Unit squares intersecting all secants of a square

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Abstract

Let S be a square of side length $s > 0$. We construct, for any sufficiently large s, a set of less than $1.994\,s$ closed unit squares whose sides are parallel to those of S such that any straight line intersecting S intersects at least one square of S. It disproves L. Fejes Tóth's conjecture that, for integral s, there is no such configuration of less than $2s-1$ unit squares.

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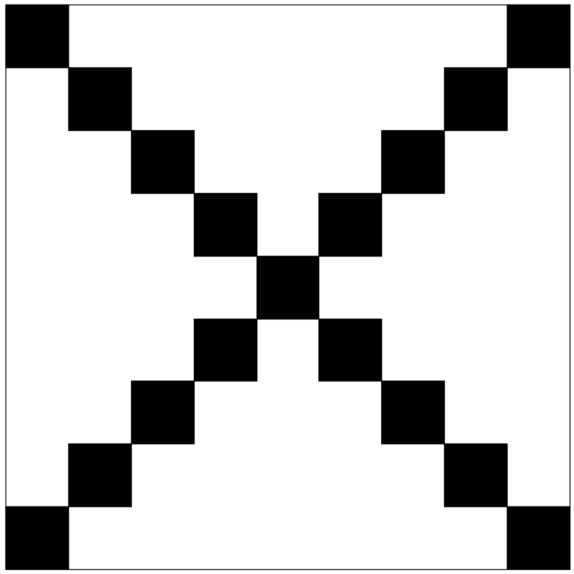

Figure 1: Line cover of size $2n-1$

1 Introduction

The following conjecture is due to L. Fejes Tóth [FT 74], [MP 85]: "Given n points in the unit square, there exists a line intersecting the unit square, which has L_∞−distance at least $1/(n+1)$ from each point."

This can obviously be restated as follows: let S denote an axis parallel square of side length s and let $\mathcal{U} = \{U_1, \ldots, U_t\}$ be any collection of axis parallel unit squares lying in S. We say that $\mathcal{U}$ is *a line cover of* S if every line which intersects S also intersects a square of $\mathcal{U}$. Let $\tau(s)$ denote the smallest $t \in \mathbb{N}$ such that there exists a line cover of S of size t. Then Fejes Tóth's conjecture states that $\tau(s) \geq 2s-1$ (the configuration in Fig. 1 gives $\tau(s) \leq 2s-1$ for s odd). We disprove this conjecture by a construction which shows $\tau(s) \leq 1.994\, s$ for s sufficiently large.

Several years ago Bárány and Fűredi studied the following version of this problem. Let $\tau'(s)$ be defined in the same way as $\tau(s)$, except that we want to cover only those lines which are parallel to one of the sides of S or to one of the diagonals of S. So we restrict our attention to lines of 4 directions. Bárány and Fűredi [BF 85] (see also [KW 90]) proved that $\tau'(s) \geq \frac{4}{3}s - \frac{1}{3}$. On the other hand, Kern and Wanka [KW 90] constructed sets which give an upper bound of $\tau'(s) \leq \frac{4}{3}s + \mathcal{O}(1)$.

2 Construction

Consider first an axis parallel square S with side length 2 centered at the origin. We define points A, B, C, D, E, F, G, O by the following relations (see Fig. 2; for technical reasons ratios of distances on it are changed):

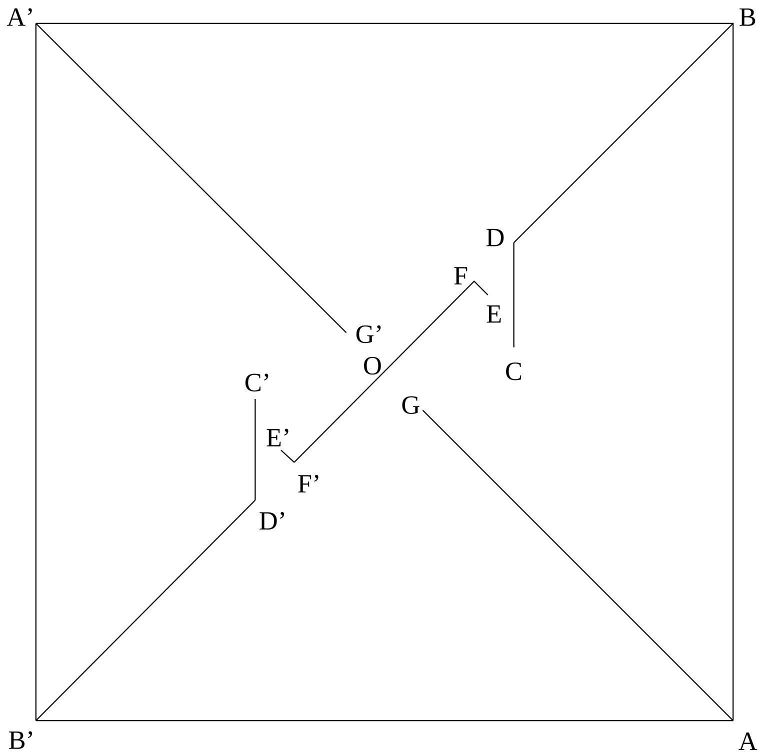

Figure 2: Points in the square S

$O = [0, 0]$, $A = [1, -1]$, $B = [1, 1]$,
O, F, D, B are collinear,
O, G, A are collinear,
CD is parallel to AB, EF is parallel to OA,
$|OD| = \frac{1}{10}|OB|$, $|FD| = \frac{1}{100}|OB|$, $|FE| = \frac{1}{1000}|OB|$,
$|OG| = \frac{1}{100}|OA|$, $|CD| = \frac{11}{1000}|AB| = 0.022$.

Thus $C = [0.1, 0.078]$, $D = [0.1, 0.1]$, $E = [0.091, 0.089]$, $F = [0.09, 0.09]$, $G = [0.01, -0.01]$. The points A', B', C', D', E', F', G' are defined as the images of the points A, B, C, D, E, F, G obtained by reflection about the origin.

Here is a technical lemma:

Lemma 1 *Each line intersecting the square S intersects some of the line segments AG, $G'A'$, $B'D'$, $D'C'$, $E'F'$, $F'F$, FE, CD, and DB.*

Proof. Due to symmetry the lemma follows from the following three statements:
1. The point C lies below the line AF.
2. The point G lies above the line $B'C$.
3. The point E lies on the line DG.

We prove the statements by the direct computation.

1. The line AF has the equation

$$y = \frac{-1.09x + 0.18}{0.91},$$

and intersects the vertical line DC in the point $[0.1, 0.78021...]$ which lies above the point C.

2. The line $B'C$ has the equation

$$y = \frac{1.078x - 0.022}{1.1},$$

and contains the point $[0.01, -0.0102]$ which lies on a vertical line below the point G.

3. The point E lies on the line DG with the equation

$$y = \frac{0.11x - 0.002}{0.09}.$$

$\square$

Now we scale the square S and all distances among points in it by the factor $\frac{s}{2}$ in order that the square S has the side length s, and construct our set as follows. We take unit squares on the line segments AG, $G'A'$, $B'D'$, $E'F'$, $F'F$, FE, DB of slope ± 1 so that two consecutive squares have always one common point (vertex) similarly as the squares on the diagonals in Fig. 1. On each of these 7 line segments we take the minimum number of unit squares so that they cover the line segment. Further we take unit squares with centers on the vertical line segments $C'D'$ and DC so that two consecutive squares have a distance 0.8 (see Fig. 3). On each of these two line segments we take the minimum number of unit squares so that their convex hull covers the line segment. We denote the obtained set of unit squares by $\mathcal{U}$.

Theorem 2 *(i) The set $\mathcal{U}$ is a line cover of S,*

(ii) the set $\mathcal{U}$ contains less than $1.994\, s$ unit squares for any sufficiently large $s > 0$.

Proof. (i) The 7 line segments of slope ± 1 are covered by squares of $\mathcal{U}$. According to Lemma 1 and due to symmetry it is sufficient to prove that any line intersecting the line segment CD intersects some square of $\mathcal{U}$.

Let p be a line intersecting the line segment CD. Since the line segments $A'G'$ and $F'F$ are covered by squares of $\mathcal{U}$ we can suppose that p does not intersect them. In this case the line p is "more vertical" than one of the three "critical" lines $A'D$, $G'F$, and $F'C$. The "critical" lines $A'D$, $G'F$, and $F'C$ have the slope greater than 0.8, equal to -0.8, and smaller than -0.8, respectively. Thus the line p has the slope outside the interval $(-0.8, 0.8)$. Consequently, it intersects some square of $\mathcal{U}$ with the center on CD.

(ii) The number of squares of the set $\mathcal{U}$ is $0.99\, s + \mathcal{O}(1)$ on each of the diagonals, $0.0005\, s + \mathcal{O}(1)$ on both the line segments $E'F'$ and EF, and $0.011 \cdot \frac{1}{1.8}\, s + \mathcal{O}(1)$ on both the line segments $C'D'$ and CD. All together, there are $2(0.99\, s + 0.0005\, s + 0.011 \cdot \frac{1}{1.8}\, s) + \mathcal{O}(1) = 1.9932...s + \mathcal{O}(1)$ squares in $\mathcal{U}$. The statement (ii) imediately follows. $\square$

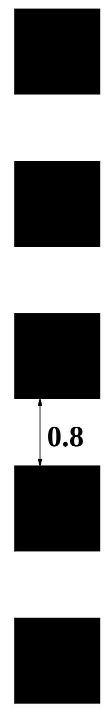

Figure 3: String of unit squares used in the construction

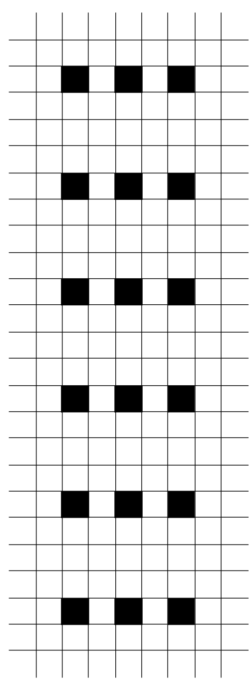

Figure 4: String of unit squares embeddable to the chess board

3 Remarks

1. The constant 1.994 in Theorem 2 can obviously still be improved. It is better to consider some curves instead of the line segments on Fig. 2, and put on them unit squares with "continuously" changing intervals among them. In this way we can get a constant smaller than 1.99. On the other hand, Bárány and Fűredi [BF 85] showed the lower bound $\tau(s) > 1.43\,s + \mathcal{O}(1)$.

2. Consider L. Fejes Tóth's problem with the following restriction. The number s is integral and the set $\mathcal{U}$ may contain only integral unit squares, i.e., unit squares whose vertices have integral coordinates. In other words, we are choosing squares of the chess board $S = s \times s$. The solution in Fig. 1 is again not optimal:

Theorem 3 *For any sufficiently large s there is a set $\mathcal{U}$ of less than $1.998\,s$ integral unit squares which is a line cover of S.*

Proof. We proceed similarly as in Section 2 with the exception that along the vertical line segments $C'D'$ and CD we place squares as shown in Fig. 4 instead of the string of squares in Fig. 3. Along both of the line segments $C'D'$ and CD we choose the length and the exact placement of the string so that the unit squares of this string are integral and their convex hull covers the line segment. We can find such strings containing $\frac{3}{4}|C'D'|+\mathcal{O}(1)$ and $\frac{3}{4}|CD|+\mathcal{O}(1)$ squares, respectively. No line of slope from the interval $\langle -\frac{5}{3}, -\frac{3}{5}\rangle$ or $\langle \frac{3}{5}, \frac{5}{3}\rangle$ passes through the sample of squares in Fig. 4. All lines of other slopes which intersect the line segment CD intersect some of the line segments $A'G'$, GA, $F'F$. Analogous arguments hold for the line segment $C'D'$. Thus $\mathcal{U}$ is a line cover of S. Finally, we count that the set $\mathcal{U}$ contains $2(0.99\,s + 0.0005\,s + 0.011 \cdot \frac{3}{4}\,s) + \mathcal{O}(1) = 1.9975\,s + \mathcal{O}(1)$ squares. $\square$

3. There is another question about the minimum size of a line cover $\mathcal{U}$ if $\mathcal{U}$ may contain arbitrary unit squares (not necessarily axis parallel). This question is related to a question about the minimum total length of curves inside a unit square which intersect all lines intersecting the unit square. According to our knowledge the four line segments in Fig. 5 of total length $\frac{1}{2}\sqrt{2}(2+\sqrt{3}) = 2.638...$ are conjectured to form the optimal solution. Note that Akman [A 87] mentions without the proof that the conjecture holds. If we put unit squares along these four line segments we get a line cover of S of size $\frac{1}{2}(2+\sqrt{3})s + \mathcal{O}(1) = 1.866...s + \mathcal{O}(1)$ (see Fig. 6). This might be the optimal solution of the problem.

References

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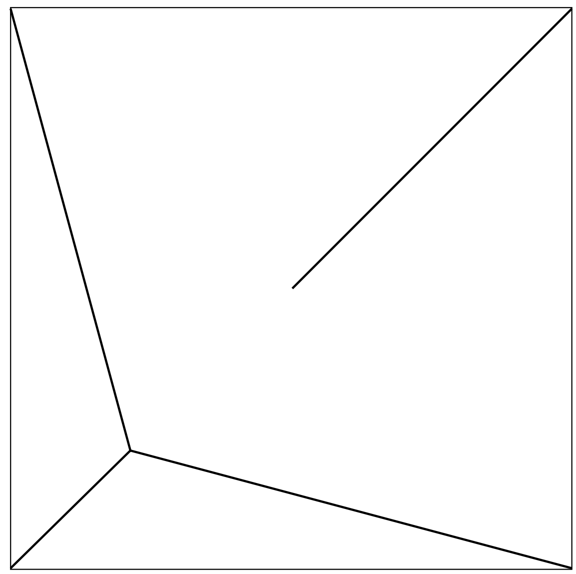

Figure 5: Line cover of a square by line segments

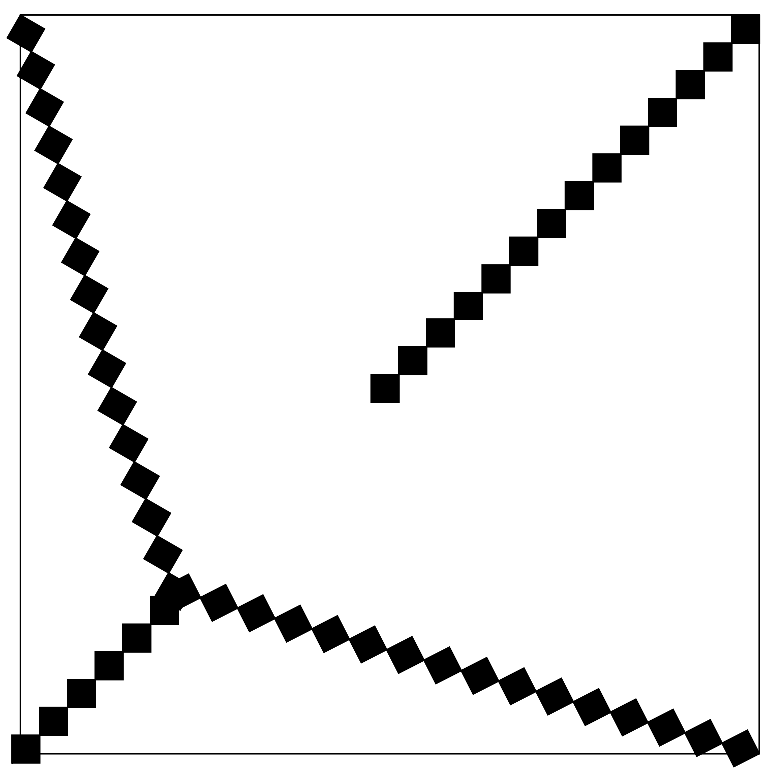

Figure 6: Line cover of a square by unit squares

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