Chapter 2

Preliminaries

In this chapter we collect known concepts and results which will be used several times within this thesis. In Section 2.1 we define and discuss the curves and surfaces that will be considered in this thesis. In Section 2.2 we define the Hausdorff distance and state known results on its computability. Next we consider the Fréchet distance between polygonal curves in Section 2.3. Several aspects of the algorithm for polygonal curves will be used by our algorithms for computing the weak Fréchet distance between triangulated surfaces and the Fréchet distance between simple polygons. Finally, in Section 2.4, we discuss the model of computation that we assume.

2.1 Curves and Surfaces

In this thesis we are interested in discrete parameterized curves and surfaces. We will assume that parameterizations of the curves and surfaces of the form

$$f\colon [0,1]^k \to \mathbb{R}^d,$$

for fixed $k \leq d$ are given. That is, we consider k-dimensional shapes in d-dimensional space. Mostly we will consider one-dimensional curves in the plane which are parameterized over the unit interval and two-dimensional surfaces in 3-space parameterized over the unit square. For simplicity, we will often denote the curves and surfaces themselves by their parameterizations f and g, as well.

We will assume the curves and surfaces to be *discrete* in the sense that the parameterizations f and g are *simplicial*. That is, the curves or surfaces are given by functions

$$f\colon |K| \to \mathbb{R}^d,$$

where K denotes a k-dimensional simplicial complex and $|K|$ its underlying space which equals $[0,1]^k$. The function f is simplicial if it is linear on simplices of K. In the one-dimensional case this implies that we are considering *polygonal curves*. In the two-dimensional case the underlying simplicial complex in parameter space is a triangulation of the unit square. The image of the triangulation under a simplicial parameterization can be a *triangulated surface*. However, a simplicial function may also “collapse” a simplex on a lower dimensional simplex, i.e., map a triangle to an edge or vertex. Our results hold also in this case, we are however interested in the case of triangulated surfaces and will consider only these in the following.

2.2 Hausdorff Distance

The Hausdorff distance is often used as a distance measure for geometric shapes. Let P, Q be two compact point sets in $\mathbb{R}^d$. Their Hausdorff distance is defined as

$$\delta_H(P,Q) = \max\big(\delta_{H,dir}(P,Q), \delta_{H,dir}(Q,P)\big),$$

where

$$\delta_{H,dir}(P,Q) \;=\; \max_{p\in P}\min_{q\in Q} \operatorname{dist}(p,q)$$

is the *directed Hausdorff distance* between P and Q and $\operatorname{dist}(\cdot,\cdot)$ the underlying metric in $\mathbb{R}^d$.

Assuming the underlying metric to be computable in constant time, the following results on computing the Hausdorff distance for finite point sets are known. For disjoint convex polygons, it can be computed in linear time [12]. For general sets of n points in $\mathbb{R}^2$ it can be computed in $O(n \log n)$ time using Voronoi diagrams [2]. For sets of n points in $\mathbb{R}^3$ it can be computed in $O(n^{4/3+\varepsilon})$ time using a data structure of Agarwal and Matoušek [1]. For two sets of m and n k-dimensional simplices in $\mathbb{R}^d$ the directed Hausdorff distance can be computed in $O(nm^{2+k})$ time [4].

2.3 Fréchet Distance

As stated in the introduction the Fréchet distance between two curves or surfaces $f, g\colon [0,1]^k \to \mathbb{R}^d, k \le d$, is defined as

$$\delta_F(f,g) := \inf_{\sigma \text{ hom}} \; \sup_{t\in[0,1]^k} \; \operatorname{dist}(f(t), g(\sigma(t)))$$

where $\sigma\colon [0,1]^k \to [0,1]^k$ ranges over all orientation-preserving homeomorphisms and $\operatorname{dist}(\cdot,\cdot)$ denotes the underlying metric on $\mathbb{R}^d$. Because the parameter spaces $[0,1]^k$ are compact, the supremum is attained and we can can replace the supremum with the maximum.

The Fréchet distance has also been defined without requiring the homeomorphisms to be orientation-preserving [26, 46]. We give our results for orientation-preserving homeomorphisms but they hold also for orientation-reversing homeomorphisms and can be extended to general homeomorphisms by considering both cases.

The Fréchet distance is defined for parameterized shapes and it is invariant under reparameterization. Therefore, the parameterizations need not be part of the input if they can be generated by the algorithms. For example, polygonal curves given by the ordered list of their endpoints can be parameterized by piecewise linear parameterizations over the unit interval. Implicit curves can often be given a parameterization. Furthermore, for simple polygons and for the weak Fréchet distance we do not necessarily need parameterizations of the surfaces, see Chapters 4 and 5.

The Fréchet distance can be used with different underlying metrics in $\mathbb{R}^d$. For the results in this thesis, we require that the underlying metric is computable, usually in polynomial time. More specifically, for the result in Chapter 3 we require that it is computable, for all other results, that it is computable in polynomial time. In the run time analyses we will assume the metric to be computable in constant time. In fact, we will analyze the run times for the three common metrics $\mathrm{d}_1, \mathrm{d}_2$, and d_∞. These are all metrics in $\mathbb{R}^d$ defined as $\mathrm{d}_p(x,y) = \|x-y\|_p$ where $\|\cdot\|_p$ denotes the p-norm. The p-norm is defined as $\|x\|_p := \big(\sum_{i=1}^d |x_i|^p\big)^{1/p}$ for a real number $p \ge 1$ and $\|x\|_\infty := \max_{i=1\dots d} |x_i|$. Usually, the metrics d_1 and d_∞ are easier to handle than the Euclidean metric d_2 because d_1 and d_∞ are given by linear equations and d_2 by quadratic equations. In the remainder of this thesis, we will use $\|x - y\|$ to denote the distance $\mathrm{dist}(x, y)$.

Two natural variants of the Fréchet distance between curves are the Fréchet distance between *closed curves* and the *weak Fréchet distance*. Closed curves are curves parameterized over the unit circle. For applying the algorithm for the Fréchet distance between non-closed curves, closed curves can be parameterized over the unit interval by choosing a fixed common starting and endpoint. For the Fréchet distance to be independent of the chosen starting point, it is defined as the infimum over all possible starting points for both curves and the Fréchet distance between the curves parameterized with these starting points.

The weak Fréchet distance is a relaxation of the Fréchet distance. It uses surjective, continuous functions as reparameterizations instead of homeomorphisms. For curves, it is also called the *non-monotone Fréchet distance* [6].

In the following sections we sketch the main ingredients for computing the Fréchet distance and its variants for polygonal curves which were given by Alt and Godau [6]. We refer to their work for a more detailed discussion.

The algorithms for computing the Fréchet distance and its variants all follow the same paradigm: first an algorithm for deciding the Fréchet distance is developed based on a geometric structure called the *free space diagram*. Then the decision algorithm is extended to a computation algorithm by searching a set of *critical values*.

We will describe the free space diagram in Section 2.3.1, and an extension of it, the *reachability structure*, in Section 2.3.2. In Section 2.3.3 we describe the decision algorithm and in Section 2.3.4 the computation algorithm.

2.3.1 Free Space Diagram

For a given real value $\varepsilon > 0$ the *free space diagram* of two continuous curves $f, g\colon [0,1] \to \mathbb{R}^d$ is defined as

$$F_\varepsilon(f,g) := \big\{(s,t) \in [0,1]^2 \;\big|\; \|f(s) - g(t)\| \le \varepsilon\big\}.$$

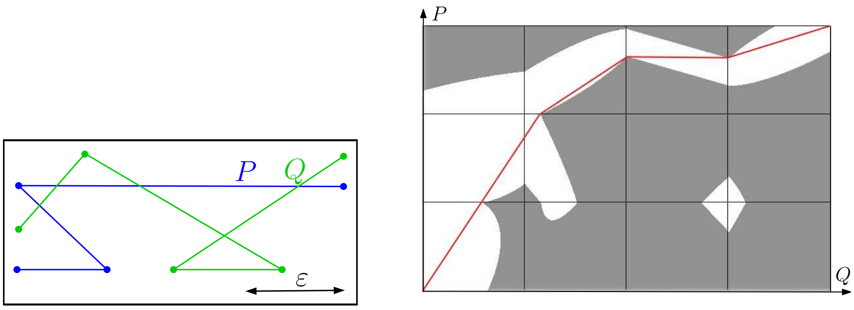
We will use the terms free space diagram and *free space* interchangeably. If f and g are polygonal curves with m and n vertices, respectively, then the free space diagram can be partitioned into n columns and m rows, giving a total of mn *cells*. See Figure 2.1 for an example of a free space diagram of two curves[1]. Each cell of the free space is the free space of two segments. The lower boundary of the free space diagram is considered to correspond to the parameter space of f and the left boundary to correspond to the parameter space of g. An important property of the free space diagram is that cells of the free space are convex.

For computing the Fréchet distance between closed polygonal curves, the *double free space diagram* is used. It is obtained by concatenating two copies of the (single) free space diagram and thus consists of $2mn$ cells partitioning $[0,2] \times [0,1]$.

2.3.2 Reachability Structure

The decision problem for closed polygonal curves is solved by computing the *reachability structure* which is based on the double free space diagram. It is a partition of the boundary of the double free space diagram into $O(mn)$ intervals. Each interval on the lower or left boundary is labeled according to whether any part of the upper or right boundary is reachable by a monotone path in the free space originating

[1] Thanks to Fabian Stehn for his program for computing graphical representations of free space diagrams.



(a) Polygonal curves P, Q and $\varepsilon > 0$

(b) A monotone path in the free space diagram of P and Q

Figure 2.1: Fréchet distance between polygonal curves.

from this interval. If there is such a path then the interval is also labeled with two pointers, one to the highest and one to the lowest reachable point from that interval. This is done analogously for intervals on the upper and right boundary.

The reachability structure has complexity $O(mn)$ and can be computed in $O(mn \log(mn))$ time by a divide and conquer algorithm. Given the reachability structure one can check in constant time whether there exists a monotone path in the free space between two given points on the boundary of the free space.

2.3.3 Decision Algorithm for Polygonal Curves

The decision algorithm for polygonal curves is based on a lemma stating that a Fréchet distance less than ε is equivalent to the existence of a monotone path in the free space from $(0, 0)$ to $(1, 1)$. Thus, for deciding the Fréchet distance, the free space is computed and it is determined whether such a path exists. For computing the free space diagram, the convexity of cells is used, i.e., only the cell boundaries are computed. The existence of a monotone path is then determined by deciding if an appropriate sequence of non-empty cell boundaries exist.

For the Fréchet distance between closed curves the reachability structure is used. The Fréchet distance between closed curves is less than ε if there is a monotone path in the double free space diagram from a point $(t, 0)$, $t \leq 1$, to $(t+1, 1)$. For this, the reachability structure is computed and then for each interval on the first half of the lower boundary, it is tested if the same interval shifted by $(1, 1)$ is in its reachability.

The weak Fréchet distance is less than ε if there is any (not necessarily monotone) path in the free space from $(0, 0)$ to $(1, 1)$. This is tested in a similar way as for the Fréchet distance.

The run time of the decision algorithms for the Fréchet distance and the weak Fréchet distance between polygonal curves is $O(mn)$, where m, n are the number of vertices of the polygonal curves. The run time of the decision algorithm for closed curves is $O(mn \log(mn))$.

2.3.4 Computation Algorithm for Polygonal Curves

The computation algorithm for the Fréchet distance between polygonal curves searches over a set of *critical values* of ε for the decision algorithm. Critical values are values which the Fréchet distance may attain. For polygonal curves, there are three types of critical values. Characterized by their effect on the free space, these are

1. $(0, 0)$ and $(1, 1)$ enter the free space
2. a cell boundary becomes non-empty
3. a horizontal or vertical passage opens.

A horizontal passage in the free space is given by a vertical interval that is part of the free space for a set of neighboring horizontal cell boundaries. A passage that opens for a parameter ε is a passage that exists in the free space of $F_\varepsilon(f, g)$ but not in any free space $F_{\varepsilon'}(f, g)$ for $\varepsilon' < \varepsilon$. Therefore, in $F_\varepsilon(f, g)$ the interval for the opening passage consists only of a point.

It has been shown [6], that the Fréchet distance between polygonal curves is attained at one of these values. An algorithm for computing the Fréchet distance is the following (Algorithm 2 of Alt and Godau [6]).

Algorithm 1: ComputeFréchet(f, g)

Input: Polygonal curves f, g
Output: $\delta_F(f, g)$

1 Determine all critical values of ε
2 Sort them
3 Do a binary search on the sorted sequence of critical values in each step solving the decision problem, continuing with the half containing smaller critical values if it has a positive answer and with the half containing larger values otherwise

The run time of this algorithm is $O((m^2 n + mn^2) \log(mn))$ and it is dominated by the sorting of the critical values in line 2. By using parametric search [39, 44], the run time can be improved to $O(mn \log(mn))$. We will use the same principle for computing the weak Fréchet distance between triangulated surfaces in Chapters 4 and the Fréchet distance between simple polygons in Chapter 5.

2.3.5 Further Results on the Fréchet Distance of Curves

The algorithm for computing the Fréchet distance between polygonal curves by Alt and Godau [6] was the first result on computing the Fréchet distance. Algorithms for matching [8, 21, 36, 52, 55] polygonal curves under the Fréchet distance have also been developed. The algorithm for computing the Fréchet distance between polygonal curves has been extended to smooth algebraic curves [47]. Furthermore, graph matching [5], and the Fréchet distance between sets of curves [19] have been analyzed. For a restricted class of curves, *κ-bounded* curves, the Fréchet distance can be approximated by the Hausdorff distance [9].

Furthermore, algorithms for variants of the Fréchet distance, the discrete Fréchet distance, the weak Fréchet distance, and the average Fréchet distance, have been considered, which we will discuss in later chapters. An open question is whether the Fréchet distance can be decided in sub-quadratic time or whether the decision problem is 3-sum hard [27]. A lower bound of $\Omega(n \log n)$ is known for deciding the Fréchet distance between curves in the plane [15].

2.3.6 Discrete Fréchet Distance

Several authors have considered a discrete version of the Fréchet distance between curves, the *discrete Fréchet distance* [11, 22, 32, 41]. For the discrete Fréchet distance the polygonal curves are modeled as the ordered sequences of their vertices. Instead of taking the infimum over all homeomorphism on the parameter spaces of the curves the infimum is taken over discrete mappings between the vertices.

In the following we give the definitions and results of Eiter and Mannila [22]. Let P, Q be two polygonal curves given by the ordered sequences of their endpoints $\langle p_0, \ldots, p_m \rangle, \langle q_0, \ldots, q_n \rangle$. A *coupling* of P and Q is an ordered sequence of pairs of vertices in P, Q, i.e., $C = \langle c_0, \ldots, c_k \rangle$ and each c_i has the form $c_i = (p, q)$ with $p \in P$ and $q \in Q$, fulfilling: $(0,0), (m,n) \in C$ and

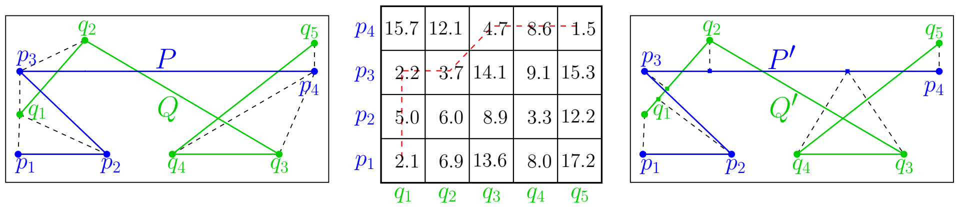
$$c_l = (p_i, q_j) \;\Rightarrow\; c_{l+1} \in \big\{(p_i + 1, q_j), (p_i, q_j + 1), (p_i + 1, q_j + 1)\big\}.$$

An example of a coupling is shown in Figure 2.2 (a).

The discrete Fréchet distance between polygonal curves P, Q is defined as

$$\delta_{dF}(P, Q) \;=\; \min_{C \text{ coupling}} \; \max_{(p_i, q_j) \in C} \; \|p_i - q_j\|.$$

It is a metric for polygonal curves.



(a) Coupling of the polygonal curves P and Q

(b) Path for the coupling in (a)

(c) Coupling of refinements of P and Q

Figure 2.2: Discrete Fréchet distance between polygonal curves.

A coupling of the vertices can be extended to a limit of homeomorphisms on the parameter spaces of the curves. This implies that the Fréchet distance is smaller or equal to the discrete Fréchet distance. Furthermore, for any homeomorphism there exists a coupling which yields a distance that is not more than the distance of the homeomorphism plus half the length of the longest edge in P or Q. In formulas,

$$\delta_F(P, Q) \;\le\; \delta_{dF}(P, Q) \;\le\; \delta_F(P, Q) + \min\big\{\nu(P), \nu(Q)\big\},$$

where $\nu(P) = \max_{i=1\ldots m} \|p_i - p_{i-1}\|$. This inequality implies that by refining P and Q such that $\nu(P)$ and $\nu(Q)$ tend to zero, the discrete Fréchet distance will tend to the Fréchet distance.

It can also be observed that for polygonal curves P, Q refinements P', Q' exist, each with at most $m + n$ vertices, such that $\delta_F(P, Q) = \delta_{dF}(P', Q')$. This follows from the fact that the Fréchet distance is always attained at a vertex. Such refinements can be constructed by adding the images of vertices of P, Q under a realizing map σ and an inverse realizing map τ, respectively. With a *realizing map* we mean a limit of homeomorphisms $\sigma_n, n \in \mathbb{N}$, such that $\delta_F(f, g) = \lim_{n \in \mathbb{N}} \max_{t \in [0,1]^k} \|f(t) - g(\sigma_n(t))\|$. See Figure 2.2 (c) for an example of a refinement P', Q' for which $\delta_F(P, Q) = \delta_{dF}(P', Q')$.

The discrete Fréchet distance can be computed using a discrete structure similar to the free space diagram for the (continuous) Fréchet distance: Take the $m \times n$ grid and in each grid cell c_{ij} write the value $\|p_i - q_j\|$. A coupling corresponds to a path in the grid from cell c_{00} to cell c_{mn} containing only edges going up, up-right, or right. I.e. it contains only edges $(c_{ij}, c_{i+1j}), (c_{ij}, c_{ij+1})$, or (c_{ij}, c_{i+1j+1}). A coupling is optimal for the discrete Fréchet distance if it minimizes the maximal value in a cell along its path. See Figure 2.2 (b) for an example. Using this representation the discrete Fréchet distance can be computed in $O(mn)$ time by dynamic programming. Thus, although the algorithm for computing the discrete Fréchet distance is less involved than the algorithm for computing the (continuous) Fréchet distance between polygonal curves, the asymptotic run time of both algorithms is $O(mn)$.

In some applications, where the input is inherently discrete, the discrete Fréchet distance can be a more appropriate distance measure than the (continuous) Fréchet distance. Two such applications are protein structures [33] and vehicle tracking data [13].

We will in Chapter 3 give a definition for the discrete Fréchet distance between triangulated surfaces. This will not be a direct generalization of the discrete Fréchet distance between curves because the definition for curves uses the monotonicity of the homeomorphisms which is not given in higher dimensions. For sampling density tending to zero, however, we will see that the two definitions of the discrete Fréchet distance will coincide with each other as well as the (continuous) Fréchet distance.

In Chapter 6 we will consider possible definitions for an *average* and *summed Fréchet distance*. We will consider also an discrete average and summed Fréchet distances which will in some cases be computable where the continuous average or summed Fréchet distance is not known to be computable.

2.4 Model of Computation

As model of computation we assume the RAM model, i.e., a random access machine. The input to all our algorithms will be rational, i.e., the vertices of the triangulated surfaces and the coefficients of the simplicial parameterizations are all rational. We assume that the random access machine can do the arithmetic operations $+, -, \times, /$ in constant time. In Chapter 3 these operations suffice, i.e., we can work in the integer RAM model.

The algorithms in Chapters 4 and 5 need to compute the intersections of ellipses, i.e., compute square and quartic roots. In the runtime analysis we assume that we can do this in constant time. These roots can be compared in constant time [23]. A fixed number of digits of such a root can be computed in time linear in the number of digits, i.e., in constant time for a fixed precision. See also [42] for a general discussion on algebraic issues in computational geometry.

In Chapters 4 and 5 we show that the weak Fréchet distance between triangulated surfaces and the Fréchet distance between simple polygons, respectively, are polynomial time computable on a RAM with the unit cost model. Our algorithms have a computation depth of constant size and do only a constant number of operations on algebraic numbers of constant degree. Therefore, the algorithms also have a polynomial run time on the RAM model with the logarithmic cost model. This implies that they run in polynomial time on a Turing machine, i.e., these decision problems lie in P.