CONNECTEDNESS, DISCONNECTEDNESS, AND LIGHT FACTORIZATION STRUCTURES IN A FUZZY SETTING

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Dedicated in friendship to Guillaume Brümmer and Keith Hardie on the occasion of their 75th and 80th birthdays respectively

Abstract: Connectedness, disconnectedness, and light factorization structures are studied in the realm of the topological constructs **FPUConv** and **FSUConv** of fuzzy preuniform convergence spaces and fuzzy semiuniform convergence spaces respectively which have been introduced by the author in [23] using fuzzy filters in the sense of Eklund and Gähler [7]. The presented theory profits from the fact that both constructs have hereditary quotients. Additionally, there are special features, e.g. a product theorem for the investigated connectedness concept and the existene of a proper class of light factorization structures on **FPUConv** as well as on **FSUConv**.
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0 Introduction

The usual connectedness concept in General Topology goes back to Riesz [25], Lennes [13] and Hausdorff [11]. In 1967 Preuss [18] studied a generalization of connectedness, called $\mathcal{E}$-connectedness, where $\mathcal{E}$ is a class of topological spaces and a topological space X is called $\mathcal{E}$-connected iff each continuous map $f : X \to E$ is constant for each $E \in \mathcal{E}$. Since $\mathcal{E}$ stands for the german word "Eigenschaft" (english: property) we write $\mathcal{P}$ instead of $\mathcal{E}$ in the following.

$\mathcal{P}$-connectedness has been characterized by means of internal properties by Arhangel'skii and Wiegandt [2] in 1975 and even earlier (1972) by Salicrup and Vázquez [26] where the paper of the latter authors was not recognized by many mathematicians since it was written in spanish. If $\mathcal{P}$ consists only of the two-point discrete topological space, then $\mathcal{P}$-connectedness means connectedness in the usual sense. Though $\mathcal{P}$-connectedness makes sense not only for the category **Top** of topological spaces but also for every category $\mathcal{C}$ (where $\mathcal{P}$ is subclass of the object class $|\mathcal{C}|$ of $\mathcal{C}$) it has a special flavour for topological constructs since there exists a two-point discrete object, e.g. if **Unif** denotes the topological construct **Unif** of uniform spaces then $\mathcal{P}$-connectedness generalizes the concept of uniform connectedness introduced by Mrowka and Pervin [16] in 1964 (note: uniform connectedness = $\mathcal{P}$-connectedness provided that $\mathcal{P}$ consists only of the two-point discrete uniform space). Already in 1883 Cantor [4] introduced a concept of connectedness which is applicable to metric spaces. It turned out that uniform connectedness in metric spaces (regarded as uniform spaces) is nothing else than Cantor-connectedness. In particular, the rationals regarded as a uniform space are uniformly connected but they are not connected as a topological space.

Concerning factorization of continuous maps connection properties in **Top** are extremely useful. In 1934 Eilenberg [6] introduced (monotone quotient, light)-factorization as a useful tool for the investigation of connectedness properties of compact metric spaces where a continuous map $f : X \to Y$ between topological spaces is called monotone (light) iff for each $y \in Y$, $f^{-1}(y)$ is connected (totally disconnected). Whyburn [28], Ponomarev [17], Bauer [3], Michael [15], Strecker [27], Salicrup and Vázquez [26], and Dykhoff [5] extended their range to arbitrary T_1-spaces and introduced various modifications and generalizations. But unfortunately (monotone quotient, light) factorizations in their classical sense do not exist in **Top**.

Recently, **Top** (resp. the construct $\mathbf{Top}_S$ of symmetric topological spaces [= R_0-spaces]) and **Unif** have been embedded into better behaved supercategories, e.g. into the strong topological universe **PUConv** (resp. **SUConv**) of preuniform convergence spaces (resp. semiuniform convergence spaces), where **SUConv** is mainly studied in the realm of Convenient Topology and **PUConv** in the framework of non-symmetric convenient topology (cf. [20],[21] and [22]). Thus, many deficiencies of topological and uniform spaces can be remedied.

In the present paper we study connectedness, disconnectedness and light factorization structures in the framework of the strong topological universes **FSUConv** und **FPUConv** of fuzzy semiuniform convergence spaces and fuzzy preunifrom convergence spaces respectively introduced in [23], where a concept of fuzzy filter due to Eklund and Gähler [7] is used which fuzzyficates the membership of filter elements too. By means of a result of Herrlich, Salicrup and Vázquez [26] from 1979 it follows that in topological constructs with hereditary quotients light factorization structures always exist and the only connectedness concept for which this is true is $\mathcal{P}$-connectedness. The relation of $\mathcal{P}$-connectedness and other connectedness concepts in the construct **FTop** of fuzzy topological spaces has been studied in detail by Lowen [14].

It has been mentioned in [19] that an internal characterization of connectedness and disconnectedness can be developed in topological constructs with hereditary quotients analogously to the result of Arhangel'skii and Wiegandt [2] in **Top**. It is proved in this paper that the class of $\mathcal{P}$-disconnected objects in a topological construct $\mathcal{C}$ with hereditary quotients is the object class of the $\{\mathcal{P}$-monotone quotient $\}$ - reflective hull of the full subconstruct $\mathcal{A}$ of $\mathcal{C}$ defined by $|\mathcal{A}| = \mathcal{P}$ – a new result. Since **FSUConv** and **FPUConv** have hereditary quotients (because they are strong topological universes) all these results can be applied to them. Furthermore, it is shown that there is a proper class of light factorization structures on **FSUConv** as well as on **FPUConv**.

Last but not least a product theorem for $\mathcal{P}$-connectedness is proved as a special feature of fuzzy preuniform convergence spaces.

By the way, fundamental constructions in the constructs **FUnif** and **FQUnif** of fuzzy uniform spaces and fuzzy quasi-uniform spaces in the sense of Gähler et al. [10] respectively are performed such as products, subspaces and suprema and the construction of the underlying fuzzy uniform space and the underlying fuzzy quasiuniform space of a fuzzy preuniform convergence space.

Since our results always include the non-fuzzy case by considering fuzzy filters with respect to a frame L (with distinct least element 0 and greatest element 1) only for the trivial case $L = \{0, 1\}$, it should be emphasized that the presentation of connectedness, disconnectedness and light factorization structures is new even for (non-fuzzy) preuniform convergence spaces.

The terminology of this paper corresponds to [1] and [21].

1 Special topics of categorical topology

In this section $\mathcal{C}$ denotes always a topological construct (see [21] for the definition).

1.1 Definition. A class $\mathcal{K} \subset |\mathcal{C}|$ is called a *connection class* (in $\mathcal{C}$) provided the the following are satisfied:

(1) $\{(X,\xi) \in |\mathcal{C}| : \mathrm{card}(X) \leq 1\} \subset \mathcal{K}$.

(2) if $f : (X,\xi) \to (Y,\gamma)$ is a surjective $\mathcal{C}$-morphism and $(X,\xi) \in \mathcal{K}$, then $(Y,\gamma) \in \mathcal{K}$.

(3) Let $(X,\xi) \in \mathcal{C}$ and let $(A_i)_{i\in I}$ be a family of subsets of X with $\bigcap\limits_{i\in I} A_i \neq \emptyset$ such that the subspaces (A_i,ξ_i) of (X,ξ) belong to $\mathcal{K}$ for each $i \in I$. Then $\bigcup\limits_{i\in I} A_i$ (regarded as a subspace of (X,ξ)) belongs to $\mathcal{K}$.

1.2 Remark. Because of (2) and (3), (1) is already fulfilled if $\mathcal{K}$ contains a non-emtpy space. Furthermore, (3) implies that for each $(X,\xi) \in |\mathcal{C}|$ there are maximal subspaces of (X,ξ) belonging to $\mathcal{K}$, the so called $\mathcal{K}$-*components* of (X,ξ) which form a partition of (X,ξ). Reasonable connectedness concepts for $\mathcal{C}$ should constitute a connection class. An important *example* is obtained as follows:
Let $\mathcal{P} \subset |\mathcal{C}|$. Then $(X,\xi) \in |\mathcal{C}|$ is called $\mathcal{P}$-*connected* iff each $\mathcal{C}$-morphism $f : (X,\xi) \to (Y,\gamma)$ is constant for each $(Y,\gamma) \in \mathcal{P}$. The class $C\mathcal{P}$ of all $\mathcal{P}$-connected $\mathcal{C}$-objects is a connection class.

1.3 Definition. Let $\mathcal{K}$ be a connection class in $\mathcal{C}$.

1) A quotient map $f : (X,\xi) \to (Y,\gamma)$ in $\mathcal{C}$ is called $\mathcal{K}$-*monotone* provided that each fibre of f (i.e. each $f^{-1}(y)$ with $y \in Y$ regarded as a subspace of (X,ξ)) belongs to $\mathcal{K}$. The class of all $\mathcal{K}$-monotone quotient maps in $\mathcal{C}$ is denoted by $M\mathcal{K}$.

2) A source $(f_i : (X,\xi) \to (X_i,\xi_i))_{i\in I}$ in $\mathcal{C}$ is called $\mathcal{K}$-*light* iff the fibres of $(f_i)_{i\in I}$ are totally $\mathcal{K}$-disconnected (Note 1° $(Z,\xi) \in \mathcal{C}$ is called totally $\mathcal{K}$-*disconnected* iff for each $z \in Z$ the $\mathcal{K}$-component containing z is singleton $\{z\}$. 2° For each $x \in X$ the subspace of (X,ξ) determined by $\bigcap\limits_{i\in I} f_i^{-1}(f_i(x))$ is called a *fibre* of $(f_i)_{i\in I}$).
The class of all $\mathcal{K}$-light sources in $\mathcal{C}$ is denoted by $L\mathcal{K}$.

3) A factorization structure $(\mathcal{E}, \mathcal{M})$ for sources in $\mathcal{C}$ (cf. [1] for the definition) is called *light* iff $\mathcal{E} = M\mathcal{K}$ and $\mathcal{M} = L\mathcal{K}$ for some connection class $\mathcal{K}$ in $\mathcal{C}$.

1.4 Theorem. (Herrlich, Salicrup and Vázquez [12]) *If quotients in $\mathcal{C}$ are hereditary and $\mathcal{K}$ is a connection class in $\mathcal{C}$, then the following are equivalent:*

(1) $(M\mathcal{K}, L\mathcal{K})$ *is a factorization structure on $\mathcal{C}$.*

(2) $\mathcal{K} = C\mathcal{P}$ for some $\mathcal{P} \subset |\mathcal{C}|$.

1.5 Remark. If $\mathcal{C}$ is extensional, i.e. $\mathcal{C}$ has one-point extensions, then every quotient in $\mathcal{C}$ is hereditary and the above theorem is valid.

Usually, a *connectedness* is a class $\mathcal{K} \subset |\mathcal{C}|$ such that $\mathcal{K} = C\mathcal{P}$ for some $\mathcal{P} \subset |\mathcal{C}|$. The above theorem implies that *there is a bijection between the class of all light factorization structures on $\mathcal{C}$ and the class of all connectednesses in $\mathcal{C}$, whenever $\mathcal{C}$ is extensional.* Since in this paper we are only interested in extensional topological constructs, we make the following **convention**: If the connection class $\mathcal{K}$ is equal to $C\mathcal{P}$ for some $\mathcal{P} \subset |\mathcal{C}|$, then we say *$\mathcal{P}$-monotone, $\mathcal{P}$-light, totally $\mathcal{P}$-disconnected, and $\mathcal{P}$-component* instead of $\mathcal{K}$-monotone, $\mathcal{K}$-light, totally $\mathcal{K}$-disconnected, and $\mathcal{K}$-component respectively.

1.6 Theorem. *Let $\mathcal{C}$ have hereditary quotients. A class $\mathcal{K} \subset |\mathcal{C}|$ is a connectedness iff the following are satisfied:*

(1) *$\mathcal{K}$ is a connection class.*

(2) *If $f : (X, \xi) \to (Y, \gamma)$ is a $\mathcal{K}$-monotone quotient map in $\mathcal{C}$ such that $(Y, \gamma) \in \mathcal{K}$, then $(X, \xi) \in \mathcal{K}$.*

Proof. Similarly to [19; 2.12] or [21; 5.1.17].

1.7 Proposition. *For each $\mathcal{P} \in |\mathcal{C}|$, let $D\mathcal{P} = \{(X, \xi) \in |\mathcal{C}| :$ each $\mathcal{C}$-morphism $f : (Y, \gamma) \to (X, \xi)$ is constant for each $(Y, \gamma) \in \mathcal{P}\}$. Then $D_H\mathcal{P} = DC\mathcal{P}$ is the class of all totally* $\mathcal{P}$-disconnected $\mathcal{C}$-objects.

Proof. Similarly to [21; 5.2.3].

1.8 Definition. A subclass $\mathcal{K} \subset |\mathcal{C}|$ is called a *disconnectedness* provided that there is some $\mathcal{Q} \subset |\mathcal{C}|$ such that $\mathcal{K} = D\mathcal{Q}$.

1.9 Remark. 1. By 1.7, the class of all totally $\mathcal{P}$-disconnected $\mathcal{C}$-objects is a disconnectedness (choose $\mathcal{Q} = C\mathcal{P}$).

2. The operators C and D have the following properties:

1) **(a)** $\mathcal{P} \subset \mathcal{Q} \subset |\mathcal{C}|$ implies $\alpha)$ $C\mathcal{P} \supset C\mathcal{Q}$ and $\beta)$ $D\mathcal{P} \supset D\mathcal{Q}$.
(b) $\mathcal{P} \subset DC\mathcal{P}$ and $\mathcal{P} \subset CD\mathcal{P}$ for each $\mathcal{P} \subset |\mathcal{C}|$.

2) $CDC = C$ and $DCD = D$

3) $C_H = CD$ and $D_H = DC$ are *hull operators*, i.e. C_H and D_H are extensive (cf. 1)(b)), isotonic ($\mathcal{P} \subset \mathcal{Q} \subset |\mathcal{C}|$ implies $C_H\mathcal{P} \subset C_H\mathcal{Q}$ and $D_H\mathcal{P} \subset D_H\mathcal{Q}$), and idempotent ($C_HC_H = C_H$ and $D_HD_H = D_H$).

1.10 Definition. A subclass $\mathcal{K} \subset |\mathcal{C}|$ is called *C_H-closed* (*D_H-closed*) provided that $\mathcal{K} = C_H\mathcal{K}$ ($\mathcal{K} = D_H\mathcal{K}$).

1.11 Proposition. *A subclass $\mathcal{K} \subset |\mathcal{C}|$* is *$C_H$-closed ($D_H$-closed) iff it is a connectedness (disconnectedness).*

1.12 Theorem. *There exists a one-one-correspondence between the connectednesses and disconnectednesses of $\mathcal{C}$ which converts the inclusion relation (Galois correspondence) and is obtained by the operators C and D.*

Proof. By means of C one obtains a one-one-correspondence which assigns to each disconnectedness, i.e. D_H-closed subclass of $\mathcal{C}$, a connectedness, i.e. a $\mathcal{C}_H$-closed subclass of $|\mathcal{K}|$. The inverse correspondence is obtained by D. It follows from 1.9.2.1)(a) (α) and (β) that the inclusion relation is converted.

1.13 Theorem. *Let $\mathcal{C}$ have hereditary quotients and let $\mathcal{P} \subset |\mathcal{C}|$. Then the class $D_H\mathcal{P}$ of all totally $\mathcal{P}$-disconnected $\mathcal{C}$-objects is the object class of the $\mathcal{E}$-reflective hull $\mathcal{E}(\mathcal{A})$ of the full subconstruct $\mathcal{A}$* of *$\mathcal{C}$ defined by $|\mathcal{A}| = \mathcal{P}$, where $\mathcal{E}$ consists of all $\mathcal{P}$-monotone quotient maps in $\mathcal{C}$.*

Proof. Let $\mathcal{E}' = \{\mathcal{P}\text{-submonotone quotient maps in } \mathcal{C}\}$, where a $\mathcal{C}$-morphism $f : (X, \xi) \to (Y, \gamma)$ is called $\mathcal{P}$-submonotone iff each fibre of f is contained in some $\mathcal{P}$-component of (X, ξ). Analogously to [19;3.9], $D_H\mathcal{P}$ is the object class of the $\mathcal{E}'$-reflective hull of $\mathcal{A}$ ($\mathcal{C}$ has hereditary quotients!). Since a $\mathcal{P}-$monotone quotient map is a $\mathcal{P}$-submonotone quotient map, one obtains

1. $|\mathcal{E}'(\mathcal{A})| = D_H\mathcal{P} \subset |\mathcal{E}(\mathcal{A})|$, where $\mathcal{E}(\mathcal{A})$ exists because $\mathcal{C}$ has hereditary quotients (and thus ($\mathcal{P}$-monotone quotients, $\mathcal{P}$-light sources) is a

factorization structure for $\mathcal{C}$). If $\mathcal{B}$ denotes the full and isomorphism-closed subconstruct of $\mathcal{C}$ defined by $|\mathcal{B}| = D_H\mathcal{P}$, then $\mathcal{B}$ is extremal epireflective in $\mathcal{C}$ and the fibres of the $\mathcal{B}$-reflection r_X of $(X,\xi) \in |\mathcal{C}|$ are exactly the $\mathcal{P}$-components of (X,ξ) (analogously to [19;3.6], since $\mathcal{C}$ has hereditary quotients), i.e. r_X is a $\mathcal{P}$-monotone quotient map. Consequently, $\mathcal{B}$ is a ($\mathcal{P}$-monotone quotient)-reflective subconstruct of $\mathcal{C}$ which contains $\mathcal{A}$. Therefore, $\mathcal{E}(\mathcal{A}) \subset \mathcal{B}$, i.e.

2. $|\mathcal{E}(\mathcal{A})| \subset D_H\mathcal{P}$.

The assertion follows immediately from 1. and 2.

1.14 Corollary. *Let $\mathcal{C}$ have hereditary quotients and let $\mathcal{P} \subset |\mathcal{C}|$. Then the following are equivalent:*

1. $(X,\xi) \in D_H\mathcal{P}$.

2. *(X,ξ) is a $\mathcal{P}$-light subobject*[1] *of a product of objects from $\mathcal{P}$.*

3. *There is a $\mathcal{P}$-light source $(f_i : (X,\xi) \to (X_i,\xi_i))_{i\in I}$ such that $(X_i,\xi_i) \in \mathcal{P}$ for each $i \in I$, where I is a class.*

Proof. Since $\mathcal{C}$ is a ($\mathcal{P}$- monotone quotient, $\mathcal{P}$-light source)-category which has products and is ($\mathcal{P}$-monotone quotient)-co-wellpowered, the assertion follows form 1.13 and the general theory of reflections (cf. [1; 16.8 and 16.22]).

1.15 Theorem. *Let $\mathcal{C}$ have hereditary quotients and let $\mathcal{P}$ be an isomorphism-closed subclass of $|\mathcal{C}|$ containing a non-empty space. Then the following are equivalent:*

1. *$\mathcal{P}$ is a disconnectedness.*

2. $\mathcal{P} = D_H\mathcal{P}$.

3. *$\mathcal{P}$ is closed under formation of $\mathcal{P}$-light subobjects*[1] *and products in $\mathcal{C}$.*

4. *$\mathcal{P}$ is closed under formation of $\mathcal{P}$-light sources in $\mathcal{C}$, i.e. whenever $(f_i : (X,\xi) \to (X_i,\xi_i))_{i\in I}$ is a $\mathcal{P}$-light source in $\mathcal{C}$ such that $(X_i,\xi_i) \in \mathcal{P}$ for each $i \in I$, then $(X,\xi) \in \mathcal{P}$.*

5. (a) *$\mathcal{P}$ is closed under formation of products and subspaces in $\mathcal{C}$.*

[1] (X,ξ) is a *$\mathcal{P}$-light subobject of* (Y,γ) *means that there is a $\mathcal{P}$-light $\mathcal{C}$-morphism* $f : (X,\xi) \to (Y,\gamma)$.

(b) *Whenever* $f : (X, \xi) \to (Y, \gamma)$ *is a surjective* $\mathcal{C}$*-morphism such that* $(Y, \gamma) \in \mathcal{P}$ *and each fibre of* f *belongs to* $\mathcal{P}$, *then* $(X, \xi) \in \mathcal{P}$.

Proof. Since $\mathcal{C}$ is a ($\mathcal{P}$-monotone quotient, $\mathcal{P}$-light source)-category which has products and is ($\mathcal{P}$-monotone quotient)-co-wellpowered, the full and isomorphism-closed subconstruct $\mathcal{A}$ of $\mathcal{C}$ defined by $|\mathcal{A}| = \mathcal{P}$ is ($\mathcal{P}$-monotone quotient)-reflective iff one of the two equivalent conditions 3. and 4. is fulfilled (cf. [1; 16.8]. By 1.13, this is true iff $\mathcal{P} = D_H\mathcal{P}$. The equivalence of $\mathcal{P} = D_H\mathcal{P}$ and 5. is proved analogously to [19;3.10] because $\mathcal{C}$ has hereditary quotients. The equivalence of 1. and 2. is obvious (cf. 1.11).

2 Some fuzzy concepts and results

In this section and the following let L be a frame with different least element 0 and greatest element 1, e.g. $L = \{0, 1\}$ or $L = [0, 1]$(= closed unit interval). For each set X, the *characteristic function* $\chi_A : A \to L$ from a subset A of X to L is defined by $\chi_A(x) = \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ if } x \in X \setminus A \end{cases}$.

2.1 Remark. For each set X, L^X can be endowed with a partial order $\leq$ defined as follows:

$$f \leq g \text{ iff } f(x) \leq g(x) \text{ for each } x \in X.$$

As in L, for infima and suprema in L^X the symbols $\wedge$ and $\bigwedge$ as well as $\vee$ and $\bigvee$ will be used respectively, e.g. for each pair $(f, g) \in L^X \times L^X$ and each $x \in L$, $(f \wedge g)(x) = f(x) \wedge g(x)$ and $(f \vee g)(x) = f(x) \vee g(x)$.

2.2 Definition. An *L-fuzzy filter* (shortly: a *fuzzy filter*) on a non-empty set is a map $\mathcal{F} : L^X \to L$ such that the following are satisfied:

FFil$_1$ $\mathcal{F}(\bar{l}) = l$ for each $l \in L$, where $\bar{l} : X \to L$ is defined by $\bar{l}(x) = l$ for each $x \in X$.

FFil$_2$ $\mathcal{F}(f \wedge g) = \mathcal{F}(f) \wedge \mathcal{F}(g)$ for all $f, g \in L^X$.

The set of all L-fuzzy filters on X is denoted by $F_L(X)$, where $F_L(\emptyset) = \emptyset$.

2.3 Remark. 1. If $\mathcal{F}$ is a fuzzy filter on X, then $\mathcal{F}(f) \leq \mathcal{F}(g)$ for all $f, g \in L^X$ such that $f \leq g$. Furthermore, for each $f \in L^X$, $\mathcal{F}(f) \leq \sup f = \sup\{f(x) : x \in X\}$.

2. For each $x \in X$, there is a fuzzy filter $\dot{x} : L^X \to L$ defined by $\dot{x}(f) = f(x)$ for each $x \in X$.

3. If $\mathcal{F}$ and $\mathcal{G}$ are fuzzy filters on X, then $\mathcal{F}$ is called *coarser* than $\mathcal{G}$ (or $\mathcal{G}$ is called *finer* then $\mathcal{F}$) denoted by $\mathcal{F} \subset \mathcal{G}$, iff $\mathcal{F}(f) \leq \mathcal{G}(f)$ for each $f \in L^X$.

2.4 Definition. A *fuzzy filter base* on a non-empty set X is a non-empty subset $\mathcal{B}$ of L^X such that the following are satisfied:

FB$_1$ $\bar{l} \in \mathcal{B}$ for each $l \in L$.

FB$_2$ For each $(f, g) \in \mathcal{B} \times \mathcal{B}$ there is some $h \in \mathcal{B}$ such that $h \leq f \wedge g$ and $\sup h = \sup f \wedge \sup g$.

2.5 Remark. Each fuzzy filter base $\mathcal{B}$ on X generates a fuzzy filter $\mathcal{F}$ on X defined by

$$\mathcal{F}(f) = \bigvee_{g \leq f, g \in \mathcal{B}} \sup g \text{ for each } f \in L^X .$$

Conversely, each fuzzy filter $\mathcal{F}$ on X can be generated by a fuzzy filter base on X, even a greatest one, denoted by base$\mathcal{F}$, where base$\mathcal{F} = \{f \in L^X : \mathcal{F}(f) = \sup f\}$.

2.6 Proposition. *Let $f : X \to Y$ be a map, $\mathcal{F}$ a fuzzy filter on X, and $\mathcal{B}$ a* base *of $\mathcal{F}$. Define for each $g \in L^X$, $f[g] \in L^Y$* by $f[g](y) = \bigvee_{x \in f^{-1}(y)} g(x)$. *Then $\{f[g] : g \in \mathcal{B}\} \cup \{\bar{l} : l \in L\}$ is a* base *of the fuzzy filter $f(\mathcal{F})$, defined by $f(\mathcal{F})(h) = \mathcal{F}(h \circ f)$ for each $h \in L^X$, where $f(\mathcal{F})$ is called the image of $\mathcal{F}$ under f. If f is surjective, then $\{f[g] : g \in \mathcal{B}\}$ is a* base *of $f(\mathcal{F})$.*

2.7 Definition. Let $f : X \to Y$ be a map and $\mathcal{F}$ a fuzzy filter on Y. Then the *inverse image of $\mathcal{F}$ under f* is the coarsest fuzzy filter $\mathcal{G}$ on X such that $\mathcal{F} \subset f(\mathcal{G})$ provided that it exists. Usually, we write $f^{-1}(\mathcal{F})$ instead of $\mathcal{G}$. If $X \subset Y$ and $i : X \to Y$ denotes the inclusion map, then $i^{-1}(\mathcal{F})$ is also called the *trace* of $\mathcal{F}$ on X.

2.8 Proposition. (cf. [8; proposition 9]). *Let $f : X \to Y$ be a map, $\mathcal{F}$ a fuzzy filter on Y, and $\mathcal{B}$* a base *of $\mathcal{F}$. Then $f^{-1}(\mathcal{F})$ exists iff* $\sup g = \sup g \circ f$ *for each $g \in \mathcal{B}$.* If *$f^{-1}(\mathcal{F})$ exists, then $\{g \circ f : g \in \mathcal{B}\}$ is a* base *of $f^{-1}(\mathcal{F})$.*

2.9 Definition. 1. Let M be a non empty set of fuzzy filters on X. Then a fuzzy filter $\bigcap_{\mathcal{F} \in M} \mathcal{F}$, called the *intersection* of all $\mathcal{F} \in M$, is defined by $\bigcap_{\mathcal{F}\in M}\mathcal{F}(f)=\bigwedge_{\mathcal{F}\in M}\mathcal{F}(f)$ for each $f\in L^X$.

2. Let $(X_i)_{i\in I}$ be a non-empty family of non-empty sets, and $\mathcal{F}_i$ a fuzzy filter on X_i for each $i\in I$. If $p_i:\prod_{i\in I}X_i\to X_i$ denotes the i-th projection, then the coarsest fuzzy filter $\mathcal{F}$ on $\prod_{i\in I}X_i$ such that $p_i(\mathcal{F})=\mathcal{F}_i$ for each $i\in I$ is called the *product of* $(\mathcal{F}_i)_{i\in I}$, where $\prod_{i\in I}\mathcal{F}_i$ is written instead of $\mathcal{F}$, or $\mathcal{F}_1\times\mathcal{F}_2$ in case $I=\{1,2\}$.

2.10 Proposition. (cf. [8; proposition 19]). *If I is a non-empty set and for $i\in I$, $\mathcal{F}_i$ is a fuzzy filter on X_i, and $\mathcal{B}_i$ is a* base of $\mathcal{F}_i$, *then $\mathcal{B}=\{\bigwedge_{j\in J}f_j\circ p_j : J\subset I$ finite and $f_j\in B_j$ for all $j\in J\}$ is a fuzzy filter* base *on $\prod_{i\in I}X_i$ generating the product $\prod_{i\in I}\mathcal{F}_i$* of $(\mathcal{F}_i)_{i\in I}$.

2.11 Definitions. 1. A *fuzzy topological space* is a pair (X,t), where X is a set and $t\subset L^X$ a *fuzzy topology* on X, i.e. the following are satisfied:

FTop$_1$ All constant maps from X to L belong to t (this includes the empty map $\emptyset:\emptyset\to L$ in case $X=\emptyset$).

FTop$_2$ $f,g\in t$ imply $f\wedge g\in t$.

FTop$_3$ $s\subset t$ implies $\bigvee s\in t$.

If t is a fuzzy topology on X, the elements of t are called *fuzzy open sets* of X.

2. A map $f:(X,t)\to(X',t')$ between fuzzy topological spaces is called *fuzzy continuous* provided that $f^{-1}(g')=g'\circ f\in t$ for each $g'\in t'$.

3. Let (X,t) be a fuzzy topological space, and $f\in L^X$. Then the *interior* of f with respect to t, denoted by $int_t f$ is defined as follows:

$$int_t f=\bigvee_{g\leq g,g\in t} g.$$

2.12 Remarks. 1. If (X,t) is a fuzzy topological space, then for each $x\in X$, a fuzzy filter $\mathcal{U}_t(x):L^X\to L$ is defined as follows:

$$U_t(x)(f)=(int_t f)(x)\text{ for each }f\in L^X,$$

called the *fuzzy neighborhood filter* of x *with respect to* t.

2. If $(X, \mathcal{X})$ is an ordinary topological space and L is a complete chain, then $t_{\mathcal{X}} = \{f \in L^X : f$ lower semicontinuous $\}$ is a fuzzy topology on X ($f \in L^X$ is lower semicontinuous iff for each $\alpha \in L, \{x \in X : f(x) > \alpha\}$ is open in $(X, \mathcal{X})$ i.e. $f : (X, \mathcal{X}) \to (L, \mathcal{L})$ is continuous, where the socalled *lower topology* $\mathcal{L}$ on L has the set $\{\{\beta \in L : \beta > \alpha\} : \alpha \in L\}$ as a subbase .).

2.13 Definition. A fuzzy topological space (X, t) is called a *topological fuzzy topological space* provided that there is a topology $\mathcal{X}$ on X such that $t_{\mathcal{X}} = t$.

2.14 Theorem. *Let L be a complete chain. Then the construct* **Top** *of topological spaces (and continuous maps) is (concretely) isomorphic to the construct* **TFTop** *of topological fuzzy topological spaces (and fuzzy continuous maps).*

Proof. 1. Let $(X, \mathcal{X})$ be a topological space. Then $O \in \mathcal{X}$ iff its characteristic function $\chi_O : X \to L$ is lower semicontinuous.

2. A map $f : (X, \mathcal{X}) \to (Y, \mathcal{Y})$ between topological spaces is continuous iff $f : (X, t_{\mathcal{X}}) \to (Y, t_{\mathcal{Y}})$ is fuzzy continuous.

3. The functor F :**Top** $\to$ **TFTop** defined by $F(X, \mathcal{X}) = (X, t_{\mathcal{X}})$ and $F(f) = f$ (cf. 2.) is a concrete isomorphism:
If for each fuzzy topological space (X, t) a topological space $(X, \mathcal{X}_t)$ is defined by $\mathcal{X}_t = \{0 \subset X : \chi_O \in t\}$,then a functor G : **TFTop** $\to$ **Top** is defined by $G(X, t) = (X, \mathcal{X}_t)$ and $G(f) = f$ (cf. 2. and note that by 1., $\mathcal{X}_{t_{\mathcal{X}}} = \mathcal{X}$ for each topology $\mathcal{X}$ on X). Obviously, $G \circ F = I_{\mathbf{Top}}$ and $F \circ G = I_{\mathbf{TFTop}}$, where for each construct $\mathcal{C}$, $I_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ denotes the identity functor.

2.15 Definition. 1. (a) Let X be a on empty set and $\mathcal{U}$ a fuzzy filter on $X \times X$. Consider the following conditions:

FU$_1$) $\mathcal{U} \subset (\dot{x, x})$ for each $x \in X$.

FU$_2$) $\mathcal{U} = \mathcal{U}^{-1}$, where $\mathcal{U}^{-1}(u) = \mathcal{U}(u^{-1})$ for each $u \in L^X$ and $u^{-1}(x, y) = u(y, x)$ for each $(x, y) \in X \times X$.
If FU$_1$) is fulfilled, consider also

FU$_3$) $\mathcal{U} \subset \mathcal{U} \circ \mathcal{U}$ where $\mathcal{U} \circ \mathcal{U}$ is the fuzzy filter on $X \times X$ defined by $\mathcal{U} \circ \mathcal{U}(u) = \bigvee_{v \in \text{base}\, \mathcal{U}, v \circ v \leq u} \mathcal{U}(v)$ for each $u \in L^{X \times X}$, and for each $v \in L^{X \times X}$ the composition $v \circ v$ is defined by $v \circ v(x, y) =$

$\bigvee_{x \in X} v(x,z) \wedge v(z,y)$ for each $(x,y) \in X \times X$. If $\mathcal{U}$ fulfills FU_1) and FU_3), then $\mathcal{U}$ is called a *fuzzy quasiuniformity* on X, and if $\mathcal{U}$ fulfills all three conditions FU_1), FU_2) and FU_3) it is called a *fuzzy uniformity* on X.

(b) A *fuzzy (quasi) uniformity* on the empty set $\emptyset$ is a map $\mathcal{U} : L^\emptyset \to L$, where $L^\emptyset = \{\emptyset\}$, such that $\mathcal{U}(\emptyset) = 1$.

(c) A *fuzzy (quasi) uniform space* is a pair $(X,\mathcal{U})$ where X is a set and $\mathcal{U}$ is a fuzzy (quasi) uniformity on X.

2. A map $f : (X,\mathcal{U}) \to (X',\mathcal{U}')$ between fuzzy quasiuniform spaces is called *fuzzy uniformity continuous* iff $\mathcal{U}' \subset (f \times f)(\mathcal{U})$.

2.16 Remarks. 1. In case $L = \{0,1\}$ fuzzy (quasi) uniform spaces may be identified with the usual (quasi) uniform spaces.

2. As in the non-fuzzy case the composition of fuzzy filters $\mathcal{F}$ and $\mathcal{G}$ on $X \times X$ need not exist as a fuzzy filter. But if there are $x, y, z \in X$ such that $\mathcal{F} \subset (\dot{x,y})$ and $\mathcal{G} \subset (\dot{y,z})$ then a fuzzy filter $\mathcal{G} \circ \mathcal{F}$ on $X \times X$, called the *composition of* $\mathcal{F}$ and $\mathcal{G}$, can be defined according to [10] as follows:

$$\mathcal{G} \circ \mathcal{F}(w) = \bigvee_{v \circ u \leq w} \mathcal{F}(u) \wedge \mathcal{G}(v) \text{ for each } w \in L^{X \times X}$$

where the composition $v \circ u \in L^{X \times X}$ of $u, v \in L^{X \times X}$ is defined by

$$v \circ u(x',y') = \bigvee_{z \in X} u(x',z) \wedge v(z,y') \text{for each } (x',y') \in X \times X.$$

If $\mathcal{F} \subset (\dot{x,x})$ and $\mathcal{G} \subset (\dot{x,x})$ for each $x \in X$, then for each $w \in L^{X \times X}$,

$$\mathcal{G} \circ \mathcal{F}(w) = \bigvee_{u \in \text{ base } \mathcal{F}, v \in \text{ base } \mathcal{G}, v \circ u \leq w} \sup(v \circ u),$$

where $\sup(v \circ u) = \sup u \wedge \sup v$ (cf. [10;12.12]). If furthermore, $\mathcal{F} = \mathcal{G}$ such that $\mathcal{F} \subset (\dot{x,x})$ for each $x \in X$, then

$$\mathcal{F} \circ \mathcal{F}(w) = \bigvee_{u' \in \text{ base } \mathcal{F}, u' \circ u' \leq w} \sup u'$$

which implies that $\{u' \circ u' : u' \in \text{ base} \mathcal{F}\}$ is a base of $\mathcal{F} \circ \mathcal{F}$, since $\sup u' \circ u' = \sup u'$.

2.17 Proposition. *Let $f : X \to Y$ be a map and $\mathcal{F}, \mathcal{G}$ fuzzy filters on X such that there are $x, y, z \in X$ with $\mathcal{F} \subset \dot{(x,y)}$ and $\mathcal{G} \subset \dot{(y,z)}$. Then $(f \times f)(\mathcal{G} \circ \mathcal{F}) \supset (f \times f)(\mathcal{G}) \circ (f \times f)(\mathcal{F})$.*

Proof. Let $w' \in L^{Y \times Y}$. Then $(f \times f)(\mathcal{G} \circ \mathcal{F})(w') = \mathcal{G} \circ \mathcal{F}(w' \circ (f \times f))$
$= \bigvee_{v \circ u \leq w' \circ (f \times f)} \mathcal{F}(u) \wedge \mathcal{G}(v) \geq \bigvee_{v' \circ u' \leq w'} \mathcal{F}(u' \circ (f \times f)) \wedge \mathcal{G}(v \circ (f \times f))$
$= (f \times f)(\mathcal{G}) \circ (f \times f)(\mathcal{F})(w')$ since $v' \circ u' \leq w'$ implies $v \circ u \leq w' \circ (f \times f)$ with $u = u' \circ (f \times f)$ and $v = v' \circ (f \times f)$. The existence of $(f \times f)(\mathcal{G}) \circ (f \times f)(\mathcal{F})$ is obvious.

2.18 Definition.

1. A *fuzzy preuniform convergence space* is a pair (X, FJ_X) where X is a set and FJ_X a set of fuzzy filters on $X \times X$ such that the following are satisfied:

   FUC$_1$) $\dot{(x,x)} \in FJ_X$ for each $x \in X$.

   FUC$_2$) $\mathcal{F} \in FJ_X$ whenever $\mathcal{G} \in FJ_X$ and $\mathcal{G} \subset \mathcal{F}$.

2. A fuzzy preuniform convergence space (X, FJ_X) is called a *fuzzy semiuniform convergence space* provided that the following is satisfied:

   FUC$_3$) $\mathcal{F} \in FJ_X$ implies $\mathcal{F}^{-1} \in FJ_X$.

3. A map $f : (X, FJ_X) \to (Y, FJ_Y)$ is called *fuzzy uniformly continuous* iff $(f \times f)(\mathcal{F}) \in FJ_Y$ for each $\mathcal{F} \in FJ_X$.

4. The construct of fuzzy preuniform convergence spaces (and fuzzy uniformly continuous maps) and its full subconstruct of fuzzy semiuniform convergence spaces is denoted by **FPUConv** and **FSUConv** respectively.

5. (a) A fuzzy preuniform convergence space (X, FJ_X) is called *(quasi) uniform* provided that there is a (quasi) uniformity $\mathcal{U}$ on X such that $FJ_X = [\mathcal{U}]$, where $[\mathcal{U}] = \{\mathcal{F} \in F_L(X \times X) : \mathcal{U} \subset \mathcal{F}\}$.

   (b) A fuzzy preuniform convergence space (X, FJ_X) is called *topological* provided the there is a fuzzy topology t on X such that $FJ_X = \{\mathcal{F} \subset F_L(X \times X)$: there is some $x \in X$ with $\mathcal{F} \supset \dot{x} \times \mathcal{U}_t(x)\}$.

2.19 Remark.

1. Obviously, every uniform fuzzy preuniform convergence space is a fuzzy semiuniform convergence space.

2. It has been mentioned in [24] that the construct **FUnif** (**FQUnif**) of fuzzy uniform (fuzzy quasiuniform) spaces (and fuzzy uniformly continuous maps) is concretely isomorphic to the construct **UFPU-Conv** (**QUFPUConv**) of uniform (quasiuniform) fuzzy preuniform convergence spaces (and fuzzy uniformly continuous maps), where the proove corresponds to the non-fuzzy case, and it has been proved that the construct **FTop** of fuzzy topological spaces (and fuzzy continuous maps) is concretely isomorphic to the construct **TFPUConv** of topological fuzzy preuniform convergence spaces. In the following we will prove that **FUnif** and **FQunif** are topological constructs and that every fuzzy preuniform convergence space has an underlying fuzzy quasi uniform space and underlying fuzzy uniform space. By [23], the initial structures in **FPUConv** are formed as follows: If X is a set, $(X_i, FJ_{X_i})_{i\in I}$ a family of fuzzy preuniform convergence spaces, and $(f_i : X \to X_i)_{i\in I}$ a family of maps, then $FJ_X = \{\mathcal{F} \in F_L(X \times X) : (f_i \times f_i) \in FJ_{X_i}$ for each $i \in I\}$ is the initial **FPUConv**-structure on X with respect to the given data.

2.20 Proposition. *Let $(Y, \mathcal{V})$ be a fuzzy (quasi) uniform space, X a non-empty set and $f : X \to Y$ a map. Then $(f \times f)^{-1}(\mathcal{V})$ exists and is the coarsest fuzzy (quasi) uniformity on X such that $f : (X, (f \times f)^{-1}(\mathcal{V})) \to (Y, \mathcal{V})$ is fuzzy uniformly continuous.*

Proof. Let $x \in X$. Since $\mathcal{V}$ fulfills FU_1), $\mathcal{V} \subset (f(x) \dot{,} f(x)) = f(\dot{x}) \times f(\dot{x}) = f(\dot{x}) \times f(\dot{x}) = (f \times f)(\dot{x} \times \dot{x})$. Consequently, $(f \times f)^{-1}(\mathcal{V})$ exists (cf. [24; 1.11]). If $\mathcal{V}$ is a quasiuniformity, then we obtain the following:

FU_1) For each $x \in X$, $\mathcal{V} \subset (f(x) \dot{\times} f(x)) \subset (f \times f)(\dot{x} \times \dot{x})$, which implies

$(f \times f)^{-1}(\mathcal{V}) \subset (f \times f)^{-1}((f \times f)(\dot{x} \times \dot{x})) \subset \dot{x} \times \dot{x} = (x \dot{\times} x)$.
FU_3) By 2.17,
$(f \times f)((f \times f)^{-1}(\mathcal{V}) \circ (f \times f)^{-1}(\mathcal{V})) \supset (f \times f)((f \times f)^{-1}(\mathcal{V})) \circ (f \times f)((f \times f)^{-1}(\mathcal{V})) \supset \mathcal{V} \circ \mathcal{V} = \mathcal{V}$ since $\mathcal{V} \circ \mathcal{V} \subset \mathcal{V}$ because of $\mathcal{V} \circ \mathcal{V}(u) = \bigvee_{v \circ v \leq u, v \in \text{base}\,\mathcal{V}} \mathcal{V}(v) \leq \mathcal{V}(u)$ [note: $v \leq v \circ v$ for each $v \in \text{base}\,\mathcal{V}$] for each $u \in L^{Y \times Y}$ and $\mathcal{V} \subset \mathcal{V} \circ \mathcal{V}$ by assumption. Thus, $(f \times f)^{-1}(\mathcal{V}) \subset (f \times f)^{-1}(\mathcal{V}) \circ (f \times f)^{-1}(\mathcal{V})$. If $\mathcal{V}$ is a fuzzy uniformity, then $\mathcal{V} = \mathcal{V}^{-1}$ and wo obtain additionally the following:
FU_2) $((f \times f)^{-1}(\mathcal{V}))^{-1} = (f \times f)^{-1}(\mathcal{V}^{-1}) = (f \times f)^{-1}(\mathcal{V})$ [since $\{h^{-1} : h \in \text{base}\,\mathcal{V}\}$ is a base of $(f \times f)^{-1}(\mathcal{V}^{-1})$ as well as a base of $((f \times f)^{-1}(\mathcal{V}))^{-1}$ because $(h \circ (f \times f))^{-1} = h^{-1} \circ (f \times f)$ for each $h \in$ base $\mathcal{V}$].
The remaining part of the proove is obvious.

2.21 Corollary. *Let $(X,[\mathcal{U}])$ be a (quasi) uniform fuzzy preuniform convergence space, $Y \subset X$ and $i : Y \to X$ the inclusion map. Then the subspace (Y, FJ_Y) of $(X,[\mathcal{U}])$ formed in* **FPUConv** *is (quasi) uniform. In particular, $(i \times i)^{-1}(\mathcal{U})$ is a (quasi) uniformity on Y and $FJ_Y = [(i \times i)^{-1}(\mathcal{U})]$.*

Proof. It suffices to prove that $(i \times i)^{-1}(\mathcal{U})$ exists and is a fuzzy (quasi) uniformity on Y. But this follows from 2.20 in case $f = i$ provided that Y is non-empty. If $Y = \emptyset$ there is nothing to prove.

2.22 Proposition. *Let $((X_i,[\mathcal{U}_i]))_{i\in I}$ be a family of (quasi) uniform fuzzy preuniform convergence spaces. Then the product space (X, FJ_X) of this family formed in* **FPUConv** *is (quasi) uniform. In particular, if all X_i are non-empty, then $FJ_X = [j(\prod\limits_{i\in I}\mathcal{U}_i)]$, where $j : \prod\limits_{i\in I} X_i \times X_i \to \prod\limits_{i\in I} X_i \times \prod\limits_{i\in I} X_i$ is the canonical isomorphism with $(p_i \times p_i) \circ j = p'_i$ for each $i \in I$ and $p_i : \prod\limits_{i\in I} X_i \to X_i$, $p'_i : \prod\limits_{i\in I} X_i \times X_i \to X_i \times X_i$ denote the i-th projections.*

Proof. It suffices to prove that $j(\prod \mathcal{U}_i)$ is a (quasi) uniformity on X: If all $\mathcal{U}_i$ are fuzzy quasiuniformities, we have the following:

FU$_1$) By assumption, for each $i \in I$, $\mathcal{U}_i \subset (x_i \dot{,} x_i)$ for each $x_i \in X_i$. Thus,

$\prod \mathcal{U}_i \subset \prod (x_i \dot{,} x_i) \subset ((x_i \dot{,} x_i)) = j^{-1}(((x_i)\dot{,}(x_i)))$, which implies

$j(\prod \mathcal{U}_i) \subset (((x_i)\dot{,}(x_i)))$ for each $(x_i) \in \prod X_i$ (note: $\prod \dot{x}_i \subset (\dot{x_i})$ is always valid, since $p_i((\dot{x_i})) = \dot{x}_i$ for each $i \in I$ and $\prod \dot{x}_i$ is the coarsest filter with this property).

FU$_3$) By 2.17 we have for each $i \in I$,
$p'_i(j^{-1}(j(\prod \mathcal{U}_i) \circ j(\prod \mathcal{U}_i))) = p_i \times p_i(j(\prod \mathcal{U}_i) \circ j(\prod \mathcal{U}_i))$
$\supset p_i \times p_i(j(\prod \mathcal{U}_i)) \circ p_i \times p_i(j(\prod \mathcal{U}_i) = p'_i((\prod \mathcal{U}_i) \circ p'_i((\prod \mathcal{U}_i)$
$= \mathcal{U}_i \circ \mathcal{U}_i \supset \mathcal{U}_i$, which implies
$\prod \mathcal{U}_i \subset \prod p'_i(j^{-1}(j(\prod \mathcal{U}_i) \circ j(\prod \mathcal{U}_i))) \subset j^{-1}(j(\prod \mathcal{U}_i) \circ j(\prod \mathcal{U}_i))$.
Thus, $j(\prod \mathcal{U}_i) \circ j(\prod \mathcal{U}_i) \supset j(\prod \mathcal{U}_i)$.
If all $\mathcal{U}_i$ are fuzzy uniformities, we obtain additionally the following:
FU$_2$): Consider the following base $\mathcal{B}$ of $\prod \mathcal{U}_i$:
$\mathcal{B} = \{ \bigwedge\limits_{k\in K} u_k \circ p'_k : K \subset I \text{ finite}, u_k \in \text{base}\,\mathcal{U}_k \text{ for all } k \in K\}$.
Then $\mathcal{B}^* = \{j[u] : u \in \mathcal{B}\}$ is a base of $j(\prod \mathcal{U}_i)$, where $j[u]((x_i),(y_i)) = u((x_i, y_i))$ since j is bijective. Furthermore, $\mathcal{B}^{*-1} = \{(j[u])^{-1} : u \in \mathcal{B}\}$ is a base of $(j(\prod \mathcal{U}_i))^{-1}$. Since for each $i \in I$, $\mathcal{U}_i = \mathcal{U}_i^{-1}$,

$$(*)\ u_i \in \text{base}\,\mathcal{U}_i \text{ iff } u_i^{-1} \in \text{base}\,\mathcal{U}_i.$$

If $K \subset I$ is finite and $u_k \in \text{base}\,\mathcal{U}_k$ for each $k \in K$,

$$(j[\bigwedge_{k\in K} u_k \circ p'_k])^{-1} = j[\bigwedge_{k\in K} u_k^{-1} \circ p'_k]$$

which is easily checked (note: $j[\bigwedge_{k\in K} u_k \circ p'_k]((x_i),(y_i)) = \bigwedge_{k\in K} u_k(x_k, y_k)$). Consequently, $v \in \mathcal{B}^*$ implies $v^{-1} \in \mathcal{B}^*$ since $(*)$ is valid. Hence, $\mathcal{B}^* = \mathcal{B}^{*-1}$ (note: $(v^{-1})^{-1} = v$) and $j(\prod \mathcal{U}_i) = (j \prod \mathcal{U}_i)^{-1}$. The remaining cases $I = \emptyset$ or $X_i = \emptyset$ for some $i \in I$ are obvious.

2.23 Proposition. ([10;13.5]). *Let $\mathcal{B} \subset L^{X\times X}$. Then $\mathcal{B}$ is the greatest* base *of a fuzzy uniformity $\mathcal{U}$ on X, i.e. $\mathcal{B} = \text{base}\,\mathcal{U}$, if and only if the following are satisfied:*

1. *$\bar{l} \in \mathcal{B}$ for each $l \in L$.*
2. *$f, g \in \mathcal{B}$ imply $f \wedge g \in \mathcal{B}$ and $\sup(f \wedge g) = \sup f \wedge \sup g$.*
3. *$f \in L^{X\times X}$ and $\bigvee_{g\in\mathcal{B}, g\leq f} \sup g = \sup f$ imply $f \in \mathcal{B}$.*
4. *$u \in \mathcal{B}$ and $x \in X$ imply $\sup u = u(x,x)$.*
5. *$u \in \mathcal{B}$ implies $u^{-1} \in \mathcal{B}$.*
6. *$u \in \mathcal{B}$ implies $\bigvee_{v\in\mathcal{B}, v\circ v\leq u} \sup v = \sup u$.*

2.24 Remark. Obviously, $\mathcal{B} \subset L^{X\times X}$ is the greatest base of a fuzzy quasiuniformity $\mathcal{U}$ on X iff the conditions 1.-4. and 6. of the above proposition are fulfilled.

2.25 Proposition. *All indiscrete* **FPUConv**-*objects are fuzzy uniform (and thus fuzzy quasiuniform).*

Proof. Let $(X, FJ_X) \in |\mathbf{FPUConv}|$ be indiscrete, i.e. $FJ_X = F_L(X \times X)$. Then $\mathcal{B} = \{\bar{l} \in L^{X\times X} : l \in L\}$ is a fuzzy filter base on $X \times X$ and the fuzzy filter $\mathcal{F}$ generated by it is contained in each $\mathcal{H} \in F_L(X \times X)$. Furthermore, $\mathcal{F}$ is a fuzzy uniformity on X: Let $\mathcal{B}^* = \text{base}\,\mathcal{F}$. Then the conditions 1.-3. in 2.23 are fulfilled since they are always satisfied by the greatest base of a fuzzy filter. The remaining conditions 4.-6. are easily checked.

2.26 Theorem. **FUnif** *and* **FQUnif** *are bireflectively embedded in* **FSU-Conv** *and* **FPUConv** *respectively.*

Proof. Since **FSUConv** is bireflective in **FPUConv** (cf. [23; 2.7]) it suffices to prove that **UFPUConv** ($\cong$ **FUnif**) and **QUFPUConv** ($\cong$ **QUnif**) are bireflective in **FPUnif**. But this follows from 2.21, 2.22 and 2.25.

2.27 Corollary. **FUnif** *and* **FQUnif** *are topological constructs.*

Proof. This follows from 2.26 since **FSUConv** and **FPUConv** are topological constructs.

2.28 Definition. Let $1_X : (X, FJ_X) \to (X, [\mathcal{V}])$ be the bireflection of $(X, FJ_X) \in |\mathbf{FPUConv}|$ w.r.t. **UFPUConv** (resp. **QUFPUConv**). Then $(X, \mathcal{V})$ is called *the underlying fuzzy uniform space* (resp. *the underlying fuzzy quasiuniform space*) of (X, FJ_X).

2.29 Theorem. *If $(X, \mathcal{V})$ is the underlying fuzzy (quasi) uniform space of $(X, FJ_X) \in$ **FPUConv**, then $\mathcal{V}$ is the finest fuzzy (quasi) uniformity which is contained in each $\mathcal{F} \in FJ_X$.*

Proof. Put $N = \{\mathcal{U} : \mathcal{U}$ is a fuzzy (quasi) uniformity on X with $\mathcal{U} \subset \mathcal{F}$ for each $\mathcal{F} \in FJ_X\}$. Then $N \neq \emptyset$ since the fuzzy filter $\mathcal{F}$ generated by $\{\bar{l} \in L^{X \times X} : l \in L\}$ belongs to N. Let $\mathcal{W}$ be the supremum of N in the set of all fuzzy (quasi) uniformities on X partially ordered by "$\subset$" ($\mathcal{W}$ exists because of 2.27!). Then $\mathcal{W}$ is the finest fuzzy (quasi) uniformity which is contained in each $\mathcal{F} \in FJ_X$. It suffices to prove that $\mathcal{W} \in N$. By [9;3.4], $\mathcal{B} = \{f_1 \wedge \cdots \wedge f_n : \{f_1, \ldots, f_n\} \subset \bigcup_{\mathcal{U} \in N} \text{base}\,\mathcal{U}$ non-empty and finite$\}$ is a base of $\mathcal{W}$. Since $\mathcal{W}$ is initial with respect to the family $(1_X^{\mathcal{U}} : X \to (X, \mathcal{U}))_{\mathcal{U} \in N}$ of identity maps it is a fuzzy (quasi) uniformity on X. Obviously, $\mathcal{B} \subset \text{base}\,\mathcal{F}$ for each $\mathcal{F} \in FJ_X$. Thus, $\mathcal{W} \subset \mathcal{F}$ for each $\mathcal{F} \in FJ_X$, i.e. $\mathcal{W} \in N$. In order to prove $\mathcal{W} = \mathcal{V}$, it suffices to show that $1_X : (X, FJ_X) \to (X, [\mathcal{W}])$ is a bireflection: Let $f : (X, FJ_X) \to (X', [\mathcal{W}'])$ be a fuzzy uniformly continuous map from (X, FJ_X) into a (quasi) uniform fuzzy preuniform convergence space $(X', [W'])$.

1. $1_X : (X, FJ_X) \to (X, [W])$ is fuzzy uniformly continuous, since $\mathcal{W} \in N$.

2. In order to prove that $f : (X, [\mathcal{W}]) \to (X', [\mathcal{W}'])$ is fuzzy uniformly continuous, it suffices to show that

$$(*)\, (f \times f)(\mathcal{W}) \supset \mathcal{W}'.$$

By assumption,

$$(**)\,(f\times f)(\mathcal{F})\supset\mathcal{W}' \text{ for each } \mathcal{F}\in FJ_X.$$

Thus, $(f\times f)^{-1}(\mathcal{W}')$ exists (cf. [24; 1.11]) and is a (quasi) uniformity by 2.20, which belongs to N since by $(**)$,
$\mathcal{F}\supset(f\times f)^{-1}((f\times f)(\mathcal{F}))\supset(f\times f)^{-1}(\mathcal{W}')$ for each $\mathcal{F}\in FJ_X$. Consequently, $(f\times f)^{-1}(\mathcal{W}')\subset\mathcal{W}$. This implies
$\mathcal{W}'\subset(f\times f)((f\times f)^{-1}(\mathcal{W}'))\subset(f\times f)(\mathcal{W})$, i.e. $(*)$ is fulfilled.

2.30 Proposition. *Let I be a non-empty set and $(\mathcal{F}_i)_{i\in I}$* a *family of fuzzy filters on a non-emtpy set X. If $M=\{\mathcal{F}_i : i\in I\}$ has a supremum $\mathcal{S}$ in $(F_L(X),\subset)$ and $\mathcal{B}_i$ is a* base *of $\mathcal{F}_i$ for each $i\in I$, then*
$\mathcal{B}=\{f_{i_1}\wedge\cdots\wedge f_{i_n} : \{i_1,\ldots,i_n\}\subset I$ finite and $f_i\in\mathcal{B}_i$ for each $i\in I\}$ is a base *of $\mathcal{S}$.*

Proof. 1. $\mathcal{B}$ is a fuzzy filter base on X:

(a) Since $\bar{l}\in B_i$ for each $i\in I$ and each $l\in L$, $\bar{l}\in\mathcal{B}$ for each $l\in L$.

(b) Since $\mathcal{B}_i\subset$ base$\mathcal{S}$ for each $i\in I$, $\mathcal{B}\subset$ base$\mathcal{S}$. If $u,v\in\mathcal{B}$, then $u\wedge v\in\mathcal{B}$ and $\sup(u\wedge v)=\sup u\wedge\sup v$ because $u,v\in$ base $\mathcal{S}$.

2. Let $\mathcal{F}$ be the fuzzy filter generated by $\mathcal{B}$.

(a) By definition of $\mathcal{B}$, $\mathcal{B}_i\subset\mathcal{B}$ for each $i\in I$. Thus, $\mathcal{F}_i\subset\mathcal{F}$.

(b) Let $\mathcal{U}\in F_L(X)$ such that $F_i\subset\mathcal{U}$ for each $i\in I$. Then $\mathcal{B}_i\subset$ base $\mathcal{F}_i\subset$ base$\mathcal{U}$ for each $i\in I$. Consequently $\mathcal{B}\subset$ base$\mathcal{U}$, i.e. $\mathcal{F}\subset\mathcal{U}$.

3. It follows from 2.(a) and 2.(b) that $\mathcal{F}=\mathcal{S}$.

2.31 Corollary. *Let X be a non-empty set, $((X_i,\mathcal{U}_i))_{i\in I}$ a family of fuzzy (quasi) uniform spaces, B_i a* base *of $\mathcal{U}_i$ for each $i\in I$, and $(f_i : X\to X_i)_{i\in I}$ a family of maps. In case $I\neq\emptyset$,*
$\mathcal{B}=\{\bigwedge_{j\in J} u_j\circ(f_j\times f_j) : J\subset I$ non-empty and finite, $u_j\in\mathcal{B}_j$ for each $j\in J\}$ is a base of the initial fuzzy (quasi) uniformity on X with respect to the given data. In case $I=\emptyset$, $\mathcal{B}=\{\bar{l}\in L^{X\times X} : l\in L\}$ is a base *of the initial fuzzy (quasi) uniformity with respect to the given data, i.e. a* base *of the indiscrete fuzzy (quasi) uniformity on X.*

Proof. By 2.20, for each $i\in I$, $\mathcal{U}_{f_i}=(f_i\times f_i)^{-1}(\mathcal{U}_i)$ is the coarsest fuzzy (quasi) uniformity on X such that f_i is uniformly continuous, i.e. the initial fuzzy (quasi) uniformity on X with respect to f. Then the initial fuzzy (quasi) uniformity on X with respect to $(f_i)_{i\in I}$ is the supremum of $(\mathcal{U}_{f_i})_{i\in I}$ in the set of all fuzzy (quasi) uniformities on X partially ordered by "$\subset$". Now the assertion is proved by applying 2.30 since $\{u_i \circ (f_i \times f_i) : u_i \in \mathcal{B}_i\}$ is a base of $\mathcal{U}_{f_i}$ for each $i \in I$. Concerning $I = \emptyset$, see the proove of 2.25.

2.32 Remark. The unique fuzzy uniformity on the empty set is initial.

3 Special features of fuzzy preuniform convergence spaces

3.1 Theorem. ([24; 2.4 and 4.3]). **FPUConv** *and its full subconstruct* **FSUConv** *are strong topological universes, i.e. they are topological constructs which are cartesian closed and extensional and in which products of quotients are quotients.*

3.2 Remark. It follows from 3.1 that all results of part 1 of this paper are applicable to **FPUConv** and **FSUConv**. In particular, in order to obtain light factorization structures for **FPUConv** and **FSUConv** only those connection classes which are connectednesses can be considered (cf. 1.4). But there are further results on connectedness and light factorization structures for **FPUConv** and **FSUConv** which will be presented in the following.

3.3 Definitions. 1. $(X, FJ_X) \in |\mathbf{FPUConv}|$ is called *fuzzy connected* provided that each fuzzy uniformly continuous map $f : (X, FJ_X) \to (\{0,1\}, \{\dot{0}\times\dot{0}, \dot{1}\times\dot{1}\})$ from (X, FJ_X) into the two-point discrete fuzzy preuniform convergence space $(\{0,1\}, \{\dot{0}\times\dot{0}, \dot{1}\times\dot{1}\})$ is constant.

2. A fuzzy preuniform convergence space (X, FJ_X) is called *diagonal* provided that $\bigcap_{x\in X} \dot{x}\times\dot{x} \in FJ_X$ if X is non-empty.

3.4 Remark. If X is a non-empty set and $g \in L^X$, then the *principal fuzzy filter* (g) generated by g has the base $\{g \wedge \bar{\alpha} : \alpha \in L\} \cup \{\bar{\alpha} : \alpha \in L\}$. For each non-empty subset M of X we get $(\chi_M) = \bigcap_{x\in M} \dot{x}$ (cf. [8; (16)]). Thus, $\bigcap_{x\in X} \dot{x}\times\dot{x} = (\chi_\triangle)$ where $\triangle$ denotes the "diagonal" of $X \times X$, i.e. $\triangle = \{(x,x) : x \in X\}$.

3.5 Example. Each fuzzy (quasi) uniform preuniform convergence space

$(X, [\mathcal{U}])$ is diagonal since for each $x \in X$, $\mathcal{U} \subset \dot{x} \times \dot{x}$ which implies $\mathcal{U} \subset \bigcap_{x \in X} \dot{x} \times \dot{x}$, i.e. $\bigcap_{x \in X} \dot{x} \times \dot{x} \in [\mathcal{U}]$.

3.6 Proposition. *Each diagonal fuzzy preuniform convergence space* (X, FJ_X) *is connected.*

Proof. Let $f : (X, FJ_X) \to (\{0,1\}, \{\dot{0} \times \dot{0}, \dot{1} \times \dot{1}\})$ be fuzzy uniformly continuous. Then $(f \times f)(\bigcap_{x \in X} \dot{x} \times \dot{x}) = \bigcap_{x \in X} f(\dot{x}) \times f(\dot{x}) = \bigcap_{x \in X} \dot{f(x)} \times \dot{f(x)}$ contains $\dot{0} \times \dot{0}$ or $\dot{1} \times \dot{1}$, i.e. $\dot{f(x)} \times \dot{f(x)} \supset \dot{0} \times \dot{0}$ for each $x \in X$ or $\dot{f(x)} \times \dot{f(x)} \supset \dot{1} \times \dot{1}$ for each $x \in X$. Thus, $f(x) = 0$ for each $x \in X$ or $f(x) = 1$ for each $x \in X$ since $\dot{0}$ and $\dot{1}$ are fuzzy ultrafilters, i.e. maximal elements in $(F_L(\{0,1\}), \subset)$ (cf. [7;4.8]).

3.7 Remark. For each non-empty set X, $\bigcap_{x \in X} \dot{x} \times \dot{x}$ is the discrete fuzzy uniformity on X, i.e. the finest fuzzy uniformity on X (Use 2.23 for base $\bigcap_{x \in X} \dot{x} \times \dot{x} = \bigcap_{x \in X}$ base $\dot{x} \times \dot{x}$, then conditions 1.-5. are obvious and concerning 6., for each $u \in$ base $(\bigcap_{x \in X} \dot{x} \times \dot{x}) \setminus \{\bar{0}\}$ there is some $v \in$ base $(\bigcap_{x \in X} \dot{x} \times \dot{x}) \setminus \{\bar{0}\}$ with $\sup u = \sup v$ and $v \circ v \leq u$, where v is defined by

$$v(x, y) = \begin{cases} \sup u \text{ if } x = y \\ 0 \text{ otherwise.} \end{cases} .).$$

In order to have a connectedness concept by means of which fuzzy (quasi) uniform spaces (= (quasi) uniform fuzzy preuniform convergence spaces) may be distinguished the two-point discrete fuzzy uniform space $(\{0,1\}, \dot{0} \times \dot{0} \cap \dot{1} \times \dot{1})$ will be helpful as the following definition shows.

3.8 Definition. A fuzzy preuniform convergence space (X, FJ_X) is called *uniformly connected* provided that each fuzzy uniformly continuous map $f : (X, FJ_X) \to (\{0,1\}, [\dot{0} \times \dot{0} \cap \dot{1} \times \dot{1}])$ from (X, FJ_X) into the two-point discrete uniform fuzzy preuniform convergence space $(\{0,1\}, [\dot{0} \times \dot{0} \cap \dot{1} \times \dot{1}])$ is constant.

3.9 Remark. In order to develop a common theory of connectedness and uniform connectedness for **FPUConv** the notion of $\mathcal{P}$-connectedness as defined under 1.2 is useful, where $\mathcal{P} \subset |\mathbf{FPUConv}|$. But there are close connections between connectedness and $\mathcal{P}$-connectedness in **FPUConv** which will be examined now where at first another characterization of connectedness is proved.

3.10 Definition. Let $(X, FJ_X) \in |\mathbf{FPUConv}|$. Then $A \subset X$ is called a *partition set* (in (X, FJ_X)) iff for each $\mathcal{F} \in FJ_X$, $\chi_{A\times A} \in$ base $\mathcal{F}$ or $\chi_{(X\backslash A)\times(X\backslash A)} \in$ base $\mathcal{F}$.

3.11 Proposition. *A fuzzy preuniform convergence space (X, FJ_X) is connected iff the empty set $\emptyset$ and X are the only partition sets in (X, FJ_X).*

Proof. "$\Rightarrow$" (indirectly). If there is a partition set $A \subset X$ such that $A \neq \emptyset$ and $A \neq X$, then $f : (X, FJ_X) \to (\{0,1\}, \{\dot{0}\times\dot{0}, \dot{1}\times\dot{1}\})$, defined by $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \in X\backslash A$, is uniformly continuous: Let $\mathcal{F} \in FJ_X$. Then 1) $\chi_{A\times A} \in$ base $\mathcal{F}$ or $\chi_{(X\backslash A)\times(X\backslash A)} \in$ base $\mathcal{F}$. Concerning the first case $(f\times f)[\chi_A] = \chi_{f[A]\times f[A]} = \chi_{\{0\}\times\{0\}} = \chi_{\{(0,0)\}} \in$ base $(f\times f)(\mathcal{F})$ which implies $[\chi_{\{(0,0)\}}] = \dot{0}\times\dot{0} \subset (f\times f)(\mathcal{F})$. Thus, $(f\times f)(\mathcal{F}) = \dot{0}\times\dot{0}$ since $\dot{(0,0)} = \dot{0}\times\dot{0}$ is an ultrafilter. Analogously, $(f\times f)(\mathcal{F}) = \dot{1}\times\dot{1}$ in the second case. Since f is non-constant, (X, FJ_X) is not connected.
"$\Leftarrow$". Let $f : (X, FJ_X) \to (\{0,1\}, \{\dot{0}\times\dot{0}, \dot{1}\times\dot{1}\})$ be fuzzy continuous, i.e. if $\mathcal{F} \in FJ_X$ then 1) $(f\times f)(\mathcal{F}) = \dot{0}\times\dot{0}$ or 2) $(f\times f)(\mathcal{F}) = \dot{1}\times\dot{1}$. Concerning the first case, $\chi_{\{(0,0)\}} \in$ base $(f\times f)(\mathcal{F})$. Because of $(f\times f)^{-1}((f\times f)(\mathcal{F})) \subset \mathcal{F}$, $\chi_{\{(0,0)\}} \circ (f\times f) = \chi_{f^{-1}(0)\times f^{-1}(0)}$ belongs to a base of $(f\times f)^{-1}((f\times f)(\mathcal{F}))$ and thus to base $\mathcal{F}$.
Analogously, $\chi_{f^{-1}(1)\times f^{-1}(1)} = \chi_{(X\backslash f^{-1}(0))\times(X\backslash f^{-1}(0))} \in$ base $\mathcal{F}$. Consequently, $f^{-1}(0)$ is a partition set in (X, FJ_X) which implies, by assumption, that $f^{-1}(0) = \emptyset$ or $f^{-1}(0) = X$, i.e. $f = \bar{1}$ or $f = \bar{0}$. Hence, (X, FJ_X) is connected.

3.12 Corollary. *Each $\mathcal{P}$-connected fuzzy preuniform convergence space is connected provided that $\mathcal{P}$ contains a space with at least two points.*

Proof. If $(X, FJ_X) \in |\mathbf{FPUConv}|$ is not connected, then there is a partition set $A \subset X$ such that $A \neq \emptyset$ and $A \neq X$. Let $P \in \mathcal{P}$ such that there are $a, b \in P$ with $a \neq b$. Thus, $f : X \to P$, defined by $f(x) = a$ if $x \in A$ and $f(x) = b$ if $x \in X\backslash A$, is fuzzy uniformly continuous (analogously to the first part of the proove of 3.11, where 0 and 1 have to be substituted by a and b respectively) and non-constant, i.e. (X, FJ_X) is not $\mathcal{P}$-connected.

3.13 Remark. In [23] and [24] generalized convergence spaces have been fuzzyficated, where a *fuzzy generalized convergence space* is a pair (X, q) such that X is a set and $q \subset F_L(X)\times X$ satisfies the following conditions:

$\mathbf{FC}_1$ $(\dot{x}, x) \in q$ for each $x \in X$.

$\mathbf{FC}_2$ $(\mathcal{F}, x) \in q$ whenever $(\mathcal{G}, x) \in q$ and $\mathcal{G} \subset \mathcal{F}$.

A map $f : (X, q) \to (X', q')$ is called here *fuzzy continuous* iff $(f(\mathcal{F}), f(x)) \in q'$ for each $(\mathcal{F}, x) \in q$. The construct of fuzzy generalized convergence spaces (and fuzzy continuous maps) is denoted by **FGConv**. Instead of $(\mathcal{F}, x) \in q$ we say $\mathcal{F}$ *converges to* x w.r.t. q and write $\mathcal{F} \xrightarrow{q} x$ (or shortly $\mathcal{F} \to x$).

3.14 Definition. Let $(X, q) \in |\mathbf{FGConv}|$, $A \subset X$. If $i_A : A \to X$ denotes the inclusion map, then the *closure of* A w.r.t q, denoted by $cl_q A$, is defined by $cl_q A = \{x \in X$: there is some $\mathcal{F} \in F_L(X)$ with $\mathcal{F} \xrightarrow{q} x$ and $i_A^{-1}(\mathcal{F})$ exists$\}$.

3.15 Proposition. *Let* $(X, q) \in |\mathbf{FGConv}|$, $A \subset X$, *and let* $i_A : A \to X$ *be the inclusion map. Then*

1. $cl_q A = \{x \in X$*: there is some* $\mathcal{F} \in F_L(A)$ *with* $i_A(\mathcal{F}) \xrightarrow{q} x\}$
2. *If* (X, q) *is topological (i.e. there is a fuzzy topology* t *on* X *such that* $(\mathcal{F}, x) \in q$ *iff* $\mathcal{F} \supset \mathcal{U}_t(x)$*), then* $cl_q A = \{x \in X : i_A^{-1}(\mathcal{U}_t(x))$ *exists*$\}$.

3.16 Definition. A fuzzy generalized convergence space (X, q) is called a T_1*-space* iff for each pair $(x, y) \in X \times X$, $(\dot{x}, y) \in q$ implies $x = y$.

3.17 Proposition. $(X, q) \in |\mathbf{FGConv}|$ *is a* T_1*-space iff for each* $x \in X$, $cl_q\{x\} = \{x\}$.

Proof. "$\Rightarrow$". Let $y \in cl_q\{x\}$. Then there is some $\mathcal{F} \in \mathcal{F}_L(X)$ with $\mathcal{F} \xrightarrow{q} y$ such that $i_{\{x\}}^{-1}(\mathcal{F})$ exists, i.e. for $f \in$ base $\mathcal{F}$, $\sup f = \sup f|_{\{x\}} = f(x)$ $= \dot{x}(f)$, which implies $\mathcal{F} \subset \dot{x}$. Thus, $\dot{x} \xrightarrow{q} y$. By assumption, $x = y$. Hence, $cl_q\{x\} = \{x\}$.
"$\Leftarrow$". Let $\dot{x} \xrightarrow{q} y$. Then $y \in cl_q\{x\}$ since $i_{\{x\}}^{-1}(\dot{x})$ exists because $f \in$ base $\dot{x}$, i.e. $\sup f = f(x)$, implies $\sup f = \sup f|_{\{x\}}$. By assumption, $x = y$.

3.18 Remark. For each $(X, FJ_X) \in |\mathbf{FPUConv}|$ there are two natural fuzzy generalized convergence spaces (X, q_{FJ_X}) and $(X, q_{\gamma_{FJ_X}})$, where $(\mathcal{F}, x) \in q_{FJ_X}$ iff $\dot{x} \times \mathcal{F} \in FJ_X$ and $(\mathcal{F}, x) \in q_{\gamma_{FJ_X}}$ iff $(\mathcal{F} \cap \dot{x}) \times (\mathcal{F} \cap \dot{x}) \in FJ_X$. Convergence in (X, q_{FJ_X}) is called *preconvergence in* (X, FJ_X) and convergence in $(X, q_{\gamma_{FJ_X}})$ is called *convergence in* (X, FJ_X).

3.19 Definition. A fuzzy preuniform convergence space (X, FJ_X) is called a T_1*-space* (resp. *pre-*T_1*-space*) iff $(X, q_{\gamma_{FJ_X}})$ (resp. (X, q_{FJ_X})) is a T_1-space.

3.20 Proposition. *Let* (X, q) *be a fuzzy generalized convergence space,* $A \subset X$ *a dense subset, i.e.* $cl_q A = X$, *and* $f : (X, q) \to (X', q')$ *a fuzzy con-*

tinuous map from (X,q) into a fuzzy generalized convergence space which is a T_1-space, such that the restriction f to A is constant. Then f is constant.

Proof. If $A=\emptyset$ there is nothing to prove. Thus, let $A\neq\emptyset$ and $f[A]=\{x'_0\}$ with $x'_0\in X'$. Define: $g:X\to X'$ by $g(x)=x'_0$ for each $x\in X$. In order to prove $f=g$, let $K=\{x\in X: f(x)=g(x)\}$. Then $cl_qK=K$:
Obviously, $K\subset cl_qK$. Let $x\in cl_qK$, i.e. there is some $(\mathcal{F},x)\in q$ such that $i_K^{-1}(\mathcal{F})$ exists. Hence, $\{h|_K : h\in \text{ base}\,\mathcal{F}\}$ is a base of $i_K^{-1}(\mathcal{F})$ and $\{i_K[h|_K] : h\in \text{ base}\,\mathcal{F}\}\cup\{\bar{l}\in L^X\}$ is a base of $\mathcal{F}'=i_K(i_K^{-1}(\mathcal{F}))$. Since $\mathcal{F}'\supset\mathcal{F}$, $(\mathcal{F}',x)\in q$. Consequently, $(f(\mathcal{F}'),f(x))\in q'$ and $(g(\mathcal{F}'),g(x))\in q'$ because f and g are fuzzy continuous. Furthermore,
$B=\{f[i[h|_K]] : h\in \text{ base}\,\mathcal{F}\}\cup\{f[\bar{l}] : l\in L\}\cup\{\bar{l}\in L^{X'} : l\in L\}$ and $B'=\{g[i[h|_K]] : h\in \text{ base}\,\mathcal{F}\}\cup\{g[\bar{l}] : l\in L\}\cup\{\bar{l}\in L^{X'} : l\in L\}$ are bases of $f(\mathcal{F}')$ and $g(\mathcal{F}')$ respectively. But for each $x'\in X'$, $f[i[h|_K]](x')=\bigvee_{x\in f^{-1}(x')} i[h|_K](x)=\bigvee_{x\in K\cap f^{-1}(x')} i[h|_K](x)=\bigvee_{x\in g^{-1}(x')} i[h|_K](x)=g[i[h|_K]](x')$, i.e. for each $h\in$ base $\mathcal{F}$, $f[i[h|_K]]=g[i[h|_K]]$. Additionally, for each $l\in L$, $g[\bar{l}]\leq f[\bar{l}]$. Thus, $f[\bar{l}]\in$ base $g(\mathcal{F}')$ since $g[\bar{l}]\in B'$ and $\sup g[\bar{l}]=\sup f[\bar{l}]=l$. Consequently, $B\subset$ base $g(\mathcal{F}')$ since $B'\subset$ base $g(\mathcal{F}')$. This implies $f(\mathcal{F}')\subset g(\mathcal{F}')$. Since $g(\mathcal{F}')=\dot{x}'_0$ (note: for each $u\in L^{X'}$, $u\circ g=\overline{u(x'_0)}$ and $g(\mathcal{F}')(u)=\mathcal{F}'(u\circ g)=\mathcal{F}'(\overline{u_0(x'_0)})=u(x'_0)=\dot{x}'_0(u)$), $\dot{x}'_0$ converges to $f(x)$ and $g(x)$ which implies $x'_0=f(x)$ and $x'_0=g(x)$ by assumption on (X',q'), i.e. $x\in K$.
Finally, from $A\subset K\subset X$ follows: $X=cl_qA\subset cl_qK=K\subset X$, i.e. $K=X$. Thus, f is constant.

3.21 Corollary. *Let (X,FJ_X) be a fuzzy preuniform convergence space and $A\subset X$ a dense (resp. pre-dense) subset of X, i.e. $cl_{q_{\gamma_{FJ_X}}}A=X$ (resp. $cl_{q_{FJ_X}}A=X$). If $f:(X,FJ_X)\to(P,FJ_P)$ is a fuzzy uniformly continuous map from (X,FJ_X) into a T_1-space (resp. pre-T_1-space) $(P,FJ_P)\in$ |**FPUConv**| such that the restriction of f to A is constant, then f is constant.*

Proof. Since $f:(X,FJ_X)\to(P,FJ_P)$ is fuzzy uniformly continuous, $f:(X,q_{\gamma_{FJ_X}})\to(P,q_{\gamma_{FJ_P}})$ and $f:(X,q_{FJ_X})\to(P,q_{FJ_P})$ are fuzzy continuous and the assertion follows from 3.20.

3.22 Definition. Let $(X,FJ_X)\in$ |**FPUConv**|. Then a subset A of X is called *$\mathcal{P}$-connected* iff it is $\mathcal{P}$-connected as a subspace of (X,FJ_X).

3.23 Proposition. *The statement "If a subset A of a fuzzy preuniform convergence space (X, FJ_X) is $\mathcal{P}$-connected, then $cl_{q_{\gamma_{FJ_X}}} A$ (resp. $cl_{q_{FJ_X}} A$) is $\mathcal{P}$-connected" is true if and only if $\mathcal{P}$ is a class of T_1-spaces (resp. pre-T_1-spaces).*

Proof. "$\Leftarrow$". Let $f : cl_{q_{\gamma_{FJ_X}}} A \to P$ (resp. $f : cl_{q_{FJ_X}} A \to P$) be a fuzzy uniformly continuous map where P is a T_1-space (resp. pre-T_1-space). By assumption $f|_A$ is constant, since A is dense (resp. predense) in $cl_{q_{\gamma_{FJ_X}}} A$ (resp. $cl_{q_{FJ_X}} A$) [use 3.15.1.], f is constant by 3.21. Thus, $cl_{q_{\gamma_{FJ_X}}} A$ (resp. $cl_{q_{FJ_X}} A$) in $\mathcal{P}$-connected.
"$\Rightarrow$". Let $P \in \mathcal{P}$ and $x \in P$. If C_x is the $\mathcal{P}$-component of P containing x, then the inclusion map $i : C_x \to P$ is fuzzy uniformly continuous and thus constant, i.e. $C_x = \{x\}$. By assumption and the fact that C_x is a maximal $\mathcal{P}$-connected subspace, $cl_{q_{\gamma_{FJ_X}}} \{x\} = \{x\}$ (resp. $cl_{q_{FJ_X}} \{x\} = \{x\}$). Thus, P is a T_1-space (resp. a pre-T_1-space).

3.24 Remarks. 1. Since convergence implies preconvergence, for each $(X, FJ_X) \in |\mathbf{FPUConv}|$ and each $A \subset X$,

$$A \subset cl_{q_{\gamma_{FJ_X}}} A \subset cl_{q_{FJ_X}} A \subset X$$

and thus, A is dense implies A is predense, and if A is *preclosed*, i.e. $A = cl_{q_{FJ_X}} A$, then A *is closed*, i.e. $A = cl_{q_{\gamma_{FJ_X}}} A$. Furthermore, T_1 is weaker than pre-T_1, i.e. pre-T_1 implies T_1.

2. If $\mathcal{P} \subset |\mathbf{FPUConv}|$ is not a class of T_1-spaces, then the concept "$\mathcal{P}$-connectedness" becomes trival:

   (a) If $\mathcal{P}$ is the empty class, then $\mathcal{P}$ is a class of T_1-spaces and $C\mathcal{P} = |\mathbf{FPUConv}|$.

   (b) Let $\mathcal{P} \subset |\mathbf{FPUConv}|$ be not a class of T_1-spaces. Since $\mathcal{P}$ is non-empty, there is some $(P, FJ_P) \in \mathcal{P}$ with at least two points such that (P, FJ_P) is not a T_1-space, i.e. there are two distinct points $a, b \in P$ with $(\dot{a}, b) \in q_{\gamma_{FJ_P}}$ or $(\dot{b}, a) \in q_{\gamma_{FJ_P}}$. Thus, each $(X, FJ_X) \in |\mathbf{FPUConv}|$ with at least two points is not $\mathcal{P}$-connected: Let $x, y \in X$ with $x \neq y$. Then $f : (X, FJ_X) \to (P, FJ_P)$ defined by $f(x) = a$ and $f[X \backslash \{x\}] = \{b\}$ is non-constant but fuzzy uniformly continuous: Let $\mathcal{F} \in FJ_X$. By definition of f, $f[X] \times f[X] = \{a, b\} \times \{a, b\}$ and $\chi_{f[X] \times f[X]} = (f \times f)[\bar{1}] \in B$ where $B = \{(f \times f)[u] : u \in \text{base}\mathcal{F}\} \cup \{\bar{\alpha} \subset L^{P \times P} :$

$\alpha \in L\}$ is a base of $(f\times f)(\mathcal{F})$. By 3.4, $(\chi_{f[X]\times f[X]}) = \dot{a}\times\dot{a}\cap\dot{a}\times\dot{b}\cap\dot{b}\times\dot{a}\cap\dot{b}\times\dot{b}$. Furthermore, $C = \{(f\times f)[\bar{1}]\wedge\bar{\alpha} : \alpha \in L\}\cup\{\bar{\alpha} : \alpha \in L\}$ is a base of $(f \times f[\bar{1}])$ (cf. 3.4) where $C \subset \text{base}(f \times f)(\mathcal{F})$ [note: $((f \times f)[\bar{1}] \wedge \bar{\alpha})(p, p') = \begin{cases} 1, \text{ if } (p, p') \in f[X] \times f[X] \\ 0 \text{ otherwise} \end{cases}$ and thus $(f \times f)(\mathcal{F})(((f \times f)[\bar{1}] \wedge \bar{\alpha}) = (f \times f)(\mathcal{F})(f \times f[\bar{1}]) \wedge (f \times f)(\mathcal{F})(\bar{\alpha}) = \sup(f \times f)[\bar{1}] \wedge \alpha = 1 \wedge \alpha = \alpha = \sup((f \times f)[\bar{1}] \wedge \bar{\alpha})$, i.e. $(f\times f)[\bar{1}]\wedge\bar{\alpha} \in \text{base}(f\times f)(\mathcal{F})$]. Consequently, $\dot{a}\cap\dot{b}\times\dot{a}\cap\dot{b} \subset \dot{a}\times\dot{a}\cap\dot{a}\times\dot{b}\cap\dot{b}\times\dot{a}\cap\dot{b}\times\dot{b} = (\chi_{f[X]\times f[X]}) \subset (f \times f)(\mathcal{F})$, which implies $(f\times f)(\mathcal{F}) \in FJ_P$ since by assumption $\dot{a}\cap\dot{b}\times\dot{a}\cap\dot{b} \in FJ_P$. Hence, $C\mathcal{P}$ consists of all one-point spaces and the empty space.

3.25 Corollary. *Let $\mathcal{P} \subset |\mathbf{FPUConv}|$ be a class of T_1-spaces (resp. pre-T_1-spaces). Then the following are satisfied:*

1. *The $\mathcal{P}$-components of each fuzzy preuniform convergence space are closed (resp. preclosed).*
2. *If a subset M of a fuzzy preuniform convergence space (X, FJ_X) is $\mathcal{P}$-connected, then each subset N of X with $M \subset N \subset cl_{q_{\gamma FJ_X}} M$ (resp. $M \subset N \subset cl_{q_{FJ_X}} M$) is $\mathcal{P}$-connected.*

Proof. 1. Since $\mathcal{P}$-components are maximal $\mathcal{P}$-connected subspaces the assertion is proved by means of 3.23.

2. M may be regarded as a $\mathcal{P}$-connected subspace of the subspace (N, FJ_N) of (X, FJ_X). Since $cl_{q_{\gamma FJ_N}} M = (cl_{q_{\gamma FJ_X}} M)\cap N = N$ (resp. $cl_{q_{FJ_N}} M = (cl_{q_{FJ_X}} M) \cap N = N$) the proove is finished by using 3.23.

In order to prove a product theorem for $\mathcal{P}$-connectedness some preparations are necessary.

3.26 Proposition. *Let $((X_i, t_i))_{i\in I}$ be a family of fuzzy topological spaces and $(\prod_{i\in I} X_i, t)$ its product in* **FTop**. *For each $x = (x_i) \in \prod X_i$, the topological neighborhood filter* $\mathcal{U}_t(x)$ of x *is the product* $\prod_{i\in I} \mathcal{U}_{t_i}(x_i)$ *of the family $(\mathcal{U}_{t_i}(x_i))_{i\in I}$ of the topological neighborhood filters of the i-th coordinates x_i of x.*

Proof. Since **FTop** is bireflective in **FGConv** (cf. [24, 3.10]) initial structures in **FTop** are formed as in **FGConv**. Thus, for each $\mathcal{F} \in F_L(\prod X_i)$

and each $x = (x_i) \in \prod X_i$,
$\mathcal{F} \supset \mathcal{U}_t(x) \Leftrightarrow p_i(\mathcal{F}) \supset \mathcal{U}_{t_i}(x_i)$ for each $i \in I \Leftrightarrow \mathcal{F} \supset \prod\limits_{i\in I} \mathcal{U}_{t_i}(x_i)$. Hence, $\mathcal{U}_t(x) = \bigcap\limits_{\mathcal{F}\supset\mathcal{U}_t(x)} \mathcal{F} = \prod\limits_{i\in I} \mathcal{U}_{t_i}(x_i)$.

3.27 Proposition. *Let $((X_i, t_i))_{i\in I}$ be a family of fuzzy topological spaces and $(\prod X_i, t)$ its product in* **FTop**. *If $x_{(0)} \in \prod X_i$ then $U = \{x \in \prod X_i : p_i(x) = p_i(x_{(0)})$ for all but finitely many $i \in I\}$ is dense in $\prod\limits_{i\in I} X_i$, i.e. $cl_{q_t} U = X$, where $(\mathcal{F}, x) \in q_t$ iff $\mathcal{F} \supset \mathcal{U}_t(x)$ and $p_i : \prod\limits_{i\in I} X_i \to X_i$ denotes the i-th projection.*

Proof. Let $x = (x_i) \in \prod X_i$. It must be proved that $i_U^{-1}(\mathcal{U}_t(x))$ exists (cf. 3.15.2.), where $\mathcal{U}_t(x) = \prod\limits_{i\in I} \mathcal{U}_{t_i}(x_i)$ (cf. 3.26). $B = \{\bigwedge\limits_{j\in J} f_j \circ p_j : J \subset I$ finite and $f_j \in$ base$\mathcal{U}_{t_j}(x_j)$ for each $j \in J\}$ is a base of $\prod\limits_{i\in I} \mathcal{U}_{t_i}(x_i)$. Let $f_{i_1} \circ p_{i_1} \wedge \cdots \wedge f_{i_n} \circ p_{i_n} \in B$. Then have to prove $\sup f_{i_1} \circ p_{i_1} \wedge \cdots \wedge f_{i_n} \circ p_{i_n} = \sup(f_{i_1} \circ p_{i_1} \wedge \cdots \wedge f_{i_n} \circ p_{i_n}|_U)$:

1. "$\geq$" is obvious.

2. "$\leq$". Let $A = \{f_{i_1}(x_{i_1}) \wedge \cdots \wedge f_{i_n}(x_{i_n}) : x = (x_i) \in \prod X_i\}$ and $A' = \{f_{i_1}(x_{i_1}) \wedge \cdots \wedge f_{i_n}(x_{i_n}) : x = (x_i) \in U\}$. If $a \in A$, then there exists some $x = (x_i) \in \prod X_i$ such that $a = f_{i_1}(x_{i_1}) \wedge \cdots \wedge f_{i_n}(x_{i_n})$. Define $x' = (x'_i) \in \prod X_i$ by $x'_i = x_i$ for each $i \in \{i_1, \ldots, i_n\}$ and $x'_i = p_i(x_{(0)})$ for each $i \in I \backslash \{i_1, \ldots, i_n\}$. Then $x' \in U$ and $f_{i_1}(x'_{i_1}) \wedge \cdots \wedge f_{i_n}(x'_{i_n}) = f_{i_1}(x_{i_1}) \wedge \cdots \wedge f_{i_n}(x_{i_n}) = a$, i.e. $a \in A'$. Consequently, $A \subset B$ and $\sup A \leq \sup B$. Hence, $i_U^{-1}(\mathcal{U}_t(x))$ exists for each $x \in \prod X_i$, i.e. U is dense in $\prod X_i$.

3.28 Proposition. *Let $((X_i, q_i))_{i\in I}$ be a family of fuzzy generalized convergence spaces. If $x_{(0)}$ is a point of the product space (X, q) of this family in* **FGConv**, *then $U = \{x \in X : p_i(x) = p_i(x_{(0)})$ for all but finitely many $i \in I\}$ is dense in X.*

Proof. Let t_i be the discrete fuzzy topology on X_i, i.e. $t_i = L^{X_i}$, for each $i \in I$, and (X, t) be the product space of $((X_i, t_i))_{i\in I}$ in **FGConv**. Then (X, t) is a fuzzy topological space since **FTop** is bireflective in **FGConv** by [24; 3.10]. By 3.27, $cl_{q_t} U = X$. Furthermore, the identity $1_{X_i} : (X_i, q_{t_i}) \to (X_i, q_i)$ is fuzzy continuous for each $i \in I$ which implies that $\prod\limits_{i\in I} 1_{X_i} = 1_X : (X, q_t) \to$

(X, q) is fuzzy continuous, i.e. $q_t \leq q$. Thus, $cl_{q_t}U = X \subset cl_q U \subset X$, i.e. $cl_q U = X$.

3.29 Theorem. *Let $((X_i, FJ_{X_i}))_{i \in I}$ be a family of non-empty fuzzy preuniform convergence spaces. Then the product space (X, FJ_X) of this family in* **FPUConv** *is $\mathcal{P}$-connected iff (X_i, FJ_{X_i}) is $\mathcal{P}$-connected for each $i \in I$.*

Proof. "$\Rightarrow$". Since the projections $p_i : X \to X_i$ are surjective and fuzzy uniformly continuous for each $i \in I$, it follows from 1.6 (cf. 1.2.(2)) that (X_i, FJ_{X_i}) is $\mathcal{P}$-connected for each $i \in I$.
"$\Leftarrow$".

a) If $\mathcal{P}$ is not a class of T_1-spaces, then by 3.24.2, $\mathcal{CP}$ is trivial and the product theorem is valid.

b) Let $\mathcal{P}$ be a class of T_1spaces and $x_{(0)} \in X$. If $x_{(0)}$ and $x_{(n)} \in X$ differ by at most $n < \infty$ coordinates, then $x_{(0)}$ and $x_{(n)}$ lie in a $\mathcal{P}$-connected subset of X which is proved by induction on the number n of differing coordinates in the following manner:

α) If $n = 1$ the assertion is correct; namely if $x_{(1)}$ and $x_{(0)}$ differ e.g. in the i-th coordinate, then $Y = X_i \times \prod_{k \neq i} p_k(x_{(0)}) \subset X$ is isomorphic to X_i and hence it is $\mathcal{P}$-connected and $x_{(0)}$ and $x_{(1)}$ lie in Y.

β) Let the assertion be valid for all $x_{(n-1)} (n \geq 2)$. If $x_{(n)}$ is given, then $x_{(n-1)}$ can be found such that $x_{(n-1)}$ and $x_{(0)}$ differ by one coordinate. By α), $x_{(n)}$ and $x_{(n-1)}$ lie in a $\mathcal{P}$-connected subset C_1, and by inductive hypothesis, $x_{(n-1)}$ and $x_{(0)}$ lie in a $\mathcal{P}$-connected subset C_2. Since $x_{(n-1)} \in C_1 \cap C_2$, i.e. $C_1 \cap C_2 \neq \emptyset$, $C = C_1 \cup C_2$ is the desired $\mathcal{P}$-connected subset containing $x_{(0)}$ and $x_{(n)}$.

Let us denote by $C_{x_{(0)}}$ the $\mathcal{P}$-component of X containing $x_{(0)}$. Thus, $U = \{x \in X : x \text{ and } x_{(0)} \text{ differ by at most finitely many coordinates}\} \subset C_{x_{(0)}} \subset X$. By 3.28, $cl_{q_{\gamma_{FJ_X}}} U = X$. Hence $C_{x_{(0)}}$ is dense in X which implies $X = C_{x_{(0)}}$ (cf. 3.25.1.), i.e. X is $\mathcal{P}$-connected.

3.30 Proposition. *Let $\mathcal{P} \subset |$***FPUConv***$|$ and $\mathcal{A}$ a bireflective (full and isomorphism-closed) subconstruct of* **FPUConv** *containing $\mathcal{P}$ (i.e. $\mathcal{P} \subset |\mathcal{A}|$). If $R :$* **FPUConv** *$\to \mathcal{A}$ denotes the bireflector,*
*then $(X, FJ_X) \in |$***FPUConv***$|$ is $\mathcal{P}$-connected iff $R((X, FJ_X))$ is $\mathcal{P}$-connected.*

Proof. Use 1.2.2. for $K = C\mathcal{P}$ and the defining property of a bireflection.

3.31 Corollary. *A fuzzy preuniform convergence space is uniformly connected iff its underlying fuzzy (quasi) unifom space is uniformly connected.*

Proof. Use 2.26, 3.7 and 3.8 and apply 3.30.

3.32 Definition. A fuzzy generalized convergence space (X, q) is called *symmetric* iff $(\mathcal{F}, x) \in q$ and $\mathcal{F} \subset \dot{y}$ for some $y \in X$ imply $(\mathcal{F}, y) \in q$.

3.33 Proposition. *Each fuzzy generalized convergence space which is a T_1-space is symmetric.*

3.34 Remark. A fuzzy topological space (X, t) may be regarded as a fuzzy generalized convergence space (X, q_t), where $(\mathcal{F}, x) \in q_t$ iff $\mathcal{F} \supset \mathcal{U}_t(x)$ (cf. [24; 3.10]). Thus, a fuzzy topological space (X, t) is called *symmetric* iff (X, q_t) is symmetric. The construct $\mathbf{FTop}_s$ of symmetric fuzzy topological spaces (and fuzzy continuous maps) is concretely isomorphic to a full subconstruct of **FSUConv** whose objects are those fuzzy semiuniform convergence spaces (X, FJ_X) for which there is a fuzzy topology t on X such that $FJ_X = \{\mathcal{F} \subset F_L(X \times X) : \text{there is some } x \in X \text{ with } \mathcal{F} \supset \mathcal{U}_t(x) \times \mathcal{U}_t(x)\}$. (similar to the non-fuzzy case), i.e. $\mathbf{FTop}_s$ *may be regarded as a full subconstruct of* **FSUConv**.

3.35 Proposition. *Let L be a complete chain and $(X, \mathcal{X})$ a topological space which is a T_1-space. Then the fuzzy topological space $(X, t_{\mathcal{X}})$ is a T_1-space (i.e the fuzzy generalized convergence space $(X, q_{t_{\mathcal{X}}})$* is a *T_1-space).*

Proof. Let $x \in X$ and $y \in cl_{q_{t_{\mathcal{X}}}}\{x\}$. Since $(X, q_{t_{\mathcal{X}}})$ is a topological fuzzy generalized convergence space, $\mathcal{U}_{t_{\mathcal{X}}}(x)$ has a trace on $\{x\}$ (cf. 3.15.2.). Thus, since each $f \in t_{\mathcal{X}}$ with $\sup f = f(y)$ belongs to base $\mathcal{U}_{t_{\mathcal{X}}}(y)$, we obtain $f(x) = \sup f|\{x\} = \sup f = f(y)$ for such an f. If y were unequal to x there would be an open set $O \subset X$ with $y \in O$ and $x \notin O$ by assumption. Since $\chi_O \in t_{\mathcal{X}}$ such that $\chi_O(y) = 1 = \sup \chi_O$, $\chi_O(x) = \chi_O(y) = 1$, i.e. $x \in O$ – a contradiction.

3.36 Remark. If L is a complete chain, then by means of 3.35, the embedding of **Top** into **FTop** (cf. 2.14) leads to an embedding of the construct $\mathbf{Top}_1$ of topological T_1-spaces (and continuous maps) into the construct $\mathbf{FTop}_1$ of fuzzy topological T_1-spaces (and fuzzy continuous maps). Furthermore, $\mathbf{FTop}_1$ is a full subconstruct of $\mathbf{FTop}_s$ (cf. 3.33) which can be embedded into **FSUConv** (cf. 3.34).

3.37 Theorem. *Let L be a complete chain. Then there is a proper class* *of light factorization structures on* **FPUConv** *as well as on* **FSUConv**.

Proof. By [26] there is a proper class of connectednesses in $\mathbf{Top}_1$ and thus in **FSUConv** since $\mathbf{Top}_1$ may be regarded as a full subconstruct of **FSUConv** (3.36). Since **FSUConv** is a full subconstruct of **FPUConv**, there is also a proper class of connectednesses in **FPUConv**. By 1.4, there is a proper class of light factorization structures on **FSUConv** as well as on **FPUConv**.

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