Model Fractional Chern Insulators

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We devise local lattice models whose ground states are model fractional Chern insulators—Abelian and non-Abelian topologically ordered states characterized by exact ground state degeneracies at any finite size and infinite entanglement gaps. Most saliently, we construct exact parent Hamiltonians for two distinct families of bosonic lattice generalizations of the $\mathcal{Z}_k$ parafermion quantum Hall states: (i) color-entangled fractional Chern insulators at band filling fractions $\nu = k/(\mathcal{C}+1)$ and (ii) nematic states at $\nu = k/2$, where $\mathcal{C}$ is the Chern number of the lowest band. In spite of a fluctuating Berry curvature, our construction is partially frustration free: the ground states reside entirely within the lowest band and exactly minimize a local $(k+1)$ body repulsion term by term. In addition to providing the first known models hosting intriguing states such as higher Chern number generalizations of the Fibonacci anyon quantum Hall states, the remarkable stability and finite-size properties make our models particularly well suited for the study of novel phenomena involving, e.g., twist defects and proximity induced superconductivity, as well as being a guide for designing experiments.

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Introduction.—The prospect of lattice-scale fractional quantum Hall (FQH) phenomena at high temperatures, without the need for a strong magnetic field, has attracted ample recent attention to the theory of fractional Chern insulators (FCIs) [1–3]. While experimental realizations of FCIs are becoming increasingly realistic in the light of the recent realizations of integer Chern insulators with unit Chern number in solid state materials [4] and cold atom systems [5], the theoretical frontier has turned towards strongly correlated states in bands with higher Chern numbers [6–27]. This is due to the fact that, although notable differences compared to the continuum setting have been established [2,28,29], all FCIs discovered in Chern number $\mathcal{C} = 1$ bands have direct continuum FQH analogs to which the adiabatic continuity has been explicitly established in several important cases [30–35]. Of special value is the early work by Kapit and Mueller who provided a natural lattice discretization of the continuum lowest Landau level and showed that a two-body on-site interaction leads to a perfect lattice version of the bosonic $\nu = 1/2$ Laughlin state [30]. Related work has established the existence of lattice parent Hamiltonians of non-Abelian states [36–38].

There is, however, a glaring lack of similar models describing $\mathcal{C} > 1$ systems, despite intriguing progress with long-ranged lattice models [39] and approximative mappings to continuum models with unusual boundary conditions [14]. Given the importance of solvable models in the theory of topological and strongly correlated states of matter—ranging from the Affleck-Kennedy-Lieb-Tasaki (AKLT) model for the Haldane spin chain [40] and the Kitaev chain describing a one-dimensional p-wave superconductor [41] to the model wave functions for the continuum FQH effect [42–44] and their concomitant pseudopotential parent Hamiltonians [45]—finding such has remained an outstanding challenge in the theory of FCIs. This is particularly pressing given the accumulating numerical evidence that $\mathcal{C} > 1$ systems feature an even richer phenomenology than continuum Landau levels.

In the present Letter, we bridge this divide and provide exact lattice parent Hamiltonians for a large class of Abelian as well as non-Abelian *model* FCIs in bands carrying any Chern number $\mathcal{C}$. We explicitly verify that the ground state multiplets are exactly degenerate at any finite size, that the gap to excited states remains finite in the thermodynamic limit, and that there is an infinite gap in the particle entanglement spectrum [46,47].

Flatband model.—We begin by constructing a family of multiorbital models possessing exactly flat lowest bands with arbitrary Chern number $\mathcal{C}$. For definitiveness, we describe our construction on a square lattice with an effective magnetic flux $\phi = 1/q$ piercing each elementary plaquette [48], although it can be generalized to any Bravais lattice and rational flux.

We assign M internal orbitals to each lattice site i with real-space coordinates (x_i, y_i), where M must be a factor of q. The site-dependent orbital index to the lattice site i is $s_i = x_i \mathrm{mod}(q/M) + m(q/M)$, with $m = 0, 1, \ldots, M-1$. The single-particle physics is governed by

$$H_0 = \sum_{j,s_j} \sum_{k,s_k} t_{j,k}^{s_j,s_k} a_{j,s_j}^{\dagger} a_{k,s_k}, \tag{1}$$

where $a^{\dagger}_{j,s_j}$ (a_{j,s_j}) creates (annihilates) a particle on the orbital s_j at site j. To achieve an exactly flat lowest Chern band we choose the hopping amplitudes as [49]

$$t^{s_j,s_k}_{j,k} = \delta^{\mathrm{mod}q}_{s_j-x_j,s_k-x_k}(-1)^{x+y+xy} e^{-(\pi/2)(1-\phi)|z|^2} e^{-i\pi\phi(\tilde{x}_j+\tilde{x}_k)y}, \tag{2}$$

where $z_j = x_j + iy_j$, $z = z_j - z_k$, and $\tilde{x}_j = x_j + (s_j - x_j)\mathrm{mod}q$. The hopping amplitudes decay as a function of the distance between site k and site j like a Gaussian. The hopping phase factor depends on both x_i and s_i according to the definition of $\tilde{x}_i$. The unit cell of our model contains q/M sites in the x direction. The q orbitals in a unit cell lead to q bands, and the lowest thereof is exactly flat and carries Chern number $\mathcal{C} = M$. For $M = 1$, our construction equations (1) and (2) reduce to the Kapit-Mueller model [30] in the Landau gauge. Similar multi-orbital models have also been studied in Refs. [10,18], albeit with different choices for the hopping amplitudes.

Although it is generally impossible to have an exactly flat band with nonzero Chern number and strictly finite hopping [50], our model is local in the sense of being at least exponentially bounded. Truncating the hopping at a distance of $d = 2$ lattice constants already gives a high flatness ratio between the band gap and bandwidth: e.g., for $\phi = 1/6$ and $\mathcal{C} = 2$ or 3 one finds $f \approx 85$ or 73, respectively. The efficiently quenched kinetic energy amplifies the importance of interaction effects and we will now proceed to show that local interactions indeed generate model FCIs.

Color-entangled FCIs.—We begin by considering N particles with the $(k+1)$ body on-site repulsion on a finite lattice of $N_x \times N_y$ unit cells with periodic boundary conditions. The interaction Hamiltonian reads

$$H_{\mathrm{int}} = \sum_i \sum_{\sigma_0 \le \sigma_1 \le \cdots \le \sigma_k \in \{s_i\}} : n_{i,\sigma_0} n_{i,\sigma_1} \cdots n_{i,\sigma_k} :, \tag{3}$$

where $n_{i,\sigma}$ is the occupation operator on the orbital σ at lattice site i, and $:\cdots:$ enforces the normal ordering. For $\mathcal{C} = M = 1$, the single-particle wave functions of the lowest band of Eq. (1) have the structure of a discretized lowest Landau level, and lattice analogs of the $\mathcal{Z}_k$ Read-Rezayi states are unique zero-energy ground states of Eq. (3) at $\nu = N/(N_x N_y) = k/2$ up to an exact $(k+1)$-fold degeneracy, when the number of particles is a multiple of k, because the exact clustering properties of these wave functions [44] carry over directly to the present lattice setting. This is astonishing given that many other properties such as the fluctuating Berry curvature in reciprocal space and the excitation spectrum already deviate from that in the continuum since the discretized Landau level orbitals are nonorthogonal. Furthermore, if the wave functions are written in a properly orthogonalized Wannier basis [31–33], they differ from the continuum model states [19,38]. Nevertheless, we find that these states are characterized by an infinite gap in the *particle* entanglement spectrum (PES) which probes the quasihole excitations of the system [46,47]. Remarkably, we find that this scenario generalizes to any $\mathcal{C} = M > 1$: at filling fractions $\nu = k/(\mathcal{C}+1)$, there are $\binom{\mathcal{C}+k}{k}$-fold exactly degenerate zero-energy ground states when the number of particles is a multiple of k, and their PES has an infinite gap.

To establish this, we project the interaction Hamiltonian Eq. (3) for a large number of samples onto the lowest band, and compute the many-body eigenvalues and eigenstates by exact diagonalization. Indeed, we always observe the expected number of zero-energy modes in the energy spectrum [48,51], which in turn implies that the band projection leaves the ground states unchanged. We also find the number of zero-energy modes is robust against the flux insertion (twisted boundary conditions). To demonstrate the infinite entanglement gap in the particle entanglement spectrum (PES), we truncate the hopping range in Eq. (1) at distance d, then track the evolution of the PES with increasing d. While the lowest band is dispersive for finite d, we study the band projected version thus ignoring the band dispersion. For a system of N particles described by the density matrix $\rho = (1/\mathcal{D})\sum_{\alpha=1}^{\mathcal{D}} |\Psi_\alpha\rangle\langle\Psi_\alpha|$, where $|\Psi_\alpha\rangle$ is the αth state in the ground state manifold with degeneracy $\mathcal{D}$ [52], the PES levels ξ are defined as $\xi \equiv -\ln\lambda$, where the λ are the eigenvalues of the reduced density matrix ρ_A of N_A particles obtained by tracing out $N_B = N - N_A$ particles from the whole system, i.e., $\rho_A = \mathrm{Tr}_B\rho$ [47]. Each PES level can be labeled by the total two-dimensional quasimomentum (K^A_x, K^A_y) of part A. When the PES levels are clearly divided into low-lying and higher excited parts, we define the entanglement gap as $\Delta_\xi \equiv \xi_{i+1} - \xi_i$, where ξ_i (ξ_{i+1}) is the highest (lowest) level in the low-lying (excited) part.

Typical PES at different truncations are shown in Fig. 1 for the $\nu = 1/2$ non-Abelian state in a $\mathcal{C} = 3$ band. Including only nearest and next-nearest neighbor hopping, i.e., $d = \sqrt{2}$ [Fig. 1(a)], we observe a clear entanglement gap of $\Delta_\xi \approx 5$ which is already larger than most of previously reported results in the literature [6,12,53]. Increasing d further elevates the nonuniversal part of the PES and quickly enlarges the entanglement gap [Figs. 1(b) and 1(c)]. Our capability of tracking the growth of the entanglement gap is limited only by the machine precision, which determines that the PES levels can be computed reliably at most up to $\xi_c \approx 36.7$, which corresponds to an exponentially small amplitude, of order $O(e^{-\xi_c/2})$, in the ground state wave function. When the nonuniversal levels merge into the numerical noise, it is impossible to identify the entanglement gap [Fig. 1(d)] accurately. This happens when the machine error dominates the high-energy part in the PES, the entanglement gap has already grown to

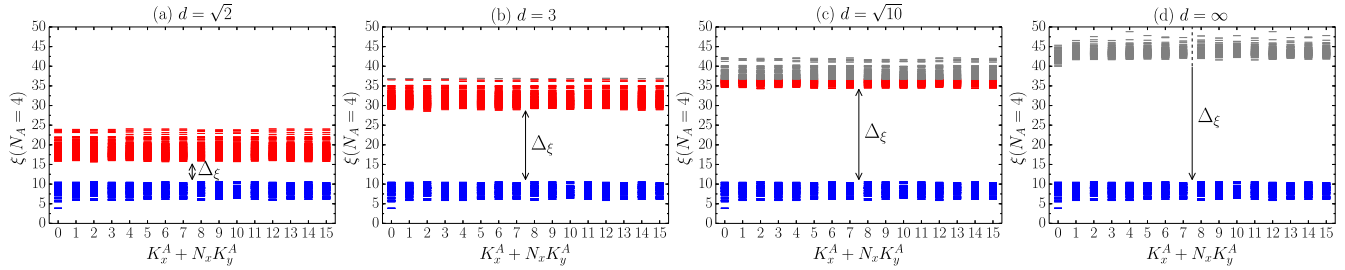

FIG. 1. Typical particle entanglement spectra (PES), here displayed for the $k = 2$, $\nu = 2/3$ non-Abelian state in a $\mathcal{C} = M = 3$ band with $N = 8$, $N_x \times N_y = 4 \times 4$, $\phi = 1/3$ and hoppings truncated at (a) $d = \sqrt{2}$, (b) $d = 3$, (c) $d = \sqrt{10}$, and (d) $d = \infty$ lattice constants. Generally, a PES obtained by numerical diagonalization includes three parts: the low-lying levels with quasihole excitation information (blue), the high nonuniversal levels (red), and numerical noise (gray) set by the double precision above $\xi_c \approx \ln(2^{-53}) \approx 36.7$. The high, nonuniversal levels merge into numerical noise for large d, preventing us from further numerically tracking the growth of the entanglement gap, which we argue to increase without bound with increasing d.

$\Delta_\xi \approx 30$, which is much larger than previously reported values. We observe a similar growth of the entanglement gap when $1/d \to 0$ in all investigated samples, as shown in Fig. 2. Δ_ξ reaches $5 \lesssim \Delta_\xi \lesssim 7$ at $d = \sqrt{2}$, exceeding most of the previously reported results, and quickly increases to $\Delta_\xi \approx 30$ at $d \approx 4$, where numerical noise starts to prevent us from further tracking the growth of Δ_ξ. However, the rapid growing of Δ_ξ and extrapolating the data to $1/d = 0$ clearly suggest infinite entanglement gaps of model FCIs. A short-range truncation of Eq. (1) is enough to get FCIs which are essentially indistinguishable from model FCIs with infinite entanglement gaps.

While the ground states and quasihole excitations have identically zero interaction energy in our model, the gap Δ_E—measured at a fixed particle number corresponding to a particle-hole excitation pair—is in principle size dependent. The projection of the interaction to the lowest band will not affect the many-body gap as long as the band gap is much larger than the interaction strength, since low-lying excitations are purely determined by the interaction and the projection, excluding excitations caused by hopping from lower to higher bands. In Fig. 3, we plot Δ_E versus the inverse particle number $1/N$ for various model FCIs. In each case we find that the gap clearly extrapolates to a finite value in the thermodynamic limit, and, compared to other FCI models, the gap is remarkably insensitive to the system size (cf., e.g., Ref. [28]).

Having established the ideal nature of FCIs in our model, we now turn to its color-entangled nature. If we interpret $m = 0, 1, \ldots, M-1$ in the orbital indices s_i as "layers" or "colors," Eq. (1) on an infinite lattice is equivalent to a shifted stacking of M layers of the infinite $M = 1$ model. However, for a *finite* lattice of $N_x \times N_y$ unit cells with periodic boundary conditions, the corresponding stacking has color-entangled boundary conditions [7,14,18] in the x direction in the sense that the hopping across the boundary may occur between orbitals belonging to different layers (usual periodic boundary conditions apply in the y direction) [48]. Crucially, each layer is not necessarily a complete $M = 1$ model with integer number of unit cells and periodic boundary conditions. Instead, one finds that Eq. (1) can be mapped to $\gcd(N_x, M)$ copies of complete $M = 1$ model with usual periodic boundary conditions. Reference [14] provided a color-entangled basis built from

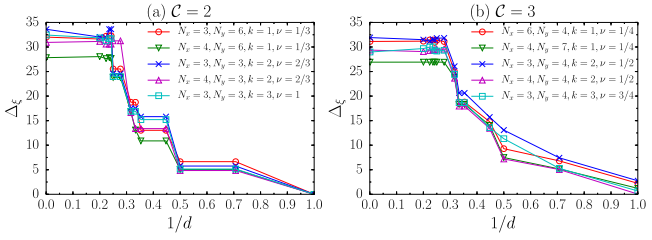

FIG. 2. The entanglement gap Δ_ξ in the $N_A = \lceil N/2 \rceil$ sector versus the inverse hopping distance $1/d$ in the (a) $\mathcal{C} = 2$ and (b) $\mathcal{C} = 3$ band. For each $\mathcal{C}$, we consider both Abelian states at $\nu = 1/(\mathcal{C}+1)$ and non-Abelian states at $\nu = 2/(\mathcal{C}+1)$ and $\nu = 3/(\mathcal{C}+1)$, with lattice geometry of either $\gcd(N_x, \mathcal{C}) = \mathcal{C}$ or $\gcd(N_x, \mathcal{C}) = 1$. ϕ is chosen as $1/\mathcal{C}$. For d longer than three or four lattice constants, the size of Δ_ξ cannot be tracked further due to the limitation of machine precision.

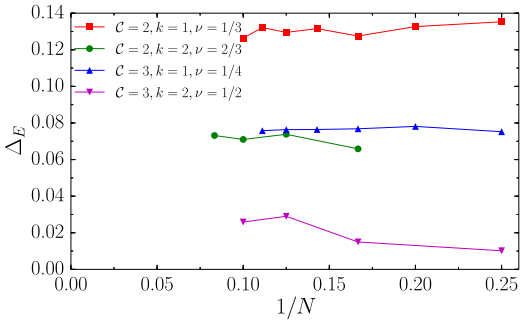

FIG. 3. The finite-size scaling of the energy gap for the Abelian $\nu = 1/(\mathcal{C}+1)$ states and non-Abelian $\nu = 2/(\mathcal{C}+1)$ states in $\mathcal{C} = 2$ and $\mathcal{C} = 3$ bands. Note that the spread of the ground state manifold is exactly zero here. N_x and N_y are appropriately chosen to make samples as isotropic as possible. ϕ is chosen as $1/\mathcal{C}$.

continuum Landau levels and showed a promising approach to the $\nu = k/(\mathcal{C}+1)$ FCIs by providing numerical evidence for a few states with small $\mathcal{C}$ and k [6,12,14]. When N_x is divisible by M, the FCIs correspond to color-dependent magnetic-flux inserted versions of the Halperin [54] or non-Abelian spin singlet states [55,56]. Our construction extends this list of color-entangled FCIs and, by contrast, gives an exact construction directly in the real-space lattice.

Nematic states.—The Hamiltonian equation (3) includes interactions within the same orbital and between different orbitals. Now we consider the zero-energy states in the presence of only on-site $(k+1)$ body intraorbital repulsion, i.e.,

$$H_{\mathrm{int}} = \sum_i \sum_{\sigma_0=\sigma_1=\cdots=\sigma_k \in \{s_i\}} :n_{i,\sigma_0} n_{i,\sigma_1} \cdots n_{i,\sigma_k}: . \qquad (4)$$

As discussed above, the single-particle problem can be mapped to $\gcd(N_x, M)$ copies of the $M = 1$ model. Because the interaction Hamiltonian Eq. (4) does not couple different copies, the many-body physics in this case is equivalent to distributing N on-site interacting particles in $\gcd(N_x, M)$ decoupled copies of $M = 1$ models, each with $[N_x/\gcd(N_x, M)] \times N_y$ unit cells. We can count the zero-energy states straightforwardly by this many-body mapping. For $\nu = k/2$ with N is divisible by $\gcd(N_x, M)$, we have $N/\gcd(N_x, M)$ on-site interacting particles at $\nu = k/2$ in each copy of the $M = 1$ model. If $N/\gcd(N_x, M)$ is divisible by k, this gives us $k+1$ zero-energy states obeying the same exclusion rule as the Read-Rezayi states within each copy, and hence a total of $(k+1)^{\gcd(N_x,M)}$ zero-energy states. However, when N is not divisible by $\gcd(N_x, M)$, the filling fraction is larger than $k/2$ in at least one copy, thus there are no zero-energy states [48]. That the degeneracy depends on N_x but not on N_y is a striking signature of the nematic nature of these states [7]. Furthermore, the number of particles N does not necessarily need to be a multiple of $\mathcal{C} = M$, which further distinguishes the nematic model FCIs from their continuum multilayer relatives.

Other states.—Following the constructions detailed above, it is straightforward to construct parent Hamiltonians for an entire zoo of new model FCIs. For instance, for $M = 2$ and even N_x, Eq. (1) has a bilayer FQH system as the continuum counterpart. Thus, with an on-site interorbital three-body repulsion in combination with a two-body intraorbital repulsion is expected to mimic the parent Hamiltonian for the coupled Moore-Read state [57,58] in the continuum. The degeneracy of this state on the torus is $2N + 3$ for an even number of particles [59], which is consistent with our numerics [48].

Discussion.—In this Letter, we have introduced model fractional Chern insulators—topologically ordered states with an infinite entanglement gap—and constructed their concomitant local parent Hamiltonians directly in the lattice. Our construction provides natural FCI counterparts of the AKLT model for the Haldane spin chain, the Kitaev chain, and the model quantum Hall wave functions and their associated continuum parent Hamiltonians. In analogy with these models, our construction also carries a notion of frustration freeness in that the ground states reside entirely within the lowest band, and exactly minimize strictly local $(k+1)$ body repulsions term by term. However, rooted in the impossibility of local Wannier functions for Chern bands [60], it appears impossible to write a gapped parent Hamiltonian, including both interactions and kinetic part, entirely as a sum of positive local terms such that each of them is minimized by the model FCIs in the two-dimensional limit, $N_x, N_y \to \infty$ [61–63].

We have conclusively shown that our model provides an infinite gap in the particle entanglement spectrum and that the energy gap remains finite in the thermodynamic limit. These results are particularly remarkable considering the strong lattice effects, e.g., reflected in a strongly nonuniform Berry curvature at small ϕ, and underscores the incompleteness of any long-wave-length description of FCIs [1].

Our construction extends the list of FCIs far beyond those in the existing literature. Notably, the $k = 3$ states (Fig. 2) are the first reported $\mathcal{C} > 1$ higher Chern number generalizations of the Fibonacci anyon quantum Hall states. Moreover, the nematic higher Chern number states provide a particularly promising basis for investigating lattice dislocations, which have been predicted to behave like non-Abelian wormholes in nematic Abelian parent states [7]. Abelian nematic states have been previously found in Refs. [18,39]; our construction is giving an ideal realization in contrast to Ref. [18], and is much simpler than the proposed parent Hamiltonian of Ref. [39], which includes a long tail of two-body interactions. Our non-Abelian nematic states are entirely new.

Our model also provides an ideal ground for investigating alternative platforms for Fibonacci anyons, deriving from more elementary Abelian FCIs in proximity to superconductors [64]. In particular, our $\mathcal{C} = 2$, $k = 1$, $\nu = 1/3$ provides an ideal lattice version of the (221) Halperin state which is a key ingredient in the construction considered in Ref. [64].

Finally, our construction provides a guide for designing experimental implementations of FCIs, particularly in cold atom or molecule systems [65] or in arrays of qubits or nonlinear optical resonators [66,67]. The very recent observation of higher Chern numbers in photonic crystals is highly encouraging in this respect [68].

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[52] Each eigenstate can be labeled by a two-dimensional total quasimomentum (K_x, K_y). The ground state degeneracy is not always possible to resolve for $d = 1$. In that case, we select the same number of lowest states in each (K_x, K_y) sector as the $d = \infty$ case to construct the ground state manifold $\{|\Psi_\alpha\rangle\}$.
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