SERIE B — INFORMATIK

Splitting formulas for Tutte polynomials

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Abstract

We present two splitting formulas for calculating the Tutte polynomial of a matroid. The first one is for a generalized parallel connection across a 3-point line of two matroids and the second one is applicable to a 3-sum of two matroids. An important tool used is the bipointed Tutte polynomial of a matroid, an extension of the pointed Tutte polynomial introduced by Thomas Brylawski in [Bry71].

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1 Introduction

It is known that determining the Tutte polynomial of a matroid (or even of a planar graph) is $\#P$-hard, and therefore also many other quantities such as the chromatic and flow polynomials of a graph or the Jones and Kauffman bracket polynomials of an alternating link ([JVW90], [BO91], [Wel93]). The natural question is how can we restrict the considered class of matroids in order to obtain polynomial-time algorithms for computing the Tutte polynomials.

The paper [OW92] of Oxley and Welsh shows that one such class are the accessible matroids of bounded width (for example the series-parallel matroids). The crucial idea of their paper was to break down an input matroid M in some not too small pieces (derived from parts of a 2-sum) and to calculate the Tutte polynomial of M from the Tutte polynomials of these pieces. For each piece the same procedure is applied recursively. A splitting formula is actually an arithmetic rule, which states how to obtain the Tutte polynomial of M from the Tutte polynomials of these smaller matroids. In [OW92] a splitting formula for 2-sums is used.

In this paper we obtain a splitting formula for a generalized parallel connection across a 3-point line and later a splitting formula for 3-sums. A 2-sum cannot be 3-connected but is 2-connected, while a 3-sum cannot be 4-connected but may be 3-connected. Therefore we can obtain more complex matroids by successive applications of the operations of 3-sums, 2-sums and direct sums than by applying only the operations of 2-sums and direct sums. Thus, the presented splitting formulas are tools for faster computation of the Tutte polynomials of matroid class more complex than given in [OW92].

It is worth mentioning that in case of Tutte polynomials of graphs Seiya Negami ([Neg87]) has obtained splitting formulas for graphs of any (fixed) connectivity.

This paper is organized as follows. In Section 3 we introduce the Tutte polynomial and some fundamental definitions linked to its computation. The next Section 4 restates briefly the notion of the pointed Tutte polynomial of Thomas Brylawki ([Bry71]) and gives the splitting formulas for a parallel connection and for a 2-sum of two matroids. The Section 5 introduces the bipointed Tutte polynomial of a matroid M, culminating in a formula for calculating it from some minors of M. Finally, the Section 6 uses all previous results in order to obtain the main splitting formulas. It also exhibits analogous splitting formulas communicated by James Oxley ([Oxl95]).

2 Basic definitions

Our notation follows [Oxl92]. We denote as $E(M)$ the set of points of a matroid M and as $r(\cdot)$ its rank operator, writing $r(M)$ for $r(E(M))$. We will denote a single-point matroid whose only element is an isthmus or a loop as I or L, respectively. For the sake of brevity we call each point which is not an isthmus nor a loop a *circuit point*. The *status* of a point is one of its three possible properties of being a loop, an isthmus or a circuit point.

We write $\mathcal{C}(M)$ to describe the family of circuits of a matroid M. For a given matroid M and $T \subseteq E(M)$ the set $\mathcal{C}' = \{C \subseteq E(M) - T : C \in \mathcal{C}(M)\}$ is the set of circuits of a matroid M' on $E(M) - T$. We say that M' has been obtained by *deletion* of T from M and so we think of deletion as an operation characterized by M and M' (the standard definition says that the deletion is the matroid M' itself). M' is denoted by $M \backslash T$. For $S \subseteq E(M)$, the matroid $M \backslash (E(M) - S)$ is called the *restriction* of M to S and denoted as $M|S$.

Analogously we define the *contraction* of M to $E(M) - T$ for a matroid M and $T \subseteq E(M)$ as an operation resulting in a matroid M' whose circuits are exactly the minimal nonempty elements of the set $\{C - T : C \in \mathcal{C}(M)\}$. M' is denoted by M/T.

Each sequence of contractions and deletions is called a *reduction*. We write $e_1^{/} e_3^{\backslash} e_2^{/}$ to describe a reduction of *length* three, in which point e_1 is contracted in M, point e_3 is deleted in the matroid resulting from the first operation and e_2 is contracted in the matroid obtained from the second operation. The case when the reduction has length one is called a *single reduction*. A matroid resulting from a reduction R of M is called a *minor* of M and denoted by $M(R)$. The operations of deletion and contraction commute both with each other and with themselves. ([Oxl92, Prop. 3.1.26, p. 109]). Thus, we may write each minor as $M \backslash T/S$ where $S \subseteq E(M) - T$ is the set of all points contracted by R and $T \subseteq E(M) - S$ is the set of all points deleted by R.

Let M_1, M_2 be matroids with $E(M_1) \cap E(M_2) = \{r\}$ and $|E(M_1)|, |E(M_2)| \geq 3$. If r is not a loop or an isthmus in M_1 or M_2, then the *2-sum* $M_1 \oplus_2 M_2$ of M_1 and M_2 is the matroid on $E(M_1) \cup E(M_2) - \{r\}$ whose set of circuits is the union of the families of circuits $\mathcal{C}(M_1 \backslash r)$, $\mathcal{C}(M_2 \backslash r)$ and $\{C_1 \cup C_2 - \{r\} : C_i$ is a circuit of M_i with $r \in C_i$, $i = 1, 2\}$.

Following [Bry71], we define a *pointed matroid* (M, r) as a pair consisting of a matroid M and a distinguished point r in $E(M)$. If $r = E_1(M_1) \cap E_2(M_2)$ is neither a loop nor an isthmus in M_1 or M_2, then the *parallel sum* (M, r) of (M_1, r) and (M_2, r) is a pointed matroid with the family of circuits

$$\begin{aligned} \mathcal{C}(M) \quad = \quad & \{C_1 \cup C_2 - \{r\} : \ C_i \text{ is a circuit of } M_i \text{ with } r \in C_i,\ i = 1, 2\} \\ & \cup \mathcal{C}(M_1) \cup \mathcal{C}(M_2). \end{aligned}$$

In [Oxl92] Oxley denotes the underlying matroid M of (M, r) as $P(M_1, M_2)$ and calls it the *parallel connection* of M_1 and M_2.

The important link to the definition of a 2-sum is that if M_1 and M_2 create a 2-sum, then this 2-sum is exactly $P(M_1, M_2) \backslash r$.

We use the following definition of a *generalized parallel connection (GPC)* taken from [Oxl92, p. 419]. Let M_1 and M_2 be two matroids and $E_1 = E(M_1)$, $E_2 = E(M_2)$. A GPC $P_N(M_1, M_2)$ of M_1 and M_2 is a matroid on $E_1 \cup E_2$ whose flats are those subsets on $X \subseteq E_1 \cup E_2$ such that $X \cap E_1$ is a flat of M_1 and $X \cup E_2$ is a flat of M_2. We write $T = E_1 \cap E_2$ and $N = M_1|T$. We call N a *connecting matroid*. We say that a GPC is *across* a 3-point line, if a connecting minor is a 3-circuit. From this definition it follows that the simple matroid associated with $M_i|T$ must be a modular flat of the simple matroid associated with M_i for at least one $i \in \{1,2\}$. This condition is sufficient for the existence of a GPC of M_1 and M_2 (see [Oxl92] for more details).

Closely related to the GPC is the notion of the 3-sum of two matroids. The idea of a 3-sum for binary matroids has been used by P.D. Seymour in his decomposition theorem for regular matroids ([Sey80]). We introduce here a generalization of his definition which also apply to non-binary matroids.

Let M_1, M_2 be matroids, $E_1 = E(M_1)$, $E_2 = E(M_2)$, $T = E_1 \cap E_2$ and $N = M_1|T$. If $M_1|T$ and $M_2|T$ are 3-circuits and the GPC $P_N(M_1, M_2)$ exists, then the *3-sum* of M_1 and M_2 equals to $P_N(M_1, M_2)\backslash T$. In other words, the sufficient conditions for the existence of $M_1 \oplus_3 M_2$ are that $M_1|T$ and $M_2|T$ are 3-circuits and that for at least one $i \in \{1,2\}$ the simple matroid associated with $M_i|T$ is a modular flat of the simple matroid associated with M_i.

(Seymour imposes additional conditions necessary in his applications: both M_1 and M_2 must be binary; both E_1 and E_2 should have more than six elements and T must not contain a cocircuit of M_1 and M_2).

3 The Tutte Polynomial and its calculation

The *Tutte polynomial* $t(M; x, y)$ of a matroid M is defined by

$$t\,(M; x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)}$$

We will use throughout the variables x and y, writing $t(M)$. The next proposition gives us an alternative definition of the Tutte polynomial.

Proposition 1 *([BO91]) The Tutte polynomial of a matroid on the empty ground set is* 1*. Otherwise, let M be a matroid on a non-empty ground set and $e \in E(M)$. Then its Tutte polynomial $t(M)$ is given by following recursive rules*

(R1) $t(M) = t(M\backslash e) + t(M/e)$ *if e is a circuit point,*

(R2) $t(M) = x\,t(M\backslash e)$ *if e is a an isthmus,*

(R3) $t(M) = y\,t(M\backslash e)$ *if e is a loop.*

We may apply the rules R1, R2, and R3 until we reach minors with an empty ground set, or in the more general case we may only reduce points in a subset E_1 of $E\,(M)$, stopping when a given minor does not contain any point in E_1.

Obviously there is a tree T associated with the process of application of R1, R2, and R3 to points in E_1. The vertex set of T is a certain subset of all reductions of points in E_1, i.e. reductions of the form $e_{i_1}^{r_1} \dots e_{i_n}^{r_n}$ with $n = |E_1|$, $r_j \in \{\backslash, /\}$ and $e_{i_j} \in E_1$ for $j = 1, \dots, n$. We create T when applying R1, R2, R3 to points in E_1 using the following algorithm, called the *calculation algorithm*:

1. Let $\mathcal{R}_0$ be the set whose only element is the empty reduction. Let $i := 0$.

2. We construct $\mathcal{R}_{i+1}$ as following: for each $R \in \mathcal{R}_i$, consider a minor $M(R)$. We choose a point $e \in E(M(R)) \cap E_1$ and apply the corresponding rule R1, R2, or R3 to e. If R2 or R3 is applied, a minor $M(R)\backslash e$ is created and so we put $Re^{\backslash}$ into $\mathcal{R}_{i+1}$. If R1 is applied, the minors $M(R)\backslash e$ and $M(R)/e$ are created and so we put $Re^{\backslash}$ and $Re^{/}$ into $\mathcal{R}_{i+1}$. We call $Re^{\backslash}$ (and possibly $Re^{/}$) a *child* of R, and R is the *parent* of $Re^{\backslash}$ (and possibly of $Re^{/}$). Thus, $\mathcal{R}_{i+1} = \{\text{child}(R) : R \in \mathcal{R}_i\}$.

3. Put $i := i + 1$; if $i < |E_1|$, go to 2.

Now the edges of T link each parent with its one or two children. As $\mathcal{R}_i = \{R : Re^r \in \mathcal{R}_{i+1} \text{ for some } e \in E_1,\ r \in \{\backslash, /\}\} = \{\text{parent}(R) : R \in \mathcal{R}_{i+1}\}$, we can obtain all $\mathcal{R}_i$'s from $\mathcal{R}_n$ for $i = 1, \ldots, n-1$. Thus, the set $\mathcal{R}_n$ completely characterizes T. We call $\mathcal{R}(E_1, T) = \mathcal{R}_n$ the *calculation* of $t(M)$ on E_1 and the tree T the *calculation tree*. The only source of differences between two such trees is the choice of point e for each minor $M(R)$ in step 2. Furthermore, let $\mathcal{R}_i(E_1, T) = \mathcal{R}_i$ and $\mathcal{M}_i(E_1, T) = \{M(R) : R \in \mathcal{R}_i(E_1, T)\}$ for $i = 1, \ldots, n$. We see that $\mathcal{M}_i(E_1, T)$ is the set of minors obtained in the ith application of step 2 of the above algorithm.

For $R \in \mathcal{R}(E_1, T)$, $R = e_{i_1}^{r_1} \ldots e_{i_n}^{r_n}$ with $r_j \in \{\backslash, /\}$ for $j = 1, \ldots, n$, we can track which of the rules R1, R2, R3 was applied during the single reduction of each e_{i_j}. If s is the number of applications of R2 and t is the number of applications of R3 during the execution of R, we call $rc(R) = x^s y^t$ the *reduction coefficient* of the reduction R. By the distributive law and induction we have:

$$t(M) = \sum_{R \in \mathcal{R}(E_1, T)} rc(R)\, t(M(R)). \tag{1}$$

Now two different reductions R_i, $R_j \in \mathcal{R}(E_1, T)$ might produce identical minors, i.e. $M(R_i) = M(R_j)$. Let $\mathcal{M}(E_2)$ be the set of all minors of M on the ground set $E_2 = E(M) - E_1$ and $N \in \mathcal{M}(E_2)$. We define the *minor coefficient* $mc_T(N)$ as the sum of the reduction coefficients of all $R \in \mathcal{R}(E_1, T)$ which produce N, i.e.:

$$mc_T(N) = \begin{cases} \displaystyle\sum_{R \in \mathcal{R}(E_1,T)\,:\, M(R) = N} rc(R) & \text{if } N \in \mathcal{M}_n(E_1, T) \\ 0 & \text{otherwise.} \end{cases}$$

Clearly

$$t(M) = \sum_{N \in \mathcal{M}(E_2)} mc_T(N)\, t(N) \tag{2}$$

which is a version of (1) with all $t(N)$ factored out for every $N \in \mathcal{M}_n(E_1, T)$.

The proof that the minor coefficients do not depend on a particular calculation tree T is very similar to the proof of Lemma 6.7 in [Bry71], (that the Tutte polynomial does not depend on the order of applications of the rules R1, R2, R3) and is left to the reader.

The idea of the splitting formula for a generalized parallel connection of two matroids is to use (2) to calculate $t(M)$. We will show in section 6 that if M is a generalized parallel connection of M_1, M_2 with ground sets E_1, E_2, respectively, then the set of minors of M on E_2 has no more than five elements and these five minors can be found easily. Moreover, there are formulas for computation of the minor coefficients for each of these minors. The only input these formulas require are the Tutte polynomials of five easily obtainable minors of M on the ground set $E_1 - E_2$. In other words, we can compute $t(M)$ only using the Tutte polynomials of the minors of M on $E_1 - E_2$ and of the minors of M on E_2.

4 The pointed Tutte polynomial and simple splitting formulas

The goal of this section is to present a splitting formula for the Tutte polynomial of a 2-sum of matroids M_1 and M_2. Such splitting formula (also stated in [OW92]) can be immediately derived from the following propositions obtained by Thomas Brylawski ([Bry71]).

Let (M, r) be a pointed matroid, $E_1 = E(M) - \{r\}$ and let $\mathcal{R}(E_1)$ be a calculation on E_1. Let $\mathcal{M} = \mathcal{M}(\{r\})$ be the set of all M-minors on $\{r\}$, then by (2) we have:

$$t(M) = \sum_{N \in \mathcal{M}} mc(N)\, t(N).$$

Clearly the only minors in $\mathcal{M}$ are N_1 and N_2, where r is an isthmus in N_1 and r is a loop in N_2. We write $t_x(M) = mc(N_1)$ and $t_y(M) = mc(N_2)$. The *pointed Tutte polynomial* $t_r(M)$ of a pointed matroid (M, r) is the polynomial $\bar{x}\, t_x(M) + \bar{y}\, t_y(M)$ on four variables x, y, $\bar{x}$, $\bar{y}$ (where $\bar{x}$ and $\bar{y}$ have only the function of distinguishing $t_x(M)$ and $t_y(M)$).

The following remark illustrates the properties of the pointed Tutte polynomial.

Remark 2 *Let (M, r) be any pointed matroid.*

(a) *We notice that by R1, R2, R3 and by the definition of mc( ) we have, for a given (M, r) and $e \in E(M) - \{r\}$,*

- $t_r(M) = t_r(M \backslash e) + t_r(M/e)$ *if e is a circuit point in M,*
- $t_r(M) = x\, t_r(M \backslash e)$ *if e is an isthmus in M,*
- $t_r(M) = y\, t_r(M \backslash e)$ *if e is a loop in M.*

(b) *From (2) it follows immediately that for the Tutte polynomial $t(M)$ of the underlying matroid M we have*

$$t(M) = x\, t_x(M) + y\, t_y(M).$$

(c) *A following formula holds for the pointed Tutte polynomial of* (M, r)*: if* $M = M_1 \oplus M_2$ *and* $r \in M_2$*, then* $t_r(M) = t(M_1)\, t_r(M_2)$.

Assume that we know $t_x(M)$ and $t_y(M)$ for a pointed matroid (M, r). Deleting r, we obtain a (non-pointed) matroid $M\backslash r$. Can we easily express $t(M\backslash r)$ in terms of $t_x(M)$ and $t_y(M)$? The answer is yes; moreover, $t(M/r)$ can also be calculated in this way.

Lemma 3 *([Bry71, Lemma 6.13, p. 15]). Let* (M, r) *be a pointed matroid with pointed Tutte polynomial* $t_r(M) = \bar{x}\, t_x(M) + \bar{y}\, t_y(M)$*. If* r *is a circuit point in* M*, then*

$$t(M\backslash r) = (x-1)\, t_x(M) + t_y(M) \tag{3}$$

and

$$t(M/r) = t_x(M) + (y-1)\, t_y(M). \tag{4}$$

Proposition 4 *([Bry71, Corollary 6.14, p. 16]) Let* (M, r) *be a pointed matroid. If we know the Tutte polynomials* $t(M/r)$ *and* $t(M\backslash r)$*, then we can compute the pointed Tutte polynomial* $t_r(M)$*:*

$$t_x(M) = [t(M/r) - (y-1)\, t(M\backslash r)] \,/\, (x + y - x\, y)\,,$$

$$t_y(M) = [t(M\backslash r) - (x-1)\, t(M/r)] \,/\, (x + y - x\, y)\,.$$

Proof: We obtain $t_x(M)$ and $t_y(M)$ solving the equations $t(M/r) = t_x(M) + (y-1)\, t_y(M)$ and $t(M\backslash r) = (x-1)\, t_x(M) + t_y(M)$ obtained in Lemma 3. □

We would like to sketch how the last proposition can be used to obtain a splitting formula for the Tutte polynomial of a 2-sum of two matroids. In Theorem 6.15, p.17 of his paper Brylawski gives the following formula for the pointed Tutte polynomial of (M, r) in terms of the pointed Tutte polynomials of M_1 and M_2 (for the case that neither M_1 nor M_2 has $\{r\}$ as a loop or an isthmus):

$$\begin{aligned} t_r(M) \;=\; & \bar{x}\, [t_x(M_1)\, t_x(M_2)] + \bar{y}\, [(y-1)\; t_y(M_1)\, t_y(M_2) \\ & t_x(M_1)\, t_y(M_2) + t_y(M_1)\, t_x(M_2)]\,. \end{aligned} \tag{5}$$

Now using Proposition 4 and the fact that for a pointed matroid (M, r) we have $t(M) = x\, t_x(M) + y\, t_y(M)$ we obtain **the splitting formula for the Tutte polynomial of a parallel connection** $M' = P(M_1, M_2)$ of matroids M_1 and M_2 (for the case that neither M_1 nor M_2 has r as a loop or an isthmus):

$$t(M') = \frac{1}{x\, y - x - y} \left[\; t\,(M_1/r) \quad t\,(M_1\backslash r)\; \right] \mathbf{B}' \left[\begin{array}{c} t\,(M_2/r) \\ t\,(M_2\backslash r) \end{array} \right]$$

where

$$\mathbf{B}' = \begin{bmatrix} x\,y - y - 1 & -1 \\ -1 & y - 1 \end{bmatrix}.$$

Applying the relations found in Proposition 4 and applying (3) to (5) we obtain the **splitting formula for the Tutte polynomial of a 2-sum** $M'' = M_1 \oplus_2 M_2$ of M_1 and M_2:

$$t(M'') = \frac{1}{x\,y - x - y} \begin{bmatrix} t\,(M_1/r) & t\,(M_1\backslash r) \end{bmatrix} \mathbf{B}'' \begin{bmatrix} t\,(M_2/r) \\ t\,(M_2\backslash r) \end{bmatrix}$$

where

$$\mathbf{B}'' = \begin{bmatrix} x - 1 & -1 \\ -1 & y - 1 \end{bmatrix}.$$

5 The bipointed Tutte polynomial

A *bipointed matroid* is an ordered triple (M, p, s) where M is a matroid and p, s are two points in $E\,(M)$.

Let $E_1 = E\,(M) \setminus \{p, s\}$. Let $\mathcal{R}(E_1)$ be a calculation of $t(M)$ on E_1 and $\mathcal{M} \supseteq \mathcal{M}_{|E_1|}(E_1)$ the set of all M-minors on $\{p, s\}$, then by (2) we have:

$$t(M) = \sum_{N \in \mathcal{M}} mc(N)\, t(N).$$

Obviously there are at most the following five minors in $\mathcal{M}$:

- N_1, in which both p and s are isthmi;
- N_2, where p is an isthmus and s is a loop;
- N_3, where p is a loop and s is an isthmus;
- N_4, where p and s are both loops;
- N_5, where $\{p, s\}$ is a circuit.

Our notation associates the symbols x, y, c with an isthmus, a loop and a circuit point respectively and so we write $t_{xx}(M) = mc(N_1)$, $t_{xy}(M) = mc(N_2)$, $t_{yx}(M) = mc(N_3)$, $t_{yy}(M) = mc(N_4)$ and $t_{cc}(M) = mc(N_5)$.

In following the symbols $\bar{x}\bar{x}$, $\bar{x}\bar{y}$, $\bar{y}\bar{x}$, $\bar{y}\bar{y}$ and $\bar{c}\bar{c}$ denote single variables. For a given bipointed matroid (M, p, s), we define the *bipointed Tutte polynomial* as the polynomial

$$t_{ps}\,(M) = \bar{x}\bar{x}\, t_{xx}(M) + \bar{x}\bar{y}\, t_{xy}(M) + \bar{y}\bar{x}\, t_{yx}(M) + \bar{y}\bar{y}\, t_{yy}(M) + \bar{c}\bar{c}\, t_{cc}(M)$$

on seven variables $\bar{x}\bar{x}$, $\bar{x}\bar{y}$, $\bar{y}\bar{x}$, $\bar{y}\bar{y}$, $\bar{c}\bar{c}$, x, y over the integers. In a way analogous to the case of a pointed Tutte polynomial, the first five variables are only used to separate the terms $t_{xx}(M), \ldots, t_{cc}(M)$.

The following remark will be used in the proofs later on. Simultaneously it illustrates the relations between $t_{ps}(M)$ and both the pointed and non-pointed Tutte polynomials.

Remark 5 **(a)** *For any $e \in E(M) \setminus \{p, s\}$, the following hold:*

- $t_{ps}(M) = t_{ps}(M \backslash e) + t_{ps}(M/e)$ *if e is a circuit point in M,*
- $t_{ps}(M) = x\, t_{ps}(M \backslash e)$ *if e is an isthmus in M,*
- $t_{ps}(M) = y\, t_{ps}(M \backslash e)$ *if e is a loop in M.*

(b) *With $t(N_5) = t(C_2) = x + y$ and (2) we obtain:*

$$t(M) = x^2\, t_{xx}(M) + xy\, t_{xy}(M) + yx\, t_{yx}(M) + y^2\, t_{yy}(M) + (x + y)\, t_{cc}(M).$$

(c) *Given any calculation of $t(M)$ which yields $t_{ps}(M)$, we can continue it by reducing p in each minor $N_1, \ldots, N_5$, obtaining the (single) pointed Tutte polynomial*

$$\begin{aligned} t_s(M) &= \bar{x}\, [x\, t_{xx}(M) + y\, t_{yx}(M) + t_{cc}(M)] \\ &\quad + \bar{y}\, [x\, t_{xy}(M) + y\, t_{yy}(M) + t_{cc}(M)]\,. \end{aligned}$$

By symmetry we have

$$\begin{aligned} t_p(M) &= \bar{x}\, [x\, t_{xx}(M) + y\, t_{xy}(M) + t_{cc}(M)] \\ &\quad + \bar{y}\, [x\, t_{yx}(M) + y\, t_{yy}(M) + t_{cc}(M)]\,. \end{aligned}$$

(d) *If $M = M_1 \oplus M_2$ and $p, s \in E(M_2)$, then $t_{ps}(M) = t(M_1)\, t_{ps}(M_2)$.*
It holds because each reduction of all $E(M_1)$-points yields M_2. Therefore for each $j = 1, \ldots, 5$ the set $\{R \in \mathcal{R}(E(M) - \{p, s\}) : M(R) = N_j\}$ of all reductions leading to one of the minors $N_1, \ldots, N_5$ is a Cartesian product of the set of all M-reductions of $E(M_1)$-points and of the set of all M_2-reductions of $(E(M_2) - \{p, s\})$-points yielding N_j. By distributivity and the definition of $mc(\,)$ we have

$$t_{xx}(M) = mc(M_1)\, t_{xx}(M_2), \ldots, t_{cc}(M) = mc(M_1)\, t_{cc}(M_2).$$

It is not hard to see that $mc(M_1) = t(M_1)$, so the statement holds.

(e) *If $M = M_1 \oplus M_2$ with $p \in E(M_1)$ and $s \in E(M_2)$, we may write $M = (M_1, p) \oplus (M_2, s)$. We expand our notation and write $t_p(M_1) = \bar{x}_p\, t_x(M_1) + \bar{y}_p\, t_y(M_1)$ and $t_s(M_2) = \bar{x}_s\, t_x(M_2) + \bar{y}_s\, t_y(M_2)$. We want to show (by similar arguments as in the previous remark) that $t_{ps}(M) = t_p(M_1)\; t_s(M_2)$ where $\bar{x}\bar{x} = \bar{x}_p\, \bar{x}_s$, $\bar{x}\bar{y} = \bar{x}_p\, \bar{y}_s$, $\bar{y}\bar{x} = \bar{y}_p\, \bar{x}_s$ and $\bar{y}\bar{y} = \bar{y}_p\, \bar{y}_s$.*
Let $\mathcal{R}(E_1, T)$ be a calculation such that in every reduction all points from $E(M_1) - \{p\}$ are reduced before any point from $E(M_2) - \{s\}$. Obviously $\mathcal{R}(E_1)$ can be written as $\mathcal{R}_1 \times \mathcal{R}_2$ where $\mathcal{R}_1$ is a set of certain M_1-reductions

of the points in $E(M_1) - \{p\}$ *and* $\mathcal{R}_2$ *is a set of certain* M_2*-reductions of the points in* $E(M_2) - \{s\}$*. If* $\mathcal{R}_1^I \subseteq \mathcal{R}_1$ *is a set of all* M_1*-reductions making* p *an isthmus and* $\mathcal{R}_2^L \subseteq \mathcal{R}_2$ *is the set of all* M_2*-reductions making* s *a loop, then* $\mathcal{R}_1^I \times \mathcal{R}_2^L \subseteq \mathcal{R}(E_1)$ *is obviously the set of all* M*-reductions yielding minor* N_2*, (in which* p *is an isthmus and* s *is a loop). Now*

$$t_x(M_1) = \sum_{R \in \mathcal{R}_1^I} rc(R), \quad t_y(M_2) = \sum_{R \in \mathcal{R}_2^L} rc(R), \quad \textit{and}$$

$$t_{xy}(M) = \sum_{R \in \mathcal{R}_1^I \times \mathcal{R}_2^L} rc(R)$$

so by definition of $rc\,()$ *we have* $t_x(M_1)\, t_y(M_2) = t_{xy}(M)$*. The three other cases follow analogously. As* p, s *cannot be in a circuit in* M*, we have* $t_{cc}(M) = 0$*, so the statement is proven.*

Given a GPC M^+ of matroids M_1 and M_2 across a 3-point line $\{p, s, q\}$, we may delete q in M^+, obtaining a bipointed matroid M. The following lemma gives us an important link between the polynomials $t_{xx}(M), \ldots, t_{cc}(M)$ and the minor coefficients of certain matroids obtained from the connecting minor N of M^+ in the process of a calculation of $t(M^+)$.

Lemma 6 *Let* (M, p, s) *be a bipointed matroid and let* M^+ *be a matroid on* $E(M) \cup \{q\}$ *such that* $M^+ \backslash q = M$ *and* $\{p, q, s\}$ *is a circuit in* M^+*. Let* $\mathcal{R}^+(E_1)$ *be a calculation of* $t(M^+)$ *on* $E_1 = E(M^+) - \{p, q, s\}$*. We denote by* $\mathcal{M}^+$ *the set of all* M^+*-minors on* $\{p, q, s\}$ *and by* $\mathcal{M}$ *the set of all* M*-minors on* $\{p, s\}$*. Then* $\phi : \mathcal{M}^+ \to \mathcal{M}$ *with* $\phi(N^+) = N^+ \backslash q$ *is a bijection and has the property that* $mc(N^+) = mc(\phi(N^+))$*. Note that* $mc(\phi(N^+))$ *is one of the* $t_{xx}(M), \ldots, t_{cc}(M)$*.*

Proof: The following table shows the bijection:

$N^+ \in \mathcal{M}^+$	$\phi(N^+) = N^+ \backslash q$
$N_1^+ : \{p, q, s\}$ is a circuit	$N_1 : p, s$ are isthmi
$N_2^+ : s$ is a loop, $\{p, q\}$ is a circuit	$N_2 : p$ is an isthmus, s is a loop
$N_3^+ : p$ is a loop, $\{s, q\}$ is a circuit	$N_3 : p$ is a loop, s is an isthmus
$N_4^+ : p, q, s$ are loops	$N_4 : p$ and s are loops
$N_5^+ : q$ is a loop, $\{p, s\}$ is a circuit	$N_5 : \{p, s\}$ is a circuit

Table 1: The bijection ϕ.

To obtain $\mathcal{M}^+$ we used the fact that the circuit $\{p, q, s\}$ is only changed when one of p, q, s is in the closure of an E_1-point to be contracted. The table contains all three cases when exactly one of p, q, s becomes a loop; furthermore, the cases when none of p, q, s is loop and when all three of p, q, s are loops. Notice that if two of the points become loops the remaining point must also become a loop as $\{p, q, s\}$ is a circuit, so each such matroid is N_4^+.

Now to show that $mc(N^+) = mc(\phi(N^+))$ for any $N^+ \in \mathcal{M}^+$ we will prove first that if M_1^+ is a M^+-minor which occurred during the calculation $\mathcal{R}^+(E_1)$, then for each $e \in E(M_1) - \{p, q, s\}$ in the corresponding M-minor M_1 (obtained by same reduction as M_1^+) e has the same status. As the order of single reductions does not affect the resulting matroid, we have $M_1 = M_1^+ \backslash q$. Now if $e \in E(M_1) - \{p, s\}$ is a loop or isthmus in M_1^+, then it clearly has the same status in M_1. If e is a circuit point, the deletion of q in M_1^+ might only affect e if all circuits containing e would also contain q. But this cannot be the case: if q is a loop in M_1^+, it is not contained in any other circuit. If q is not a loop, there must be a circuit $\{p, q, s\}$ or $\{p, q\}$ or $\{q, s\}$ in M_1^+. Thus, the application of the strong circuit elimination axiom on any circuit containing both e and q and on one of the three listed circuits yields a circuit with e but without q.

The last claim ensures that in each M_1^+, M_1 we may apply the same rule R1, R2, R3 to e, so $\mathcal{R}^+(E_1)$ is also a calculation for M. We see that $mc(N^+) = mc(\phi(N^+))$ for each $N^+ \in \mathcal{M}^+$. □

Lemma 7 *Let (M, p, s) be a bipointed matroid and M^+ a matroid on $E(M) \cup \{q\}$ such that $M^+ \backslash q = M$ and $\{p, q, s\}$ is a circuit. Denote by M^c the matroid M^+/q and by M^- the single pointed matroid $M^c \backslash s$. Then*

$$t_{ps}(M^c) = \bar{y}\bar{y}\, t_y(M^-) + \bar{c}\bar{c}\, t_x(M^-) \tag{6}$$

and

$$\begin{aligned} t_{ps}(M^c) \;\; = \;\; & \bar{y}\bar{y}\, [(y-1)t_{yy}(M) + t_{xy}(M) + t_{yx}(M)] \\ & + \bar{c}\bar{c}\, [(y-1)\, t_{cc}(M) + t_{xx}(M)] \, . \end{aligned} \tag{7}$$

Proof: We use the same notation as in the proof of Lemma 6. As $\{p, q, s\}$ is a circuit in M^+, $\{p, s\}$ is a circuit in M^c. Thus, only minors N_4 and N_5 may occur in M^c and so only $t_{yy}(M^c)$ and $t_{cc}(M^c)$ might be nonzero. Given any M^c-minor M_1 and $e \in E(M^c) - \{p, s\}$ we notice that status of e is the same in both M_1 and $M_1 \backslash s$. This is due to the fact that for any circuit C_1 of M_1 with $e, s \in C_1$, by the circuit elimination axiom $(C_1 \cup \{p\}) - \{s\}$ is also a circuit containing e, i.e. deletion of s doesn't affect the status of any M_1-circuit point. Clearly loops and isthmi also retain their status, so any calculation $\mathcal{R}(E(M^c) - \{p, s\})$ of M^c is also a $M^c \backslash s = M^-$-calculation. Thus, (with minor coefficients with respect to calculations for M and M^-) $t_{yy}(M) = mc(N_4) = mc(N_4 \backslash s) = t_y(M^-)$, as p is a loop in the M^--minor $N_4 \backslash s$, and $t_{cc}(M^c) = mc(N_5) = mc(N_5 \backslash s) = t_x(M^-)$ as p is an isthmus in the M^--minor $N_5 \backslash s$.

Now for the proof of (7) recall that in the calculation $\mathcal{R}^+(E_1)$ of Lemma 6 we did not reduce the point q, i.e. the minors $N_1^+, \ldots, N_5^+$ have point set $\{p, q, s\}$. Now we want to extend this calculation by reducing q to obtain the bipointed Tutte polynomial $t_{ps}(M^+)$. Observe that by Lemma 6 the minor coefficients $mc(N_1^+), \ldots, mc(N_5^+)$ are respectively $mc(N_1) = t_{xx}(M), \ldots, mc(N_5) = t_{cc}(M)$, and therefore we can express $t_{ps}(M^+)$ in terms of $t_{xx}(M), \ldots, t_{cc}(M)$. We have:

- $N_1^+\backslash q = N_1$ and $N_1^+/q = N_5$, so $t(N_1^+) = t(N_1) + t(N_5)$,
- $N_2^+\backslash q = N_2$ and $N_2^+/q = N_4$, so $t(N_2^+) = t(N_2) + t(N_4)$,
- $N_3^+\backslash q = N_3$ and $N_3^+/q = N_4$, so $t(N_3^+) = t(N_3) + t(N_4)$,
- $N_4^+\backslash q = N_4$ and q is a loop in N_4^+, so $t(N_4^+) = y\, t(N_4)$,
- $N_5^+\backslash q = N_5$ and q is a loop in N_5^+, so $t(N_5^+) = y\, t(N_5)$.

Using (2) and, as mentioned, Lemma 6, we obtain

$$\begin{aligned} t\left(M^+\right) &= t(N_1)\, mc(N_1) + t(N_2)\, mc(N_2) + t(N_3)\, mc(N_3) + \\ &\quad t(N_4)\, [mc(N_2) + mc(N_3) + y\, mc(N_4)] + t(N_5)\, [mc(N_1) + y\, t(N_5)] \end{aligned}$$

i.e. by the definition of a bipointed Tutte polynomial,

$$\begin{aligned} t_{ps}\left(M^+\right) &= \bar{x}\bar{x}\, t_{xx}(M) + \bar{x}\bar{y}\, t_{xy}(M) + \bar{y}\bar{x}\, t_{yx}(M) + \\ &\quad \bar{y}\bar{y}\, [t_{xy}(M) + t_{yx}(M) + y\, t_{yy}(M)] + \bar{c}\bar{c}\, [t_{xx}(M) + y\, t_{cc}(M)]\,. \end{aligned}$$

By Remark 5(a), $t_{ps}\,(M^+) = t_{ps}\,(M^+\backslash q) + t_{ps}\,(M^+/q) = t_{ps}\,(M) + t(M^c)$, and thus, $t_{ps}\,(M^c) = t_{ps}\,(M^+) - t_{ps}\,(M)$, which yields (7). □

Lemma 8 (a) *If (C^n, p, s) is a circuit on n points (including p and s), then $t_{ps}\,(C^n) = \bar{x}\bar{x}\,(x^{n-3} + \ldots + 1) + \bar{c}\bar{c}$ (with $x^{n-3} + \ldots + 1 = 0$ for $n < 3$).*

(b) *If (C_n, p, s) is a cocircuit on n points (including p and s), i.e. a matroid without loops in which all other $n-1$ points are in the closure of p (and so also of s), then $t_{ps}\,(C_n) = \bar{y}\bar{y}\;(y^{n-3} + \ldots + 1) + \bar{c}\bar{c}$ (with $y^{n-3} + \ldots + 1 = 0$ for $n < 3$).*

Proof: By induction.

(a) For $n = 2$ clearly $t_{ps}\,(C^2) = \bar{c}\bar{c}$. If $n > 2$, then for $e \in E\,(C^n) - \{p, s\}$ we have, by Remark 5(a), $t_{ps}\,(C^n) = t_{ps}\,(C^n\backslash e) + t_{ps}\,(C^n/e) = x^{n-3}\,\bar{x}\bar{x} + t_{ps}\,(C^{n-1}) = x^{n-3}\,\bar{x}\bar{x} + \bar{x}\bar{x}\,(x^{n-4} + \ldots + 1) + \bar{c}\bar{c} = \bar{x}\bar{x}\,(x^{n-3} + x^{n-4} + \ldots + 1) + \bar{c}\bar{c}$.

(b) Again in case $n = 2$ we obtain $t_{ps}\,(C_2) = \bar{c}\bar{c}$. If $n > 2$, then for $e \in E\,(C_n) - \{p, s\}$ the application of Remark 5(a) gives $t_{ps}\,(C_n) = t_{ps}\,(C_n\backslash e) + t_{ps}\,(C_n/e) = t_{ps}\,(C_{n-1}) + y^{n-3}\,\bar{y}\bar{y} = \bar{y}\bar{y}(y^{n-4} + \ldots + 1) + \bar{c}\bar{c} + y^{n-3}\,\bar{y}\bar{y} = \bar{y}\bar{y}\,(y^{n-3} + \ldots + 1) + \bar{c}\bar{c}$.

□

The formulas (3) and (4) express $t(M\backslash r)$ and $t(M/r)$ in terms of $t_x(M)$ and $t_y(M)$. Now by deleting or contracting p from (M, p, s) we obtain (single) pointed matroids $(M\backslash p, s)$ and $(M/p, s)$ respectively. We will show below how to find

$t_x(M\backslash p)$ and $t_y(M\backslash p)$, i.e the pointed Tutte polynomial of $(M\backslash p, s)$ from the bi-pointed Tutte polynomial of (M, p, s). An analogous formula will be obtained for $(M/p, s)$. By symmetry the formulas for the (single) pointed Tutte polynomials of $(M\backslash s, p)$ and $(M/s, p)$ will also be deduced.

Lemma 9 *Let (M, p, s) be a bipointed matroid with bipointed Tutte polynomial $t_{ps}(M)$. If p is a circuit point in M, then*

$$\begin{aligned} t_s(M\backslash p) &= \bar{x}\,[(x-1)t_{xx}(M) + t_{yx}(M) + t_{cc}(M)] + \\ &\quad \bar{y}\,[(x-1)t_{xy}(M) + t_{yy}(M)] \end{aligned} \tag{8}$$

and

$$\begin{aligned} t_s(M/p) &= \bar{x}\,[t_{xx}(M) + (y-1)t_{yx}(M)] + \\ &\quad \bar{y}\,[t_{xy}(M) + (y-1)t_{yy}(M) + t_{cc}(M)]\,. \end{aligned} \tag{9}$$

By symmetry, if s is a circuit point in M, then

$$t_p(M\backslash s) = \bar{x}\,[(x-1)t_{xx}(M) + t_{xy}(M) + t_{cc}(M)] + \bar{y}\,[(x-1)t_{yx}(M) + t_{yy}(M)]$$

and

$$t_p(M/s) = \bar{x}\,[t_{xx}(M) + (y-1)t_{xy}(M)] + \bar{y}\,[t_{yx}(M) + (y-1)t_{yy}(M) + t_{cc}(M)]\,.$$

Proof: We show (8) by induction on $n = |E(M)|$. Cases A, B and C handle the basis and some special cases, while Case D handles the general induction step.

Case A. $M = (M_1, p) \oplus (M_2, s)$.
We have by Remark 2 (c), $t_s(M\backslash p) = t(M_1\backslash p)\, t_s(M_2)$ and by (3) $t(M_1\backslash p) = (x-1)\, t_x(M\backslash p) + t_y(M\backslash p)$, and so, by the fact that $t_{cc}(M) = 0$ and using Remark 5(e),

$$\begin{aligned} t_s(M\backslash p) &= (x-1)\, t_x(M_1)\, t_s(M_2) + t_y(M_1)\, t_s(M_2) \\ &= \bar{x}\,[(x-1)\, t_{xx}(M) + t_{yx}(M) + t_{cc}(M)] + \\ &\quad \bar{y}\,[(x-1)\, t_{xy}(M) + t_{yy}(M)]\,. \end{aligned}$$

In all of the following cases p and s are in a common circuit.

Case B. $M = (C^n, p, s) \oplus N$.
By Lemma 6(a) $t_{ps}(C^n) = \bar{x}\bar{x}\,(x^{n-3} + \ldots + 1) + \bar{c}\bar{c}$ and, because $t_s(C^n\backslash p) = \bar{x}\, x^{n-2}$, we have

$$\begin{aligned} t_s(C^n\backslash p) &= \bar{x}\,[(x-1)t_{xx}(C^n) + t_{cc}(C^n)] \\ &= \bar{x}\,[(x-1)\, t_{xx}(C^n) + t_{yx}(C^n) + t_{cc}(C^n)] + \\ &\quad \bar{y}\,[(x-1)\, t_{xy}(C^n) + t_{yy}(C^n)] \end{aligned}$$

as $t_{yx}(C^n) = t_{xy}(C^n) = t_{yy}(C^n) = 0$. By Remark 2(c) we have $t_s(M\backslash p) = t_s(C^n\backslash p)\, t(N)$ and by Remark 5 (d) $t_{ps}(M) = t_{ps}(C^n)\; t(N)$, and so the statement is proven.

Case C. $M = (C_n, p, s) \oplus N$.
By Lemma 6(b) $t_{ps}(C_n) = \bar{y}\bar{y}\,(y^{n-3} + \ldots + 1) + \bar{c}\bar{c}$ and, with $t_s(C_n \backslash p) = t_s(C_{n-1}) = \bar{x} + \bar{y}\,(1 + y + \ldots + y^{n-3})$, we have

$$\begin{aligned} t_s(C_n \backslash p) &= \bar{x}\, t_{cc}(C_n) + \bar{y}\, t_{yy}(C_n) \\ &= \bar{x}\, [(x-1)\, t_{xx}(C_n) + t_{yx}(C_n) + t_{cc}(C_n)] + \\ &\qquad \bar{y}\, [(x-1)\, t_{xy}(C_n) + t_{cc}(C_n)]\,. \end{aligned}$$

Again $t_s(M \backslash p) = t_s(C_n \backslash p)\, t(N)$ and $t_{ps}(M) = t_{ps}(C_n)\; t(N)$ and so the statement follows.

Case D. In all matroids M such that p and s are in a common circuit and which are not of the types treated in cases B and C, p must be in two different circuits such that at least one of them has more than two elements (otherwise we would have case C). Thus, there is a circuit point $e \in E(M) - \{p, s\}$ such that p is not in the closure of e and there is a circuit containing p but not e. So p is a circuit point in both $M \backslash e$ and M/e and e is a circuit point in $M \backslash p$. With $(M \backslash p) \backslash e = (M \backslash e) \backslash p$, $(M \backslash p)/e = (M/e) \backslash p$ and by Remark 5(a) and the induction hypothesis,

$$\begin{aligned} t_s(M \backslash p) &= t_s((M \backslash e) \backslash p) + t_s((M/e) \backslash p) \\ &= \bar{x}\, [(x-1)\, (t_{xx}(M \backslash e) + t_{xx}(M/e)) + (t_{yx}(M \backslash e) + \\ &\qquad t_{yx}(M/e)) + (t_{cc}(M \backslash e) + t_{cc}(M/e))] + \\ &\quad \bar{y}\, [(x-1)\ (t_{xy}(M \backslash e) + t_{xy}(M/e)) + \\ &\qquad (t_{yy}(M \backslash e) + t_{yy}(M/e))]\,. \end{aligned}$$

By Remark 5(a), we have $t_{xx}(M) = t_{xx}(M \backslash e) + t_{xx}(M/e)$ etc. so (8) is proven.

Finally, we show (9). By Remark 2(a), $t_s(M) = t_s(M \backslash p) + t_s(M/p)$ and, by Remark 5(c),

$$t_s(M) = \bar{x}\, [x\, t_{xx}(M) + y\, t_{yx}(M) + t_{cc}(M)] + \bar{y}\, [x\, t_{xy}(M) + y\, t_{yy}(M) + t_{cc}(M)]$$

so

$$\begin{aligned} t_s(M/p) &= t_s(M) - t_s(M \backslash p) \\ &= \bar{x}\, [(x - (x-1))\, t_{xx}(M) + (y-1)\, t_{yx}(M) + (1-1)\, t_{cc}(M)] + \\ &\qquad \bar{y}\, [(x - (x-1))\, t_{xy}(M) + (y-1)\, t_{yy}(M) + t_{cc}(M)] \end{aligned}$$

and we are done. □

In a way analogous to Proposition 4, the following theorem shows how to calculate the bipointed Tutte polynomial of a bipointed matroid M from the Tutte polynomials of minors of a matroid M^+ (which is the GPC of of M_1 and M_2). It is the main result of this section.

Theorem 10 *Let (M,p,s) be a bipointed matroid such that p and s are both circuit points and furthermore s is a circuit point in both $M\backslash p$ and M/p. Let M^+ be a matroid on $E(M)\cup\{q\}$ such that $M = M^+\backslash q$ and $\{p,q,s\}$ is a circuit in M^+. We define five minors of M^+ : $Q_1 = M^+\backslash p\backslash s\backslash q$, $Q_2 = M^+\backslash p/s\backslash q$, $Q_3 = M^+/p\backslash s\backslash q$, $Q_4 = M^+/p/s/q$, $Q_5 = M^+\backslash p\backslash s/q$. Let $\vec{q}$ be the vector*

$$\vec{q} = [t(Q_1), t(Q_2), t(Q_3), t(Q_4), t(Q_5)]^T ,$$

and $\vec{t}$ the vector

$$\vec{t} = [t_{xx}(M), t_{xy}(M), t_{yx}(M), t_{yy}(M), t_{cc}(M)]^T .$$

Then $\vec{t} = C\,\vec{q}$, where C is a matrix:

$$\frac{1}{-1-x-y+x\,y}\begin{bmatrix} \frac{(1-y)^2}{-x-y+xy} & \frac{1-y}{-x-y+xy} & \frac{1-y}{-x-y+xy} & \frac{2}{-x-y+xy} & \frac{1-y}{-x-y+xy} \\ \frac{1-y}{-x-y+xy} & 1 & \frac{1}{-x-y+xy} & \frac{1-x}{-x-y+xy} & \frac{1}{-x-y+xy} \\ \frac{1-y}{-x-y+xy} & \frac{1}{-x-y+xy} & 1 & \frac{1-x}{-x-y+xy} & \frac{1}{-x-y+xy} \\ \frac{2}{-x-y+xy} & \frac{1-x}{-x-y+xy} & \frac{1-x}{-x-y+xy} & \frac{(1-x)^2}{-x-y+xy} & \frac{1-x}{-x-y+xy} \\ \frac{1-y}{-x-y+xy} & \frac{1}{-x-y+xy} & \frac{1}{-x-y+xy} & \frac{1-x}{-x-y+xy} & 1 \end{bmatrix}$$

Thus, we can compute the bipointed Tutte polynomial of M from the Tutte polynomials of the minors $Q_1, \dots, Q_5$ of M^+.

Proof: Let (M^-, p) be a single pointed matroid with $M^- = M^+\backslash s/q$. First we will prove that if $\vec{m}$ is the vector

$$\vec{m} = \left[t(M\backslash p\backslash s), t(M\backslash p/s), t(M/p\backslash s), t(M/p/s), t_x(M^-)\right]^T$$

and **A** the matrix:

$$\mathbf{A} = \begin{bmatrix} (x-1)^2 & x-1 & x-1 & 1 & x-1 \\ x-1 & (x-1)(y-1) & 1 & y-1 & 1 \\ x-1 & 1 & (x-1)(y-1) & y-1 & 1 \\ 1 & y-1 & y-1 & (y-1)^2 & y-1 \\ 1 & 0 & 0 & 0 & y-1 \end{bmatrix}$$

then $\vec{m} = \mathbf{A}\,\vec{t}$. By (8) we have

$$t_s(M\backslash p) = \bar{x}\,[(x-1)\,t_{xx}(M) + t_{yx}(M) + t_{cc}(M)] + \\ \bar{y}\,[(x-1)\,t_{xx}(M) + t_{yy}(M)] .$$

By assumption s is a circuit point in $M\backslash p$, so we may apply (3), yielding

$$t(M\backslash p\backslash s) = (x-1)\,[(x-1)\,t_{xx}(M) + t_{yx}(M) + t_{cc}(M)] + \\ (x-1)\,t_{xy}(M) + t_{yy}(M)$$

which proves the first of the five equations of $\vec{m} = \mathbf{A}\,\vec{t}$. Equations two to four are proved analogously and the last equation is obtained by comparing the coefficients of $\bar{c}\bar{c}$ in (6) and (7). (A comparison of the coefficients of $\bar{y}\bar{y}$ yields the equation $t_y(M^-) = (y-1)\,t_{yy}(M) + t_{xy}(M) + t_{yx}(M)$, which might be interchanged with the last equation of $\vec{m} = \mathbf{A}\vec{t}$. This new set of equations gives us an alternative way to compute $\vec{t}$).

Now $M = M^+\backslash q$, so $Q_1 = M\backslash p\backslash s$, $Q_2 = M\backslash p/s$ and $Q_3 = M/p\backslash s$. In $M^+/p/s$ q must be a loop because $\{p, s, q\}$ is a circuit of M^+, so the deletion of q is the same as its contraction and we see that $Q_4 = M/p/s$. By same conclusion $M^-/p = M^+/q\backslash s/p = Q_4$, furthermore $M^-\backslash p = Q_5$ and by Proposition 4

$$t_x(M^-) = \frac{1}{x+y-x\,y}\left(t\left(M^-/p\right) - (y-1)\;t\left(M^-\backslash p\right)\right).$$

Using all these identities we see that

$$\vec{m} = \left[t(Q_1), t(Q_2), t(Q_3), t(Q_4), \frac{1}{x+y-x\,y}\left(t\left(Q_4\right) - (y-1)\;t\left(Q_5\right)\right)\right]^{T.}$$

Defining

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{x+y-x\,y} & \frac{1-y}{x+y-x\,y} \end{bmatrix}$$

we can write $\vec{m} = \mathbf{B}\,\vec{q}$ and so we obtain

$$\mathbf{B}\,\vec{q} = \mathbf{A}\,\vec{t}.$$

Furthermore, the inverse $\mathbf{A^{-1}}$ of $\mathbf{A}$ exists and is the matrix

$$\begin{bmatrix} \frac{(1-y)^2}{-x-y+xy} & \frac{1-y}{-x-y+xy} & \frac{1-y}{-x-y+xy} & \frac{1}{-x-y+xy} & -1 \\ \frac{1-y}{-x-y+xy} & 1 & \frac{1}{-x-y+xy} & \frac{1}{1-y} & \frac{1}{-1+y} \\ \frac{1-y}{-x-y+xy} & \frac{1}{-x-y+xy} & 1 & \frac{1}{1-y} & \frac{1}{-1+y} \\ \frac{2}{-x-y+xy} & \frac{1-x}{-x-y+xy} & \frac{1-x}{-x-y+xy} & \frac{1-x}{1-y} & \frac{1-x}{-1+y} \\ \frac{1-y}{-x-y+xy} & \frac{1}{-x-y+xy} & \frac{1}{-x-y+xy} & \frac{1}{(-1+y)(x+y-xy)} & \frac{-x-y+xy}{-1+y} \end{bmatrix}$$

times $(-1-x-y+x\,y)^{-1}$, so $\vec{t} = \mathbf{C}\vec{q}$, where $\mathbf{C} = \mathbf{A^{-1}B}$. $\square$

6 Splitting formulas for a GPC and a 3-sum of two matroids

Throught this section we use the following notations. Let M_1, M_2 be matroids, $E_1 = E(M_1)$, $E_2 = E(M_2)$, $T = E_1 \cap E_2$. We assume that $N = M_1|T$ is a 3-circuit and the GPC $M = P_N(M_1, M_2)$ of M_1 and M_2 exists. The matroid previously denoted as M^+ is now written as M (and represents a GPC) due to the fact that in this section the GPC is in the center of our interest. In general, we drop the superscript "+" when describing matroids containing all three elements $p, s, q \in T$.

A *looping* of a matroid M on $e \in E(M)$ is the following operation which yields a *looped* matroid denoted as $M|_\ell e$: if e is a loop, then $M|_\ell e = M$. Otherwise we adjoin to M an element e' parallel to e and then contract e'.

For each connecting matroid there is a set $\mathcal{N}$ of matroids obtained from N by loopings on zero or more elements of N. For example, using the definition from Table 1, if N is a 3-circuit on $\{p, s, q\}$ then $\mathcal{N} = \{N_1^+, \ldots, N_5^+\}$. To the end of this section we will write $N_i \in \mathcal{N}$ for the matroid N_i^+ given in Table 1 for $i = 1, \ldots, 5$. Obviously we can imitate the influence of a reduction R (of some elements of $E_1 - T$ of M_1 occurring during a calculation on M_1) on N by applying zero or more loopings to the connecting minor N.

Applying the same loopings to M_2 on elements in T as when obtaining a $N_i \in \mathcal{N}$ we obtain a matroid M_2^i on $E(M_2)$. It is obvious that M_2^i is completely determined by M_2 and by N_i only. We will denote the set of such matroids by $\mathcal{M}_2$ and order them in such a way that M_2^i corresponds to $N_i \in \mathcal{N}$.

Let cl_1 be the closure operator of M_1. The following lemma shows some basic properties of the GPC.

Lemma 11 *([Oxl92, p. 419]) The GPC has the following properties:*

(a) *$P_N(M_1, M_2)|E_1 = M_1$ and $P_N(M_1, M_2)|E_2 = M_2$.*

(b) *If $e \in E_1 - T$, then $P_N(M_1, M_2)\backslash e = P_N(M_1\backslash e, M_2)$.*

(c) *If $e \in E_1 - cl_1(T)$, then $P_N(M_1, M_2)/e = P_N(M_1/e, M_2)$.*

(d) *If $e \in T$, then $P_N(M_1, M_2)/e = P_{N/e}(M_1/e, M_2/e)$.*

Lemma 12 *Let $\mathcal{R}(E_1 - T)$ be a calculation on M_1 yielding the sum*

$$t(M_1) = \sum_{i=1}^{|\mathcal{N}|} mc(N_i)\, t(N_i).$$

Then $\mathcal{R}(E_1 - T)$ is applicable to $M = P_N(M_1, M_2)$, and we have

$$mc(N_i) = mc(M_2^i) \qquad \text{for } i = 1 \ldots 5,$$

what implies

$$t(M) = \sum_{i=1}^{|\mathcal{N}|} mc(N_i)\, t(M_2^i).$$

Proof: Let R be a reduction occurring during the calculation $\mathcal{R}(E_1 - T)$ and $M' = M(R)$ and $M_1' = M_1(R)$ the minors of M and of M_1, resp., obtained by R. We have shown that $\mathcal{R}(E_1 - T)$ is applicable to both M and M_1 if we can show that for any reduction R occurring during $\mathcal{R}(E_1 - T)$ we have $M'|E_1 = M_1'$. Also from this it follows that if R has length $|E_1 - T|$ then $M'|E_1 = M_1' \in \mathcal{N}$ and so $mc(N_i) = mc(M_2^i)$ for $i = 1, \ldots, 5$ by definition of the minor coefficients.

We will show the statement $M'|E_1 = M_1'$ by induction on the length of R. The base case holds as $M|E_1 = M_1$ by (a) of Lemma 11. For the induction step assume that $M'|E_1 = M_1'$ holds for a reduction R. R might have changed N into a matroid $N' = M_1'|T$. Obviously we can obtain N' applying to N some loopings from a set H, say. Let M_2' be a matroid obtained by applying all loopings in H to M_2 (thus, $M_2'|T = N'$ i.e. we imitate the influence of R onto M_2 considered as a part of M).

Now let $e \in E_1 - T$ be an element to be reduced in the next step of the calculation. We want to show that

$$M'/e = P_{N'}(M_1'/e, M_2'),$$

and

$$M'\backslash e = P_{N'}(M_1'\backslash e, M_2')$$

as then $(M'/e)|E_1 = M_1'/e$ and $(M'\backslash e)|E_1 = M_1'\backslash e$ by (a) of Lemma 11, what would complete the induction.

But, by (b), (c) and (d) of Lemma 11 we have

- if $e \in E(M_1') - T$, then $M'\backslash e = P_{N'}(M_1'\backslash e, M_2')$,
- if $e \in E(M_1') - cl_1(T)$, then $M'/e = P_{N'}(M_1'/e, M_2')$,
- if $e \in cl_1(T)$ and e will be contracted, we have the following two cases
  - if e is a loop then we can delete e and so again $M'\backslash e = P_{N'}(M_1'\backslash e, M_2')$,
  - otherwise e is parallel to some element $e' \in T$. We contract e' and by (d) of the mentioned lemma we have

    $$P_{N'}(M_1', M_2')/e' = P_{N'/e}(M_1'/e, M_2'/e).$$

    To suffice the condition that only elements in $E_1 - T$ are reduced, we relabel e and e'. The effect of these operations equals a looping on $e' \in T$.

□

Theorem 13 (Splitting formula for GPC with N being a 3-circuit) *Let M be a GPC $P_N(M_1, M_2)$ with N being a 3-circuit. We write $E_1 = E(M_1)$, $E_2 = E(M_2)$, $E(N) = T = \{p, s, q\}$ and require that in M_1 there is a circuit $U_1 \cup \{s\}$ with $U_1 \subseteq E_1 - T$ and a circuit $U_2 \cup \{p\}$ with $U_2 \subseteq E_1 - T$ (otherwise M can be represented as a parallel connection or a direct sum of M_1 and M_2).*

Let

$$Q_1 = M_1\backslash p\backslash s\backslash q,\ Q_2 = M_1\backslash p/s\backslash q,\ Q_3 = M_1/p\backslash s\backslash q,\ Q_4 = M_1/p/s/q,$$
$$Q_5 = M_1\backslash p\backslash s/q$$

be five minors of M_1 *on* $E_1 - T$ *and let*

$$M_2^1 = M_2,\ M_2^2 = M_2|_\ell s,\ M_2^3 = M_2|_\ell p,\ M_2^4 = M_2|_\ell p|_\ell s,\ M_2^5 = M_2|_\ell q$$

be five looped matroids of M_2 *on* E_2*. Furthermore we define the vectors:*

$$\vec{q} = [t(Q_1), t(Q_2), t(Q_3), t(Q_4), t(Q_5)]^T \text{ and}$$
$$\vec{p} = [t(M_2^1), t(M_2^2), t(M_2^3), t(M_2^4), t(M_2^5)]^T .$$

Then

$$t(M) = \vec{q}^T \mathbf{C}\, \vec{p}$$

where $\mathbf{C}$ *is the symmetric matrix given in Theorem 10, i.e.:*

$$\frac{1}{-1-x-y+x\,y}\begin{bmatrix} \frac{(1-y)^2}{-x-y+xy} & \frac{1-y}{-x-y+xy} & \frac{1-y}{-x-y+xy} & \frac{2}{-x-y+xy} & \frac{1-y}{-x-y+xy} \\ \frac{1-y}{-x-y+xy} & 1 & \frac{1}{-x-y+xy} & \frac{1-x}{-x-y+xy} & \frac{1}{-x-y+xy} \\ \frac{1-y}{-x-y+xy} & \frac{1}{-x-y+xy} & 1 & \frac{1-x}{-x-y+xy} & \frac{1}{-x-y+xy} \\ \frac{2}{-x-y+xy} & \frac{1-x}{-x-y+xy} & \frac{1-x}{-x-y+xy} & \frac{(1-x)^2}{-x-y+xy} & \frac{1-x}{-x-y+xy} \\ \frac{1-y}{-x-y+xy} & \frac{1}{-x-y+xy} & \frac{1}{-x-y+xy} & \frac{1-x}{-x-y+xy} & 1 \end{bmatrix}.$$

Proof: Put $M_1^- = M_1\backslash q$. By assumption both p and s are circuit points in M_1^- and s is a circuit point in both $M_1^-\backslash p$ and M_1^-/p. Thus, we may apply Theorem 10 to M_1, yielding:

$$\vec{q}^T\mathbf{C} \;=\; \vec{t}^{\,T} \;=\; \Big[t_{xx}(M_1^-), t_{xy}(M_1^-), t_{yx}(M_1^-), t_{yy}(M_1^-), t_{cc}(M_1^-)\Big].$$

It remains to show that the minor coefficients $mc(M_2^1), \ldots, mc(M_2^5)$ are $t_{xx}(M_1^-)$, $t_{xy}(M_1^-)$, $t_{yx}(M_1^-)$, $t_{yy}(M_1^-)$ and $t_{cc}(M_1^-)$, respectively, as then

$$t(M) = \sum_{i=1}^{5} mc(M_2^i)\, t(M_2^i) = \vec{t}^{\,T}\, \vec{p} = \vec{q}^T \mathbf{C}\, \vec{p}$$

by (2) and by the fact that $|\mathcal{N}| = 5$ what can be seen from Table 1 (where $\mathcal{M}^+ = \mathcal{N}$).

By Lemma 6 $t_{xx}(M_1^-)$, $t_{xy}(M_1^-)$, $t_{yx}(M_1^-)$, $t_{yy}(M_1^-)$ and $t_{cc}(M_1^-)$ are exactly $mc(N_1), \ldots, mc(N_5)$, respectively (the matroids $N_1, \ldots, N_5$ are denoted in Lemma 6 as $N_1^+, \ldots, N_5^+$, respectively). We apply Lemma 11 to see that the minor coefficients $mc(N_1), \ldots, mc(N_5)$ are exactly $mc(M_2^1), \ldots, mc(M_2^5)$, respectively. □

The following theorem gives a splitting formula for 3-sums of matroids.

Theorem 14 (Splitting formula for 3-sums of matroids) *Let* $M' = M_1 \oplus_3 M_2$ *be a 3-sum of the matroids* M_1 *and* M_2*. We write* $E_1 = E(M_1)$, $E_2 = E(M_2)$, $E(N) = T = \{p, s, q\}$ *and* $N = M_1|T = M_2|T$*. We also require that in* M_1 *there is a circuit* $U_1 \cup \{s\}$ *with* $U_1 \subseteq E_1 - T$ *and a circuit* $U_2 \cup \{p\}$ *with* $U_2 \subseteq E_1 - T$ *and that in* M_2 *there is a circuit* $V_1 \cup \{s\}$ *with* $V_1 \subseteq E_2 - T$ *and a circuit* $V_2 \cup \{p\}$ *with*

$V_2 \subseteq E_2 - T$ i.e. we require that M' cannot be represented as a 2-sum or a direct sum of M_1 and M_2. Then the following splitting formula holds.

Let

$$Q_1 = M_1 \backslash p \backslash s \backslash q,\ Q_2 = M_1 \backslash p / s \backslash q,\ Q_3 = M_1 / p \backslash s \backslash q,\ Q_4 = M_1 / p / s / q,$$
$$Q_5 = M_1 \backslash p \backslash s / q$$

be five minors of M_1 on $E_1 - T$ and let

$$P_1 = M_2 \backslash p \backslash s \backslash q,\ P_2 = M_2 \backslash p / s \backslash q,\ P_3 = M_2 / p \backslash s \backslash q,\ P_4 = M_2 / p / s / q,$$
$$P_5 = M_2 \backslash p \backslash s / q$$

be five minors of M_2 on $E_2 - T$. Furthermore we define the vectors:

$$\vec{q} = \left[t\left(Q_1\right), t\left(Q_2\right), t\left(Q_3\right), t\left(Q_4\right), t\left(Q_5\right)\right]^T \text{ and}$$
$$\vec{p}' = \left[t\left(P_1\right), t\left(P_2\right), t\left(P_3\right), t\left(P_4\right), t\left(P_5\right)\right]^T.$$

Then

$$t\left(M\right) = \vec{q}^T \mathbf{C}\, \vec{p}'$$

where $\mathbf{C}$ is the symmetric matrix given in Theorem 10 and in Theorem 13.

Proof: We apply R1, R2, and R3 to the elements in T of $M = P_N(M_1, M_2)$, yielding:

$$\begin{aligned} t(M_1 \oplus_3 M_2) &= t(P_N(M_1, M_2)) - t(M/s\backslash q\backslash p) - t(M/p\backslash q\backslash s) - t(M/q\backslash p\backslash s) \\ &\quad - t(M/s/q\backslash p) - t(M/p/q\backslash s) - y\, t(M/p/s\backslash q) \end{aligned} \tag{10}$$

(clearly $M/s/q\backslash p = M/p/q\backslash s = M/p/s\backslash q$).

Except for $M_1 \oplus_3 M_2$, each of the matroids occurring on the right-hand side of the above equation is a direct sum or a 2-sum of two matroids on $E_1 - T$ and on $E_2 - T$, respectively. Therefore $t(M/s/q\backslash p) = t(M/p/q\backslash s) = t(M/p/s\backslash q) = t(Q_4)\, t(P_4)$ and, for example

$$t(M/s\backslash q\backslash p) = \frac{1}{x\,y - x - y} \begin{bmatrix} t(Q_4) & t(Q_2) \end{bmatrix} \begin{bmatrix} x-1 & -1 \\ -1 & y-1 \end{bmatrix} \begin{bmatrix} t(P_4) \\ t(P_2) \end{bmatrix}$$

where we write

$$P_1 = M_2 \backslash p \backslash s \backslash q,\ P_2 = M_2 \backslash p / s \backslash q,\ P_3 = M_2 / p \backslash s \backslash q,\ P_4 = M_2 / p / s / q,$$
$$P_5 = M_2 \backslash p \backslash s / q.$$

Thus, we can represent each of the polynomials on the right-hand side of the equation (10) except for $t(P_N(M_1, M_2))$ in terms of the Tutte polynomials of $Q_1, \ldots Q_5$ and $P_1, \ldots P_5$.

Also the splitting formula of the GPC for $t(P_N(M_1, M_2))$ can be given in terms of the Tutte polynomials of $Q_1, \ldots Q_5$ and $P_1, \ldots P_5$, as $\vec{p}$ can be easily calculated from the Tutte polynomials of $P_1, \ldots P_5$ applying the rules R1, R2, and R3:

$$\vec{p} = \begin{bmatrix} t(P_1) + t(P_2) + t(P_3) + (2+y)\,t(P_4) + t(P_5) \\ y\,(t(P_2) + (1+y)\,t(P_4)) \\ y\,(t(P_3) + (1+y)\,t(P_4)) \\ y^3\,t(P_4) \\ y\,(t(P_5) + (1+y)\,t(P_4)) \end{bmatrix}.$$

In the final step we multiply out the right-hand side of the equation (10) and recollect the coefficients of the products $t(Q_i)\,t(P_j)$ for $i, j \in \{1, \ldots, 5\}$.

It turns out that the splitting formula for $M_1 \oplus_3 M_2$ is identical to the one for the GPC but $\vec{p}$ is exchanged with

$$\vec{p}\,' = [t(P_1), t(P_2), t(P_3), t(P_4), t(P_5)]^T$$

i.e.

$$t(M_1 \oplus_3 M_2) = \vec{q}^{\,T}\,\mathbf{C}\,\vec{p}\,'.$$

□

Remark 15 *James Oxley has obtained similar splitting formulas for a 3-sum and for a GPC across a 3-point line. In place of matroids $Q_1, \ldots, Q_5$ Oxley's formulas require the matroids $M_1, M_1/s, M_1/p, M_1/q$ and $M_1/p/s/q$ and they require the matroids $M_2, M_2/s, M_2/p, M_2/q$ and $M_2/p/s/q$ in place of $M_2^1, \ldots, M_2^5$ (splitting formula for 3-sums) or $P_1, \ldots, P_5$ (splitting formulas for a GPC). The author restates two most important formulas as communicated ([Oxl95]), but using the notations from Theorem 14 and putting*

$$A_1 = t(M_1/s) + t(M_1/p) + t(M_1/q)$$

and

$$A_2 = t(M_2/s) + t(M_2/p) + t(M_2/q).$$

Splitting formula for a GPC of matroids M_1 and M_2 with the connecting minor N being a 3-circuit:

$$\begin{aligned} t(P_N(M_1, M_2)) = {} & (x\,y - x - y)^{-1}\,(x\,y - x - y - 1)^{-1}\,((x\,y - x - y - 1)\,y\cdot \\ & [\,t(M_1/s)\,t(M_2/s) + t(M_1/p)\,t(M_2/p) + t(M_1/q)\,t(M_2/q)] + \\ & 2\,y^3\,[t(M_1)\,t(M_2/s/p/q) + t(M_2)\,t(M_1/s/p/q)] + y^2\,A_1\,A_2 + \\ & y\,(1-y)\,[t(M_1)\,A_2 + t(M_2)\,A_1] - \\ & y^3\,(1+x)\,[t(M_1/s/p/q)\,A_2 + t(M_2/s/p/q)\,A_1] + \\ & y^3\,(x^2 + x + y + 3\,x\,y)\,t(M_1/s/p/q)\,t(M_2/s/p/q) + (y-1)^2\,t(M_1)\,t(M_2)\big). \end{aligned}$$

Splitting formula for the 3-sum of matroids M_1 and M_2:

$$t(M_1 \oplus_3 M_2) = (x\,y - x - y)^{-1}\,(x\,y - x - y - 1)^{-1}\,((x\,y - x - y - 1)\cdot \\ [\,t(M_1/s)\,t(M_2/s) + t(M_1/p)\,t(M_2/p) + t(M_1/q)\,t(M_2/q)] + \\ 2\,y^3\,[t(M_1)\,t(M_2/s/p/q) + t(M_2)\,t(M_1/s/p/q)] + y^2\,A_1\,A_2\; + \\ y\,(\,1 - y\,)\;[t(M_1)\,A_2\; + t(M_2)\,A_1] - \\ y^2\,(\,1 + x + 2\,y\,)\;[t(M_1/s/p/q)\,A_2 + t(M_2/s/p/q)\,A_1] + \\ t(M_1/s/p/q)\,t(M_2/s/p/q)\;\Big[\,4\,y^4 + 5\,y^3 + 3\,y^2 + y + x + x^2 + 3\,x\,y + \\ 3\,x\,y^2 + 3\,x\,y^3\,\Big] + \,(\,y - 1\,)^2\,t(M_1)\,t(M_2)\Big)\,.$$

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