Emergence of spontaneous symmetry breaking in dissipative lattice systems

Henrik Wilming, Michael J. Kastoryano, Albert H. Werner, and Jens Eisert

Citation: Journal of Mathematical Physics **58**, 033302 (2017); doi: 10.1063/1.4978328
View online: http://dx.doi.org/10.1063/1.4978328
View Table of Contents: http://aip.scitation.org/toc/jmp/58/3
Published by the American Institute of Physics

JOURNAL OF MATHEMATICAL PHYSICS **58**, 033302 (2017)

Emergence of spontaneous symmetry breaking in dissipative lattice systems

Henrik Wilming,[1] Michael J. Kastoryano,[2] Albert H. Werner,[2,3] and Jens Eisert[1]
[1]*Dahlem Center for Complex Quantum Systems, Freie Universität Berlin, 14195 Berlin, Germany*
[2]*NBIA, Niels Bohr Institute, University of Copenhagen, 2100 Copenhagen, Denmark*
[3]*QMATH, Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, 2100 Copenhagen, Denmark*

(Received 16 September 2016; accepted 25 February 2017; published online 17 March 2017)

A cornerstone of the theory of phase transitions is the observation that many-body systems exhibiting a spontaneous symmetry breaking in the thermodynamic limit generally show extensive fluctuations of an order parameter in large but finite systems. In this work, we introduce the dynamical analog of such a theory. Specifically, we consider local dissipative dynamics preparing an equilibrium steady-state of quantum spins on a lattice exhibiting a discrete or continuous symmetry but with extensive fluctuations in a local order parameter. We show that for all such processes, there exist asymptotically stationary symmetry-breaking states, i.e., states that become stationary in the thermodynamic limit and give a finite value to the order parameter. We give results both for discrete and continuous symmetries and explicitly show how to construct the symmetry-breaking states. Our results show in a simple way that, in large systems, local dissipative dynamics satisfying detailed balance cannot uniquely and efficiently prepare states with extensive fluctuations with respect to local operators. We discuss the implications of our results for quantum simulators and dissipative state preparation. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4978328]

I. INTRODUCTION

One of the backbones of modern physics is the theory of phase transitions, whereby a phase transition is accompanied by a change of an order parameter reflecting the spontaneous breakdown of a symmetry.[1] Although this paradigm has been enriched by the existence of topological phases of matter, there still remains a lot to be learned about these more conventional types of phase transitions.

Usually, thermal phase transitions are studied from a *kinematic* point of view: While at high temperatures the Gibbs state is unique,[2] below a critical temperature several thermal states, corresponding to the different symmetry-broken phases, might exist in the thermodynamic limit. In systems of finite volume, the thermal state at any finite temperature is always unique and order parameters associated with a symmetry of the Hamiltonian vanish due to the corresponding symmetry of the Gibbs state. Nevertheless, phase transitions can be associated with extensive fluctuations of the order parameter and can therefore already be witnessed in finite systems. More concretely, the value of order parameters in symmetry-breaking thermal states in the thermodynamic limit, which arise due to infinitesimal symmetry-breaking fields, can be lower bounded by the magnitude of fluctuations in systems of large but finite volumes without symmetry-breaking fields.[3–5]

Such kinematic results do not say anything about how the different phases of matter are *prepared* by a physical mechanism and whether they are stable against dissipation. In this work, we provide such a dynamical picture: we consider the preparation of equilibrium states with extensive fluctuations of a local order parameter in large volumes by dissipative open-system dynamics, generated by local Liouvillians. We then show that there are always symmetry-breaking sequences of asymptotically stationary states, which converge to steady states in the thermodynamic limit. Furthermore, we demonstrate, in the case of continuous symmetries, that if the Liouvillian commutes with the charge operator generating the symmetry, there exist dissipative Goldstone-modes on top of symmetry-broken steady-states.

Similar results have been shown in the case of ground-states of local Hamiltonians by Koma and Tasaki,[5] i.e., closed quantum many-body systems: Extensive fluctuations in order parameters in ground-states of local Hamiltonians lead to symmetry-breaking ground-states in the thermodynamic limit.

Our results show that natural dissipative processes cannot uniquely prepare a state with density fluctuations on reasonable time scales. In particular, if the target steady-state is a Gibbs state with a temperature below a symmetry-breaking phase transition, symmetry-breaking phases will become steady-states in the thermodynamic limit.

Recently, time scales of equilibration and decoherence on the one hand and the closing of the dissipative gap of open many-body systems out of equilibrium on the other have received a lot of attention.[6–11] Our results significantly contribute to these discussions by connecting such a critical slowing down in open many-body systems to a fundamental physical phenomenon, namely, symmetry-breaking phase transitions. We do so by explicitly constructing the corresponding symmetry-breaking asymptotically stationary states and rigorously estimating their equilibration time scales.

Apart from the interpretation of our results in terms of the theory of phases in many-body systems and dissipative phase transitions,[6,12] the findings may also have implications on the feasibility of Gibbs state preparation at low temperatures. A key aim of *quantum simulations* is to explore unknown *zero* temperature phase diagrams of local Hamiltonians that are beyond the reach of classical computers.[13,14] At best, such a quantum simulation can hope to prepare Gibbs states at low temperatures, effectively through some dissipative process, to infer the zero temperature behaviour. However, the present results constitute an obstacle against such a procedure—a fact that has thus far largely been overlooked.

A. Structure of the document

First, we introduce our basic setup in Section II, where we also briefly discuss the notion of detailed balance. Section III collects our results: We first treat the case of discrete symmetries in the simplest setting in Section III A, sketching the essential ingredients of the proof and stating our main theorem. Before going over to continuous symmetries in Section III C and the discussion of Goldstone modes (Sec. III D), we present rigorous bounds on equilibration time scales in our setup (Sec. III B). The proofs of the general statements are contained in Section IV. Finally, we summarize and discuss our findings and point to open problems in Section V.

II. SETUP

For simplicity, we consider sequences of systems defined on finite cubic lattices $\Lambda \subset \mathbb{Z}^d$ of increasing volume L^d, where we associate with every point in Λ a finite-dimensional quantum-system with Hilbert space $\mathcal{H}_x$. Our results can, however, also be transferred to other regular lattices and our findings equally well apply to *fermionic* open systems[6,9,12] as the required notions of locality carry over immediately. The total system is then described by the Hilbert space $\mathcal{H}_\Lambda = \otimes_{x\in\Lambda}\mathcal{H}_x$. In the following, we will often be concerned with the total magnetisation in the z-direction on a region $X \subseteq \Lambda$ as measured by the observable

$$S^z_X := \sum_{x\in X} S^z_{\{x\}} \tag{1}$$

as well as its (global) density $S^z_\Lambda/|\Lambda|$. If we consider a lattice system of spin-1/2 particles, we therefore have $S^z_{\{x\}} = \sigma^z_{\{x\}}/2$. More generally we refer to operators that are sums over local operators supported around individual lattice sites as *extensive quantities*.

The dissipative time-evolution in the Heisenberg picture is generated by a local Liouvillian super-operator $\mathcal{L}^\Lambda$,

$$A(t) = \mathrm{e}^{t\mathcal{L}^\Lambda}[A], \quad \mathcal{L}^\Lambda = \sum_{x\in\Lambda}\mathcal{L}^\Lambda_x, \tag{2}$$

where square brackets indicate the action of a super-operator and each $\mathcal{L}_x^\Lambda$ acts on an observable A as[15]

$$\mathcal{L}_x^\Lambda[A] = \mathrm{i}[H_x, A] + \sum_i 2\left((L_x^i)^\dagger A L_x^i - \left\{(L_x^i)^\dagger L_x^i, A\right\}\right), \tag{3}$$

with $\{L_x^i\}$ being the Lindblad operators. Throughout this work, we will assume that the terms $\mathcal{L}_x^\Lambda$ modelling the dissipative process are strictly local, i.e., all operators H_x and L_x^i are supported exclusively on a ball $B_r(x)$ of radius r centered around x (with respect to the standard metric of the lattice). However, our results also carry over to the setting of approximately local Liouvillians. We will always assume periodic boundary conditions and uniformly bounded dynamics, i.e., $\left\|\mathcal{L}_x^\Lambda[A]\right\| \leq b\,\|A\|$ for some constant $b > 0$ independent of x and Λ.

A steady-state of the dynamics is any state of the system ω whose expectation values are time-independent, i.e., satisfy

$$\omega\left(\mathcal{L}^\Lambda[A]\right) = 0, \tag{4}$$

for any observable A supported in Λ. Here, we use the notation $\omega(A) = \mathrm{Tr}(\rho_\omega A)$ if ω is represented by the density matrix ρ_ω. Steady-states play a similar role in open systems as ground-states do in closed systems. If the steady-state is unique, any initial state will eventually converge to it in the infinite-time limit and any observable A will approach $\omega(A)\mathbf{1}$.

The locality of the dynamics ensures that the time evolution of (quasi-)local observables is well defined in the thermodynamic limit via

$$A(t) := \lim_{\Lambda \nearrow \mathbb{Z}^d} \mathrm{e}^{\mathcal{L}^\Lambda t}[A], \tag{5}$$

for any observable A in the algebra of (quasi-)local observables. This can be seen using Lieb-Robinson bounds, which can also be proven for local Liouvillian dynamics.[16–20]

Since we are ultimately interested in the thermodynamic limit, we will restrict our attention to local observables, such as order parameters. We will mostly be interested in sequences of states, for which the expectation value of any fixed local observable becomes constant over time as we go to the thermodynamic limit. In other words, the time it takes to reach stationarity from such states diverges with the system size.

Definition 1 (Asymptotically stationary states). We call a sequence of states ω_Λ (one for each volume Λ) asymptotically stationary if it satisfies

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \omega_\Lambda(\mathcal{L}^\Lambda[A]) = 0 \tag{6}$$

for all local operators A.

Importantly, note that we only require *local* expectation values to be time-independent. However, in the thermodynamic limit, these are also the only ones which we can measure and meaningfully talk about.

A. Detailed balance

We are interested in lattice systems that are in equilibrium with their environment and therefore consider states ω for which the Liouvillian is in *detailed balance* (or reversible), in the standard sense of this term. Furthermore, we will focus on the purely dissipative part of the Liouvillian and therefore neglect any unitary contribution in the Liouvillian. This is because we are interested in the *preparation* of the state and not so much in their free evolution. In the quantum setting, detailed-balance for such a purely dissipative Liouvillian on a finite system is then usually formulated in the Heisenberg picture by requiring[21–27]

$$\omega_\Lambda(A\mathcal{L}^\Lambda[B]) = \omega_\Lambda(\mathcal{L}^\Lambda[A]\,B), \tag{7}$$

for any two observables A and B. Since trace preservation requires $\mathcal{L}[\mathbf{1}] = 0$ for any Liouvillian, the assumption of detailed balance already implies that ω is a steady-state.[28] Indeed, this is a most natural property, and many of the most important classes of Liouvillians satisfy detailed balance.[21,23,27] This is in particular true for dynamics describing a weak coupling to a thermal bath. In particular, for Gibbs states of local commuting Hamiltonians at arbitrary temperature, such Liouvillians can be constructed explicitly.[27] For non-commuting local Hamiltonians, such explicit constructions constitute an open problem. For the convenience of the reader, we also show in the Appendix how the above notion of detailed balance generalises the classical notion for Markov chains.

Our key result demonstrates that the symmetry-breaking states that we construct satisfy detailed balance in a weaker sense, namely, only for local observables and with an error that vanishes with the system-size. We call this the weaker condition of detailed balance *asymptotic reversibility.*

Definition 2 (Asymptotically reversible states). Let $\mathcal{L}^\Lambda$ be a sequence of Liouvillians and ω_Λ a sequence of states. We call ω_Λ asymptotically reversible if

$$\lim_{\Lambda\nearrow\mathbb{Z}^d}\left(\omega_\Lambda(\mathcal{L}^\Lambda[A]B)-\omega_\Lambda(A\mathcal{L}^\Lambda[B])\right)=0 \tag{8}$$

for any two local operators A, B.

In particular, asymptotically reversible states are automatically asymptotically stationary. Hence, we will *assume* the standard definition of detailed balance for the steady state and *prove* the asymptotic reversibility of the constructed symmetry breaking states.[29]

III. RESULTS

A. Discrete symmetries

It is well known that thermal states on large but finite lattice systems exhibit extensive fluctuations in order parameters, e.g., the magnetisation density, below the critical temperature. Associated with such fluctuations are long-range correlations and the existence of several distinct symmetry-breaking phases in the thermodynamic limit.[4] We will now consider the case of a steady-state with a $\mathbb{Z}_2$-symmetry, such as spin-flip along the z-direction. Our main result shows that finite density fluctuations of the order parameters at arbitrary large volumes in the steady-state of a reversible Liouvillian imply the existence of at least two additional asymptotically stationary states, which explicitly break the symmetry.

We will illustrate the proof of these results in a simple example: the spin-1/2 Ising model. We will assume that we have a reversible strictly local Liouvillian preparing the Ising model at zero temperature, whose state we write as $\omega=(\omega^+ + \omega^-)/2$, where ω^+ and ω^- are the states with all spins pointing up or down, respectively. We do not need to assume that ω is the unique steady-state of the dynamics. We will now first write the states $\omega^\pm$ in a different way, making use of the fact that ω has fluctuations in the magnetisation density. Then we show that $\omega^\pm$ both have to be asymptotically stationary. It will be clear from the arguments given that also at non-zero temperature below the phase-transition, there are symmetry-breaking asymptotically stationary states (of course a lattice-dimension larger than one is needed for this to happen). It is however not clear, whether these symmetry-breaking states correspond exactly to the pure thermodynamic phases described by extremal Kubo-Martin-Schwinger (KMS)-states in the thermodynamic limit.

In the following, by some abuse of notation, we identify A with its support and call $|X|$ the cardinality of the set X. Therefore $|A|$ denotes the volume of the support of A. For convenience, we also set $|\Lambda|=N$ in the following and omit Λ-subscripts on states and operators. Due to the fact that the all-up and all-down states are product-eigenstates of the total magnetisation, we have $\omega^\pm(S^zA)=\omega^\pm(AS^z)=\pm N\omega^\pm(A)/2$ and $\omega^\pm(S^zAS^z)=N^2\omega^\pm(A)/4$. Defining

$$\tilde{O}^\pm := \frac{1}{\sqrt{2}}\left(\mathbf{1}\pm\frac{S^z}{\omega((S^z)^2)^{1/2}}\right), \tag{9}$$

one finds

$$\omega^\pm(A)=\omega\left(\tilde{O}^\pm A\tilde{O}^\pm\right). \tag{10}$$

More generally, the symmetry of ω under spin-flips together with its fluctuations in S^z alone is sufficient to show that we can use Eq. (10) as the *definition* of candidate symmetry-breaking states, with non-vanishing magnetisation density: If

$$\omega((S^z)^2) \geq (\frac{1}{2}\mu N)^2 \tag{11}$$

is satisfied for some $\mu > 0$, it follows that

$$\begin{aligned} |\omega^{\pm}(S^z)| &= \left| \frac{1}{2} \left(\frac{\omega(S^z)}{2} + \frac{\omega((S^z)^3)}{\omega((S^z)^2)} \right) \pm \frac{\omega((S^z)^2)}{\omega((S^z)^2)^{1/2}} \right| \\ &\geq \frac{1}{2}\mu N, \end{aligned} \tag{12}$$

since the terms with odd-parity under spin-flips vanish.

For $\omega^{\pm}$ to be asymptotically stationary, we see from (10) that $\omega(S^z \mathcal{L}[A] S^z)$ has to grow slower than N^2 and that $\omega(S^z \mathcal{L}[A])$ has to grow slower than N as we increase the volume. We will only prove the former as the latter follows by a fully analogous argument.

First we point out that $\omega(S^z A S^z) = \omega(S^z [A, S^z]) + \omega((S^z)^2 A)$. The first term is clearly of order N, since $[A, S^z]$ is at most of order $|A|$ due to the locality of S^z and A. We can therefore neglect this term. We will now assume that the Liouvillian satisfies a certain *approximate Leibniz-rule* and that it implies the asymptotic stationarity of $\omega^{\pm}$. In the second step, we will prove this property. Hence, assume for a moment that

$$\mathcal{L}\left[(S^z)^2 A\right] = \mathcal{L}\left[(S^z)^2\right] A + (S^z)^2 \mathcal{L}[A] + O(N). \tag{13}$$

Combining this with the reversibility and stationarity of ω, we obtain

$$\begin{aligned} \omega(\mathcal{L}\left[(S^z)^2\right] A) &= \omega((S^z)^2 \mathcal{L}[A]) & (14) \\ &= \omega(\mathcal{L}\left[(S^z)^2 A\right]) - \omega(\mathcal{L}\left[(S^z)^2\right] A) + O(N) & (15) \\ &= -\omega(\mathcal{L}\left[(S^z)^2\right] A) + O(N). \end{aligned}$$

Thus $\omega((S^z)^2 \mathcal{L}[A]) = 0$ up to order N, which finishes the proof. What is left to show is Eq. (13). To do that, we define $\tilde{A}$ as the smallest region such that $\mathcal{L}[A] = \mathcal{L}_{\tilde{A}}[A]$, where $\mathcal{L}_{\tilde{A}}$ contains only those terms of $\mathcal{L}$ that are supported within $\tilde{A}$. We obtain

$$\begin{aligned} \mathcal{L}\left[(S^z)^2 A\right] &= (\mathcal{L} - \mathcal{L}_{\tilde{A}})\left[(S^z)^2\right] A + \mathcal{L}_{\tilde{A}}\left[(S^z)^2 A\right] \\ &= \mathcal{L}\left[(S^z)^2\right] A + (S^z)^2 \mathcal{L}[A] \\ &\quad + \mathcal{L}_{\tilde{A}}\left[(S^z)^2 A\right] - \mathcal{L}_{\tilde{A}}\left[(S^z)^2\right] A - (S^z)^2 \mathcal{L}_{\tilde{A}}[A], \end{aligned} \tag{16}$$

where we have used $(\mathcal{L} - \mathcal{L}_{\tilde{A}})[XA] = (\mathcal{L} - \mathcal{L}_{\tilde{A}})[X] A$ for any operator X. Writing $S^z = Q + R$, where Q is supported on the complement of $\tilde{A}$ and R is supported on $\tilde{A}$, we see that the term with Q^2 cancels out, as $\mathcal{L}_{\tilde{A}}\left[Q^2 X\right] = Q^2 \mathcal{L}_{\tilde{A}}[X]$ for arbitrary X. The operator norm of the remaining terms are either zero due to $\mathcal{L}[\mathbf{1}] = 0$ or of order N, since $\mathcal{L}_{\tilde{A}}$ is of order $|\tilde{A}|$, which only differs from $|A|$ by some constant factor due to the locality of the Liouvillian. This finishes the proof.

Note that the argument works for any local order parameter instead of S^z and does not depend on the local dimension of the lattice-model or on any specific detail of the Liouvillian. In fact it turns out that the states $\omega^{\pm}$ are not only asymptotically stationary but asymptotically reversible. We will state this result as a general theorem.

Theorem 3 (Reversibility from fluctuations). *Let $\mathcal{L}^{\Lambda}$ be a sequence of local Liouvillians that are reversible with respect to a sequence of states ω_{Λ}, fulfilling Eq. (11) with respect to some extensive quantity and having a vanishing expectation value on the extensive quantity. Then the corresponding states $\omega^{\pm}_{\Lambda}$, defined through Eq. (10), are asymptotically reversible and thus asymptotically stationary.*

We stress that the theorem holds without any requirement on how the order parameter transforms under some symmetry and applies also to non-translationally invariant order parameters. The transformation properties are only necessary to show that the states $\omega^{\pm}$ are symmetry-breaking. Furthermore the theorem also applies to Liouvillians whose interactions decay as a power-law with exponent β

provided that $\beta > 2d$. The proof of this general case is completely analogous to the one given above; however, some technicalities arise due to the approximate locality and the stronger statement about reversibility.

B. Time scales

In general, asymptotically stationary states will relax to a steady-state after a sufficiently long time if the system is finite. We can estimate the scaling of this survival time t_{eq} of the symmetry-breaking states $\omega_\Lambda^\pm$ in terms of the system size. From the fact that the states are symmetry-breaking, we can lower-bound the equilibration time by the time it takes for the order parameter to relax to the steady state value. Using Lieb-Robinson bounds, we find in the case of finite-range interactions that the equilibration time t_{eq} scales at least as

$$t_{\mathrm{eq}} \geq cL^{d/d+1}, \tag{17}$$

for some constant $c > 0$. Thus, the survival time diverges with the system-size, which implies that for large systems and reasonable time scales, the symmetry-breaking states will not relax into the steady-state.

C. Continuous symmetries

Let us now turn to continuous symmetries, where our results can be further strengthened. We now assume the existence of an extensive self-adjoint quantity C, which we call charge and generates the symmetry. Furthermore we assume the existence of extensive order parameters $O_\Lambda^{(1,2)}$, satisfying the commutation relations

$$[C_\Lambda, O_\Lambda^{(1)}] = \mathrm{i}O_\Lambda^{(2)}, \quad [C_\Lambda, O_\Lambda^{(1)}] = -\mathrm{i}O_\Lambda^{(2)}. \tag{18}$$

The simplest example to keep in mind is again given by ferromagnetism, choosing $C_{\{x\}} = S^z_{\{x\}}$ and $O^{(1)}_{\{x\}} = S^{(x)}_{\{x\}}, O^{(2)}_{\{x\}} = S^{(y)}_{\{x\}}$, but we could also deal, for example, with staggered magnetic fields. We will from now on consider steady-states ω_Λ represented by density matrices ρ_Λ commuting with the charge, i.e.,

$$[\rho_\Lambda, C_\Lambda] = 0. \tag{19}$$

This implies that the state is not symmetry-breaking: $\omega_\Lambda(O_\Lambda^{(i)}) = 0$ for $i = 1, 2$. As previously, we now assume that ω_Λ exhibits extensive fluctuations in the order parameters,

$$\omega_\Lambda\left((O_\Lambda^{(1)})^2\right) = \omega_\Lambda\left((O_\Lambda^{(2)})^2\right) \geq (\mu o|\Lambda|)^2, \tag{20}$$

for some $\mu > 0$ and all system sizes.

With a construction similar to (10) in terms of the order parameters $O_\Lambda^{(i)}$, Koma and Tasaki[5] constructed a family of states $\{\omega_\Lambda^{(M)}; M \leq |\Lambda|\}$, which under the above assumptions are asymptotically symmetry breaking in the sense that

$$\omega_\Lambda^{(M)}\left(O_\Lambda^{(2)}\right) = 0, \tag{21}$$

$$\lim_{M\to\infty}\lim_{\Lambda\nearrow\mathbb{Z}^d}\frac{1}{|\Lambda|}\omega_\Lambda^{(M)}\left(O_\Lambda^{(1)}\right) \geq \sqrt{2}\mu o. \tag{22}$$

For details of the construction, see Theorem 4 in the section containing the proofs. As in the case of discrete symmetries, we can hence explicitly construct a family of symmetry breaking states. Furthermore it is clear that we can "rotate them around" using the charge C_Λ as a generator of rotations. We thus obtain a whole $U(1)$-manifold of symmetry-breaking states in the thermodynamic limit.

Theorem 4 (Asymptotic stationarity of symmetry breaking states). *Under the assumption of* Eqs. (19) and (20)*, let $\mathcal{L}^\Lambda$ be a sequence of local Liouvillians that are in detailed balance with respect to ω_Λ. Then for any M, the states $\omega_\Lambda^{(M)}$ are asymptotically reversible and asymptotically stationary.*

Note, that we require the steady state ω_Λ and not the dynamics, i.e., $\mathcal{L}^\Lambda$, to be symmetric with respect to the charge. Symmetry of the dynamics would imply instead[30]

$$\mathcal{L}^\Lambda\,[[C_\Lambda, A]] = \left[C_\Lambda, \mathcal{L}^\Lambda\,[A]\right] \tag{23}$$

for any observable A. If $\mathcal{L}^\Lambda$ is symmetric, however, there exists at least one steady-state that is symmetric in the sense of Eq. (19). In particular if the steady-state of $\mathcal{L}^\Lambda$ is unique, the symmetry of the Liouvillian ensures that the steady-state is also symmetric and our theorem applies.

The proof of Theorem 4 uses the same strategy as the one of Theorem 3 and also generalises to Liouvillians whose interactions decay faster than any polynomial: First we prove an approximate Leibniz-rule similar to Eq. (13), which, together with reversibility, implies the result.

D. Goldstone-modes

In closed systems, Goldstone's theorem shows the existence of spin-waves of arbitrarily small energy above symmetry-broken states if the Hamiltonian locally commutes with the charge.[31] The physical intuition is that a global rotation of all spins does not cost any energy and a spin-wave with very long wavelengths has a locally almost constant magnetisation. Since the Hamiltonian is local, the energetic cost of such a spin-wave is very low and goes to zero as the wave-length goes to infinity. The analogous intuition holds also true in the case of open systems if the Liouvillian is local and symmetric in the sense of Eq. (23). We give an explicit construction of such dissipative Goldstone-modes in the section containing the proofs.

IV. PROOFS

A. General proof for discrete symmetry breaking

In this section, we prove Theorem 3 for the general case of approximately local Liouvillians. The essential ideas are the same as in the proof for compactly supported Liouvillians presented in the example of the Ising model, but we have to estimate the corrections due to the fact that the Liouvillians are not compactly supported. Again we always assume periodic boundary conditions for simplicity. Let us first properly define Liouvillians with non-compact support. Then we will precisely formulate the theorem and prove it. Informally, we say that a Liouvillian is approximately local if each term $\mathcal{L}_x^\Lambda$ may be well approximated by a compactly supported term $\tilde{\mathcal{L}}_x^\Lambda$ with support in a ball $B_l(x)$ of radius l around x. The error is quantified by a function f.

Definition 5 (f-local Liouvillian). Let $f:\mathbb{Z}^d \to \mathbb{R}$ with $f(0) = 1$ be given. A sequence of Liouvillians $\mathcal{L}^\Lambda = \sum_{x\in\Lambda}\mathcal{L}_x^\Lambda$ is f-local if there exists a sequence of compactly supported Liouvillians $\tilde{\mathcal{L}}^\Lambda = \sum_x \tilde{\mathcal{L}}_x^\Lambda$ such that

$$\left\|\mathcal{L}_x^\Lambda\,[A] - \tilde{\mathcal{L}}_x^\Lambda\,[A]\right\| \le c\,\|A\|\,f(l), \tag{24}$$

where $\tilde{\mathcal{L}}_x^\Lambda$ is supported within $B_l(x)$ and $b > 0$ is a constant.

Definition 6 (Approximately local Liouvillian). We will say that $\mathcal{L}^\Lambda$ is approximately local if it is f-local and f decays at least as fast as

$$f(l) = \frac{1}{1 + l^\beta}, \quad \beta > 2d. \tag{25}$$

Instead of considering the order parameter S^z, we will from now on consider an arbitrary order-parameter $O_\Lambda = \sum_{x\in\Lambda} O_{\{x\}}$, where O_x is compactly supported around lattice site x and $\|O_{\{x\}}\| \le o$ for all $x \in \mathbb{Z}^d$. This defines the constant $o > 0$. Given a state ω_Λ, we define the states

$$\omega_\Lambda^\pm(A) := \omega(\tilde{O}_\Lambda^\pm A \tilde{O}_\Lambda^\pm) \quad \text{with} \quad \tilde{O}_\Lambda^\pm := \frac{1}{\sqrt{2}}\left(\mathbf{1} \pm \frac{O_\Lambda}{\omega((O_\Lambda)^2)^{1/2}}\right). \tag{26}$$

The precise theorem that we want to prove now is the following.

Theorem 7 (Reversibility from fluctuations). *Let $\mathcal{L}^\Lambda$ be an approximately local Liouvillian that is in detailed balance with respect to the sequence of states ω_Λ. Assume the existence of a $\mathbb{Z}_2$-symmetry U_Λ such that*

$$\omega_\Lambda(A) = \omega_\Lambda(U_\Lambda A U_\Lambda^{-1}), \quad O_x = -U_\Lambda O_x U_\Lambda^{-1} \tag{27}$$

and that there exists a constant $0 < \mu < 1$ such that

$$\omega_\Lambda(O_\Lambda^2) \geq (\mu o |\Lambda|)^2. \tag{28}$$

Then the states $\omega_\Lambda^\pm$ are asymptotically reversible and hence asymptotically stationary.

Proof: For simplicity, we will drop the Λ labels on all operators and states; in particular we will write O instead of O_Λ and ω instead of ω_Λ. We will also set $N := |\Lambda|$. It will be useful to introduce the following quantity, which measures how far the action of a Liouvillian deviates from a derivation, i.e., fulfils the Leibniz-rule,

$$\Gamma_{\mathcal{L}}(X, Y) := \mathcal{L}[XY] - \mathcal{L}[X]\,Y - X\mathcal{L}[Y]. \tag{29}$$

We have to prove that the states $\omega^\pm$ are asymptotically reversible, i.e.,

$$\Delta(A, B) := \omega^\pm(A\mathcal{L}[B]) - \omega^\pm(\mathcal{L}[A]\,B) \to 0 \tag{30}$$

as the system size increases. To do that, we will show separately that

$$\lim_{\Lambda\nearrow\mathbb{Z}^d} \frac{\omega\left(O^\dagger O(A\mathcal{L}[B] - \mathcal{L}[A]\,B)\right)}{|\Lambda|^2} = 0, \ \lim_{\Lambda\nearrow\mathbb{Z}^d} \frac{\omega\left(O(A\mathcal{L}[B] - \mathcal{L}[A]\,B)\right)}{|\Lambda|} = 0.$$

Let us first show that, due to reversibility, it suffices to show that for any local operators A, B we have

$$\lim_{\Lambda\nearrow\mathbb{Z}^d} \frac{\omega\left(\Gamma_{\mathcal{L}}(O^\dagger O, A)B\right)}{|\Lambda|^2} = 0, \quad \lim_{\Lambda\nearrow\mathbb{Z}^d} \frac{\omega\left(\Gamma_{\mathcal{L}}(O, A)B\right)}{|\Lambda|} = 0. \tag{31}$$

Indeed, suppose the two properties are true. Then we can use reversibility to write

$$\begin{aligned}\frac{\omega\left(O^\dagger O(A\mathcal{L}[B] - \mathcal{L}[A]\,B)\right)}{|\Lambda|^2} &= \frac{\omega\left((\mathcal{L}\left[O^\dagger OA\right] - \mathcal{L}[A])B\right)}{|\Lambda|^2}\\ &= \frac{\omega(\Gamma_{\mathcal{L}}(O^\dagger O, A)B)}{|\Lambda|^2} + \frac{\omega(\mathcal{L}\left[O^\dagger O\right]AB)}{|\Lambda|^2}.\end{aligned} \tag{32}$$

By our assumption (31), the first term on the right hand side vanishes in the thermodynamic limit and we obtain

$$\lim_{\Lambda\nearrow\mathbb{Z}^d} \frac{\omega\left(O^\dagger O(A\mathcal{L}[B] - \mathcal{L}[A]\,B)\right)}{|\Lambda|^2} = \lim_{\Lambda\nearrow\mathbb{Z}^d} \frac{\omega(\mathcal{L}\left[O^\dagger O\right]AB)}{|\Lambda|^2}. \tag{33}$$

We will now use two different ways to evaluate this equation. On the one hand, we can use reversibility to obtain

$$\lim_{\Lambda\nearrow\mathbb{Z}^d} \frac{\omega\left(O^\dagger O(A\mathcal{L}[B] - \mathcal{L}[A]\,B)\right)}{|\Lambda|^2} = \lim_{\Lambda\nearrow\mathbb{Z}^d} \frac{\omega(O^\dagger O\mathcal{L}[AB])}{|\Lambda|^2}. \tag{34}$$

On the other hand, we can write

$$\begin{aligned}\omega(\mathcal{L}\left[O^\dagger O\right]AB) &= -\omega(\Gamma_{\mathcal{L}}(O^\dagger O, AB)) + \omega(\mathcal{L}(O^\dagger OAB)) - \omega(O^\dagger O\mathcal{L}[AB])\\ &= -\omega(\Gamma_{\mathcal{L}}(O^\dagger O, AB)) - \omega(O^\dagger O\mathcal{L}[AB]).\end{aligned} \tag{35}$$

But since AB is also a local operator, we obtain from assumption (31) that

$$\begin{aligned}\lim_{\Lambda\nearrow\mathbb{Z}^d} \frac{\omega\left(O^\dagger O(A\mathcal{L}[B] - \mathcal{L}[A]\,B)\right)}{|\Lambda|^2} &= -\lim_{\Lambda\nearrow\mathbb{Z}^d} \frac{\omega(\Gamma_{\mathcal{L}}(O^\dagger O, AB))}{|\Lambda|^2} - \lim_{\Lambda\nearrow\mathbb{Z}^d} \frac{\omega(O^\dagger O\mathcal{L}[AB])}{|\Lambda|^2}\\ &= -\lim_{\Lambda\nearrow\mathbb{Z}^d} \frac{\omega(O^\dagger O\mathcal{L}[AB])}{|\Lambda|^2}.\end{aligned}$$

In other words, we have

$$-\lim_{\Lambda\nearrow\mathbb{Z}^d}\frac{\omega(O^\dagger O\mathcal{L}[AB])}{|\Lambda|^2}=\lim_{\Lambda\nearrow\mathbb{Z}^d}\frac{\omega\left(O^\dagger O(A\mathcal{L}[B]-\mathcal{L}[A]B)\right)}{|\Lambda|^2}=\lim_{\Lambda\nearrow\mathbb{Z}^d}\frac{\omega(O^\dagger O\mathcal{L}[AB])}{|\Lambda|^2},$$

which just means

$$\lim_{\Lambda\nearrow\mathbb{Z}^d}\frac{\omega\left(O^\dagger O(A\mathcal{L}[B]-\mathcal{L}[A]B)\right)}{|\Lambda|^2}=0. \tag{36}$$

Essentially the same argument also works to show

$$\lim_{\Lambda\nearrow\mathbb{Z}^d}\frac{\omega\left(O(A\mathcal{L}[B]-\mathcal{L}[A]B)\right)}{|\Lambda|}=0 \tag{37}$$

and therefore $\lim_{\Lambda\nearrow\mathbb{Z}^d}\Delta(A,B)=0$.

What is left is to prove the properties given in Eq. (31). To do that, first we approximate each term $\mathcal{L}_x$ in the Liouvillian by a truncated Liouvillian $\tilde{\mathcal{L}}_x$ that is supported on a ball of radius L^α around x, where $0<\alpha<1$ is to be chosen later. By assumption, for each term this introduces an error given by

$$\left\|\mathcal{L}_x[X]-\tilde{\mathcal{L}}_x[X]\right\|\le\|X\|\,cf(L^\alpha)\le\|X\|\,c\frac{1}{1+L^{\alpha\beta}}. \tag{38}$$

We will collect the error terms in a Liouvillian $\mathcal{R}$, so that $\mathcal{L}=\tilde{\mathcal{L}}+\mathcal{R}$. For any local operator, we will denote by $\tilde{\mathcal{L}}_{\tilde{A}}$ the Liouvillian containing all terms of $\tilde{\mathcal{L}}$ whose support has overlap with A. Denote the support of this Liouvillian by $\tilde{A}$. We can then make use of the following useful lemma.

Lemma 8 (Approximate derivation). For any operator X, any local operator A, and any strictly local Liouvillian $\tilde{\mathcal{L}}$, we have

$$\Gamma_{\tilde{\mathcal{L}}}(X,A)=\Gamma_{\tilde{\mathcal{L}}_{\tilde{A}}}(X,A). \tag{39}$$

Proof. This follows immediately from $(\tilde{\mathcal{L}}-\tilde{\mathcal{L}}_{\tilde{A}})[XA]=(\tilde{\mathcal{L}}-\tilde{\mathcal{L}}_{\tilde{A}})[X]A$. □

Since Γ is linear in the Liouvillian, we can write

$$\Gamma_{\mathcal{L}}(O^\dagger O,A)=\Gamma_{\mathcal{R}}(O^\dagger O,A)+\Gamma_{\tilde{\mathcal{L}}_{\tilde{A}}}(O^\dagger O,A). \tag{40}$$

By assumption, $\omega(O^\dagger O)$ is of the order $|\Lambda|^2$ and therefore we are done once we can show

$$\frac{||\Gamma_{\mathcal{R}}(O^\dagger O,A)B||}{L^{2d}}\to0,\qquad\frac{||\Gamma_{\tilde{\mathcal{L}}_{\tilde{A}}}(O^\dagger O,A)B||}{L^{2d}}\to0 \tag{41}$$

in the limit $L\to\infty$. For the first term, using sub-multiplicativity of the norm and the triangle-inequality, we get

$$\begin{aligned}\frac{\left\|\Gamma_{\mathcal{R}}(O^\dagger O,A)B\right\|}{L^{2d}}&=\frac{\left\|\tilde{\mathcal{L}}_{\mathcal{R}}\left[O^\dagger OA\right]B-\tilde{\mathcal{L}}_{\mathcal{R}}\left[O^\dagger O\right]AB-O^\dagger O\tilde{\mathcal{L}}_{\mathcal{R}}[A]B\right\|}{L^{2d}}\\&\le\frac{3|\Lambda|\left\|O^\dagger O\right\|\,||A||\,||B||}{L^{2d}}cf(L^\alpha)\\&\le\frac{3o^2|\Lambda|^3\,||A||\,||B||}{L^{2d}}cf(L^\alpha),\end{aligned} \tag{42}$$

making use of $O=\sum_{x\in\Lambda}O_x$. Therefore,

$$\frac{\left\|\Gamma_{\mathcal{R}}(O^\dagger O,A)B\right\|}{L^{2d}}\le3o^2\,||A||\,||B||\,c\frac{L^d}{1+L^{\alpha\beta}}. \tag{43}$$

Thus, we see that the term vanishes in the thermodynamic limit as long as $\beta>d/\alpha$. For the second term, we first decompose O as $O=Q+R$, where Q is supported on the complement of $\tilde{A}$ and R is supported on $\tilde{A}$. Then we have $\Gamma_{\tilde{\mathcal{L}}_{\tilde{A}}}(Q,X)=Q\Gamma_{\tilde{\mathcal{L}}_{\tilde{A}}}(\mathbf{1},X)=0$, since $\Gamma_{\mathcal{L}}(\mathbf{1},X)=0$ for any Liouvillian $\mathcal{L}$ and operator X. This implies

$$\Gamma_{\tilde{\mathcal{L}}_{\tilde{A}}}(O^\dagger O,A)B=2Q\Gamma_{\tilde{\mathcal{L}}_{\tilde{A}}}(R,A)B+\Gamma_{\tilde{\mathcal{L}}_{\tilde{A}}}(R^2,A)B. \tag{44}$$

Therefore, a norm-estimate gives

$$\frac{\left\|\Gamma_{\tilde{\mathcal{L}}_{\tilde{A}}}(O^\dagger O, A)B\right\|}{L^{2d}} \leq Ko^2 \, \|A\| \, \|B\| \, \frac{|\tilde{A}|^2}{L^d} = Ko^2 |A| L^{(2\alpha-1)d}, \tag{45}$$

where K is some positive constant. The term thus converges to zero for $\alpha < 1/2$. By essentially the same arguments, we can bound the quantities $\Gamma_{\mathcal{R}}(O, A)$ and $\Gamma_{\tilde{\mathcal{L}}_{\tilde{A}}}(O, A)$, which yield the same constraints on α and β. Concluding, we see the theorem holds true for any $\beta > 2d$.

1. Survival time scale

In this section, we prove the lower bound on the survival time of the symmetry-breaking states. For simplicity, we will consider finite-range Liouvillians of range r with the steady-state ω. It should be clear, however, that the same argument can also be applied to approximately local Liouvillians with exponentially decaying tails. The proof will combine our techniques for the proof of asymptotic stationarity with Lieb-Robinson bounds. From the proof of asymptotic stationarity in the main text, it is clear that

$$\left|\omega_\Lambda^\pm(\mathcal{L}^\Lambda \, [A])\right| \leq k_1 \frac{\|A\| \, |\tilde{A}|}{|\Lambda|}, \tag{46}$$

for some constant $k_1 > 0$ and any local operator A. Dissipative Lieb-Robinson bounds[20] tell us that we can approximate time-evolved local observables by observables which are supported in the finite *Lieb-Robinson cone*. Let A be a local observable, then we denote the time-evolved observable on the volume Λ by

$$\exp(t\mathcal{L}^\Lambda) \, [A] =: A^\Lambda(t). \tag{47}$$

Lieb-Robinson bounds are valid for local dissipative systems in a very similar way as they hold for local Hamiltonian systems.[20] They give rise to a Lieb-Robinson velocity $v > 0$ that depends only on the dimension d of the lattice (here chosen to be a cubic lattice) as well as the range r and the strength of the Liouvillian. They can be used to show that $A^\Lambda(t)$ can be approximated by an observable $A^\vee(t)$ that is supported within a set that only contains lattice sites at most $\tilde{v}t \leq L$ away from A, as long as $\tilde{v} > v$, up to an error of approximation that is exponentially small in $\tilde{v}$. More specifically,

$$\left\|A^\Lambda(t) - A^\vee(t)\right\| \leq k_2 \, \|A\| \, (\tilde{v}t)^{d-1} \exp(-(\tilde{v} - v)t), \tag{48}$$

again for a constant $k_2 > 0$ depending on d, r, and the norm of the Liouvillian. Combining this with the previous estimate, we get

$$\left|\omega_\Lambda^\pm(\mathcal{L}\left[A^\Lambda(t) - A^\vee(t)\right])\right| \leq k_1 k_2 \, \|A\| \, (\tilde{v}t)^{d-1} \exp(-(\tilde{v} - v)t). \tag{49}$$

Notice that the bound is independent of the system size and the right hand side can be made arbitrarily small, uniformly in t, by suitably increasing $\tilde{v}$. The dependence on the dimension d in this bound is made more explicit in Refs. 9 and 20.

With these ingredients, we now bound the minimal time $t_{\text{eq}} > 0$ that it takes to change the expectation value of an *on-site* observable A, such as the order-parameter, by a fixed value ΔA. In order to arrive at a bound for this minimal time, we write

$$\begin{aligned}
\Delta A < \left|\omega_\Lambda^\pm(A(t_{\text{eq}}) - A(0))\right| &\leq \int_0^{t_{\text{eq}}} \left|\omega_\Lambda^\pm\left(\frac{\mathrm{d}A(s)}{\mathrm{d}s}\right)\right| \mathrm{d}s = \int_0^{t_{\text{eq}}} \left|\omega_\Lambda^\pm\left(\mathcal{L}^\Lambda\left[A^\Lambda(s)\right]\right)\right| \mathrm{d}s \\
&\leq \int_0^{t_{\text{eq}}} \left|\omega_\Lambda^\pm\left(\mathcal{L}^\Lambda\left[A^\vee(s)\right]\right)\right| \mathrm{d}s \\
&\quad + \int_0^{t_{\text{eq}}} \left|\omega_\Lambda^\pm\left(\mathcal{L}^\Lambda\left[A^\Lambda(s) - A^\vee(s)\right]\right)\right| \mathrm{d}s \\
&\leq k_1 \int_0^{t_{\text{eq}}} \frac{\|A\| \, ((2l+1) + 2\tilde{v}s))^d}{|\Lambda|} \mathrm{d}s \\
&\quad + k_1 k_2 \, \|A\| \int_0^{t_{\text{eq}}} (\tilde{v}s)^{d-1} \exp(-(\tilde{v} - v)s) \mathrm{d}s \\
&\leq \|A\| \left(k_1' \frac{(\tilde{v}t_{\text{eq}})^{d+1}}{|\Lambda|} + k_1 k_2 \delta(\tilde{v}, v, d)\right),
\end{aligned} \tag{50}$$

for a suitable constant $k_1' > 0$ independent of the system size. Here,

$$\delta(\tilde{v}, v, d) := \int_0^\infty (\tilde{v}s)^{d-1} \exp(-(\tilde{v} - v)s)\mathrm{d}s = \frac{(d-1)!}{(1 - v/\tilde{v})^d} \frac{1}{\tilde{v}} > 0 \tag{51}$$

converges to zero with increasing $\tilde{v}$, and otherwise is dependent on the dimension d and the Lieb-Robinson velocity $v > 0$ but again independent of the system size. For any dimension d and any given local Liouvillian with Lieb-Robinson velocity $v > 0$, one can always choose a $\tilde{v} > 0$ such that

$$\delta(\tilde{v}, v, d) < \frac{\Delta A}{\|A\|} \frac{1}{k_1 k_2}. \tag{52}$$

Using that $|\Lambda| = L^d$, it then follows that

$$t_{\mathrm{eq}} > \frac{1}{\tilde{v}} \left(\frac{\Delta A / \|A\| - k_1 k_2 \delta(\tilde{v}, v, d)}{k_1'} \right)^{1/(d+1)} L^{d/d+1} > cL^{d/d+1} \tag{53}$$

for a suitable $c > 0$, which finishes the proof. The restriction to an on-site operator A was made for reasons of the simplicity of the argument only, and an analogous analysis holds for any strictly local operator A as well.

B. Continuous symmetry breaking

In this section, we consider the case of continuous symmetry breaking and prove a theorem which yields as corollary Theorem 4 of the main text. Compared to the case of discrete symmetry breaking, we will have to assume slightly stronger locality properties for the Liouvillian.

Definition 9 (Short-ranged Liouvillian). An f-local Liouvillian is short-ranged if f decays at least as fast as $\exp(-l^\alpha/\xi)$ *for some strictly positive constants $\alpha > 0$ and $\xi > 0$.*

As in the case of discrete symmetry breaking, we will consider explicit families of states which are symmetry-breaking in the thermodynamic limit. These families have been introduced by Koma and Tasaki. To simplify their notation, let us first introduce a family of functionals on local observables. Let m, m' be integers such that $|m|, |m'| \leq M$. Using the notation from the main text, we define the functionals

$$\chi_\Lambda^{(m,m')}(A) := \frac{\omega_\Lambda \left((O_\Lambda^-)^{m'} A (O_\Lambda^+)^m \right)}{Z(m)Z(m')}, \tag{54}$$

with $Z(m) = \omega_\Lambda \left((O_\Lambda^-)^m (O_\Lambda^+)^m \right)^{1/2}$. Here we use the shorthand $(O_\Lambda^+)^m = (O_\Lambda^-)^{-m}$ if $m < 0$ and note that $\chi_\Lambda^{(m,m')}(\mathbf{1}) = \delta_{m,m'}$ for states whose representative density matrix ρ_Λ commutes with the charge.

Theorem 10 (Symmetry breaking states[5]). *For any* $M < |\Lambda|$, we *define the states*

$$\omega_\Lambda^{(M)}(A) := \frac{1}{2M+1} \sum_{m=-M}^{M} \sum_{m'=-M}^{M} \chi_\Lambda^{(m,m')}(A). \tag{55}$$

Assume that ω_Λ are represented by density matrices commuting with the charge: $[\rho_\Lambda, C_\Lambda] = 0$. If the condition

$$\omega_\Lambda \left((O_\Lambda^{(1)})^2 \right) = \omega_\Lambda \left((O_\Lambda^{(2)})^2 \right) \geq (\mu o |\Lambda|)^2 \tag{56}$$

is fulfilled, the states $\omega_\Lambda^{(M)}$ are asymptotically symmetry breaking in the sense that

$$\omega_\Lambda^{(M)} \left(O_\Lambda^{(2)} \right) = 0, \tag{57}$$

$$\lim_{M\to\infty} \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \omega_\Lambda^{(M)} \left(O_\Lambda^{(1)} \right) \geq \sqrt{2} \mu o. \tag{58}$$

In the following, we will drop again Λ from all the operators and again set $N = |\Lambda|$ for the simplicity of notation. To state our main result about continuous symmetry breaking, we define the quantities

$$\Delta^{(m,m')}(A, B) := \chi^{(m,m')} (B \mathcal{L}[A]) - \chi^{(m,m')} (\mathcal{L}[B] A), \tag{59}$$

which measure how far the functionals $\chi^{(m,m')}$ are reversible with respect to $\mathcal{L}$.

Theorem 11 (Continuous symmetry breaking). *Suppose $\mathcal{L}$ is a short-ranged Liouvillian that satisfies detailed balance with respect to ω. Furthermore suppose that ω fulfils the assumptions (17) and (18) of the main text. Then*

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} |\Delta^{(m,m')}(A,B)| = 0, \tag{60}$$

for any two local operators A, B.

Corollary 12 (Convergence to a reversible steady state). Any state obtained from linear combinations of $\chi^{(m,m')}$ is asymptotically reversible. In particular, the states $\omega^{(M)}$ in the main text are asymptotically reversible and hence asymptotically stationary.

We will split the proof into several lemmas. The first lemma was proven by Koma and Tasaki and will turn out to be essential. The second lemma makes use of it and lets us rewrite the problem in a way which will enable us to make use of detailed balance.

Lemma 13 (Koma, Tasaki[5]). Let (17) and (18) of the main text be fulfilled for a state ω represented by ρ. Let A be some finite region and decompose O^+ as $O^+ = Q_A + R_A$, where Q_A is supported on the complement of A and R_A is supported on A. Then we have the inequalities

$$\frac{\mathrm{Tr}(Q_A^{m-k}\rho(Q_A^*)^{m-k})}{\mathrm{Tr}(Q_A^m\rho(Q_A^*)^m)} \leq (\mu o N)^{-2k} \tag{61}$$

and

$$r_A^{(M)} = \left|\frac{\mathrm{Tr}\left((O^+)^M\rho(O^-)^M\right)}{\mathrm{Tr}(Q_A^M\rho(Q_A^*)^M)}\right| \geq 2 - \exp(\frac{2|A|M}{\mu N}) \geq 2 - \mathrm{e}^{\mu/8} \tag{62}$$

for $N \geq \frac{16|A|^2}{\mu^2}$ and $|\frac{M}{N}| \leq \frac{\mu^2}{16|A|}$.

Proof. We reproduce the proof at the end of the section for the reader's convenience. □

Lemma 14 (Local observables). Let A be any local observable. Then

$$\left|\mathrm{Tr}\left(\chi^{(m,m')}A\right)\right| \leq O\left(\frac{M|A|\,\|A\|}{N}\right) + \left|\frac{\mathrm{Tr}\left(\rho(O^-)^{m'}(O^+)^m A\right)}{\mathrm{Tr}((O^+)^m\rho(O^-)^m)^{1/2}\mathrm{Tr}((O^+)^{m'}\rho(O^-)^{m'})^{1/2}}\right|.$$

Proof. First we split the expectation values into

$$\mathrm{Tr}\left(\rho(O^-)^{m'}A(O^+)^m\right) = \mathrm{Tr}\left(\rho(O^-)^{m'}[A,(O^+)^m]\right) + \mathrm{Tr}\left(\rho(O^-)^{m'}(O^+)^m A\right). \tag{63}$$

We have to show that the first term divided by the denominator is of the corresponding order. To do that let us split up O^+ as $O^+ = Q_A + R_A$, where Q_A is supported on the complement of A and R_A is supported on A. This implies that $[Q_A,A] = 0$ and $[Q_A,R_A] = 0$. Using a binomial expansion, we obtain

$$\begin{aligned} \text{1st term} &= \sum_{k=0}^{m'}\sum_{l=0}^{m}\binom{m'}{k}\binom{m}{l}\mathrm{Tr}\left(\rho(Q_A^*)^{m'-k}(R_A^*)^k[A,Q_A^{m-l}R_A^l]\right) \\ &= \sum_{k=0}^{m'}\sum_{l=1}^{m}\binom{m'}{k}\binom{m}{l}\mathrm{Tr}\left(\rho(Q_A^*)^{m'-k}(R_A^*)^k[A,R_A^l]Q_A^{m-l}\right). \end{aligned} \tag{64}$$

We now use the Schwartz inequality

$$\begin{aligned} |\mathrm{Tr}(\rho A^*BC)| &\leq [\mathrm{Tr}(\rho A^*A)\mathrm{Tr}(\rho C^*B^*BC)]^{1/2} \\ &\leq \|B\|\,[\mathrm{Tr}(\rho A^*A)\mathrm{Tr}(\rho C^*C)]^{1/2}, \end{aligned} \tag{65}$$

together with inequality (61) to obtain

$$\begin{aligned}\left|\frac{\text{1st term}}{\mathrm{Tr}(Q_A^m\rho(Q_A^*)^m)^{1/2}\mathrm{Tr}(Q_A^{m'}\rho(Q_A^*)^{m'})^{1/2}}\right| &\leq 2\,\|A\|\sum_{k=0}^{m'}\sum_{l=1}^{m}\binom{m'}{k}\binom{m}{l}\left(\frac{|A|}{\mu N}\right)^{k+l}\\ &\leq 2\,\|A\|\exp\left(\frac{|A|m'}{\mu N}\right)\left(\exp\left(\frac{|A|m}{\mu N}\right)-1\right)\\ &\leq 2\,\|A\|\exp\left(\frac{|A|M}{\mu N}\right)\left(\exp\left(\frac{|A|M}{\mu N}\right)-1\right)\\ &\leq 2\,\|A\|\,\frac{16|A|}{\mu^2}\mathrm{e}^{\mu/16}(\mathrm{e}^{\mu/16}-1)\frac{M}{N},\end{aligned} \tag{66}$$

where we assumed $N \geq \frac{16|A|^2}{\mu^2}$ and $|\frac{M}{N}| \leq \frac{\mu^2}{16|A|}$. Multiplying with the ratio (62), we obtain the desired bound. □

Let us now turn to the proof of the theorem. We note that the proof does not depend on the symmetry of the Liouvillian, just on the symmetry and long-range order of ρ, and the locality and reversibility of the dynamics. Without the loss of generality, we can assume that $m, m' \geq 0$ since otherwise we merely have to exchange O^+ and O^- and some operators with their adjoints in the proof.

By the above lemma, we have

$$\begin{aligned}\Delta^{(m,m')}(A,B) &\simeq \frac{\omega\left((O^-)^{m'}(O^+)^m(\mathcal{L}[A]B - A\mathcal{L}[B])\right)}{\omega\left((O^+)^m(O^-)^m\right)^{1/2}\omega\left((O^+)^{m'}(O^-)^{m'}\right)^{1/2}}\\ &=: \omega(\Omega^{(m,m')}(\mathcal{L}[A]B - A\mathcal{L}[B])),\end{aligned} \tag{67}$$

where $\simeq$ denotes equality up to terms that vanish in the thermodynamic limit and we have introduced the operator

$$\Omega^{(m,m')} := \frac{(O^-)^{m'}(O^+)^m}{\omega\left((O^+)^m(O^-)^m\right)^{1/2}\omega\left((O^+)^{m'}(O^-)^{m'}\right)^{1/2}}. \tag{68}$$

We will now first approximate $\mathcal{L}$ by a strictly local Liouvillian $\tilde{\mathcal{L}}$, by approximating each local term $\mathcal{L}_x$ by a term $\tilde{\mathcal{L}}_x^l$ that is supported within the ball of radius l around x. For each term, this introduces at most an error $\mathrm{c}f(l)$. We collect the correcting terms in an error term $\mathcal{R}$, so that we have

$$\mathcal{L} = \tilde{\mathcal{L}} + \mathcal{R}. \tag{69}$$

Lemma 15 (Approximate detailed balance). The Liouvillian $\tilde{\mathcal{L}}$ satisfies approximate detailed balance with respect to ω: For any two operators, we have

$$|\omega(\tilde{\mathcal{L}}[A]B) - \omega(A\tilde{\mathcal{L}}[B])| = |\omega(\mathcal{R}[A]B) - \omega(A\mathcal{R}[B])| \leq 2|\Lambda|cf(l)\,\|A\|\,\|B\|. \tag{70}$$

Proof. The claim follows immediately from $|\omega(X)| \leq \|X\|$ for any state ω and operator X. □

Remembering that $f(l)$ decays faster than any polynomial, the above lemma shows that, even if A or B grow polynomially with the system size, $\tilde{\mathcal{L}}$ is asymptotically in detailed balance if we choose that l grows at least like L^α for some $0 < \alpha < 1$.

Similarly, if we write $\tilde{\Delta}^{(m,m')}$ for the same quantity as $\Delta^{(m,m')}$, but where we replace $\mathcal{L}$ with $\tilde{\mathcal{L}}$, we obtain

$$\left|\Delta^{(m,m')}(A,B) - \tilde{\Delta}^{(m,m')}(A,B)\right| \leq 2\,\|A\|\,\|B\|\,|\Lambda|cf(l). \tag{71}$$

In particular, if we can choose $l \propto L^\alpha$ for some constant $0 < \alpha < 1$, this error vanishes in the thermodynamic limit. We will therefore now consider $\tilde{\Delta}^{(m,m')}$ and show that it vanishes in the thermodynamic limit as long as we choose $\alpha < 1/2$.

So suppose from now on that $l = L^\alpha$ with $\alpha < 1/2$. We will again use an approximate derivation-property of the Liouvillian $\tilde{\mathcal{L}}$ together with the fact that it is asymptotically reversible with respect to ω. To do that, we will denote by $\tilde{\mathcal{L}}_{\tilde{A}}$ the Liouvillian containing all terms of $\tilde{\mathcal{L}}$ whose support has overlap with A. Due to the locality of $\tilde{\mathcal{L}}$, there are at most $|\tilde{A}| \leq |A|l^d$ such terms.

The following lemma will be, together with Lemma 13, the key result to prove the theorem. This property will be particularly useful in combination with Lemma 8.

Lemma 16 (Asymptotically local derivation). *Let A be a local observable and let $f(l)$ grow at most like L^α with $0<\alpha<1$ with the system size. Then*

$$\lim_{\Lambda\nearrow\mathbb{Z}^d}\omega(\Gamma_{\tilde{\mathcal{L}}}(\Omega^{(m,m')},A))=\lim_{\Lambda\nearrow\mathbb{Z}^d}\omega(\Gamma_{\tilde{\mathcal{L}}_{\tilde{A}}}(\Omega^{(m,m')},A))=0. \tag{72}$$

Before we give the proof of this lemma, we will show how it implies the main theorem. The steps are essentially the same as in the case of discrete symmetry breaking. Let A, B be local operators. We first use approximate detailed balance together with the approximate derivation property to rewrite $\tilde{\Delta}^{(m,m')}(A,B)$,

$$\tilde{\Delta}^{(m,m')}(A,B)\simeq\omega\left(\Omega^{(m,m')}\left(A\tilde{\mathcal{L}}[B]-\tilde{\mathcal{L}}[A]B\right)\right) \tag{73}$$

$$\simeq\omega\left(\left(\tilde{\mathcal{L}}(\Omega^{(m,m')}A)-\Omega^{(m,m')}\tilde{\mathcal{L}}[A]\right)B\right) \tag{74}$$

$$\simeq\omega\left(\tilde{\mathcal{L}}\left[\Omega^{(m,m')}\right]AB\right)\simeq\omega\left(\Omega^{(m,m')}\tilde{\mathcal{L}}[AB]\right), \tag{75}$$

where again $\simeq$ denotes equality up to terms that vanish in the thermodynamic limit and where we have used approximate detailed balance in the last step. On the other hand, since AB is also a local observable, we can also use the approximate derivation property to show

$$\omega\left(\tilde{\mathcal{L}}\left[\Omega^{(m,m')}\right]AB\right)\simeq\omega\left(\tilde{\mathcal{L}}\left[\Omega^{(m,m')}AB\right]\right)-\omega\left(\Omega^{(m,m')}\tilde{\mathcal{L}}[AB]\right) \tag{76}$$

$$\simeq-\omega\left(\Omega^{(m,m')}\tilde{\mathcal{L}}[AB]\right). \tag{77}$$

Combining the two estimates with $\tilde{\Delta}^{(m,m')}(A,B)\simeq\Delta^{(m,m')}(A,B)$, we therefore get

$$-\omega\left(\Omega^{(m,m')}\tilde{\mathcal{L}}[AB]\right)\simeq\Delta^{(m,m')}(A,B)\simeq\omega\left(\Omega^{(m,m')}\tilde{\mathcal{L}}[AB]\right). \tag{78}$$

In other words,

$$\lim_{\Lambda\nearrow\mathbb{Z}^d}\Delta^{(m,m')}(A,B)=0. \tag{79}$$

Proof (Of Lemma 16). To prove the lemma, we split up O^+ as $O^+=Q+R$, where Q is supported on the complement of $\tilde{A}$ and R collects the remaining terms. In particular, this means that $\tilde{\mathcal{L}}_{\tilde{A}}[QX]=Q\tilde{\mathcal{L}}_{\tilde{A}}[X]$ for any operator X. Let us also introduce the short-hand notation

$$Z^{(m,m')}:=\omega((O^+)^m(O^-)^m)^{1/2}\omega((O^+)^{m'}(O^-)^{m'})^{1/2}. \tag{80}$$

We now use a binomial expansion to write

$$\begin{aligned}\left|\omega(\Gamma_{\tilde{\mathcal{L}}_{\tilde{A}}}(\Omega^{(m,m')},A))\right|&\le\sum_{k=0}^{m}\sum_{k'=0}^{m'}\binom{m}{k}\binom{m'}{k'}\left|\frac{\omega\left(\Gamma_{\tilde{\mathcal{L}}_{\tilde{A}}}((Q^\dagger)^{m'-k}Q^{m-k}(R^\dagger)^{k'}R^k,A)\right)}{Z^{(m,m')}}\right|\\&=\sum_{k=0}^{m}\sum_{k'=0}^{m'}\binom{m}{k}\binom{m'}{k'}\left|\frac{\omega\left((Q^\dagger)^{m'-k}Q^{m-k}\Gamma_{\tilde{\mathcal{L}}_{\tilde{A}}}((R^\dagger)^{k'}R^k,A)\right)}{Z^{(m,m')}}\right|.\end{aligned} \tag{81}$$

But

$$\Gamma_{\mathcal{L}}(\mathbf{1},X)=0 \tag{82}$$

for any operator X and any Liouvillian $\mathcal{L}$. Therefore we can neglect the term with $k'=k=0$. Combining this with another application of the Cauchy-Schwartz inequality, we get

$$\begin{aligned}&\left|\omega(\Gamma_{\tilde{\mathcal{L}}_{\tilde{A}}}(\Omega^{(m,m')},A))\right|\\&\le\sum_{k,k'}{}'\binom{m}{k}\binom{m'}{k'}\frac{\omega\left((Q^\dagger)^{m'-k}Q^{m'-k}\right)^{1/2}\omega\left((Q^\dagger)^{m-k}Q^{m-k}\right)^{1/2}}{Z^{(m,m')}}\\&\quad\times\left\|\Gamma_{\tilde{\mathcal{L}}_{\tilde{A}}}((R^\dagger)^{k'}R^k,A)\right\|,\end{aligned} \tag{83}$$

where the primed sum omits the term $k' = k = 0$. We can now use Lemma 13 to bound the fraction as

$$\frac{\omega\left((Q^\dagger)^{m'-k}Q^{m'-k}\right)^{1/2}\omega\left((Q^\dagger)^{m-k}Q^{m-k}\right)^{1/2}}{Z^{(m,m')}} \le \frac{(\mu o L^d)^{-2(k+k')}}{2-\mathrm{e}^{\mu/8}}, \tag{84}$$

provided that $L^d \ge \frac{16|\tilde{A}|^2}{\mu^2}$ and $|\frac{M}{L^d}| \le \frac{\mu^2}{16|\tilde{A}|}$, where $M \ge |m|, |m'|$. Since by assumption $|\tilde{A}| \le |A| L^{\alpha d}$, the inequalities are fulfilled for large enough system sizes as long as $\alpha < 1/2$. Similarly, by the locality of the Liouvillian, we can upper bound the norm-factor as

$$\left\|\Gamma_{\tilde{\mathcal{L}}_{\tilde{A}}}((R^\dagger)^{k'}R^k, A)\right\| \le 3b|\tilde{A}|\,\|R\|^{k+k'}\,\|A\| \le 3b|A|\,\|A\|\,L^{\alpha d}\left(o|A|L^{\alpha d}\right)^{k+k'}. \tag{85}$$

Combining the two estimates, we get

$$\begin{aligned}\left|\omega(\Gamma_{\tilde{\mathcal{L}}_{\tilde{A}}}(\Omega^{(m,m')}, A))\right| &\le \frac{3b|A|\,\|A\|}{2-\mathrm{e}^{\mu/8}} L^{\alpha d} \sum_{k,k'}{}' \binom{m}{k}\binom{m'}{k'}\left(\frac{|A|}{\mu}L^{(\alpha-1)d}\right)^{k+k'} \\ &\le \frac{3b|A|\,\|A\|}{2-\mathrm{e}^{\mu/8}} L^{\alpha d}\left(\exp(\frac{|A|}{\mu}ML^{(\alpha-1)d}) - 1\right).\end{aligned} \tag{86}$$

But $L^{\alpha d}\left(\exp\left(\frac{|A|}{\mu}ML^{(\alpha-1)d}\right) - 1\right)$ converges to zero as $L \to \infty$ as long as $\alpha < 1/2$. This finishes the proof. □

1. Proof of Lemma 13

Let $a_m := \mathrm{Tr}(Q_A^m \rho (Q_A^*)^m)$. We have to prove

$$\frac{a_m}{a_{m-1}} \ge (\mu o N)^2. \tag{87}$$

We first calculate a_1, which gives

$$\begin{aligned} a_1 &= \mathrm{Tr}((O^+ - R_A)\rho(O^- - R_A^*)) \\ &\ge \mathrm{Tr}(\rho O^- O^+) - 2\left\|O^+ R_A^*\right\| \le 2No^2|A| \\ &= \frac{1}{2}\left[\mathrm{Tr}(\rho O^+ O^-)\mathrm{Tr}(\rho O^- O^+) + \mathrm{Tr}(\rho[O^+, O^-])\right] - 2o^2 N|A| \\ &\ge \mathrm{Tr}(\rho {O^{(1)}}^2) + \mathrm{Tr}(\rho {O^{(2)}}^2) - 2o^2(1+|A|)N \\ &\ge 2o^2\mu^2 N^2\left[1 - \frac{1+|A|}{\mu^2 N}\right]. \end{aligned} \tag{88}$$

Using the bound $N \ge \frac{16|A|^2}{\mu^2}$, we have

$$1 - \frac{1+|A|}{\mu^2 N} \ge 1 - \frac{1+|A|}{16|A|^2} \ge 1 - \frac{1}{8} > 0, \tag{89}$$

since $|A| \ge 1$. Therefore $a_1 > 0$. Next we can again use the Schwartz inequality to get

$$\begin{aligned} (a_{m-1})^2 &\le \mathrm{Tr}(\rho(Q_A^*)^{m-2}Q_A^{m-2})\mathrm{Tr}(\rho(Q_A^*)^{m-1}Q_A Q_A^* Q_A^{m-1}) \\ &= a_{m-2}\left\{\mathrm{Tr}(\rho(Q_A^*)^m Q_A^m) + \mathrm{Tr}(\rho(Q_A^*)^{m-1}[Q_A, Q_A^*]Q_A^{m-1})\right\} \\ &\le a_{m-2}\left\{a_m + 4o^2 N a_{m-1}\right\}. \end{aligned} \tag{90}$$

Assuming $a_{m-2} \neq 0, a_{m-1} \neq 0$, which is true for $m = 2$, we get

$$\frac{a_m}{a_{m-1}} \ge \frac{a_{m-1}}{a_{m-2}} - 4o^2 N. \tag{91}$$

Summing up, we obtain

$$\begin{aligned}\frac{a_m}{a_{m-1}} &\geq a_1 - 4o^2N(m-2)\\ &\geq 2(\mu o N)^2\left[1-\frac{1+|A|}{\mu^2 N}-\frac{2(m-2)}{\mu^2 N}\right]\\ &\geq 2(\mu o N)^2\left[1-\frac{1+|A|}{\mu^2 N}-\frac{2M}{\mu^2 N}\right]\\ &\geq 2(\mu o N)^2\left[1-\frac{1+|A|}{16|A|^2}-\frac{1}{8|A|}\right]\\ &\geq 2(\mu o N)^2\left[\frac{16-2-2}{16}\right]=(\mu o N)^2\frac{3}{2}>(\mu o N)^2,\end{aligned} \tag{92}$$

where we have used $N \geq \frac{16|A|^2}{\mu^2}$, $|\frac{M}{N}| \leq \frac{\mu^2}{16|A|}$, and $|A| \geq 1$. The desired bound thus holds by induction. Let us now lower bound the ratio

$$r_A^{(M)} = \left|\frac{\mathrm{Tr}\left((O^+)^M \rho (O^-)^M\right)}{\mathrm{Tr}(Q_A^M \rho (Q_A^*)^M)}\right|. \tag{93}$$

We use a binomial expansion again to first obtain

$$\begin{aligned}\left|\mathrm{Tr}\left((O^+)^M\rho(O^-)^M\right)\right| &= \left|\mathrm{Tr}(Q_A^M\rho(Q_A^*)^M) + {\sum_{k,l}}' \binom{M}{k}\binom{M}{l}\mathrm{Tr}\left(\rho(Q_A^*)^{M-k}(R_A^*)^k Q_A^{M-l}R_A^{M-l}\right)\right|\\ &\geq |\mathrm{Tr}(Q_A^M\rho(Q_A^*)^M)| - {\sum_{k,l}}'\left|\binom{M}{k}\binom{M}{l}\mathrm{Tr}\left(\rho(Q_A^*)^{M-k}(R_A^*)^k Q_A^{M-l}R_A^{M-l}\right)\right|,\end{aligned} \tag{94}$$

where the primed sum goes over all $k, l = 0, \ldots, M$ except for $k = l = 0$. Using the Schwartz inequality and (61) again we get the bound

$$\begin{aligned} r_A^{(M)} &\geq 1 - {\sum_{k,l}}'(|A|o)^{k+l}(\mu o N)^{-(k+l)}\\ &\geq 1 - \left[\left(1+\frac{|A|}{\mu N}\right)^{2M} - 1\right]\\ &\geq 2 - \exp\left(\frac{2|A|M}{\mu N}\right) \geq 2 - \mathrm{e}^{\mu/8}.\end{aligned} \tag{95}$$

Note that, in particular, $r_A^{(M)} > 0$, since $0 < \mu \leq 1$

C. Goldstone modes

Here we give a sketch of how to construct dissipative Goldstone modes above a symmetry-broken steady-state if the Liouvillian is symmetric and commutes with charge in the sense of Eq. (21) of the main text. For simplicity, we will assume that the dynamics is strictly local. For any cubical volume $\Lambda \subseteq \mathbb{Z}^d$ of side-length L and local region $A \subset \Lambda$, define

$$U_A := \exp\left(\frac{2\pi\mathrm{i}}{L}\sum_{x\in A}\sum_{j=1}^{d} x_j C_x\right). \tag{96}$$

The operator U_Λ creates a spin-wave of wavelength L on the whole volume Λ. We can then define states

$$\sigma_\Lambda^{(M)}(A) = \omega_\Lambda^{(M)}\left((U_\Lambda)^\dagger A U_\Lambda\right). \tag{97}$$

For large M these describe symmetry-broken states with one spin-wave excitation in each space-direction. Now fix some local observable A. As Λ increases, we can approximate U_A by an operator V_A that effects a spatially constant rotation in the region A,

$$V_A := \exp\left(\frac{2\pi \mathrm{i}}{L}\varphi C_A\right) = \exp\left(\frac{2\pi \mathrm{i}}{L}(\sum_{j=1}^{d} x_j^{(0)})C_A\right), \tag{98}$$

where $x^{(0)}$ is a point in the center of A. A Taylor-expansion then yields

$$\begin{aligned}\left\|U_{\tilde{A}}\mathcal{L}_A^{\Lambda}[A]\,U_{\tilde{A}}^{\dagger} - V_{\tilde{A}}\mathcal{L}_A^{\Lambda}[A]\,V_{\tilde{A}}^{\dagger}\right\| &\leq \frac{2\pi}{L}\left\|\left[\sum_{y\in\tilde{A}}\sum_{j=1}^{d}(y_j - x_j^{(0)})C_y, \mathcal{L}_A^{\Lambda}[A]\right]\right\| + O(1/L^2)\\ &\leq \frac{2\pi}{L}\mathrm{diam}(\tilde{A})d|\tilde{A}|^2 2co\,\|A\| + O(1/L^2) = O(1/L),\end{aligned} \tag{99}$$

with $c > 0$ as in Definition 5 and $o = \|C_x\|$. We thus obtain

$$\begin{aligned}\sigma_{\Lambda}^{(M)}\left(\mathcal{L}^{\Lambda}[A]\right) &= \omega_{\Lambda}^{(M)}\left(U_{\tilde{A}}\mathcal{L}^{\Lambda}[A]\,U_{\tilde{A}}^{\dagger}\right) \approx \omega_{\Lambda}^{(M)}\left(V_{\tilde{A}}\mathcal{L}^{\Lambda}[A]\,V_{\tilde{A}}^{\dagger}\right)\\ &= \omega_{\Lambda}^{(M)}\left(\mathcal{L}^{\Lambda}\left[V_A^{\dagger}AV_A\right]\right) \approx 0,\end{aligned}$$

where $\approx$ denotes equality up to a difference of order $1/L$. Thus time-derivatives of local observables become vanishingly small as the system-size (and wave-length) increases. The actual expectation values instead can differ arbitrarily. In particular, for any L the order parameter $\mathbf{m}(x) = (O_x^{(1)}, O_x^{(2)})$ perfectly distinguishes the two states: Its image $\mathbf{m}(\Lambda) \subset \mathbb{R}^2$ is a single point for $\omega_{\Lambda}^{(M)}$ and an arbitrarily dense (as L increases) circle for $\sigma_{\Lambda}^{(M)}$. Note that the above arguments did not rely on the reversibility assumption but only on the locality and symmetry of the Liouvillian under the action of a locally generated symmetry.

V. DISCUSSION

The properties of local dissipative dynamics, such as *locality*,[17] *mixing times*,[9,27,32] and *stability against perturbations*[9,33] have recently attracted a great deal of interest. These results are mainly motivated by the question of whether such dissipative processes can be used for reliably storing quantum information in *quantum memories*,[34–36] performing *computations*[37,38] and *quantum simulations*,[14] or preparing certain quantum states,[39] in particular *topological phases of matter*.[12,40] Here, we have shown that they also give a dynamical view-point on the emergence of spontaneous symmetry breaking: our results show that local dissipative dynamics satisfying detailed balance with respect to a state with extensive fluctuations of an order parameter necessarily also prepares different symmetry-breaking phases in the thermodynamic limit. Thus symmetry-breaking phases are dynamically stabilised by dissipative dynamics in detailed balance. We hence also provide further insight into asymptotic stationarity in the dynamics of dissipative processes such as those involving multi-component Rydberg gases, as have recently been discussed in the literature.[41]

An important feature of our work is that it shifts the perspective of symmetry breaking phase transitions from properties of Hamiltonians to properties of quantum states. This mindset is similar to recent studies in the field of topological order, where the emphasis has been put on states described by *tensor networks* and their entanglement structure instead of Hamiltonians.[42–44]

Our results rely on the locality of the dynamics and detailed-balance. While locality is clearly necessary, the role of reversibility is not quite as clear: It is known that with simple non-reversible update rules of an asynchronous cellular automaton, it is possible to have a domain of stability in the phase diagram even though this is impossible for equilibrium statistical mechanics models.[45,46] It is an open problem whether criticality can be induced robustly out of equilibrium without the simultaneous production of asymptotically stationary states.

In the case of discrete symmetry breaking, examples of Liouvillians fulfilling our assumptions can be given easily, since such Liouvillians exist for any Gibbs state of commuting local Hamiltonians[27] as, for example, the Ising model. In the case of continuous, non-commutative symmetries, we currently cannot provide an explicit example, mirroring the lack of exactly solvable models showing continuous symmetry-breaking in the Hamiltonian case. Indeed, it is a long-standing open problem in mathematical physics to show whether reversible and quasi-local Gibbs samplers exist for non-commutative Hamiltonians, in particular below the critical temperature. While Liouvillians constructed through weak-coupling techniques always fulfill detailed balance, they are typically not quasi-local anymore. Our results imply that if non-commutative Gibbs samplers exist, they necessarily also have to prepare different phases below a phase transition temperature. Finally, it will be interesting to study whether similar results hold for discrete time (quantum) Markov processes. These would give information about the convergence of (quantum) Markov chain Monte Carlo algorithms, which are typically in detailed balance and are used in many areas of physics.

ACKNOWLEDGMENTS

This work has been supported by the ERC (TAQ), the EU (RAQUEL, AQuS, SIQS), the COST network, the DFG (Grant Nos. CRC 183 and EI 519/7-1), and the Studienstiftung des Deutschen Volkes. M.J.K. was supported by the Carlsberg foundation and the Villum foundation.

APPENDIX: DETAILED BALANCE

As detailed balance plays an important role for the present work, we briefly explain here how the notion of detailed balance that we use precisely generalises the classical notion of detailed balance. To do that let $\mathcal{L}$ be a Liouvillian in detailed balance with the quantum state ω, i.e., $\omega(\mathcal{L}(A)B) = \omega(A\mathcal{L}B)$ for any two bounded operators A, B. For simplicity, let us assume that the dynamics takes place on a finite-dimensional Hilbert-space. We can decompose ω into mutually orthogonal pure states ψ_j with associated projection operators P_j and probabilities p_j. Then we have

$$\omega(e^{t\mathcal{L}}[P_i]\,P_j) = p_j\psi_j(\mathrm{e}^{t\mathcal{L}}[P_i]) =: p_j\mathbb{P}(i,j;t), \tag{A1}$$

where $\mathbb{P}(i,j;t)$ denotes the probability to end up in the state ψ_i after time t when having started in the state j. From detailed balance, we then get (upon integrating)

$$p_j\mathbb{P}(i,j;t) = \omega(e^{t\mathcal{L}}[P_i]\,P_j) = \omega(P_i e^{t\mathcal{L}}\left[P_j\right]) = p_i\mathbb{P}(j,i;t), \tag{A2}$$

which is precisely the condition of detailed balance in a classical Markov chain defined over the states ψ_j with transition probabilities $\mathbb{P}(i,j;t)$. In particular, if ω is a Gibbs state of a non-degenerate Hamiltonian at inverse temperature β, the states ψ_j are energy-eigenstates associated with energies E_j and we get the well-known relation

$$\mathbb{P}(i,j;t) = \mathrm{e}^{-\beta(E_j-E_i)}\mathbb{P}(j,i;t). \tag{A3}$$

[1] R. K. Pathria and P. D. Beale, *Statistical Mechanics* (Elsevier Science, 2011).
[2] M. Kliesch, C. Gogolin, M. J. Kastoryano, A. Riera, and J. Eisert, "Locality of temperature," Phys. Rev. X **4**, 031019 (2014); e-print arXiv:1309.0816.
[3] R. B. Griffiths, "Spontaneous magnetization in idealized ferromagnets," Phys. Rev. **152**, 240–246 (1966).
[4] F. J. Dyson, E. H. Lieb, and B. Simon, "Phase transitions in quantum spin systems with isotropic and nonisotropic interactions," J. Stat. Phys. **18**, 335–383 (1978).
[5] T. Koma and H. Tasaki, "Symmetry breaking and finite size effects in quantum many-body systems," J. Stat. Phys. **76**, 745–803 (1994); e-print arXiv:cond-mat/9708132.
[6] J. Eisert and T. Prosen, "Noise-driven quantum criticality" (2010); e-print arXiv:1012.5013.
[7] M. Höning, M. Moos, and M. Fleischhauer, "Critical exponents of steady-state phase transitions in fermionic lattice models," Phys. Rev. A **86**, 013606 (2012); e-print arXiv:1108.2263.
[8] K. Temme, "Lower bounds to the spectral gap of davies generators," J. Math. Phys. **54**, 122110 (2013); e-print arXiv:1305.5591.
[9] M. J. Kastoryano and J. Eisert, "Rapid mixing implies exponential decay of correlations," J. Math. Phys. **54**, 102201 (2013); e-print arXiv:1303.6304.
[10] Z. Cai and T. Barthel, "Algebraic versus exponential decoherence in dissipative many-particle systems," Phys. Rev. Lett. **111**, 150403 (2013); e-print arXiv:1304.6890.
[11] M. Znidaric, "Relaxation times of dissipative many-body quantum systems," Phys. Rev. E **92**, 042143 (2015); e-print arXiv:1507.07773.
[12] S. Diehl, E. Rico, M. A. Baranov, and P. Zoller, "Topology by dissipation in atomic quantum wires," Nat. Phys. **7**, 971 (2011); e-print arXiv:1105.5947.

[13] I. Bloch, J. Dalibard, and S. Nascimbene, "Quantum simulations with ultracold quantum gases," Nat. Phys. **8**, 267 (2012).
[14] H. Weimer, M. Müller, I. Lesanovsky, P. Zoller, and H. P. Büchler, "A Rydberg quantum simulator," Nat. Phys. **6**, 382–388 (2010); e-print arXiv:0907.1657.
[15] G. Lindblad, "On the generators of quantum dynamical semigroups," Commun. Math. Phys. **48**, 119–130 (1976).
[16] E. H. Lieb and D. W. Robinson, "The finite group velocity of quantum spin systems," Commun. Math. Phys. **28**, 251–257 (1972).
[17] D. Poulin, "Lieb-robinson bound and locality for general markovian quantum dynamics," Phys. Rev. Lett. **104**, 190401 (2010); e-print arXiv:1003.3675.
[18] T. Barthel and M. Kliesch, "Quasi-locality and efficient simulation of markovian quantum dynamics," Phys. Rev. Lett. **108**, 230504 (2011); e-print arXiv:1111.4210.
[19] B. Nachtergaele, A. Vershynina, and V. A. Zagrebnov, "Lieb-Robinson bounds and existence of the thermodynamic limit for a class of irreversible quantum dynamics," AMS Contemp. Math. **552**, 161–175 (2011); e-print arXiv:1103.1122.
[20] M. Kliesch, C. Gogolin, and J. Eisert, "Lieb-Robinson Bounds and the Simulation of Time-Evolution of Local Observables in Lattice Systems," in *Many-Electron Approaches in Physics, Chemistry and Mathematics,* edited by V. Bach and L. D. Site (Springer: Mathematical Physics Studies, 2014), pp. 301–318. e-print arXiv:1306.0716.
[21] E. B. Davies, "Markovian master equations," Commun. Math. Phys. **39**, 91–110 (1974).
[22] R. Alicki, "On the detailed balance condition for non-hamiltonian systems," Rep. Math. Phys. **10**, 249–258 (1976).
[23] A. Kossakowski, A. Frigerio, V. Gorini, and M. Verri, "Quantum detailed balance and kms condition," Commun. Math. Phys. **57**, 97–110 (1977).
[24] V. Jakšić and C.-A. Pillet, "On entropy production in quantum statistical mechanics," Commun. Math. Phys. **217**, 285–293 (2001).
[25] F. Fagnola and V. Umanit, "Generators of detailed balance quantum markov semigroups," Infinite Dimens. Anal., Quantum Probab. Relat. Top. **10**, 335–363 (2007).
[26] F. Fagnola and V. Umanit, "Generators of kms symmetric Markov semigroups on B(h) symmetry and quantum detailed balance," Commun. Math. Phys. **298**, 523–547 (2010).
[27] M. J. Kastoryano and F. G. S. L. Brandao, "Quantum gibbs samplers: The commuting case," Commun. Math. Phys. **344**, 915–957 (2016); e-print arXiv:1409.3435.
[28] Unlike in most studies using the notion of detailed balance, we do not require the steady-state to be faithful (full-rank) because the proofs of our results do not require this assumption.
[29] However, it would be sufficient to assume the weaker notion of asymptotic reversibility for the reference state with long-range correlations instead of full detailed balance to obtain our results.
[30] V. V. Albert and L. Jiang, "Symmetries and conserved quantities in Lindblad master equations," Phys. Rev. A **89**, 022118 (2014); e-print arXiv:1310.1523.
[31] L. Landau, J. F. Perez, and W. F. Wreszinski, "Energy gap, clustering, and the goldstone theorem in statistical mechanics," J. Stat. Phys. **26**, 755–766 (1981).
[32] M. J. Kastoryano and K. Temme, "Quantum logarithmic sobolev inequalities and rapid mixing," J. Math. Phys. **54**, 052202 (2013); e-print arXiv:1207.3261.
[33] T. S. Cubitt, A. Lucia, S. Michalakis, and D. Perez-Garcia, "Stability of local quantum dissipative systems," Commun. Math. Phys. **337**, 1275–1315 (2015); e-print arXiv:1303.4744.
[34] K. Fujii, M. Negoro, N. Imoto, and M. Kitagawa, "Measurement-free topological protection using dissipative feedback," Phys. Rev. X **4**, 041039 (2014); e-print arXiv:1401.6350.
[35] M. Herold, M. Kastoryano, E. T. Campbell, and J. Eisert, "Fault tolerant dynamic decoders for topological quantum memories," npj Quantum Inf. **1**, 15010 (2015); e-print arXiv:1406.2338.
[36] R. Koenig and F. Pastawski, "Generating topological order: No speedup by dissipation," Phys. Rev. B **90**, 045101 (2013); e-print arXiv:1310.1037.
[37] F. Verstraete, M. M. Wolf, and J. I. Cirac, "Quantum computation and quantum-state engineering driven by dissipation," Nat. Phys. **5**, 633–636 (2009); e-print arXiv:0803.1447.
[38] M. J. Kastoryano, M. M. Wolf, and J. Eisert, Phys. Rev. Lett. **103**, 110501 (2013); e-print arXiv:1205.0985.
[39] P. D. Johnson, F. Ticozzi, and L. Viola, "General fixed points of quasi-local frustration-free quantum semigroups: From invariance to stabilization," Quantum Inf. Comp. **16**, 657–699 (2016).
[40] C.-E. Bardyn, M. A. Baranov, C. V. Kraus, E. Rico, A. Imamoglu, P. Zoller, and S. Diehl, "Topology by dissipation," New J. Phys. **15**, 085001 (2013); e-print arXiv:1302.5135.
[41] R. Gutierrez, J. P. Garrahan, and I. Lesanovsky, "Non-equilibrium fluctuations and metastability in the dynamics of dissipative multi-component Rydberg gases," New J. Phys. **18**, 093054 (2016); e-print arXiv:1603.00828.
[42] N. Schuch, D. Perez-Garcia, and J. I. Cirac, "Classifying quantum phases using matrix product states and projected entangled pair states," Phys. Rev. B **84**, 165139 (2011); e-print arXiv:1010.3732.
[43] X. Chen, Z.-C. Gu, and X.-G. Wen, "Classification of gapped symmetric phases in 1d spin systems," Phys. Rev. B **83**, 035107 (2011); e-print arXiv:1103.3323.
[44] A. M. Turner, F. Pollmann, and E. Berg, "Topological phases of one-dimensional fermions: An entanglement point of view," Phys. Rev. B **83**, 075102 (2011); e-print arXiv:1409.8616.
[45] C. H. Bennett and G. Grinstein, "Role of irreversibility in stabilizing complex and nonergodic behavior in locally interacting discrete systems," Phys. Rev. Lett. **55**, 657 (1985).
[46] G. Grinstein, "Can complex structures be generically stable in a noisy world?," IBM J. Res. Dev. **48**, 5–12 (2004).