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Relationships between widths of a convex body and of an inscribed parallelotope

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Abstract. Assume that a parallelotope P is inscribed in a three-dimensional convex body C. A conjecture says that $w_1^{-1} + w_2^{-1} + w_3^{-1} \geq 1$, where w_i is the ratio of the width of C to the width of P for the direction perpendicular to the i-th pair of parallel facets of P. We prove three weaker inequalities. One of them is $w_1^{-1} + w_2^{-1} + a_3^{-1} \geq 1$, where a_3 denotes the related axial diameter of C.

Let C be a convex body in Euclidean n-space E^n and let P be a parallelotope inscribed in C. Denote by w_i the ratio of the width of C to the width of P for the direction perpendicular to the i-th pair of parallel facets of P for $i = 1, \ldots, n$. Let us recall a conjecture from [2].

C o n j e c t u r e 1. For every convex body $C \subset E^n$ and every parallelotope P inscribed in C we have

$$\sum_{i=1}^{n} w_i^{-1} \geq 1. \tag{1}$$

For $n = 2$ Conjecture 1 holds true as shown in [2]. The note [1] announces without a proof that Conjecture 1 is true for $n = 3$ under some very specific assumptions.

We consider the three-dimensional case. We have $P = E_1 + E_2 + E_3$, where E_1, E_2, E_3 are three independent edges of P. Of course, $F_1 = E_2 + E_3$, $F_2 = E_1 + E_3$ and $F_3 = E_1 + E_2$ are three non-parallel facets of P.

We define the *i-th axial diameter* of C as the ratio a_i of the length of the longest segment in C which is parallel to E_i to the length of E_i, where $i = 1, 2, 3$. In other words, a_i is the length of the longest segment in C parallel to the i-th axis of a coordinate system whose unit vectors determine the edges E_1, E_2 and E_3. The term axial diameter was introduced by Scott [3], [4]. If P is the unit cube then a_i and w_i are just the inner and outer 1-quermasses of C.

Since $a_i \leq w_i$ for $i = 1, 2, 3$, the inequality $w_1^{-1} + w_2^{-1} + w_3^{-1} \geq 1$ is stronger than $a_1^{-1} + a_2^{-1} + a_3^{-1} \geq 1$. The last inequality has been proved by Scott [3]. It has been improved up to $w_1^{-1} + a_2^{-1} + a_3^{-1} \geq 1$ by Wills [5]. The results of Scott and Wills hold true for the more general situation of a convex body without interior lattice points.

Our aim is to prove a few inequalities which are better than the inequality of Scott but still weaker than the conjectured inequality (1).

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Theorem 1. *Assume that a parallelotope P is inscribed in a three-dimensional convex body C. Then*

$$w_1^{-1} + w_2^{-1} + a_3^{-1} \geq 1. \tag{2}$$

P r o o f. We do not make our considerations narrower assuming that P is a cube with vertices (e_1, e_2, e_3), where $e_i \in \{-1, 1\}$ for $i = 1, 2, 3$, in a rectangular coordinate system. Just we can always obtain this case by an affine transformation.

We apply Steiner symmetrization with respect to the plane $x_3 = 0$. Denote by D the image of our body C. Of course, w_1 and w_2 remain unchanged. Let H_i, H_i' be the supporting planes of D perpendicular to E_i, where $i = 1, 2$. Denote be L_i and L_i' the intersection of the plane $x_3 = 0$ with H_i and H_i', respectively, where $i = 1, 2$. Observe that H_i and H_i' touch D at some points which are in L_i and L_i', respectively, where $i = 1, 2$. Of course, the width of D in the direction perpendicular to the plane $x_3 = 0$ is a_3. Let g be the point of support of D by the supporting plane $x_3 = \frac{1}{2}a_3$.

Consider the homothety γ with the center g and the ratio $1 - a_3^{-1}$. The image of the plane $x_3 = 0$ is the plane $x_3 = 1$. The common part C_0 of C and of the plane $x_3 = 1$ is a two-dimensional convex body in the plane $x_3 = 1$. Since the cube P is inscribed in C, the square face P_0 of P which lies in the plane $x_3 = 1$ is inscribed in C_0. Denote by v_1 and v_2 the widths of C_0 in the directions perpendicular to the sides of P_0; the directions are perpendicular to H_1 and H_2, respectively. Since the two-dimensional case of Conjecture 1 holds true (see Lemma in [2]), we have $v_1^{-1} + v_2^{-1} \geq 1$. On the other hand, the width of the strip between the pair of parallel straight lines $\gamma(L_i)$, $\gamma(L_i')$ is $w_i(1 - a_3^{-1})$, where $i = 1, 2$. Since the straight lines L_1, L_1', L_2, L_2' have non-empty intersections with C, from the convexity of C it follows that the straight lines $\gamma(L_1)$, $\gamma(L_1')$, $\gamma(L_2)$, $\gamma(L_2')$ have non-empty intersections with C_0. Consequently, $w_1(1 - a_3^{-1}) \leq v_1$ and $w_2(1 - a_3^{-1}) \leq v_2$. So we have $\big(w_1(1 - a_3^{-1})\big)^{-1} + \big(w_2(1 - a_3^{-1})\big)^{-1} \geq 1$. Thus ${w_1}^{-1} + {w_2}^{-1} \geq 1 - {a_3}^{-1}$. Hence ${w_1}^{-1} + {w_2}^{-1} + {a_3}^{-1} \geq 1$. ∎

Wills [5] presents an example that (2) does not hold true in the more general situation when C is a convex body without points of an integer lattice in the interior.

When we change the roles of the axes in Theorem 2, we get ${w_1}^{-1} + {a_2}^{-1} + {w_3}^{-1} \geq 1$ and ${a_1}^{-1} + {w_2}^{-1} + {w_3}^{-1} \geq 1$. Adding those three inequalities and dividing by 3 we obtain the following corollary which, in a sense, moves us two thirds of the way from the Scott's inequality ${a_1}^{-1} + {a_2}^{-1} + {a_3}^{-1} \geq 1$ to the conjectured inequality ${w_1}^{-1} + {w_2}^{-1} + {w_3}^{-1} \geq 1$.

Corollary 1. *Assume that a parallelotope P is inscribed in a three-dimensional convex body C. Then*

$$\frac{2}{3}\left(w_1^{-1} + w_2^{-1} + w_3^{-1}\right) + \frac{1}{3}\left(a_1^{-1} + a_2^{-1} + a_3^{-1}\right) \geq 1.$$

Let us pay attention that the approach of the proof of Theorem 2 and induction arguments lead to analogical inequality ${w_1}^{-1} + {w_2}^{-1} + {a_3}^{-1} + \ldots + {a_n}^{-1} \geq 1$ for an n-dimensional convex body with an inscribed parallelotope. In analogy to Corollary 1, we also obtain $\frac{2}{n}\sum_{i=1}^{n} w_i^{-1} + \frac{n-2}{n}\sum_{i=1}^{n} a_i^{-1} \geq 1$.

By the *i-th column K_i generated by P*, where $i \in \{1, 2, 3\}$, we understand the union of all straight lines parallel to E_i which have non-empty intersection with P. Of course,

$P = K_1 \cap K_2 \cap K_3$. Consider the ratio k_i of the width of $C \cap K_i$ to the width of P in the direction perpendicular to F_i, where $i = 1, 2, 3$. Of course, k_i is the outer 1-quermass of $K_i \cap C$ in the direction of E_i.

Observe that if C is centrally symmetric, then there is no longer segment in C in the direction of E_i than the segment being the intersection of C with the straight line passing through the centers of P and of F_i. Thus if C is centrally symmetric, we have $a_i \leq k_i \leq w_i$ for $i = 1, 2, 3$. Consequently, the inequality $k_1^{-1} + k_2^{-1} + k_3^{-1} \geq 1$ is stronger than $a_1^{-1} + a_2^{-1} + a_3^{-1} \geq 1$ but weaker than $w_1^{-1} + w_2^{-1} + w_3^{-1} \geq 1$.

Theorem 2. *Assume that a parallelotope P is inscribed in a three-dimensional centrally-symmetric convex body C. Then*

$$k_1^{-1} + k_2^{-1} + k_3^{-1} \geq 1. \tag{3}$$

P r o o f. As in the proof of Theorem 1 assume that P is a cube with vertices (e_1, e_2, e_3).

Of course, if at least one of the numbers k_1, k_2, k_3 is equal to 1, then (3) holds true. Let

$$k_1 > 1, \quad k_2 > 1, \quad k_3 > 1. \tag{4}$$

The assumption of our Theorem implies, that the boundary of C contains points $x = (x_1, x_2, k_3)$, $y = (y_1, k_2, y_3)$, $z = (k_1, z_2, z_3)$ such that all the coordinates $x_1, x_2, y_1, y_3, z_2, z_3$ are between -1 and 1. What is more, we can assume that x_1 and x_2 are non-negative. Still we can choose the order and the orientation of the axes of our coordinate system which ensures that. From the central symmetry we see that the boundary of C contains also points $-x$, $-y$, $-z$ symmetric with respect to the origin to x, y, z. There are eight triples of points from amongst $x, y, z, -x, -y, -z$ such that exactly one point in each triple is selected from each of the sets $\{x, -x\}$, $\{y, -y\}$, $\{z, -z\}$. From (4) we see that our six points $x, -x, y, -y, z, -z$ are pairwise different. Thus each triple $\{e_1x, e_2y, e_3z\}$ of points, where $e_1, e_2, e_3 \in \{1, 2\}$, determines exactly one plane $\Pi\{e_1x, e_2y, e_3z\}$ passing through them. We obtain eight planes; four pairs of symmetric planes.

We will prove our Theorem by indirect approach. In other words, we assume since now that

$$k_1^{-1} + k_2^{-1} + k_3^{-1} < 1 \tag{5}$$

and our aim is to show that P is not inscribed in C.

In order to prove that P is not inscribed in C it is sufficient to show that at least one of the eight vertices (e_1, e_2, e_3) of P is not in the boundary of C. We will prove this by showing that at least one of the eight planes $\Pi\{e_1x, e_2y, e_3z\}$ has empty intersection with P. Taking into account the central symmetry of C and P, it is sufficient to show that at least one of the following four determinants is positive:

$$d_1 = \det \begin{bmatrix} 1-x_1 & 1-x_2 & 1-k_3 \\ 1-y_1 & 1-k_2 & 1-y_3 \\ 1-k_1 & 1-z_2 & 1-z_3 \end{bmatrix}, \quad d_2 = \det \begin{bmatrix} 1+x_1 & 1+x_2 & 1-k_3 \\ 1-y_1 & 1-k_2 & 1+y_3 \\ 1-k_1 & 1-z_2 & 1+z_3 \end{bmatrix},$$

$$d_3 = \det \begin{bmatrix} 1-x_1 & 1+x_2 & 1-k_3 \\ 1+y_1 & 1-k_2 & 1+y_3 \\ 1-k_1 & 1+z_2 & 1-z_3 \end{bmatrix}, \quad d_4 = \det \begin{bmatrix} 1+x_1 & 1-x_2 & 1-k_3 \\ 1+y_1 & 1-k_2 & 1-y_3 \\ 1-k_1 & 1+z_2 & 1+z_3 \end{bmatrix}.$$

Let

$$p = k_3(y_1 + z_2) - x_1y_3 - x_2z_3,$$
$$q = k_2(x_1 + z_3) - x_2y_1 - y_3z_2,$$
$$r = k_1(x_2 + y_3) - x_1z_2 - y_1z_3,$$
$$k = k_1k_2k_3 - k_1k_2 - k_1k_3 - k_2k_3,$$
$$t = -(k_1 - 1)x_2y_3 - (k_2 - 1)x_1z_3 - (k_3 - 1)y_1z_2 + x_1y_3z_2 + x_2y_1z_3.$$

Since $k = k_1k_2k_3\big(1 - k_1^{-1} - k_2^{-1} - k_3^{-1}\big)$, from (5) we see that $k > 0$. A simple calculation shows that

$$\begin{aligned} d_1 &= p + q + r + k + t, \\ d_2 &= p - q - r + k + t, \\ d_3 &= -p + q - r + k + t, \\ d_4 &= -p - q + r + k + t. \end{aligned}$$

Remember that $x_1 \geq 0$ and $x_2 \geq 0$. Every of the numbers y_1, y_3, z_2, z_3 can be non-negative (which we mark by $+$) or negative (marked by $-$). We have $2^4 = 16$ cases in which every of the numbers y_1, y_3, z_2, z_3 is of a fixed sign. Some of those cases are equivalent in the following sense. We can change by rotation the coordinate system such that the oriented axes exchange their positions. But we agree only for such an exchange after which the new x_1 and x_2 are again non-negative. It is easy to see that we have exactly 6 non-equivalent cases. They are considered in Cases 1–6 below. Successive signs $+$ or $-$ mean that succesive numbers $x_1, x_2, y_1, y_3, z_2, z_3$ are non-negative or negative, respectively.

In every case, we show that at least one of the determinants d_1, d_2, d_3, d_4 is positive. In order to conclude this, we show that a positive linear combination of those four determinants is positive (sometimes the combination is reduced to a single determinant). In every case (besides Case 1), and subcase we present our linear combination as a sum of non-negative components, from which at least one is positive. This positive component is always equal or greater than δk, where $\delta > 0$. We remember that $k > 0$. The positiveness of δk and the non-negativeness of the remaining components follows always from the assumptions of the considered case (or subcase).

Case 1: $++++++$. Of course, if $x_1 = x_2 = y_1 = y_3 = z_2 = z_3 = 0$, then $d_1 = k > 0$. Moreover, the first derivatives of d_1 with respect to every of the six variables are non-negative when the variables are at most 1 (e.q. the derivative with respect to x_1 is $(1 - y_3)(1 - z_2) - (1 - k_2)(1 - z_3) \geq 0$). Thus $d_1 > 0$ in our Case.

Case 2: $++++-+$. We have $2d_1 + d_3 + d_4 = 2q + 2r + 4k + 4t = 2(k_1 - 1)(x_2 + y_3 - 2x_2y_3) + 2x_2(1 - y_1) + 2y_3 - 2x_1z_2(1 - y_3) - 2y_3z_2(1 - x_1) + 2k_2x_1(1 - z_3) + 2(k_2 - 1)z_3(1 - x_1) + 2z_3(1 - y_1) + 2x_1z_3 - 4(k_3 - 1)y_1z_2 + 4x_2y_1z_3 + 4k > 0$.

Case 3: $++++--$. Of course, $d_1 + d_4 = 2r + 2k + 2t = 2k_1(x_2 + y_3 - x_2y_3) + 2x_2y_3 - 2(k_2 - 1)x_1z_3 - 2(k_3 - 1)y_1z_2 + 2x_1z_2(y_3 - 1) + 2y_1z_3(x_2 - 1) + 2k > 0$.

Case 4: $++-+-+$. In view of (5) we have $1 - k_1^{-1} - k_2^{-1} - k_3^{-1} > 0$ and also we have $\big(3 - k_1^{-1} - k_2^{-1} - k_3^{-1}\big)k_1k_2k_3 - (k_2 - 1)x_1z_3 - (k_1 - 1)x_2y_3 - (k_3 - 1)y_1z_2 \geq 2k_1k_2k_3 - (k_2 - 1) - (k_1 - 1) - (k_3 - 1) \geq 2k_1k_2k_3 - k_1k_2 - k_2k_3 - k_2k_3 + 3 = k + k_1k_2k_3 + 3 > 0$. Those two inequalities are used below, where we show that a positive linear combination of d_1, d_2, d_3 is positive: $\big(1 - \frac{1}{2}k_1^{-1} - \frac{1}{2}k_2^{-1}\big)d_1 + \big(1 - \frac{1}{2}k_2^{-1} - \frac{1}{2}k_3^{-1}\big)d_3 + \big(1 - \frac{1}{2}k_1^{-1} - \frac{1}{2}k_3^{-1}\big)d_4 = (k_2 - 1)(x_1 + z_3 - 2x_1z_3) + (k_1 - 1)(x_2 + y_3 - 2x_2y_3) + (k_3 - 1)(-y_1 - z_2 - 2y_1z_2) + \big(1 - k_1^{-1} - k_2^{-1} - k_3^{-1}\big)\big[\big(3 - k_1^{-1} - k_2^{-1} - k_3^{-1}\big)k_1k_2k_3 - (k_2 - 1)x_1z_3 - (k_1 - 1)x_2y_3 - (k_3 - 1)y_1z_2\big] + \big(1 - k_1^{-1}\big)[x_1z_2(y_3 - 1) + y_1z_3(x_2 - 1)] + \big(1 - k_2^{-1}\big)[x_2y_1(z_3 - 1) + y_3z_2(x_1 - 1)] + \big(1 - k_3^{-1}\big)[x_1y_3(z_2 + 1) + x_2z_3(y_1 + 1)] > 0$.

Case 5: $++--+-$. We have $w = -(k_1-1)x_2y_3 - (k_2-1)x_1z_3 - (k_3-1)y_1z_2 \geq 0$.

Subcase 5.1 when $y_1 + z_2 \geq 0$ or when $x_1 + z_3 \geq 0$. The first inequality implies $d_1 + d_2 = 2p + 2k + 2t = 2w + 2x_1y_3(z_2-1) + 2x_2z_3(y_1-1) + 2k_3(y_1+z_2) + 2k > 0$. Analogically, in the second possibility we obtain that $d_1 + d_3 > 0$.

Subcase 5.2 when $x_2 + y_3 \leq 0$. We obtain that $d_2 + d_3 = -2r + 2k + 2t = 2w + 2x_1z_2(y_3+1) + 2y_1z_3(x_2+1) - 2k_1(x_2+y_3) + 2k > 0$

Subcase 5.3 when $y_1 + z_2 < 0$, $x_1 + z_3 < 0$ and $x_2 + y_3 > 0$. The assumptions of this subcase mean that $z_2 < -y_1$, $x_1 < -z_3$ and $-y_3 < x_2$. Since all the six numbers are positive, we obtain $x_1(-y_3)z_2 < (-y_1)x_2(-z_3)$. Thus $x_1y_3z_2 + x_2y_1z_3 > 0$. This, $k > 0$ and $w > 0$ imply the inequality here: $d_1+d_2+d_3+d_4 = 4k+4t = 4w+4x_1y_3z_2+4x_2y_1z_3+4k > 0$.

Case 6: $++----$. We divide Case 6 into six subcases. In subcases 6.1–6.4 we consider d_3+d_4 and we deal with the value $h = -(k_1-1)x_2y_3 - (k_2-1)x_1z_3 + x_1y_3 + x_2z_3$. We always show that $h \geq 0$. This leads to the conclusion that $d_3 + d_4 > 0$. Just we have $d_3+d_4 = -2p+2k+2t = -2k_3y_1 - 2k_3z_2 - 2(k_3-1)y_1z_2 + 2x_1y_3z_2 + 2x_2y_1z_3 + 2h + 2k > 0$.

From (5) we conclude that $k_1 + k_2 - k_1k_2 = k_1k_2\big(k_1^{-1} + k_2^{-1} - 1\big) < 0$. We will apply this inequality in Subcases 6.3–6.6.

Subcase 6.1 when $(k_1-1)^{-1}x_1 \leq x_2 \leq (k_2-1)x_1$. We have $h = y_3[x_1 - (k_1-1)x_2] + z_3[x_2 - (k_2-1)x_1] \geq 0$.

Subcase 6.2 when $(k_1-1)y_3 \leq z_3 \leq (k_2-1)^{-1}y_3$. Analogically like in preceding subcase $h = x_1[y_3 - (k_2-1)z_3] + x_2[z_3 - (k_1-1)y_3] \geq 0$.

Subcase 6.3 when $x_2 \geq (k_2-1)x_1$ and $z_3 \geq (k_2-1)^{-1}y_3$. There exists $\lambda \geq k_2 - 1$ such that $y_3 = \lambda z_3$. We have $h = x_1y_3 + x_2z_3 - (k_1-1)\lambda x_2z_3 - (k_2-1)x_1z_3 = x_1y_3 + x_2z_3[1-(k_1-1)\lambda] - (k_2-1)x_1z_3 \geq x_1y_3 + (k_2-1)x_1z_3[1-(k_1-1)\lambda] - (k_2-1)x_1z_3 = x_1y_3 - (k_1-1)(k_2-1)\lambda x_1z_3 = x_1y_3 - (k_1-1)(k_2-1)x_1y_3 = (k_1+k_2-k_1k_2)x_1y_3 \geq 0$.

Subcase 6.4 when $x_2 \leq (k_1-1)^{-1}x_1$ and $z_3 \leq (k_1-1)y_3$. There exists $\lambda \geq k_1 - 1$ such that $z_3 = \lambda y_3$. We have $h = x_1y_3 + x_2z_3 - (k_1-1)x_2y_3 - (k_2-1)x_1\lambda y_3 = x_1y_3[1 - (k_2-1)\lambda] + x_2z_3 - (k_1-1)x_2y_3 \geq (k_1-1)x_2y_3[1-(k_2-1)\lambda] + x_2z_3 - (k_1-1)x_2y_3 = -x_2z_3(k_1-1)(k_2-1)\lambda + x_2z_3 = (k_1+k_2-k_1k_2)x_2z_3 \geq 0$.

Subcase 6.5 when $x_2 \geq (k_2-1)x_1$ and $z_3 \leq (k_1-1)y_3$. Consider the function $f(u,v) = k_1u - k_2v + uv$ for $u \in [0,1]$ and $v \in [-1,0]$. Looking to the partial derivatives we deduce that if $u_1 \geq u_2$ and $v_1 \leq v_2$, then $f(u_1,v_1) \geq f(u_2,v_2)$. Thus $k_1x_2 - k_2z_3 + x_2z_3 \geq k_1(k_2-1)x_1 - k_2(k_1-1)y_3 + (k_1-1)(k_2-1)x_1y_3$. Moreover, from the assumption of our subcase we get $-(k_1-1)x_2y_3 \geq -(k_1-1)(k_2-1)x_1y_3$ and $-(k_2-1)x_1z_3 \geq -(k_1-1)(k_2-1)x_1y_3$. In view of the above three inequalities we obtain $(-k_2x_1+k_1y_3+x_1y_3)+(k_1x_2-k_2z_3+x_2z_3)-(k_1-1)x_2y_3-(k_2-1)x_1z_3 \geq -k_2x_1+k_1y_3+x_1y_3+k_1(k_2-1)x_1-k_2(k_1-1)y_3+(k_1-1)(k_2-1)x_1y_3+(k_1-1)(k_2-1)x_1y_3-2(k_1-1)(k_2-1)x_1y_3 = (k_1k_2-k_1-k_2)[x_1(1-y_3)-y_3] \geq 0$. Hence $d_4 = (-k_2x_1+k_1y_3+x_1y_3) + (k_1x_2-k_2z_3+x_2z_3) - (k_1-1)x_2y_3 - (k_2-1)x_1z_3 - (k_3-1)[(y_1+1)z_2+y_1] - (1+x_1y_3)z_2 + (x_2-1)y_1(z_3+1) - x_1z_2 + y_3z_2 + k$ is positive.

Subcase 6.6 when $x_2 \leq (k_1-1)^{-1}x_1$ and $z_3 \geq (k_2-1)^{-1}y_3$. We can write the assumptions of this subcase as $x_1 \geq (k_1-1)x_2$ and $y_3 \leq (k_2-1)z_3$. Then the considerations of the preceding case can be repeated for d_3 instead of d_4. ∎

In the course of the proof of Theorem 2, we have shown that if $k_1^{-1} + k_2^{-1} + k_3^{-1} < 1$, then the interior of the octahedron Q with the vertices $x, y, z, -x, -y, -z$ contains at least one vertex of P. Let us formulate this as the first statement in the following Corollary 2.

We only change the notation in order to unify it with the notation of the forthcoming Conjecture 2. The second statement of Corollary 2 easily follows from the first one.

Corollary 2. *Let Q be the octahedron with vertices $p_1 = (p_{11}, p_{12}, p_{13})$, $p_2 = (p_{21}, p_{22}, p_{23})$, $p_3 = (p_{31}, p_{32}, p_{33})$, $-p_1$, $-p_2$, $-p_3$ such that $p_{ii} \geq 1$ for $i = 1, 2, 3$ and $|p_{ij}| \leq 1$ for $i \neq j$. If $p_{11}^{-1} + p_{22}^{-1} + p_{33}^{-1} < 1$, then the interior of Q contains at least one point (e_1, e_2, e_3), where $e_1, e_2, e_3 \in \{-1, 1\}$. Moreover, the weak inequality implies that just Q contains at least one such a point.*

More general, consider the crosspolytope $Q = \text{conv}\{p_1, p'_1, \dots, p_n, p'_n\}$ in E^n, where $p_i = (p_{i1}, \dots, p_{in})$ and $p'_i = (p'_{i1}, \dots, p'_{in})$ such that $p_{ii} \geq 1$, $p'_{ii} \leq -1$ for $i = 1 \dots n$, and such that for every different $i, j \in \{1, \dots n\}$ the inequality $|p_{ij}| \leq 1$ holds true.

The relationship between Theorem 2 and Corollary 2 can be expressed in the more general form: the conjecture that for every convex body C and every parallelotope P inscribed in C we have $\sum_{i=1}^n k_i^{-1} \geq 1$ is equivalent to Conjecture 2 below.

C o n j e c t u r e 2. If $\sum_{i=1}^n (p_{ii} - p'_{ii})^{-1} < 1$, then the interior of the crosspolytope Q contains at least one point of the form $(e_1, \dots, e_n)$, where $e_i \in \{-1, 1\}$ for $i = 1, \dots n$.

Let us show the equivalence by the indirect approach.

Assume that there exists a convex body C and a parallelotope P inscribed in C such that $\sum_{i=1}^n k_i^{-1} < 1$. We may assume that $(e_1, \dots, e_n)$ are the vertices of P (we just apply an affine transformation). Support C by $2n$ hyperplanes parallel to the facets of P. Denote by p_i and p'_i the points of support in $C \cap K_i$ such that $p_{ii} \geq 1$ and $p'_{ii} \leq -1$. Since P is inscribed in C and since $Q \subset C$, no vertex of P in the interior of Q. So Conjecture 2 is false.

On the other hand, if Conjecture 2 is false, then there are points $p_1, p'_1, \dots, p_n, p'_n$ with $p_{ii} \geq 1$, $p'_{ii} \leq -1$ for $i = 1, \dots, n$ and with $|p_{ij}| \leq 1$ for $i \neq j$ such that $\sum_{i=1}^n (p_{ii} - p'_{ii})^{-1} < 1$ and such that the interior of Q does not contain all points of the form $(e_1, \dots, e_n)$, where $e_i \in \{-1, 1\}$. Let C be the convex hull of the set of points consisting from all the 2^n points $(e_1, \dots, e_n)$ and of points $p_1, p'_1, \dots, p_n, p'_n$. Of course, P is inscribed in C but $\sum_{i=1}^n k_i^{-1} < 1$.

From this equivalence and from the inequalities $k_i \leq w_i$ we conclude that Conjecture 1 is stronger than Conjecture 2.

It is easy to show that Conjecture 2 is equivalent to the following related conjecture: $\sum_{i=1}^n (p_{ii} - p'_{ii})^{-1} \leq 1$ implies that the crosspolytope Q contains at least one point of the form $(e_1, \dots, e_n)$, where $e_i \in \{-1, 1\}$.

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