

THE GEOMETRY OF L^p -SPACES OVER ATOMLESS MEASURE SPACES AND THE DAUGAVET PROPERTY

ENRIQUE A. SÁNCHEZ PÉREZ AND DIRK WERNER

ABSTRACT. We show that L^p -spaces over atomless measure spaces can be characterized in terms of a p -concavity type geometric property that is related with the Daugavet property.

1. INTRODUCTION

A Banach space Y is said to have the *Daugavet property* if for every rank one operator $T: Y \rightarrow Y$, the *Daugavet equation*

$$\|\text{Id} + T\| = 1 + \|T\|$$

is satisfied; in this case, it is known that the equation is satisfied for every weakly compact operator ([9, Theorem 2.3]). Although for $L^1(\mu)$ spaces over an atomless measure μ this property is always fulfilled, it is known that this equation is only satisfied for a compact operator T on L^p for $1 < p < \infty$ when its norm is an eigenvalue of T ; this result can be extended to uniformly convex or uniformly smooth Banach spaces, and also to locally uniformly convex Banach spaces (see Corollary 2.4 and Theorem 2.7 in [1] or Section 4 in [9]).

After this negative result, some efforts have been made in order to find a similar lower estimate for $\|\text{Id} + T\|$ in terms of $\|T\|$ in general Banach spaces or for the particular case of L^p spaces. Based on the early ideas of Benyamini and Lin in [4] several authors have been working in the direction of finding nice lower bounds for $\|\text{Id} + T\|$ in terms of a function $\psi: (0, +\infty) \rightarrow (1, +\infty)$ such that the inequality $\|\text{Id} + T\| \geq \psi(\|T\|)$ holds for all compact operators $T: Y \rightarrow Y$ (see for instance [6, 12, 13, 15, 17, 19] and [7] and the references therein). As in these cases, we are interested in this paper in finding a good alternative to the Daugavet equation for L^p spaces, or in a more general sense, for Banach function spaces satisfying certain p -convexity type requirements. Our aim is to give a geometrical description of $L^p(\mu)$ spaces defined over measures μ without atoms in the same geometrical terms as for spaces with the Daugavet property (slices and the geometry of the unit ball). In order to do that, we use p -convexity and p -concavity properties of quasi-Banach function spaces for developing a sort of p -convexification

Date: January 5, 2010. *File:* LpDaugavet7.tex.

2000 Mathematics Subject Classification. Primary 46B04; secondary 46B25.

Key words and phrases. Daugavet property, L_p -space.

The first author was partially supported by a grant from the Generalitat Valenciana (BEST/2009/108) and a grant from the Universidad Politécnica de Valencia (PAID-00-09/2291). Support of the Ministerio de Educación y Ciencia, under project #MTM2006-11690-C02-01 (Spain) and FEDER is also gratefully acknowledged.

technique that allows us to obtain the desired geometrical description. Regarding the Daugavet property for Banach function spaces the results that are nowadays known are in a sense negative; for instance, in the class of Orlicz spaces over atomless finite measure spaces, the spaces that satisfy the Daugavet property with respect to the Luxemburg norm are isomorphic to L^1 (see [3, Theorem 2.5]). However, it must be noted that there are Banach function spaces other than $L^1(\mu)$ and $L^\infty(\mu)$ over atomless measures μ that satisfy the Daugavet property (see for instance Section 5 in [5]; an explicit example is $c_0(L^1[0, 1])$, the c_0 -sum of $L^1[0, 1]$ -spaces).

Let us start by recalling some well-known facts and by introducing some notation. Let $1 \leq p < \infty$. A p -convex and p -concave Banach lattice can be identified isomorphically and in order with an L^p -space; if the corresponding p -convexity and p -concavity constants are equal to 1, then this identification is given by an isometry (see for instance [14, Theorem 2.7]). In this paper we provide a Daugavet type geometric property which is more restrictive than the p -concavity that is only satisfied for L^p -spaces over measure spaces without atoms (see Theorem 2.8). In fact, it characterizes this class of spaces.

We remark that we deal with a different p -version of the Daugavet property in our paper [18].

If Y is a Banach space, we denote as usual by B_Y and S_Y the (closed) unit ball and the unit sphere respectively. Y^* stands for its dual space. The slice $S(y^*, \varepsilon)$ of B_Y defined by $y^* \in B_{Y^*}$ and $\varepsilon > 0$ is given by

$$S(y^*, \varepsilon) = \{y \in B_Y : \langle y, y^* \rangle \geq 1 - \varepsilon\}.$$

Notice that for the slice to be non-trivial it is enough to require that $y^* \in S_{Y^*}$. Recall that Y has the Daugavet property if and only if the following geometric property is fulfilled: for every $y \in S_Y$, every $y^* \in S_{Y^*}$ and every $\varepsilon > 0$, there is an element $x \in S(y^*, \varepsilon)$ such that $\|y + x\| \geq 2 - \varepsilon$ (see [9, Lemma 2.1], [9, Lemma 2.2] or [8, Theorem 2.2]). The reader can find more information on the geometric description of the Daugavet property in [8, 9, 10] and in [2, Ch. 11].

We also use standard notation regarding quasi-Banach function spaces. A quasi-Banach space $(E, \|\cdot\|_E)$ is a linear space that is complete with respect to the topology induced by a quasi-norm $\|\cdot\|_E$. If E is also a linear lattice, we say that $(E, \|\cdot\|_E)$ is a quasi-Banach lattice if $\|\cdot\|_E$ is a lattice quasi-norm in E , i.e., $\|x\|_E \leq \|y\|_E$ whenever $x, y \in E$ and $|x| \leq |y|$. Let (Ω, Σ, μ) be a measure space. A quasi-Banach function space $X(\mu)$ over the measure μ is an ideal of $L^0(\mu)$, the usual μ -a.e. order is considered, which is a quasi-Banach space with a lattice quasi-norm $\|\cdot\|$ such that for every $A \in \Sigma$ of finite measure, $\chi_A \in X(\mu)$ (see for instance [11, Chapter 1.b], [14, Chapter 2.6] and [16, Chapter 2] for definitions and main results regarding these structures, but notice that the last property is not required in some of these references). In the case that $\|\cdot\|$ is a norm, we say that $(X(\mu), \|\cdot\|)$ is a Banach function space, see [11, Definition 1.b.17]. We shall write simply X for $X(\mu)$ if the measure is fixed in the context.

Let $0 < p < \infty$. A quasi-Banach lattice E is *p-convex* if there is a constant K such that for every finite sequence $(x_i)_{i=1}^n$ in E ,

$$\left\| \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \right\|_E \leq K \left(\sum_{i=1}^n \|x_i\|_E^p \right)^{1/p}.$$

A quasi-Banach lattice E is *p-concave* if there is a constant k such that for every finite sequence $(x_i)_{i=1}^n$ in E ,

$$\left(\sum_{i=1}^n \|x_i\|_E^p \right)^{1/p} \leq k \left\| \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \right\|_E.$$

The best constants in the inequalities above are denoted by $M^{(p)}(E)$ and $M_{(p)}(E)$, respectively, and are called the *p-convexity* and the *p-concavity* constants of E . Throughout the paper we will assume for a *p-convex* Banach function space that in fact $M^{(p)}(E) = 1$, and when *p-concavity* is required that $M_{(p)}(E) = 1$. To indicate this, we will say that they are constant 1 *p-convex* or constant 1 *p-concave*, respectively.

Let $0 < p < \infty$. Consider a quasi-Banach function space $X(\mu)$. Let

$$X(\mu)_{[p]} := \{h \in L^0(\mu) : |h|^{1/p} \in X(\mu)\}$$

be its *p*-th power, which is a quasi-Banach function space when endowed with the quasinorm $\|h\|_{X_{[p]}} := \||h|^{1/p}\|_X^p$, $h \in X_{[p]}$ (see [16, Ch. 2]). For $p \geq 1$, if X is *p-convex* and $M^{(p)}(X) = 1$, then $X(\mu)_{[p]}$ is a Banach function space, since in this case $\|\cdot\|_{X_{[p]}}$ is a norm. If the Banach function space is *p-convex*, but the *p-convexity* constant is not 1, then $\|\cdot\|_{X_{[p]}}$ is not a norm, but it is equivalent to a norm (see for instance [16, Prop. 2.23]). It is also well known that every *p-convex* Banach lattice can be renormed in such a way that the new norm is a lattice norm with *p-convexity* constant equal to 1 (see [11, Proposition 1.d.8]).

2. BANACH FUNCTION SPACES WITH *p*-TH POWERS HAVING THE DAUGAVET PROPERTY

Let $0 < p < \infty$ and let $X(\mu)$ be a constant 1 *p-convex* quasi-Banach function space. If $f \in X(\mu)$, we can always write it as $f = \text{sign}\{f\}|f|$. This allows us to define the (obviously non-linear) map $i_p: X(\mu) \rightarrow X(\mu)_{[p]}$ by

$$i_p(f) = \text{sign}\{f\}|f|^p.$$

Throughout the paper, we shall write

$$f^p := i_p(f)$$

for the sake of simplicity, but notice that for even integers f^p is in general not $|f|^p$. The map i_p is bijective and satisfies

$$\|i_p(f)\|_{X_{[p]}} = \|\text{sign}\{f\}|f|^p\|_{X_{[p]}} = \|f\|_X^p, \quad f \in X(\mu). \quad (2.1)$$

The inverse map $i_p^{-1}: X_{[p]} \rightarrow X$ coincides with $i_{1/p}: Y \rightarrow Y_{[1/p]}$, where $Y = X_{[p]}$.

In what follows we characterize the *p-convex* Banach function spaces whose *p*-th powers satisfy the Daugavet property. The key idea to achieve

this is to introduce the notions of $1/p$ -th power of a slice and p -convexification of an operator $T: X_{[p]} \rightarrow X_{[p]}$.

If $X(\mu)$ is a constant 1 p -convex Banach function space, let $S_{[p]}(x^*, \varepsilon)$ be a slice in $X(\mu)_{[p]}$, where $x^* \in B_{(X(\mu)_{[p]})^*}$. Consider the set

$$S_{[p]}^{1/p}(x^*, \varepsilon) := \{f \in X(\mu): f^p \in S_{[p]}(x^*, \varepsilon)\}.$$

We call it the $1/p$ -th power of the slice $S_{[p]}(x^*, \varepsilon)$.

If $T: X_{[p]} \rightarrow X_{[p]}$ is an operator, we define its p -convexification $\varphi_T: X(\mu) \rightarrow X(\mu)$ by

$$\varphi_T(f) := i_p^{-1} \circ T \circ i_p(f) = (T(f^p))^{1/p}, \quad f \in X.$$

We also define $\|\varphi_T\| := \sup_{f \in B_X} \|\varphi_T(f)\|$. Notice that

$$\begin{aligned} \|\varphi_T\| &= \sup_{f \in B_X} \|\varphi_T(f)\|_X = \sup_{f \in B_X} \|(T(f^p))^{1/p}\|_X \\ &= \sup_{h \in B_{X_{[p]}}} \|T(h)\|_{X_{[p]}}^{1/p} = \|T\|^{1/p}. \end{aligned}$$

The following two lemmas provide a geometric description of the Daugavet property for a Banach function space in terms of slices of the p -convexification. Their proofs follow the lines of the ones of Lemmas 2.1, 2.2 and 2.8 in [9]. However, we spell out the arguments that prove the main equivalences with some detail in order to show the role played by the p -convexity of the norm of $X(\mu)$.

Lemma 2.1. *Let $X(\mu)$ be a quasi-Banach function space and let $0 < p < \infty$. The following assertions are equivalent:*

(0) *The space satisfies the following.*

(i) $(X_{[p]})^* \neq \{0\}$.

(ii) *For every finite family of rank-one continuous operators $T_i:$*

$$X_{[p]} \rightarrow X_{[p]},$$

$$\sup_{f \in B_X} \left\| \left(\sum_{i=1}^n \varphi_{T_i}(f)^p \right)^{1/p} \right\| \leq \left(\sum_{i=1}^n \|\varphi_{T_i}\|^p \right)^{1/p}.$$

(iii) *For every rank-one operator $T: X_{[p]} \rightarrow X_{[p]}$,*

$$\sup_{f \in B_X} \| |f^p + \varphi_T(f)^p|^{1/p} \|_X = (1 + \|\varphi_T\|^p)^{1/p}.$$

(1) *X is constant 1 p -convex and for every rank-one operator $T: X_{[p]} \rightarrow X_{[p]}$,*

$$\sup_{f \in B_X} \| |f^p + \varphi_T(f)^p|^{1/p} \|_X = (1 + \|\varphi_T\|^p)^{1/p}. \quad (2.2)$$

(2) *X is constant 1 p -convex and for every $f \in S_X$, every $x^* \in S_{(X_{[p]})^*}$ and every $\varepsilon > 0$ there is an element $g \in S_{[p]}^{1/p}(x^*, \varepsilon)$ such that*

$$\| |f^p + g^p|^{1/p} \|_X^p \geq 2 - 2\varepsilon.$$

(3) *$X(\mu)_{[p]}$ is a Banach function space over μ with the Daugavet property.*

- (4) X is constant 1 p -convex and for every $f \in S_X$ and every slice $S(x_0^*, \varepsilon_0)$ of $B_{X_{[p]}}$ there is another non-trivial slice $S_{[p]}(x_1^*, \varepsilon_1) \subset S_{[p]}(x_0^*, \varepsilon_0)$ such that for every $g \in S_{[p]}^{1/p}(x_1^*, \varepsilon_1)$ the inequality

$$\|(f^p + g^p)^{1/p}\|_X^p \geq 2 - \varepsilon_0$$

holds.

Proof. Let us start with (0) \Rightarrow (1). Take a finite set $f_1, \dots, f_n \in X$. Since the dual space $(X_{[p]})^*$ contains a non-trivial element x^* , we can assume that $x^* \in S_{(X_{[p]})^*}$, and we can consider the operators $T_i: X_{[p]} \rightarrow X_{[p]}$ given by $T_i(h) := \langle h, x^* \rangle |f_i|^p$. They are obviously continuous and $\|T_i\| = \|f_i^p\|_{X_{[p]}} = \|f_i\|_X^p$. Thus, by (ii), we have

$$\begin{aligned} \left(\sum_{i=1}^n \|f_i\|_X^p \right)^{1/p} &= \left(\sum_{i=1}^n \|\varphi_{T_i}\|^p \right)^{1/p} \\ &\geq \sup_{f \in B_X} \left\| \left(\sum_{i=1}^n \varphi_{T_i}(f) \right)^{1/p} \right\| \\ &= \sup_{f \in B_X} \left\| \left(\sum_{i=1}^n T_i(f^p) \right)^{1/p} \right\| \\ &= \sup_{f \in B_X} \left\| \left(\sum_{i=1}^n \langle f^p, x^* \rangle |f_i|^p \right)^{1/p} \right\| \\ &= \sup_{h \in B_{X_{[p]}}} (\langle h, x^* \rangle)^{1/p} \left\| \left(\sum_{i=1}^n |f_i|^p \right)^{1/p} \right\|_X. \end{aligned}$$

Consequently, X is p -convex and $M^{(p)}(X) = 1$, and so (1) is obtained.

For the converse, since X is p -convex and has p -convexity constant equal to 1, $X_{[p]}$ is a Banach function space (see for instance [16, Proposition 2.23]), and so its dual space is non-trivial. It only remains to prove (ii). Take any finite set of rank-one operators $T_i: X_{[p]} \rightarrow X_{[p]}$, $i = 1, \dots, n$. Each of them can be written as $T_i = x_i^* \otimes f_i^p$, where $\|x_i^*\| = 1$ and $f_i \in X$. Then for every $f \in B_X$,

$$\begin{aligned} \left\| \left(\sum_{i=1}^n (\varphi_{T_i}(f))^p \right)^{1/p} \right\| &\leq \left\| \left(\sum_{i=1}^n |\langle f, x_i^* \rangle|^p |f_i|^p \right)^{1/p} \right\| \\ &\leq \left(\sum_{i=1}^n \|f_i\|^p \right)^{1/p} = \left(\sum_{i=1}^n \|\varphi_{T_i}\|^p \right)^{1/p}. \end{aligned}$$

This gives (0).

Let us now prove the equivalence of (1) and (2). First notice that (1) is equivalent to the fact that for every rank-one operator $T: X_{[p]} \rightarrow X_{[p]}$,

$$\sup_{g \in B_X} \|g^p + T(g^p)\|_X^p = 1 + \|T\|.$$

For (1) \Rightarrow (2), take $f \in S_{X(\mu)}$, $x^* \in S_{(X(\mu)_{[p]})^*}$ and $\varepsilon > 0$. Consider $T = x^* \otimes f^p$. The equality above can be written as

$$\sup_{g^p \in B_{X_{[p]}}} \|g^p + T(g^p)\|_{X_{[p]}} = 1 + \|T\|. \quad (2.3)$$

In particular, this implies that we can assume by Lemma 11.4 in [2] (or [20, p. 78]) that T and hence x^* and f are of norm one. Take $h \in S_X$ such that

$$\| |h^p + T(h^p)|^{1/p} \|_X^p \geq 2 - \varepsilon.$$

We can also assume that $\langle h^p, x^* \rangle \geq 0$ (otherwise replace h by $-h$). Notice first that since $X(\mu)$ is p -convex with constant 1,

$$1 + \langle h^p, x^* \rangle = \| |h^p|^{1/p} \|_X^p + \| |T(h^p)|^{1/p} \|_X^p \geq 2 - \varepsilon,$$

which implies that $\langle h^p, x^* \rangle \geq 1 - \varepsilon$. Consequently, $h \in S_{[p]}^{1/p}(g^*, \varepsilon)$. On the other hand, using again the constant 1 p -convexity of $X(\mu)$,

$$\begin{aligned} 2 - \varepsilon &\leq \| |h^p + T(h^p)|^{1/p} \|_X^p \\ &\leq \| |h^p + f^p|^{1/p} \|_X^p + \| |T(h^p) - f^p|^{1/p} \|_X^p \\ &= \| |h^p + f^p|^{1/p} \|_X^p + \| |(\langle h^p, x^* \rangle - 1)f^p|^{1/p} \|_X^p \\ &\leq \| |h^p + f^p|^{1/p} \|_X^p + (1 - \langle h^p, x^* \rangle) \\ &\leq \| |h^p + f^p|^{1/p} \|_X^p + \varepsilon. \end{aligned}$$

This gives (2).

For the converse, we can suppose again that T is defined as $T = x^* \otimes f^p$ for two norm one elements x^* and f . Let $\varepsilon > 0$ and $h \in S_{[p]}^{1/p}(x^*, \varepsilon)$ such that

$$\| |f^p + h^p|^{1/p} \|_X^p \geq 2 - 2\varepsilon.$$

Then, by the constant 1 p -convexity of $X(\mu)$,

$$\begin{aligned} 2 - 2\varepsilon &\leq \| |f^p + h^p|^{1/p} \|_X^p \\ &= \| |f^p - T(h^p) + T(h^p) + h^p|^{1/p} \|_X^p \\ &\leq \| |f^p - T(h^p)|^{1/p} \|_X^p + \| |T(h^p) + h^p|^{1/p} \|_X^p \\ &\leq (1 - \langle h^p, x^* \rangle) + \| |T(h^p) + h^p|^{1/p} \|_X^p \\ &\leq \varepsilon + \| |T(h^p) + h^p|^{1/p} \|_X^p. \end{aligned}$$

Since this holds for every $\varepsilon > 0$ and the converse inequality always holds, we obtain the result.

Taking into account the definition of the isometric map i_p , the definition of the norm $\|\cdot\|_{X_{[p]}}$ and (2.3), the equivalence of (1) and (3) becomes obvious. Notice that the fact that $X(\mu)_{[p]}$ is a Banach function space over μ is equivalent to the fact that $X(\mu) = (X(\mu)_{[p]})_{[1/p]}$ is constant 1 p -convex (see for instance [16, Proposition 2.23(ii)]).

Similar arguments prove (3) \Rightarrow (4); a direct proof can be given using Lemma 2.1(a) in [9], the definition of the norm in $X_{[p]}$ and the fact that every element $h \in X_{[p]}$ can be written as f^p for some $f \in X$. (4) \Rightarrow (2) is obvious. \square

Remark 2.2. $L^p(\mu)$ spaces over a non-atomic measure μ satisfy the equivalent statements of Lemma 2.1; this is a direct consequence of $(L^p(\mu))_{[p]} = L^1(\mu)$ and the well-known fact that $L^1(\mu)$ satisfies the Daugavet property (see for instance [1, Theorem 3.2], or the example after Theorem 2.3 in [9] for a simple proof). However, we can easily construct Banach function spaces which are not L^p spaces but their p -th powers have the Daugavet property. For instance, consider a σ -finite atomless measure space (Ω, Σ, μ) and an infinite measurable partition $\{A_i\}$ of Ω and take a Banach space F with a 1-unconditional normalized Schauder basis endowed with its natural Banach function space structure given by the pointwise order. Consider the Banach space X defined as the F -sum of the spaces $L^1(\mu|_{A_i})$, where $\mu|_{A_i}$ denotes the restriction of μ to A_i , $i \in \mathbb{N}$, that is, X is the space of sequences (f_i) such that $f_i \in L^1(\mu|_{A_i})$ and $(\|f_i\|) \in F$. If F has the positive Daugavet property (i.e., every positive rank one operator on F satisfies the Daugavet equation), then Theorem 5.1 in [5] ensures that the F -sum has the Daugavet property. The spaces ℓ^1 and ℓ^∞ satisfy the positive Daugavet property, but the reader can find other examples in [5, Section 5]. It is easy to see that the $1/p$ -th power of X is also a Banach function space and it can be identified isometrically with the $F_{[1/p]}$ -sum of the spaces $L^p(\mu|_{A_i})$. Since $(X_{[1/p]})_{[p]} = X$ has the Daugavet property, $X_{[1/p]}$ satisfies the assertions of Lemma 2.1.

Other examples can be constructed using the fact that spaces of Bochner integrable functions over atomless measures satisfy the Daugavet property (see again the example after Theorem 2.3 in [9]). Let Y be a Banach lattice, let μ be a measure without atoms and consider the Bochner space $L^1(\mu, Y)$. It is a Banach lattice when the natural order is considered; assume that it is also an order continuous Banach lattice with a weak unit. Then it can be represented as a Banach function space Z (see for instance [11, Theorem 1.b.14]). Since the Daugavet property is preserved under isometries, we obtain that $Z_{[1/p]}$ satisfies the statements of Lemma 2.1.

Remark 2.3. Note that although the assertions in Lemma 2.1 have been stated in terms of rank one operators, the equivalences also hold when other classes of operators satisfying the Daugavet equation in $X_{[p]}$ are considered. Therefore, it includes for instance the weakly compact operators and further classes, see for example [9] and [10].

The following ‘‘sign independent’’ inequality is crucial for the computations regarding the p -convexification of the Daugavet property.

Given $1 \leq p < \infty$, we denote by p' the conjugate exponent defined by $1/p + 1/p' = 1$. Also, we let $k(p) = 1$ if $p \geq p'$ and $k(p) = 2^{(p'/p)-1}$ if $p < p'$. It follows

$$(a^{p/p'} + b^{p/p'})^{p'/p} \leq k(p)(a + b)$$

for real numbers $a, b \geq 0$.

Lemma 2.4. *Let $1 \leq p < \infty$ and consider two elements f and g in the unit ball of the constant 1 p -convex Banach function space X . Then*

$$\| |f^p - g^p|^{1/p} \|^p \leq \|f - g\|^p + p(2k(p))^{p/p'} \|f - g\|.$$

Consequently, the map $i_p: X \rightarrow X_{[p]}$ mapping f to f^p is continuous.

Proof. Let $1 \leq p < \infty$ and consider $c^p := \text{sign}\{c\}|c|^p$ for every $c \in \mathbb{R}$. Let $a, b \in \mathbb{R}$. Then we have to take into account two cases:

1) $\text{sign}\{a\} \neq \text{sign}\{b\}$. Suppose without loss of generality that $a \geq 0$ and $b \leq 0$. Then

$$|a^p - b^p|^{1/p} = |a^p + |b|^p|^{1/p} \leq |a + |b|| = |a - b|.$$

2) $\text{sign}\{a\} = \text{sign}\{b\}$. Then it is known that

$$|a^p - b^p|^{1/p} \leq (p|a^{p-1} + b^{p-1}| \cdot |a - b|)^{1/p}$$

(see for instance [16, Section 2.2]).

Take now two functions $f, g \in B_X$ and put $A = \{\omega: \text{sign}\{f(\omega)\} \neq \text{sign}\{g(\omega)\}\}$ and $B = \{\omega: \text{sign}\{f(\omega)\} = \text{sign}\{g(\omega)\}\}$. Then by case 1)

$$\| |f^p - g^p|^{1/p} \chi_A \| \leq \| |f - g| \chi_A \|.$$

Since $p - 1 = p/p'$, by the Hölder inequality for the Banach lattice X (see for instance Proposition 1.d.2 in [11]), we obtain (see also [16, Section 2.2] for the pointwise inequalities involved)

$$\begin{aligned} \| |f^p - g^p|^{1/p} \chi_B \| &\leq p \| (|f^{p-1} + g^{p-1}| \cdot |f - g|)^{1/p} \chi_B \| \\ &\leq p \| (|f^{p/p'} + g^{p/p'}|^{p'/p})^{1/p'} \cdot |f - g|^{1/p} \chi_B \| \\ &\leq p \| (|f^{p/p'} + g^{p/p'}|^{p'/p}) \chi_B \|^{p/p'} \cdot \| |f - g| \chi_B \| \\ &\leq pk(p)^{p/p'} \| |f + g| \chi_B \|^{p/p'} \cdot \| |f - g| \chi_B \| \\ &\leq p(2k(p))^{p/p'} \| |f - g| \chi_B \|. \end{aligned}$$

Therefore, since by the constant 1 p -convexity of X the inequality

$$\| |f^p - g^p|^{1/p} \| \leq \| |f^p - g^p|^{1/p} \chi_A \| + \| |f^p - g^p|^{1/p} \chi_B \|$$

is satisfied, we obtain the result. \square

Remark 2.5. We can relate our property for rank one operators with the general ψ -Daugavet property for Banach spaces that has been quoted in the Introduction (see [6, 15, 19]). For example in Theorem 2.1 of [15] inequalities like $\|\text{Id} + T\| \geq (1 + c_p \|T\|^p)^{1/p}$ for a compact operator $T: X \rightarrow X$ are considered, where c_p is a non-negative constant. In our case we obtain the following similar estimate in terms of the p -convexification φ_T . For instance, if $X_{[p]}$ is a Banach function space with the Daugavet property and $T: X_{[p]} \rightarrow X_{[p]}$ is weakly compact, we obtain

$$(1 + \|\varphi_T\|^p)^{1/p} \leq \sup_{f \in B_X} \| |f^p + \varphi_T(f)^p|^{1/p} \|_X$$

or equivalently

$$(1 + \|T\|)^{1/p} \leq \sup_{f \in B_X} \| |f^p + T(f^p)|^{1/p} \|_X.$$

Clearly in the case of positive operators, and using the estimate given in the proof of Lemma 2.4 for the case of different signs, this inequality gives also

$$(1 + \|\varphi_T\|^p)^{1/p} \leq \sup_{f \in B_X} \| |f + \varphi_T(f)| \|_X.$$

The following lemma is similar to Lemma 2.8 in [9].

Lemma 2.6. *Suppose that $X_{[p]}$ is a Banach space with the Daugavet property. Then for every finite dimensional subspace X_0 of X , every $\varepsilon > 0$ and every $x^* \in (X_{[p]})^*$ there is an element $g \in S_{[p]}^{1/p}(x^*, \varepsilon)$ such that for every $f \in X_0$ and $t \in \mathbb{R}$*

$$\|((tg)^p + f^p)^{1/p}\|^p \geq (1 - \varepsilon)(|t|^p + \|f\|^p).$$

Proof. Take $\delta > 0$ such that $\delta^{p+p(2k(p))^{p/p'}} \delta \leq \varepsilon/2$, where $k(p)$ is defined as above, a finite dimensional subspace X_0 of X and a finite δ -net $\{f_1, \dots, f_n\}$ in S_{X_0} . Applying Lemma 2.1(4) we find a sequence of slices $S_{[p]}(x_n^*, \varepsilon_n) \subset \dots \subset S_{[p]}(x_1^*, \varepsilon_1) \subset S_{[p]}(x^*, \varepsilon)$ such that

$$\|(f_k^p + g^p)^{1/p}\|_X^p \geq 2 - \delta^p$$

for all $g \in S_{[p]}^{1/p}(x_k^*, \varepsilon_k)$, $k = 1, \dots, n$. If we consider elements g in $S_{[p]}^{1/p}(x_n^*, \varepsilon_n)$, these inequalities are true for all $k = 1, \dots, n$. Consequently, by Lemma 2.4 and the constant 1 p -convexity of X , for every $g \in S_{[p]}^{1/p}(x_n^*, \varepsilon_n)$ and $f \in S_{X_0}$ there is an index $k \in \{1, \dots, n\}$ such that

$$\begin{aligned} \|(f^p + g^p)^{1/p}\|_X^p &\geq \|(f_k^p + g^p)^{1/p}\|_X^p - \|f_k^p - f^p\|_X^p \\ &\geq 2 - \delta^p - \varepsilon/2 \geq 2 - \varepsilon. \end{aligned}$$

Now, if $0 \leq s \leq t$ are real numbers such that $t^p + s^p = 1$, then for all such g and f ,

$$\begin{aligned} \|((tg)^p + (sf)^p)^{1/p}\|^p &= \|(t^p(g^p + f^p) - |s^p - t^p|f^p)^{1/p}\|^p \\ &\geq t^p\|(g^p + f^p)^{1/p}\|^p - |s^p - t^p|\|f\|^p \\ &\geq t^p(2 - \varepsilon) + s^p - t^p = 1 - \varepsilon. \end{aligned}$$

Since the same calculations can be done for $t \leq s$, we obtain the following inequality for every $t \geq 0$ and $f \in X_0$:

$$\begin{aligned} \|((tg)^p + f^p)^{1/p}\|^p &= \left\| \left((tg)^p + \left(\|f\| \frac{f}{\|f\|} \right)^p \right)^{1/p} \right\|^p \\ &= \left\| \left(\frac{t^p g^p}{t^p + \|f\|^p} + \frac{\|f\|^p (f/\|f\|)^p}{t^p + \|f\|^p} \right)^{1/p} \right\|^p \cdot (t^p + \|f\|^p) \\ &\geq (1 - \varepsilon)(t^p + \|f\|^p). \end{aligned}$$

The symmetry of the norm allows to obtain the same inequality for every $t \in \mathbb{R}$, replacing t by $|t|$. \square

The following lemma makes it clear that the Daugavet type equation (2.2) fails in the presence of atoms.

Lemma 2.7. *Let (Ω, Σ, μ) be a measure space. Let $0 < p < \infty$, let $X(\mu)$ be a constant 1 p -convex quasi-Banach function space over μ and suppose that μ has an atom. Then there is a rank one operator $T: X_{[p]} \rightarrow X_{[p]}$ such that*

$$\sup_{f \in B_X} \|(f^p + T(f^p))^{1/p}\|_X < (1 + \|T\|)^{1/p}.$$

Proof. Recall that by the constant 1 p -convexity of X , $X_{[p]}$ is a Banach function space. Let $\{a\}$ be an atom for μ . Then $0 < \mu(\{a\}) < \infty$ and the characteristic function $\chi_{\{a\}}$ belongs to $X_{[p]}$, and defines a (continuous)

functional of $(X_{[p]})^*$ by $\langle h, \chi_{\{a\}} \rangle = \int \chi_{\{a\}} h d\mu = h(a)\mu(\{a\})$, $h \in X_{[p]}$. Let T be the non-null rank one operator $T = -\frac{\chi_{\{a\}}}{\mu(\{a\})} \otimes \chi_{\{a\}}$. Then

$$\begin{aligned} \sup_{f \in B_X} \|(f^p + T(f^p))^{1/p}\| &= \sup_{f \in B_X} \|(f^p(a)\chi_{\{a\}} + f^p\chi_{\Omega \setminus \{a\}} - f^p(a)\chi_{\{a\}})^{1/p}\| \\ &= \sup_{f \in B_X} \|f\chi_{\Omega \setminus \{a\}}\| \leq 1 < (1 + \|T\|)^{1/p}, \end{aligned}$$

as claimed. \square

The following result provides the desired geometric characterization of L^p spaces over atomless measure spaces. Recall that an abstract L^p space is a Banach lattice E for which for every couple of disjoint elements $x, y \in E$, the equality $\|x+y\|^p = \|x\|^p + \|y\|^p$ holds (see for instance [11, Definition 1.b.1]).

Theorem 2.8. *Let $1 \leq p < \infty$ and let $X(\mu)$ be a quasi-Banach function space over μ . The following statements are equivalent:*

- (i) X is an abstract L^p space such that $X_{[p]}$ has the Daugavet property.
- (ii) X is equal to $L^p(hd\mu)$ for some $0 < h \in L^1(\mu)$ and the measure μ does not have atoms.
- (iii) $(X_{[p]})^* \neq \{0\}$, X is constant 1 p -concave, and for every finite set of operators $T_i: X_{[p]} \rightarrow X_{[p]}$, $i \in \{1, \dots, n\}$ we have that

$$\sup_{f \in B_X} \left\| \left(\sum_{i=1}^n \varphi_{T_i}(f)^p \right)^{1/p} \right\| \leq \left(\sum_{i=1}^n \|\varphi_{T_i}\|^p \right)^{1/p}$$

and

$$\sup_{f \in B_X} \| |f^p + \varphi_{T_i}(f)^p|^{1/p} \|_X = (1 + \|\varphi_{T_i}\|^p)^{1/p}.$$

- (iv) X is constant 1 p -convex, constant 1 p -concave and for every rank one operator $T: X_{[p]} \rightarrow X_{[p]}$,

$$\sup_{f \in B_X} \| |f^p + \varphi_T(f)^p|^{1/p} \|_X = (1 + \|\varphi_T\|^p)^{1/p}.$$

- (v) X is constant 1 p -convex and for every slice $S_{[p]}(x^*, \delta)$, every $\varepsilon > 0$ and every finite dimensional subspace X_0 of X there is an element $g \in S_{[p]}^{1/p}(x^*, \delta)$ such that for every $f_1, \dots, f_n \in X_0$ and $\alpha_i \geq 0$ satisfying $\sum_{i=1}^n \alpha_i^p = 1$,

$$\left\| \left(\sum_{i=1}^n |(\alpha_i g)^p + f_i^p| \right)^{1/p} \right\|^p \geq (1 - \varepsilon) \left(\|g\|^p + \sum_{i=1}^n \|f_i\|^p \right).$$

- (vi) X is constant 1 p -convex and for every slice $S_{[p]}(x^*, \delta)$, every $\varepsilon > 0$ and every finite dimensional subspace X_0 of X there is an element $g \in S_{[p]}^{1/p}(x^*, \delta)$ and an element $x_0^* \in B_{(X_{[p]})^*}$ such that

$$\left(\frac{|f^p + g^p|}{\|f\|^p + \|g\|^p} \right)^{1/p} \in S_{[p]}^{1/p}(x_0^*, \varepsilon)$$

for every $f \in X_0$.

Proof. (i) \Rightarrow (ii). Since X is an abstract L^p space and $1 \leq p < \infty$, X is in particular a σ -order continuous Banach function space. Using a Maurey-Rosenthal type factorization argument (see for example Corollary 6.17 in [16, Ch. 6]), we find that there is a function $0 < g$ such that the identity map $\text{Id}: X(\mu) \rightarrow X(\mu)$ factorizes through $L^p(\mu)$ by means of the multiplication operators $M_g: X \rightarrow L^p(\mu)$ and $M_{1/g}: L^p(\mu) \rightarrow X$; in fact, M_g defines an isometry (notice that for applying the Corollary 6.17 quoted above it is necessary to take into account that the operator M_g always has dense range). Therefore, the space $X(\mu)$ can be identified isometrically and in order with $L^p(h d\mu)$, $h = g^p$, and its elements are the same functions. So, $X_{[p]} = L^1(h d\mu)$ has the Daugavet property, and therefore by Lemma 2.7 with $p = 1$ the measure $h d\mu$ does not have atoms. Consequently, μ does not have atoms either.

For (ii) \Rightarrow (i), just recall that an $L^1(\nu)$ -space over an atomless measure space has the Daugavet property (see for instance [1, Theorem 3.2] or the example after Theorem 2.3 in [9]).

By Lemma 2.1, (ii) implies the equivalent statements (iii) and (iv), taking into account that X can be written as a Banach function space over the measure $h d\mu$. Clearly, (iv) implies (i).

Let us now show (i) \Rightarrow (v). Assume that X is an L^p -space and $X_{[p]}$ has the Daugavet property. Then Lemma 2.6 provides for every finite dimensional subspace X_0 of X , every $\varepsilon > 0$ and every $x^* \in (X_{[p]})^*$ an element $g \in S_{[p]}^{1/p}(x^*, \varepsilon)$ such that for every $f \in X_0$ and $t \in \mathbb{R}$

$$\|((tg)^p + f^p)^{1/p}\|_X^p \geq (1 - \varepsilon)(|t|^p + \|f\|_X^p).$$

Thus, taking into account that X is an L^p -space (and then also constant 1 p -concave), for every finite set of elements $f_1, \dots, f_n \in X_0$ and positive real numbers α_i such that $\sum_{i=1}^n \alpha_i^p = 1$, we obtain

$$\begin{aligned} \left\| \left(\sum_{i=1}^n |f_i^p + (\alpha_i g)^p| \right)^{1/p} \right\|_X^p &= \sum_{i=1}^n \| |f_i^p + (\alpha_i g)^p|^{1/p} \|_X^p \\ &\geq (1 - \varepsilon) \left(\|g\|_X^p + \sum_{i=1}^n \|f_i\|_X^p \right). \end{aligned} \quad (2.4)$$

For (v) \Rightarrow (vi) we apply the following separation argument. Consider the convex set $B_{(X_{[p]})^*}$, which is a compact Hausdorff space when endowed with the weak* topology, and the family of all functions $\Phi_{f_1, \dots, f_n; \alpha_1, \dots, \alpha_n}: B_{(X_{[p]})^*} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, $f_1, \dots, f_n \in X_0$, $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, $\sum_{i=1}^n \alpha_i^p = 1$, defined by

$$\Phi_{f_1, \dots, f_n; \alpha_1, \dots, \alpha_n}(x^*) := (1 - \varepsilon) \left(\|g\|_X^p + \sum_{i=1}^n \|f_i\|_X^p \right) - \left\langle \sum_{i=1}^n |f_i^p + \alpha_i^p g^p|, x^* \right\rangle.$$

Each function defined in this way is clearly convex, and the family of all such functions is concave, since each convex combination of two such functions can be written again as a function of the same family; indeed for $0 \leq \beta \leq 1$ we have

$$\beta \Phi_{f_1, \dots, f_n; \alpha_1, \dots, \alpha_n} + (1 - \beta) \Phi_{\bar{f}_1, \dots, \bar{f}_m; \bar{\alpha}_1, \dots, \bar{\alpha}_m} =$$

$$\Phi_{\beta^{1/p}f_1, \dots, \beta^{1/p}f_n, (1-\beta)^{1/p}\bar{f}_1, \dots, (1-\beta)^{1/p}\bar{f}_m; \beta^{1/p}\alpha_1, \dots, \beta^{1/p}\alpha_n, (1-\beta)^{1/p}\bar{\alpha}_1, \dots, (1-\beta)^{1/p}\bar{\alpha}_m}.$$

These functions are continuous with respect to the weak* topology, and by (2.4) and the Hahn-Banach Theorem for each of them there is an $x_1^* \in B_{(X_{[p]})^*}$ such that $\Phi_{f_1, \dots, f_n; \alpha_1, \dots, \alpha_n}(x_1^*) \leq 0$, so an application of Ky Fan's Lemma (see for instance [16, Lemma 6.12]) gives an element x_0^* such that

$$\Phi_{f_1, \dots, f_n; \alpha_1, \dots, \alpha_n}(x_0^*) \leq 0$$

for all the functions. Therefore, in particular, the inclusion in (vi) is obtained.

Let us now prove (vi) \Rightarrow (ii). Take a finite set of elements $f_1, \dots, f_n \in X$ and consider the finite dimensional subspace X_0 generated by them. Take any slice $S_{[p]}(x^*, \varepsilon_0)$ generated by a norm one element x^* and an $\varepsilon > 0$. Then an application of (vi) gives a $g \in S_{[p]}^{1/p}(x^*, \varepsilon_0)$ and an element $x_0^* \in B_{(X_{[p]})^*}$ such that

$$\sum_{i=1}^n \langle |f_i^p + g^p|, x_0^* \rangle \geq (1 - \varepsilon) \left(\sum_{i=1}^n \|f_i\|_X^p + n\|g\|_X^p \right).$$

Thus,

$$\left\| \left(\sum_{i=1}^n |f_i^p| \right)^{1/p} \right\|_X^p + n\|g\|_X^p \geq (1 - \varepsilon) \left(\sum_{i=1}^n \|f_i\|_X^p + n\|g\|_X^p \right)$$

and therefore,

$$\left\| \left(\sum_{i=1}^n |f_i^p| \right)^{1/p} \right\|_X^p + \varepsilon n\|g\|_X^p \geq (1 - \varepsilon) \left(\sum_{i=1}^n \|f_i\|_X^p \right).$$

Since this construction can be done for every $\varepsilon > 0$, we obtain that

$$\left\| \left(\sum_{i=1}^n |f_i^p| \right)^{1/p} \right\| = \left(\sum_{i=1}^n \|f_i\|_X^p \right)^{1/p}.$$

Consequently, X is an abstract L^p space. Also, for $\varepsilon > 0$, taking a single function $f \in S_X$ and the subspace X_0 generated by it and an $x^* \in S_{(X_{[p]})^*}$ we obtain by (vi) an $x_0^* \in B_{(X_{[p]})^*}$ and a function $g \in S_{[p]}^{1/p}(x^*, \varepsilon)$ such that

$$\| |f^p + g^p|^{1/p} \|_X^p \geq \langle |f^p + g^p|, x_0^* \rangle \geq 2(1 - \varepsilon).$$

Thus, Lemma 2.1 gives that $X_{[p]}$ has the Daugavet property. \square

Corollary 2.9. *Let $1 \leq p < \infty$. Every separable quasi-Banach function space satisfying the equivalent statements of Theorem 2.8 is order isomorphic and isometric to $L^p([0, 1])$.*

This is a direct consequence of Theorem 2.8 and the characterization of atomless separable L^p -spaces (see [14, Theorem 2.7.3]).

Remark 2.10. Note that using Kakutani's representation theorem (see for instance [11, Theorem 1.b.2] or [14, Theorem 2.7.1]) Theorem 2.8 can be applied in a more abstract setting, without the requirement for X to be a quasi-Banach function space. If X is just a Banach lattice that is also an abstract L^p space, then X is order isometric to an $L^p(\mu)$ space over some measure space (Ω, Σ, μ) , so in this case the condition of $L^p(\mu)_{[p]} = L^1(\mu)$

having the Daugavet property, i.e., μ having no atoms, is characterized by the equivalent statements of the theorem. Therefore, Corollary 2.9 can also be stated for Banach lattices *via* the atomic properties of the representing measure that Kakutani's theorem gives.

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INSTITUTO UNIVERSITARIO DE MATEMÁTICA PURA Y APLICADA, UNIVERSIDAD POLITÉCNICA DE VALENCIA, CAMINO DE VERA S/N, 46071 VALENCIA, SPAIN.

Current address: Department of Mathematics, Freie Universität Berlin, Arnimallee 6, D-14 195 Berlin, Germany.

E-mail address: easancpe@mat.upv.es

DEPARTMENT OF MATHEMATICS, FREIE UNIVERSITÄT BERLIN, ARNIMALLEE 6, D-14 195 BERLIN, GERMANY

E-mail address: werner@math.fu-berlin.de