the divergences. This is the reason why the decomposition in Proposition 3.5 is only preliminary.

**Proof.** The first equation is just the definition Eq. (3.4). The second follows from

$$(J^t \eta J)^{\nu_1 \nu_2} = \eta^{\nu_1 \nu_2} - \frac{p^{\nu_1} e^{\nu_2} + e^{\nu_1} p^{\nu_2}}{(pe)_+} + m^2 \frac{e^{\nu_1} e^{\nu_2}}{(pe)_+^2}$$

together with the fact that  $A^P$  is traceless and conserved, hence  $p^\nu$  and  $\eta^{\nu_1 \nu_2}$  act trivially in Eq. (3.8).  $\square$

### 3.2. Decoupling in the massless limit

The massless results of this section are equivalent to results obtained recently by Plaschke and Yngvason [29, Sect. 4A]. While these authors consider Wigner intertwiners directly at  $m = 0$ , we exhibit smooth families of fields  $A^{(r)}|_{m \geq 0}$ .

We turn to the task of a complete decoupling at  $m = 0$ . We do this by a study of the 2-point functions. In a positive metric, decoupling the 2-point functions implies the decoupling of the field equations.

The 2-point functions of the massive escort fields  $a^{(r)}$  do not decouple. In order to compute them efficiently, we cast Eq. (3.8) into the form of a “generating functional”:

$$\sum_{r \leq s} \binom{s}{r} a^{(r)}(e)(f) = Z(f, e) := A^P(J_e^t f + m e I_e).$$

Here  $I_e$  is the string integration, understood in this formula as an operation acting on the field, and  $J_e$  acts by multiplication with  $J_\mu^\nu(p, e)$  and its complex conjugate on the creation resp. annihilation part of the field. Then

$${}_m M^{Z(f, -e), Z(f', e')} = \sum_{r, r'} \binom{s}{r} \binom{s}{r'} \cdot {}_m M^{a^{(r)}(-e)(f), a^{(r')}(e')(f')}.$$

Given the l.h.s. as a function of  $f$  and  $f'$ , the correlations between  $a^{(r)}$  and  $a^{(r')}$  can be read off by selecting the terms of the appropriate homogeneities in  $f$  and in  $f'$ .

In order to compute the l.h.s., we have to contract each factor  $\pi_{\mu\mu}$  in Eq. (2.2) twice with  $(J(p, e)^t f - i m e / (pe)_+)^{\mu}$ , each factor  $\pi_{\nu\nu}$  twice with  $(J(p, e')^t f + i m e' / (pe')_+)^{\nu}$ , and each factor  $\pi_{\mu\nu}$  with both vectors. Because of the first of Eq. (3.6) and Eq. (3.2), and because  $(m e / (pe)_+)^t \pi(m e / (pe)_+) = -1 + O(m^2)$ , all these contractions are of the form  $E + 1 + O(m)$  resp.  $E - 1 + O(m)$ , and one arrives at

$${}_m M^{Z(f, -e), Z(f', e')} = (-1)^s \sum_{2n \leq s} \beta_n^s (E_{ff} + 1)^n (E_{f'f'} + 1)^n (E_{ff'} - 1)^{s-2n} + O(m).$$

We get the massless 2-point functions

**Proposition 3.7.** *At  $m = 0$ , one has*

$${}_0 M^{a^{(r)}(-e)(f), a^{(r')}(e')(f')} = (-1)^r \sum_{r-2n=r'-2n'} \beta_{nn'}^{r,r'} (E_{ff})^n (E_{f'f'})^{n'} (E_{ff'})^{r-2n} \quad (3.12)$$

with

$$\binom{s}{r} \binom{s}{r'} \cdot \beta_{nn'}^{r,r'} = \sum_m \binom{m}{n} \binom{m}{n'} \binom{s-2m}{r-2n} \cdot \beta_m^s. \quad (3.13)$$

In particular,  ${}_0 M^{a^{(r)}(-e), a^{(r')}(e')} = 0$  if  $r - r'$  is odd.

**Proof.** Eq. (3.13) are the coefficients of the respective terms of homogeneity  $r$  in  $f$  and  $r'$  in  $f'$ .  $\square$

One could also have computed Eq. (3.12) by descending from Eq. (3.3) with Eq. (3.11) at  $m > 0$ , and then taking  $m \rightarrow 0$ .

We now set out to “diagonalize” the mixed 2-point functions Eq. (3.12) with the help of the operator  $E(e, e)_{\mu\nu}$  given in Eq. (1.29). We write  $E_{ff} \equiv f^t E(e, e) f$ .

**Proposition 3.8.** *The combinations*

$$A^{(r)}(f) = \sum_{2k \leq r} \alpha_k^r \cdot (-E_{ff})^k a^{(r-2k)}(f) \tag{3.14}$$

are traceless at  $m = 0$  if and only if  $\alpha_k^r = \alpha_r \cdot \gamma_k^r$ , with  $\gamma_k^r$  given in Eq. (2.8). Only the coefficient  $\alpha_r$  (that will be used later for normalization) may depend on  $s$ .

**Proof.** By applying  $\square_f$  to Eq. (3.14) and noticing that  $\text{Tr}(E) = 2 + O(m^2)$ ,  $\text{Tr}(a^{(r)}) = -a^{(r-2)}$ , and  $E_\nu^\mu a_{\mu\mu_2\dots\mu_r}^{(r)} = a_{\nu\mu_2\dots\mu_r}^{(r)} + O(m)$  because  $\partial a^{(r)} = O(m)$  and  $ea^{(r)} = 0$  (Eq. (3.11) and Corollary 3.2), one obtains the recursion

$$\alpha_k^r = -\frac{(r - 2k + 2)(r - 2k + 1)}{4k(r - k)} \alpha_{k-1}^r.$$

This is solved by  $\frac{\alpha_k^r}{\alpha_0^r} = \gamma_k^r$ .  $\square$

Because the definition Eq. (3.14) is upper triangular in  $r$ , the inverse formula is of the same form. We did, however, not succeed to compute its coefficients in closed form.

The operators  $E_{ff}$  and  $E_{f'f'}$  involved in the field definitions produce the factors denoted with the same symbols (cf. Eq. (3.3)) in the 2-point functions. Therefore, the correlations among  $A^{(r)}(f)|_{m=0}$  are of the same general form as Eq. (3.12) with different coefficients. Because  $A^{(r)}$  are traceless, the same must be true for their correlations. This implies their decoupling:

**Proposition 3.9.**

$${}_0M^{A^{(r)}(e)(f), A^{(r')}(e')(f')} = \delta_{rr'} N_r \cdot (-1)^r \sum_{2n \leq r} \gamma_n^r (E_{ff})^n (E_{f'f'})^n (E_{ff'})^{r-2n} \tag{3.15}$$

with the same coefficients  $\gamma_n^r = \frac{1}{4^n n!} \frac{r!}{(r-2n)!} \frac{1}{(1-r)_n}$  as in Eq. (2.8). The proper normalization  $N_r = 1$  can be achieved by adjusting  $\alpha_r = \alpha_0^r$ .

**Proof.** We make a general ansatz with coefficients  $\gamma_{nn'}^{rr'}$  with  $r - 2n = r' - 2n'$ . The vanishing of  $\square_f$  and of  $\square_{f'}$  gives conflicting recursions for  $\gamma_{nn'}^{rr'}$  unless  $r = r'$ . If  $r = r'$ , the recursion implies the displayed coefficients.  $\square$

While Eq. (3.14) are defined for  $m \geq 0$ , the decoupling is exact only at  $m = 0$ .

**Corollary 3.10.** *The massless symmetric tensor potentials  $A^{(r)}(x, e)$  are traceless (by construction) and conserved. They satisfy in addition the axial gauge condition*

$$e^\mu A_{\mu\mu_2\dots\mu_r}^{(r)}(x, e) = 0.$$



They are string-localized potentials given by the same formula Eq. (3.1) (with  $s$  replaced by  $r \leq s$ ) for the massless field strengths associated with the Wigner representations of helicity  $h = \pm r$  [42]. They coincide with the potentials given in [29, Sect. 4A].

**Proof.** When the divergence is taken, the derivative may be contracted with an index of  $E$  or with an index of  $a^{(r-2k)}$ . The former contributions are  $Ep = O(m^2)$ , the latter are  $O(m)$  by Eq. (3.11), hence the divergence vanishes at  $m = 0$ . The axial gauge is a consequence of Corollary 3.2 and the fact that  $e^\mu E(e, e)_{\mu\nu} = 0$ . The last statements are immediate because  $E_{\mu\nu}$  differs from  $\eta_{\mu\nu}$  by derivative terms that do not contribute to the field strengths; and the coefficients are the same as in Eq. (2.6).  $\square$

It remains to relate the normalization  $N_r$  in Eq. (3.15) (which should be  $= 1$  in the standard normalization Sect. 2.2) to  $\alpha_r = \alpha_0^r$  from Eq. (3.14). Because it is the coefficient of the purely mixed term  $(E_{ff'})^r$  in Eq. (3.12), it is easy to see from Eq. (3.14) and Eq. (3.15) that  $N_r \gamma_0^r = (\alpha_r)^2 \beta_{00}^{rr}$ , with  $\beta_{00}^{rr} = \binom{s}{r}^{-1} \sum_{2m \leq s-r} \frac{1}{4^m m!} \frac{(r-s)2m}{(\frac{1}{2}-s)_m}$  given by Eq. (3.13). So the proper normalization is fixed by

$$(\alpha_r)^2 = (\beta_{00}^{rr})^{-1} = \binom{s}{r} \frac{\Gamma(\frac{1}{2} + s)\Gamma(1 + r)}{\Gamma(\frac{1}{2} + \frac{r+s}{2})\Gamma(1 + \frac{r+s}{2})}. \tag{3.16}$$

**Remark 3.11.** (i) The decoupled massless fields  $A^{(r)}$  are independent of the spin  $s \geq r$  of the massive field in whose decomposition they emerge in the massless limit.

(ii) The axial gauge condition in Corollary 3.10 ensures the reduction of the degrees of freedom as compared to the massive representation of spin  $r$  (relevant little group  $SO(D - 2) = E(D - 2)/\mathbb{R}^{D-2}$  vs.  $SO(D - 1)$  for  $m > 0$  in  $D$  dimensions).

(iii) For the 2-point functions of the components  $A_{\mu_1 \dots \mu_r}^{(r)}$ , the factors  $E_{ff}$ ,  $E_{ff'}$  etc. in Eq. (3.15) have to be replaced by corresponding components of the tensors  $E(e, e)(p)$ :

$${}_0M^{A_{\mu_1 \dots \mu_r}^{(r)}(e), A_{\nu_1 \dots \nu_r}^{(r)}(e')} = (-1)^r \sum \tilde{\gamma}_n^r (E(e, e)_{\mu\mu})^n (E(e', e')_{\nu\nu})^n (E(e, e')_{\mu\nu})^{r-2n}$$

(notation as in Eq. (2.2)).

(iv) Taking the total curl, kills all factors  $p_\mu$  in all  $E$  tensors. Therefore the 2-point functions of the highest field strengths  $F_{[\mu_1 \nu_1] \dots [\mu_r \nu_r]}^{(r)}$  are the same as if they were derived from point-localized potentials  $A^{F(r)}$  with indefinite 2-point functions Eq. (2.6). These Feynman gauge potentials are neither traceless nor conserved.

**Corollary 3.12.** *The massless field strengths are independent of  $e$ , hence they are point-localized fields, and*

$$A_{\mu_1 \dots \mu_r}^{(r)}(x, e) = (I_e^r F_{[\mu_1 \nu_1] \dots [\mu_r \nu_r]}^{(r)})(x) e^{\nu_1} \dots e^{\nu_r}. \tag{3.17}$$

The formula Eq. (3.7) holds in the same way for the massless potentials  $A_{\mu_1 \dots \mu_r}^{(r)}$ .

**Proof.** The first statement is (iv) of Remark 3.11. Eq. (3.17) follows by the same argument as the one leading to Eq. (3.3). Eq. (3.7) for the massless potentials  $A_{\mu_1 \dots \mu_r}^{(r)}(x, e)$  follows by the same argument as in Corollary 3.3.  $\square$

Corollary 3.12 secures the string-independence of massless actions  $S_{\text{int}}(e)$  like Eq. (1.33).

### 3.3. “Fattening”

The 2-point function Eq. (3.15) with  $r = s$  is exact also for  $m > 0$ .<sup>6</sup>

Thus, if one takes the massless string-localized potential  $A^{(s)}|_{m=0}$  with 2-point function Eq. (3.15) (with  $r = s$ ) as the starting point, one can get the mass by simply changing the dispersion relation  $p^0 = \omega_m(\vec{p})$  and taking the arguments of the functions  $E_{\mu\nu}(p)$  on the mass-shell. The previous analysis, where we have derived this massive 2-point function from a positive theory, shows that this deformation preserves positivity. The fattened field brings along with it all lower rank fields  $A^{(r)}$  by virtue of the coupling through the divergence, and the Proca field  $A^P$  can be restored from the massive field  $A^{(s)}$ . This is possible because the deformation decreases the number of null states of the 2-point function, viewed as a quadratic form. Indeed, the massive potential is not conserved, and hence it can create more one-particle states.

**Remark 3.13.** The fattening allows to continuously “turn on the mass” in interactions with vector or tensor bosons without appealing to the Higgs mechanism and the “eating of the Goldstone boson”. See the comments in Sect. 1.

One can also get back the Proca potential  $A^P(x)$  as derivatives of the fattened potential  $A^{(s)}(x, e)$ :

**Proposition 3.14.** *The point-localized Proca potential can be restored from the string-localized massive helicity  $h = \pm s$  field  $A^{(s)}|_{m>0}$  by “applying the Proca 2-point function Eq. (2.2)”, regarded as a differential operator  $(\pi_{\mu\nu} = \eta_{\mu\nu} + m^{-2}\partial_\mu\partial_\nu)$ :*

$$A^P_{\mu_1\dots\mu_s}(x) = (-1)^s \cdot {}_m M^{A^P_{\mu_1\dots\mu_s}, A^{P\nu_1\dots\nu_s}} \cdot A^{(s)}_{\nu_1\dots\nu_s}|_m(x, e)$$

**Proof.** We multiply the 2-point function in the form Eq. (2.2) on  $A^{(s)}$  in the form Eq. (3.14). In the first step, we notice that every factor  $E_{\nu\nu}$  contained the field<sup>7</sup> annihilates the 2-point function because the latter is conserved and traceless. Thus, we may replace  $A^{(s)}$  by its leading term  $a^{(s)}$  ( $k = 0$  in Eq. (3.14),  $\alpha_0^s = \alpha_s = 1$ ). In the second step, we notice that every factor  $\pi^{\nu\nu}$  in the 2-point function annihilates  $a^{(s)}$  by virtue of Eq. (3.11). Thus we may replace the 2-point function by its leading term  $n = 0$  in Eq. (2.2), which is  $(-\pi)^{\otimes s}$ . The claim then follows from Eq. (3.9).  $\square$

**Proposition 3.15.** *Conversely, we have the formulae*

$$a^{(s)}_{\mu_1\dots\mu_s}(x, e) = (-1)^s \cdot {}_m M^{a^{(s)}_{\mu_1\dots\mu_s}(-e), a^{(s)\nu_1\dots\nu_s}(e)} \cdot A^P_{\nu_1\dots\nu_s}(x) \tag{3.18}$$

for  $m > 0$ , and (after taking the limit  $m \rightarrow 0$  of the regular field  $a^{(s)}$ )

$$A^{(s)}_{\mu_1\dots\mu_s}(x, e) = (-1)^s \cdot {}_0 M^{A^{(s)}_{\mu_1\dots\mu_s}(e), A^{(s)\nu_1\dots\nu_s}(e)} \cdot a^{(s)}_{\nu_1\dots\nu_s}(x, e) \tag{3.19}$$

for  $m = 0$ , to restore the massless helicity field  $A^{(s)}$  from the Proca field. In position space, the 2-point functions Eq. (3.3), Eq. (3.15) are understood as integro-differential operators, cf. Eq. (1.29).

<sup>6</sup> This is not true for the massive fields  $A^{(r)}$  with  $r < s$ . Due to their coupling to fields with  $r' > r$ , their 2-point functions are not just polynomials in  $E_{\mu\nu}(p)$ , cf. Eq. (3.12).

<sup>7</sup> We suppress sub-indices like  $E_{\nu_i\nu_j}$  in this and all similar arguments to follow.

**Proof.** For Eq. (3.18), we notice that every factor  $E^{\nu\nu}$  annihilates  $A^P$  (traceless and conserved), hence only  $n = 0$  in the 2-point function contributes, and the factors  $E_\mu^\nu$  act on  $A^P$  like  $\delta_\mu^\nu - \frac{p_\mu e^\nu}{(pe)_+} = J_\mu^\nu$ . This gives  $a^{(s)}$  by Eq. (3.8). For Eq. (3.19), we notice that at  $m = 0$ ,  $E_\mu^\nu$  acts on  $a^{(s)}$  like  $\delta_\mu^\nu$  by the first of Eq. (3.11) and Corollary 3.2, and  $E^{\nu\nu}$  acts like  $\eta^{\nu\nu}$ . Thus, the second of Eq. (3.11) implies the claim.  $\square$

#### 4. Stress–energy tensor

##### 4.1. The point-localized stress–energy tensor for $m > 0$

We refer to Appendix A for some comments on stress–energy tensors and Lagrangians for free fields of higher spin.

For our purposes here, it suffices to “read back” a suitable stress–energy tensor for the Proca field  $A_{\mu_1 \dots \mu_s}^P$  from a simple form of the Poincaré generators.

**Proposition 4.1.** *The generators of the Poincaré transformations of the Proca field can be written as*

$$P_\sigma = (-1)^s \int d^3\vec{x} \left[ -\frac{1}{4} A_{\mu_1 \dots \mu_s}^P \overset{\leftrightarrow}{\partial}_\sigma \overset{\leftrightarrow}{\partial}_0 A^{P\mu_1 \dots \mu_s} \right], \tag{4.1}$$

$$M_{\sigma\tau} = (-1)^s \int d^3\vec{x} \left[ -\frac{1}{4} \left( x_\sigma \cdot A_{\mu \times}^P \overset{\leftrightarrow}{\partial}_0 \overset{\leftrightarrow}{\partial}_\tau A^{P\mu \times} - (\sigma \leftrightarrow \tau) \right) - s A_{\sigma \times}^P \overset{\leftrightarrow}{\partial}_0 A_\tau^{P \times} \right], \tag{4.2}$$

where  $X_\times Y^\times$  stands for the contraction in  $s - 1$  indices  $\mu_2 \dots \mu_s$ .

Here and everywhere below, normal ordering is understood.

Before we give the proof, we state the corollary:

**Corollary 4.2.** *The generators Eq. (4.1) and Eq. (4.2) can be obtained from the “reduced stress–energy tensor”*

$$T_{\rho\sigma}^{\text{red}} := (-1)^s \left[ -\frac{1}{4} A_{\mu \times}^P \overset{\leftrightarrow}{\partial}_\rho \overset{\leftrightarrow}{\partial}_\sigma A^{P\mu \times} - \frac{s}{2} \partial^\mu \left( A_{\rho \times}^P \overset{\leftrightarrow}{\partial}_\sigma A_\mu^{P \times} + (\rho \leftrightarrow \sigma) \right) \right]. \tag{4.3}$$

See Appendix A for how  $T^{\text{red}}$  relates to more familiar stress–energy tensors.

Eq. (4.1) and the first term in Eq. (4.3) already appear in [10]. The second term in Eq. (4.3) does not contribute to the momenta, but it produces the last term in Eq. (4.2), which is necessary in order to get the correct infinitesimal boosts. This will become apparent in the proof of Proposition 4.1. The first term in Eq. (4.3) and the two parts of the derivative term are separately conserved w.r.t. both indices  $\rho$  and  $\sigma$  by virtue of Lemma B.1(i) resp. (ii).

**Proof of Corollary 4.2.** We have to do the integrals Eq. (1.17) at fixed  $x^0 = t$ . The first part of Eq. (4.3) obviously gives Eq. (4.1) and the first terms of Eq. (4.2). The two pieces of the second part do not contribute to  $P_\sigma$ , and they give rise to the last term of Eq. (4.2) by Lemma B.1(i) and (ii), respectively.  $\square$

**Proof of Proposition 4.1.** The argument for  $P_\sigma$  can essentially be found in [10], except that the commutator Eq. (4.4) has been guessed not quite correct [10, Eq. (4.2)]. We display the argument here because we shall use many variants of it below. See also footnote 7.

The 2-point function Eq. (2.2) fixes the commutation relation

$$[A_{\mu_1 \dots \mu_s}^P(x), A_{\nu_1 \dots \nu_s}^P(y)] = (-1)^s D_{\mu_1 \dots \mu_s, \nu_1 \dots \nu_s} \Delta_m(x - y) \tag{4.4}$$

where  $(-1)^s D_{\mu_1 \dots \mu_s, \nu_1 \dots \nu_s} = m M_{\mu_1 \dots \mu_s, \nu_1 \dots \nu_s}^{AP}$  is the 2-point function regarded as a differential operator ( $\pi_{\mu\nu} = \eta_{\mu\nu} + m^{-2} \partial_\mu \partial_\nu$ ) acting on the commutator function  $\Delta_m(x - y)$  of the scalar free field. The commutator of  $P_\sigma$  with  $A_{\nu_1 \dots \nu_s}^P$  is

$$[P_\sigma, A_{\nu_1 \dots \nu_s}^P(y)] = -\frac{1}{2} \int d^3 \vec{x} D_{\mu_1 \dots \mu_s, \nu_1 \dots \nu_s} \Delta_m(x - y) \overset{\leftrightarrow}{\partial}_0 \overset{\leftrightarrow}{\partial}_\sigma A^{P\mu_1 \dots \mu_s}(x).$$

The derivatives  $\partial_\mu$  appearing in pieces of the differential operator  $D$  can be partially integrated using Lemma B.1(i), with  $\Theta_{\rho\sigma}$  of the form  $\partial_\mu (D' \dots \Delta_m \overset{\leftrightarrow}{\partial}_\rho \overset{\leftrightarrow}{\partial}_\sigma A^{P\mu \dots})$ , suppressing further indices. After partial integration, the derivatives act on the field  $A^{P\mu \dots}(x)$  where they vanish. Thus, one may replace all operators of the form  $\pi_{\mu\nu}$  and  $\pi_{\mu\mu}$  in  $D$  by  $\eta_{\mu\nu}$  and  $\eta_{\mu\mu}$ . Because the latter also kill the field  $A^{P\mu \dots \mu}(x)$ , only the contribution  $n = 0$  of the 2-point function Eq. (2.2) (that specifies the operator  $D$ ) survives, and  $D$  may be replaced by the “identity operator”  $(\eta_{\mu\nu})^{\otimes s}$ . At this point, the integral can be immediately performed: because Eq. (4.1) integrated at  $x^0 = t$  is independent of  $t$ , one may choose  $x^0 = y^0$ , and use the equal-time properties of the scalar commutator function:  $\Delta_m(x)|_{x^0=0} = 0$  and  $\partial_0 \Delta_m(x)|_{x^0=0} = -i\delta(\vec{x})$ . We get the desired result  $[P_\sigma, A_{\nu_1 \dots \nu_s}^P(y)] = -i\partial_\sigma A_{\nu_1 \dots \nu_s}^P(y)$ .

The argument for the Lorentz generators is more involved. The commutator of the first terms in Eq. (4.2) with  $A_{\nu_1 \dots \nu_s}^P$  is

$$-\frac{1}{2} \int d^3 \vec{x} x_\sigma D_{\mu_1 \dots \mu_s, \nu_1 \dots \nu_s} \Delta_m(x - y) \overset{\leftrightarrow}{\partial}_0 \overset{\leftrightarrow}{\partial}_\tau A^{P\mu_1 \dots \mu_s}(x) - (\sigma \leftrightarrow \tau).$$

All terms involving  $\partial_\mu \partial_\mu$ , either from  $\pi_{\mu\mu}$  or from  $\pi_{\mu\nu} \pi_{\mu\nu}$  within  $D$ , vanish because  $\int d^3 \vec{x} \partial_\mu [D' \dots \Delta_m \overset{\leftrightarrow}{\partial}_0 \overset{\leftrightarrow}{\partial}_\sigma A_\tau^{P\mu \dots}] = 0$  (using Lemma B.1(i) twice). Thus, the only contributions are due to  $(\eta_{\mu\nu})^{\otimes s}$  and  $s$  terms  $(\eta_{\mu\nu})^{\otimes s-1} m^{-2} \partial_\mu \partial_\nu$ . The former give rise, if evaluated at  $x^0 = y^0$ , to the infinitesimal transformation of the point  $x$ :

$$-\frac{1}{2} \int d^3 \vec{x} x_\sigma \Delta_m(x - y) \overset{\leftrightarrow}{\partial}_0 \overset{\leftrightarrow}{\partial}_\tau A_{\nu_1 \dots \nu_s}^P = -i(x_\sigma \partial_\tau - x_\tau \partial_\sigma) A_{\nu_1 \dots \nu_s}^P.$$

The latter give rise, again by Lemma B.1(i), to the undesired term

$$-\frac{1}{2m^2} \sum_{i=1}^s \int d^3 \vec{x} \partial_{\nu_i} \Delta_m \overset{\leftrightarrow}{\partial}_0 \overset{\leftrightarrow}{\partial}_\sigma A_{\tau \nu_1 \dots \widehat{\nu_i} \dots \nu_s}^P - (\sigma \leftrightarrow \tau) = \frac{i}{m^2} \sum_{i=1}^s \partial_{\nu_i} F_{[\sigma\tau] \nu_1 \dots \widehat{\nu_i} \dots \nu_s}^P.$$

On the other hand, the commutator of the last term in Eq. (4.2) with  $A_{\nu_1 \dots \nu_s}^P$  is

$$-s \int d^3 \vec{x} D_{\sigma \mu_2 \dots \mu_s, \nu_1 \dots \nu_s} \Delta_m(x - y) \overset{\leftrightarrow}{\partial}_0 A_{\tau \nu_1 \dots \nu_s}^{P\mu_2 \dots \mu_s}(x) - (\sigma \leftrightarrow \tau).$$

Again, all terms involving  $\partial_\mu$  vanish by Lemma B.1(i), and terms involving  $\eta_{\mu\mu}$  vanish because  $A^P$  is traceless. Thus, only the terms  $\pi_{\sigma\nu} (\eta_{\mu\nu})^{\otimes s-1}$  survive:

$$= -\sum_{i=1}^s \int d^3 \vec{x} (\eta_{\sigma \nu_i} + m^{-2} \partial_\sigma \partial_{\nu_i}) \Delta_m(x - y) \overset{\leftrightarrow}{\partial}_0 A_{\tau \nu_1 \dots \widehat{\nu_i} \dots \nu_s}^P(x) - (\sigma \leftrightarrow \tau).$$

The contribution from  $\eta_{\sigma \nu_i}$  gives the infinitesimal transformation of the tensor indices

$$-i \sum_{i=1}^s (\eta_{\sigma v_i} \overset{\leftrightarrow}{\partial}_0 A^P_{\tau v_1 \dots \widehat{v}_i \dots v_s} - \eta_{\tau v_i} \overset{\leftrightarrow}{\partial}_0 A^P_{\sigma v_1 \dots \widehat{v}_i \dots v_s}).$$

The remaining contribution from  $m^{-2} \partial_\sigma \partial_{v_i}$  is

$$-\frac{1}{m^2} \sum_{i=1}^s \int d^3 \vec{x} \partial_\sigma \partial_{v_i} \Delta_m(x-y) \overset{\leftrightarrow}{\partial}_0 A^P_{\tau v_1 \dots \widehat{v}_i \dots v_s}(x) - (\sigma \leftrightarrow \tau)$$

and cancels with the previous undesired term thanks to the identity

$$\int d^3 \vec{x} \left[ X \overset{\leftrightarrow}{\partial}_0 \overset{\leftrightarrow}{\partial}_\sigma Y + 2 \partial_\sigma X \overset{\leftrightarrow}{\partial}_0 Y \right] = \int d^3 \vec{x} \partial_\sigma \left[ X \overset{\leftrightarrow}{\partial}_0 Y \right] = 0 \tag{4.5}$$

(once more by Lemma B.1(i), writing  $\partial_\sigma = \partial^\mu \eta_{\mu\sigma}$ ).  $\square$

#### 4.2. The string-localized stress–energy tensors for $m = 0$

We are going to separate “irrelevant contributions” from the reduced stress–energy tensor, that do not contribute to the generators. It is, however, more practical, to perform the corresponding partial integrations inside the generators Eq. (4.1), Eq. (4.2), and read back a resulting stress–energy tensor, as we have done before. In the first step, the partial integrations remove all terms that are singular in the massless limit.

We insert the preliminary decomposition Eq. (3.10) of the point-localized potential  $A^P$  in terms of derivatives of string-localized fields  $a^{(r)}$  into the Poincaré generators Eq. (4.1) and Eq. (4.2), and partially integrate all the derivatives of the decomposition. The result is

**Proposition 4.3.** *Expressed in terms of string-localized fields  $a^{(r)}$  ( $r \leq s$ ), the Poincaré generators are*

$$P_\sigma = \sum_{r=0}^s \binom{s}{r} (-1)^r \int d^3 \vec{x} \left[ -\frac{1}{4} a^{(r)}_{\mu_1 \dots \mu_r}(x, e) \overset{\leftrightarrow}{\partial}_0 \overset{\leftrightarrow}{\partial}_\sigma a^{(r)\mu_1 \dots \mu_r}(x, e') \right],$$

$$M_{\sigma\tau} = \sum_{r=0}^s \binom{s}{r} (-1)^r \int d^3 \vec{x} \left[ -\frac{1}{4} x_\sigma a^{(r)}_{\mu \times}(x, e) \overset{\leftrightarrow}{\partial}_0 \overset{\leftrightarrow}{\partial}_\tau a^{(r)\mu \times}(x, e') - \frac{r}{2} a^{(r)}_{\sigma \times}(x, e) \overset{\leftrightarrow}{\partial}_0 a^{(r)\times}_\tau(x, e') \right] - (\sigma \leftrightarrow \tau)$$

for any pair  $e, e'$ , and at all values of the mass  $m$ .

**Remark 4.4.** All quadratic expressions are understood as Wick products. As noticed in [23], under the Wick ordering the strings  $e, e'$  may be set equal. We retain them to be independent, because this enlarges the class of stress–energy tensors.

**Proof of Proposition 4.3.** We insert the expansion Eq. (3.10) of  $A^P(x)$  in terms of derivatives of  $a^{(r)}(e)$  resp.  $a^{(r')}(e')$  into Eq. (4.1).

It is routine work to partially integrate all the derivatives coming from Eq. (3.10), using Lemma B.1(i) again and again. The field equations Eq. (3.11) produce positive powers of the mass  $m$ , that cancel all inverse powers of the expansion: Partially integrating  $\partial_\mu a^{(r)}(e)$  against  $a^{(r')\mu \dots}(e')$ , one gets  $m a^{(r)}(e) \dots a^{(r'-1)\dots}(e')$  by Eq. (3.11), and vice versa. Partially integrating  $\partial_\mu a^{(r)}(e)$  against  $\partial^\mu a^{(r')\dots}(e')$ , one gets  $m^2 a^{(r)}(e) \dots a^{(r')\dots}(e')$  by the Klein–Gordon equation. In the expansion of the momenta Eq. (4.1), the number of terms with  $a$  contractions between derivatives,  $b$  contractions between  $a^{(r)}(e)$  and a derivative,  $b'$  contractions between a derivative

and  $a^{(r')}(e')$ , and  $c$  contractions between  $a^{(r)}(e)$  and  $a^{(r')}(e')$ , such that  $r = b + c$ ,  $r' = b' + c$  and  $a + b + b' + c = s$ , is  $\frac{s!}{a!b!b'!c!}$ . Each such term after partial integration becomes (schematically)  $(-1)^{b+b'}a^{(c)} \dots a^{(c)}$  times the same operator quadratic in  $a^{(c)}$ . Therefore the combinatorics is done by observing that  $\sum_{a+b+b'=s-c} (-1)^{b+b'} \frac{s!}{a!b!b'!c!} = (-1)^{s-c} \binom{s}{c}$ .

The expansion of the Lorentz generators Eq. (4.2) is likewise just a counting issue, where special care has to be taken with the tensor indices  $\sigma, \tau$  in the second contribution to  $M_{\sigma\tau}$ . When they are attached to derivatives, they cancel against the results of partial integrations according to Lemma B.1(i) in the first term.  $\square$

The formulae in Proposition 4.3 have the merit that they do not contain any singular fields, and one may read back a conserved and symmetric massive string-localized stress–energy tensor  $T_{\sigma\rho}^{\text{reg}}(e, e')$  that is regular at  $m = 0$ , in exactly the same way as was done in Corollary 4.2 from Proposition 4.1. The limit  $m \rightarrow 0$  can be taken directly by putting  $m = 0$ . But these steps are of little use, because the intermediate escort fields  $a^{(r)}$  do not decouple. We must in turn express  $a^{(r)}$  in Proposition 4.3 in terms of the decoupling string-localized fields  $A^{(r-2k)}$ . The following result holds only at  $m = 0$ , where the decoupling of 2-point functions is exact.

**Proposition 4.5.** *At  $m = 0$ , one has*

$$P_\sigma = \bigoplus_{r=0}^s P_\sigma^{(r)}, \quad M_{\sigma\tau} = \bigoplus_{r=0}^s M_{\sigma\tau}^{(r)} \tag{4.6}$$

where for any  $e, e'$

$$P_\sigma^{(r)} = (-1)^r \int d^3\vec{x} \left[ -\frac{1}{4} A_{\mu_1 \dots \mu_r}^{(r)}(x, e) \overset{\leftrightarrow}{\partial}_0 \overset{\leftrightarrow}{\partial}_\sigma A^{(r)\mu_1 \dots \mu_r}(x, e') \right], \tag{4.7}$$

$$M_{\sigma\tau}^{(r)} = (-1)^r \int d^3\vec{x} \left[ -\frac{1}{4} x_\sigma A_{\mu \times}^{(r)}(x, e) \overset{\leftrightarrow}{\partial}_0 \overset{\leftrightarrow}{\partial}_\tau A^{(r)\mu \times}(x, e') - \frac{r}{2} A_{\sigma \times}^{(r)}(x, e) \overset{\leftrightarrow}{\partial}_0 A_\tau^{(r)\times}(x, e') \right] - (\sigma \leftrightarrow \tau). \tag{4.8}$$

The notation in Eq. (4.6) asserts that the generators  $P_\sigma^{(r)}$  and  $M_{\sigma\tau}^{(r)}$  commute with  $A^{(r')}$  and consequently with  $P_\sigma^{(r')}$  and  $M_{\sigma\tau}^{(r')}$  ( $r' \neq r$ ), and hence generate the infinitesimal Poincaré transformations of  $A^{(r)}$  according to Eq. (1.5).

**Proof.** We insert the expansion Eq. (3.14) in terms of  $E_{\mu\mu}(e, e)^k a_{\mu \dots \mu}^{(r-2k)}(e)$  into  $A^{(r)}(e)$  in Eq. (4.7). We partially integrate the derivatives contained in the factors  $E(e, e)$  (cf. Eq. (1.29)). When they hit  $A^{(r)}(e')$ , they vanish because  $A^{(r)}$  are conserved at  $m = 0$ . The remaining contribution  $\eta_{\mu\mu}$  of  $E_{\mu\mu}$  is directly contracted with  $A^{(r)}(e')$ , and vanishes because  $A^{(r)}$  are traceless at  $m = 0$ . Thus, only the leading term  $A^{(r)}(e) = \alpha_r a^{(r)}(e) + \dots$  contributes. Now, we expand  $A^{(r)}(e')$  and partially integrate the derivatives contained in  $E^{\mu\mu}(e', e')$  onto  $a^{(r)}(e)$ , where they vanish because  $a^{(r)}$  are conserved at  $m = 0$ . But  $a^{(r)}$  are not traceless, and  $E(e', e')^k$  acts like  $\eta^k a^{(r)}(e) = (-1)^k a^{(r-2k)}(e)$  by Eq. (3.11). It remains to add up the coefficients

$$\sum_{2k \leq s-r} (\alpha_{r+2k})^2 \gamma_k^{r+2k} = \binom{s}{r}.$$

(We were not able to establish this identity for finite sums of rational numbers in closed form, but have verified it numerically until  $s = 100$ .)

Again, the case of the Lorentz generators requires a more involved combinatorics. Let us consider the first step: the partial integration of derivatives  $\partial_\mu a'(e)$  contained in  $E(e, e)^k a^{(r-2k)}(e)$  against  $A^{(r)}(e')$ . By [Lemma B.1\(i\)](#), the partial integrations within the first term in [Eq. \(4.8\)](#) give undesired non-vanishing contributions of the form

$$-\frac{1}{4} \cdot 2 \cdot \frac{r(r-1)}{2} \int d^3\vec{x} \left[ a'(x, e) \overset{\leftrightarrow}{\partial}_0 \overset{\leftrightarrow}{\partial}_\sigma A_\tau^{(r)\dots}(x, e') \right] - (\sigma \leftrightarrow \tau),$$

where the factor  $2 \cdot \frac{r(r-1)}{2}$  counts the assignments of the other contracted indices. On the other hand, when the index  $\sigma$  is attached to a factor  $E$  in the second term of [Eq. \(4.8\)](#), it gives the undesired term

$$-\frac{r}{2} \cdot (r-1) \int d^3\vec{x} \left[ \partial_\sigma a'(x, e) \overset{\leftrightarrow}{\partial}_0 A_\tau^{(r)\dots}(x, e') \right] - (\sigma \leftrightarrow \tau)$$

with another counting factor. These terms cancel each other by virtue of [Eq. \(4.5\)](#). In the second step: the partial integration of derivatives  $\partial^\mu a''(e')$  contained in  $E(e', e')$  within  $A^{(r)}(e')$  against  $a^{(r)}(e)$ , the cancellations occur with the same pattern. This shows the equality of the generators in [Proposition 4.5](#) and [Proposition 4.3](#).

The final statements are immediate:  $A^{(r)}$  mutually commute, because their mixed 2-point functions vanish. Hence the “ $r$ ” generators commute with the “ $r$ ” fields and generators. Then the “ $r$ ” generators act on the “ $r$ ” fields like the full generators  $P_\sigma$  and  $M_{\sigma\tau}$ , hence they implement the correct Poincaré transformations.  $\square$

One can now read back conserved and symmetric massless string-localized stress–energy tensors  $T_{\sigma\rho}^{(r)}(r)$  from [Eq. \(4.7\)](#), [Eq. \(4.8\)](#).

**Proposition 4.6.** *The generators [Eq. \(4.7\)](#) and [Eq. \(4.8\)](#) can be obtained from the string-localized massless stress–energy tensors for every  $r \geq 1$ :*

$$T_{\rho\sigma}^{(r)}(x, e, e') := (-1)^r \left[ -\frac{1}{4} A_{\mu\times}^{(r)}(x, e) \overset{\leftrightarrow}{\partial}_\rho \overset{\leftrightarrow}{\partial}_\sigma A^{(r)\mu\times}(x, e') \right. \\ \left. -\frac{r}{4} \partial^\mu \left( A_{\rho\times}^{(r)}(x, e) \overset{\leftrightarrow}{\partial}_\sigma A_\mu^{(r)\times}(x, e') \begin{matrix} +(e \leftrightarrow e') \\ +(\rho \leftrightarrow \sigma) \end{matrix} \right) \right]. \tag{4.9}$$

**Proof.** The argument is the same as with [Corollary 4.2](#).  $\square$

The stress–energy tensors  $T^{(r)}$  do not depend on the spin  $s \geq r$  of the reduced stress–energy tensor [Eq. \(4.3\)](#) from which they were extracted at  $m = 0$ . By [Eq. \(3.17\)](#), they can also be expressed in terms of the corresponding field strengths  $F^{(r)}$ , that are directly obtained from the massless helicity  $h = \pm r$  Wigner representations [\[42\]](#).

**Remark 4.7.** For the charge operator

$$Q = (-1)^s i \int_{x^0=t} d^3\vec{x} A_{\mu_1\dots\mu_s}^{P*}(x) \overset{\leftrightarrow}{\partial}_0 A^{P\mu_1\dots\mu_s}(x)$$

for complex potentials, one can proceed in complete analogy as with the momentum operators, and obtains string-localized massless conserved currents

$$J_{\rho}^{(r)}(x, e, e') = (-1)^r \frac{i}{2} \left( A_{\mu_1 \dots \mu_r}^{(r)*}(x, e) \overleftrightarrow{\partial}_{\rho} A^{(r)\mu_1 \dots \mu_r}(x, e') + (e \leftrightarrow e') \right). \quad (4.10)$$

The string-localized densities  $T_{\rho\sigma}^{(r)}$  and  $J_{\rho}^{(r)}$  may be averaged over the directions of their strings (cf. Remark 4.4) with test functions of arbitrarily small support. Hence, they can be localized in arbitrarily narrow spacelike cones.

## 5. Conclusion

We have introduced string-localized potentials for massive particles of integer spin  $s$ , that admit a smooth massless limit to potentials with individual helicities  $h = \pm r$ ,  $r \leq s$ . We have elaborated several remarkable properties of the massless limit, including an inverse prescription how to pass from the massless to the massive potentials via a manifestly positive deformation of the 2-point function.

As a byproduct, we could construct string-localized currents and stress–energy tensors for massless fields of any helicity, that evade the Weinberg–Witten theorem in a very conservative way.

Our results also allow to approximate string-localized fields in the massless infinite-spin Wigner representations [27] by the massive scalar escort fields  $A^{(0)}$  in the limit  $s \rightarrow \infty$ ,  $m^2 s(s+1) = \kappa^2 = \text{const.}$  [31].

The feature of string-localization arises just by multiplication operators in momentum space (of a special form), acting on the intertwiner functions that define covariant fields in terms of creation and annihilation operators of the  $(m, s)$  Wigner representations.

In particular, string-localization of the fields does not change the nature of the particles that they describe, nor does it relax any of the fundamental principles of relativistic quantum field theory. We emphasize that we regard fields (associated with a given particle) mainly as a device to formulate interaction Lagrangians. String-localized interactions are admissible whenever their string-dependence is a total derivative. In that case, string-localized fields have the primary benefit of a better UV behaviour than point-localized fields associated with the same particles. They therefore admit the formulation of interactions that are otherwise only possible at the expense of introducing states of negative norm and compensating ghost fields.

The renormalized perturbation theory of interactions mediated by string-localized fields is presently investigated. It bears formal analogies with BRST renormalization, but is more economic (by avoiding auxiliary unphysical degrees of freedom), and much closer to the fundamental principles of relativistic quantum field theory.

The necessity of using string-localized quantities to connect the vacuum state with scattering states in theories with short-range interactions was exhibited much earlier by Buchholz and Fredenhagen [4,5] investigated, in the framework of algebraic quantum field theory, the localization properties of particle states in charged sectors relative to the vacuum. Their conclusion was that, depending on the given model, the best possible localization is in an arbitrarily narrow spacelike cone, and that in the presence of a mass gap it cannot be worse in general.

The emerging renormalized perturbation theory using string-localized fields [37,38,23,26,17] is the practical realization of this insight.

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### Appendix A. Stress–energy tensors for higher spin fields

[12] and [16] give excellent discussions of how to properly define stress–energy tensors. We focus only on a few facts.

It is well-known from the example of the free Maxwell field, that the canonical definition

$$T_{\rho\sigma} = \sum \frac{\partial L}{\partial \partial^\rho \phi} \partial_\sigma \phi - \eta_{\rho\sigma} L[\phi],$$

where the sum extends over all independent fields, may not give rise to a symmetric stress–energy tensor. Consequently its Lorentz generators defined by Eq. (1.17) are not time-independent, even if  $L$  is Lorentz invariant. In the Maxwell case, the canonical stress–energy tensor is also not gauge invariant, and both defects can be cured “in one stroke” by adding the trivially conserved term  $\partial_\kappa (F^{\mu\kappa} A_\nu)$ . There are other prescriptions (e.g., [1,33]) to obtain symmetric stress–energy tensors in the general case.

The modern approach uses the Hilbert stress–energy tensor that is defined by varying a generally covariant version of the action [33,19,12,16] with respect to the metric, and then putting  $g_{\mu\nu} = \eta_{\mu\nu}$ :

$$T_{\rho\sigma}^{(\text{Hilbert})}(x) := 2 \frac{\delta S}{\delta g^{\rho\sigma}(x)} \Big|_{g=\eta}. \tag{A.1}$$

The Hilbert tensor is always symmetric and conserved. In both approaches, one first needs a Lagrangian whose Euler–Lagrange equations are the equation of motion.

This question has been addressed by Fierz and Pauli [11] and Fronsdal [15] for free massive spin  $s$  fields; they used auxiliary fields to ensure the vanishing of the divergence. When varying with respect to the metric, one may omit terms involving the divergence and the auxiliary fields that vanish by virtue of the equations of motion. For  $s = 2$ , this gives

$$L' = \frac{1}{4} F_{[\mu\nu]\kappa}^{\text{P}} F^{\text{P}[\mu\nu]\kappa} - \frac{m^2}{2} A_{\nu\kappa}^{\text{P}} A^{\text{P}\nu\kappa}. \tag{A.2}$$

The generally covariant action is

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{4} g^{\mu\mu'} g^{\nu\nu'} F_{[\mu\nu]\kappa}^{\text{P}} F_{[\mu'\nu']\kappa'}^{\text{P}} - \frac{m^2}{2} g^{\nu\nu'} A_{\nu\kappa}^{\text{P}} A_{\nu'\kappa'}^{\text{P}} \right) g^{\kappa\kappa'} \tag{A.3}$$

where  $F_{[\mu\nu]\kappa}^{\text{P}} := D_\mu A_{\nu\kappa}^{\text{P}} - D_\nu A_{\mu\kappa}^{\text{P}} = \partial_\mu A_{\nu\kappa}^{\text{P}} - \partial_\nu A_{\mu\kappa}^{\text{P}} - (\Gamma_{\mu\kappa}^\lambda A_{\nu\lambda}^{\text{P}} - \Gamma_{\nu\kappa}^\lambda A_{\mu\lambda}^{\text{P}})$ . The variation of  $g^{\mu\mu'}$  and  $g^{\nu\nu'}$  and the factor  $\sqrt{-g}$  in  $S$  give the stress–energy tensor

$$T_{\rho\sigma}^{(\text{Fierz})} = \eta^{\lambda\lambda'} F_{[\rho\lambda]}^{\text{P}\mu} F_{[\sigma\lambda']\mu}^{\text{P}} - m^2 A_{\rho}^{\text{P}\mu} A_{\sigma\mu}^{\text{P}} - \eta_{\rho\sigma} L' \tag{A.4}$$

This tensor was first considered by Fierz [10]. However, unlike the case of antisymmetrized indices, the Christoffel symbols for the indices  $\kappa, \kappa'$  do not drop out; and the contraction by  $g^{\kappa\kappa'}$  carries another dependence on the metric, so that we have

**Proposition A.1.** *The Hilbert stress–energy tensor is  $T_{\rho\sigma}^{(\text{Hilbert})} = T_{\rho\sigma}^{(\text{Fierz})} + \Delta T_{\rho\sigma}$  with*

$$\Delta T_{\rho\sigma} = -\frac{1}{2} \partial^\mu \left[ A_{\rho}^{\text{P}\lambda} F_{[\sigma\lambda]\mu}^{\text{P}} + A_{\sigma}^{\text{P}\lambda} F_{[\rho\lambda]\mu}^{\text{P}} + A_{\mu}^{\text{P}\lambda} (F_{[\lambda\rho]\sigma}^{\text{P}} + F_{[\lambda\sigma]\rho}^{\text{P}}) \right]. \tag{A.5}$$

Fierz [10] has shown that  $T^{(\text{Fierz})}$  produces the Hamiltonian

$$P_0 = -\frac{1}{4} \int d^3\vec{x} A^{P\mu\nu} \overset{\leftrightarrow}{\partial}_0 \overset{\leftrightarrow}{\partial}_0 A_{\mu\nu}^P,$$

and one easily verifies that the commutator is  $i[P_0, A_{\mu\nu}^P] = \partial_0 A_{\mu\nu}^P$ .<sup>8</sup> The same is true for all  $P_\sigma$ . Fierz has actually given a hierarchy of  $s$  linearly independent stress–energy tensors  $T^{(q)}$  for the free massive spin  $s$  field. They involve an increasing number  $q = 1, \dots, s$  of derivatives of the potential, and overall factors  $(-2m^2)^{-(q-1)}$ . They all produce the same generators  $P_\sigma$  that implement the correct infinitesimal translations  $i[P_\sigma, A_{\mu_1 \dots \mu_s}^P] = \partial_\sigma A_{\mu_1 \dots \mu_s}^P$ .

The Fierz stress–energy tensors also all produce the same generators  $M_{\sigma\tau}^{(\text{Fierz})}$ , but the latter do not implement the correct infinitesimal Lorentz transformations! E.g., for  $s = 2$ , one finds  $i[M_{0i}^{(\text{Fierz})}, A_{00}^P] = (x_0\partial_i - x_i\partial_0)A_{00}^P + A_{i0} - m^{-2}\partial_0 F_{[0i]0}^P$  rather than the correct  $i[M_{0i}, A_{00}^P] = (x_0\partial_i - x_i\partial_0)A_{00}^P + 2A_{i0}$ .

This defect is precisely cured by the correction  $\Delta T_{\rho\sigma}$  in the Hilbert stress–energy tensor, given in Eq. (A.5).

For spin  $s > 2$ , the situation is worse. As for  $s = 2$ , the Fierz stress–energy tensors do not generate the correct Lorentz generators. The covariant generalizations of higher spin stress–energy tensors involving auxiliary fields [15] suffer from inconsistencies, so that their variation w.r.t. the metric is problematic. Nevertheless, let us naively generalize Eq. (A.2):<sup>9</sup>

$$L' = (-1)^s \left[ \frac{1}{4} F_{[\mu\nu]\kappa_2 \dots \kappa_s}^P F^{P[\mu\nu]\kappa_2 \dots \kappa_s} - \frac{m^2}{2} A_{\nu\kappa_2 \dots \kappa_s}^P A^{P\nu\kappa_2 \dots \kappa_s} \right],$$

make it generally covariant as in Eq. (A.3), and compute its Hilbert tensor. The result is  $T^{(\text{Hilbert})} = T^{(\text{Fierz})} + \Delta T$  where  $T^{(\text{Fierz})}$  is exactly as in Eq. (A.4) (all additional, un-curved indices  $\kappa_3 \dots \kappa_s \equiv \times$  contracted) with the overall sign  $(-1)^s$  (due to our sign convention of the metric), and

$$\Delta T_{\rho\sigma} = -(-1)^s \frac{s-1}{2} \partial^\mu \left[ A_\rho^{P\lambda\times} F_{[\sigma\lambda]\mu\times}^P + A_\sigma^{P\lambda\times} F_{[\rho\lambda]\mu\times}^P + A_\mu^{P\lambda\times} (F_{[\lambda\rho]\sigma\times}^P + F_{[\lambda\sigma]\rho\times}^P) \right]$$

arising from the variation of the metric and Christoffel symbols associated with each of the  $s - 1$  contracted indices  $\kappa_2 \dots \kappa_s$  in  $L'$ .  $T^{(\text{Hilbert})}$  differs from the reduced stress–energy tensor Eq. (4.3) only by “irrelevant derivative terms”, in the sense that it gives the same Poincaré generators Eq. (4.1) and Eq. (4.2). Clearly, it does not have a massless limit either, because of the factors up to  $m^{-2s}$  in  ${}_m M^{A^P, A^P}$ .

The prescription just outlined does not look satisfactory. Indeed, our strategy of “reading back” a stress–energy tensor from the correct Poincaré generators, as we have done in [Corollary 4.2](#), is an alternative prescription that does not need a classical Lagrangian. Because the “correct generators” are determined by their commutation relations with the field, which in turn are dictated by the Wigner representation theory, this approach is intrinsically quantum theoretic.

(We are not aware of a general argument that the Hilbert tensor always, also in the presence of constraints, yields the correct generators. This issue is not explicitly mentioned in the literature, including the reviews [16,12].)

<sup>8</sup> See [Proposition 4.1](#).

<sup>9</sup> Much as Eq. (A.2), this is not a valid Lagrangian since it does not entail the constraints.

That  $T^{\text{red}}$  and  $T^{(\text{Hilbert})}$  differ only by irrelevant derivative terms, can be verified by hand (but we spare the reader this cumbersome exercise). One may first rewrite  $T^{(\text{Fierz})}$  with the help of the identities

$$\eta^{\lambda\lambda'} F_{[\rho\lambda]\times}^{\text{P}} F_{[\sigma\lambda']}^{\text{P}\times} - m^2 A_{\rho\times}^{\text{P}} A_{\sigma}^{\text{P}\times} = F_{[\rho\mu]\times}^{\text{P}} \partial_{\sigma} A^{\text{P}\mu\times} - \partial^{\mu} (F_{[\rho\mu]\times}^{\text{P}} A_{\sigma}^{\text{P}\times})$$

and

$$-\frac{1}{4} F^{\text{P}[\mu\nu]\times} F_{[\mu\nu]\times}^{\text{P}} + \frac{m^2}{2} A^{\text{P}\nu\times} A_{\nu\times}^{\text{P}} = -\frac{1}{2} \partial^{\mu} [A^{\text{P}\nu\times} F_{[\mu\nu]\times}^{\text{P}}],$$

then add  $\Delta T$ , and finally show that the difference from Eq. (4.3) does not contribute to the generators according to Lemma B.1(i) and (ii) (where  $\Theta$  are various contributions to the stress–energy tensor).

### Appendix B. A useful lemma

The following (rather trivial, but very useful) lemma deals with a covariant form of partial integration of four-derivatives in spatial (fixed-time) integrals.

**Lemma B.1.** *With a tensor  $\Theta_{\rho\sigma}$  we associate the “charges” (not necessarily independent of  $t$ )  $\Pi_{\sigma} := \int_{x^0=t} d^3\vec{x} \Theta_{0\sigma}$  and  $\Omega_{\sigma\tau} := \int_{x^0=t} d^3\vec{x} (x_{\sigma} \Theta_{0\tau} - x_{\tau} \Theta_{0\sigma})$ . We assume all fields or functions to have sufficiently rapid decay in spatial directions, so that boundary terms do not matter.*

(i) *If  $\Theta_{\rho\sigma}$  is of the form*

$$\Theta_{\rho\sigma} = \partial^{\mu} (Y_{\mu} \overset{\leftrightarrow}{\partial}_{\rho} Z_{\sigma})$$

(or a sum of terms<sup>10</sup> of the same structure), where  $Y$  and  $Z$  are solutions to the Klein–Gordon equation, then  $\partial^{\rho} \Theta_{\rho\sigma} = 0$  trivially. The charges  $\Pi_{\sigma} := \int_{x^0=t} d^3\vec{x} \Theta_{0\sigma} = 0$  vanish, and the charges  $\Omega_{\sigma\tau}$  are

$$\Omega_{\sigma\tau} = \int_{x^0=t} d^3\vec{x} (Y_{\tau} \overset{\leftrightarrow}{\partial}_0 Z_{\sigma} - Y_0 \overset{\leftrightarrow}{\partial}_{\tau} Z_{\sigma}) - (\sigma \leftrightarrow \tau).$$

(ii) *The same is true with*

$$\Theta_{\rho\sigma} = \partial^{\mu} X_{[\mu\rho]\sigma},$$

where  $[\mu\rho]$  stands for an anti-symmetric index pair, and

$$\Omega_{\sigma\tau} = \int_{x^0=t} d^3\vec{x} (X_{[\tau 0]\sigma} - X_{[\sigma 0]\tau}).$$

(iii) *In order for  $\Omega_{\sigma\tau}$  to vanish, the respective integrands have to be spatial derivatives.*

**Proof.**  $\Theta_{0\sigma} = \partial^{\mu} X_{[\mu 0]\sigma}$  in (ii) is a spatial derivative, because the term  $\mu = 0$  is absent by anti-symmetry. The claim follows by partial integration. (i) is a special case of (ii) by writing  $\Theta_{\rho\sigma} = \partial^{\mu} (Y_{\mu} \overset{\leftrightarrow}{\partial}_{\rho} Z_{\sigma} - Y_{\rho} \overset{\leftrightarrow}{\partial}_{\mu} Z_{\sigma})$ . The statement (iii) is trivial.  $\square$

<sup>10</sup> A term  $Y \overset{\leftrightarrow}{\partial}_{\rho} Z_{\mu\sigma} = Y \delta_{\mu}^{\lambda} \overset{\leftrightarrow}{\partial}_{\rho} Z_{\lambda\sigma}$  can be written as a sum over terms of this form.

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