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Abstract. Given a planar graph G, the *graph theoretic dilation* of G is defined as the maximum ratio of the shortest-path distance and the Euclidean distance between any two vertices of G. Given a planar point set S, the graph theoretic dilation of S is the minimum graph theoretic dilation that any triangulation of S can achieve. We study the graph theoretic dilation of the regular n-gon. In particular, we compute a simple lower bound for the graph theoretic dilation of the regular n-gon and use this bound in order to derive an efficient approximation algorithm that computes a triangulation whose graph theoretic dilation is within a factor of $1 + O\left(1/\sqrt{\log n}\right)$ of the optimum. Furthermore, we demonstrate how the general concept of *exclusion regions* applies to minimum dilation triangulations.

Keywords: Algorithms and data structures, Computational geometry, Approximation algorithm, Optimal triangulations, Exclusion region.

1 Introduction

When planning a subway network for a big city, people are faced with great challenges. Since one of the goals of a subway system is to reduce the car traffic on the streets, it has to be designed in such a way that the quality of the connection between the stations can compete with the direct route, since otherwise nobody would use the subway, but the citizens would continue to endanger the environment by using their SUVs in order to get from u to v. Thus, the subway network should not force its users to take a large detour in order to reach their destination. As an additional challenge, it is very difficult and expensive to have intersecting subway lines outside the stations. Therefore, the network should be planar. How can we design such a high-quality subway network efficiently? This is the problem that we are going to deal with in this report.

Formally, we can state the problem as follows: given a planar point set S, we would like to find a triangulation T of S such that the maximum dilation between any pair of points is minimal, where the *dilation* between a pair of points $u, v \in S$ is defined as the ratio between the shortest path distance of u, v in T and the Euclidean distance $|uv|$. Such a triangulation is called a *minimum dilation triangulation* of T.

Up to now, very little research has been done on minimum dilation triangulations, but there has been some work on estimating the dilation of certain types of triangulations that had already been studied in other contexts. Chew [1] shows that the rectilinear Delaunay triangulation has dilation at most $\sqrt{10}$. A similar result for the Euclidean Delaunay triangulation is given by Dobkin *et al.* [3] who show that the dilation of the Euclidean Delaunay triangulation can be bounded from above by $\left(\left(1+\sqrt{5}\right)/2\right)\pi \approx 5.08$. This bound was further improved to $2\pi/\left(3\cos\left(\pi/6\right)\right) \approx 2.42$ by Keil and Gutwin [7]. Das and Joseph generalize these results by identifying two properties of planar graphs such that if A is an algorithm that computes a planar graph from a given set of points and if all the graphs constructed by A meet these properties, then the dilation of all the graphs constructed by A is bounded by a constant [2]. More details on these results can be found in Eppstein's survey [6].

In spite of these results, the actual minimum dilation triangulation remains mysterious, and in this paper we will report on some new results that we were able to obtain. First, we will restrict our attention to the graph theoretic dilation of the set of nodes of a regular n-gon. Even though this seems to be a very special case, it turns out that it is even nontrivial to find an algorithm that approximates the graph theoretic dilation of the regular n-gon and to prove its correctness. Furthermore, it seems that some of our results should be generalizable to *fat point sets*, *i.e.,* planar point sets that can be sandwiched between two circles whose radii have a constant ratio. After that, we consider the general case and show how the well known concept of *exclusion-regions* [4] applies to the minimum dilation triangulation. Exclusion regions provide a local test for the inclusion of an edge in a triangulation and can be used in heuristics in order to filter out edges that do not belong to a triangulation.

The organization of this report is as follows: In section 2, we are going give the required background and definitions that are important to the ensuing discussion and mention some simple consequences of these definitions. These definitions will be filled with life in section 3, where we compute a lower bound on the graph theoretic dilation of any triangulation of the regular n-gon. Furthermore, we show that the lower bound implies that the Euclidean distance between the two vertices of any maximum dilation pair is bounded from below by a large constant. This will be used in section 4 whose purpose is to present a polynomial time approximation algorithm that approximates the graph theoretic dilation of the regular n-gon within a factor of $1+O(1/\sqrt{\log n})$. Finally, in section 5, we shall consider the general case and identify some local properties of the minimum dilation triangulation that can be used in order to identify edges that are impossible for the minimum dilation triangulation of a given point set S.

2 Preliminaries

Let S be a finite planar point set, and T a triangulation of S. If we consider two points $u, v \in S$, there are two distance metrics that are of particular practical interest: the *Euclidean distance* between u and v, $|uv|$, and the *shortest path distance* between u and v with respect to T, which we are going to call $\pi_T(u,v)$. The shortest path distance represents the minimum distance we must cover in order to travel from u to v when we are only allowed to use the edges in T.

The ratio between the shortest path distance and the Euclidean distance is called $\delta_T(u,v)$, the *dilation* between u and v with respect to T:

$$\delta_T(u,v) := \begin{cases} 1, & \text{if } u = v, \\ \dfrac{\pi_T(u,v)}{|uv|}, & \text{if } u \neq v. \end{cases}$$

Intuitively, the dilation is a measure for the quality of the connection between u and v in T. If the dilation is large, this means that we have to travel a long way along the edges in T in order to reach v from u even though the direct route would be much shorter.

The maximum over all the dilations between pairs of vertices in T is called the *graph theoretic dilation* of T, $\delta(T)$, and measures the quality of the connection between any two vertices in T. The best possible graph theoretic dilation of any triangulation of S is called the *graph theoretic dilation* of S and will be denoted by $\delta(S)$. Thus, we have

$$\delta(T) := \max_{u,v \in S} \delta_T(u,v) \text{ and } \delta(S) := \min_{T \text{ triangulation of } S} \delta(T).$$

The triangulation that achieves the graph theoretic dilation of S is called the *minimum dilation triangulation* of S.

In the first few sections, we will consider $S_n := \{s_0, s_1, \ldots, s_{n-1}\}$, the set of nodes of a regular n-gon in counter-clockwise order. We will often need to refer to the number of vertices that lie between a given pair of nodes $s_a, s_b \in S_n$. Thus, we define the *convex hull distance* between s_a and s_b, $\Delta_{S_n}(s_a, s_b)$, as

$$\Delta_{S_n}(s_a, s_b) := \min\{|b-a|, n-|b-a|\}.$$

3 A Lower Bound for the Regular n-Gon

Let $S_n = \{s_0, s_1, \ldots, s_{n-1}\}$ be the set of nodes of a regular n-gon in counter-clockwise order. Let T be any triangulation of S_n. In this section, we are going to determine lower bounds on the graph theoretic dilation $\delta(T)$ of T. The main strategy will be to look at a distinguished pair of vertices and to determine the minimum dilation between this pair of vertices that any triangulation can achieve. In the following, we will make the simplifying assumption that n is divisible by 4, but by a more detailed analysis it can be shown that the lower bound holds for all n that are larger than 74.

First, it is easily seen that in T there is a longest line segment $\ell = \overline{s_\gamma s_a}$ such that the convex hull distance $\Delta_{S_n}(s_\gamma, s_a)$ between s_γ and s_s is at least $\frac{n}{3}$ and that ℓ is adjacent to another line segment $\ell' = \overline{s_\gamma s_b}$ such that the convex hull distance $\Delta_{S_n}(s_\gamma, s_b)$ between s_γ and s_b is at least half of $n - \Delta_{S_n}(s_\gamma, s_a)$. We will assume that $\gamma = 0$. Now let $x = s_{\frac{n}{4}}$ and $y = s_{\frac{3n}{4}}$. In the following we shall compute a lower bound the dilation between x and y (see figure 1).

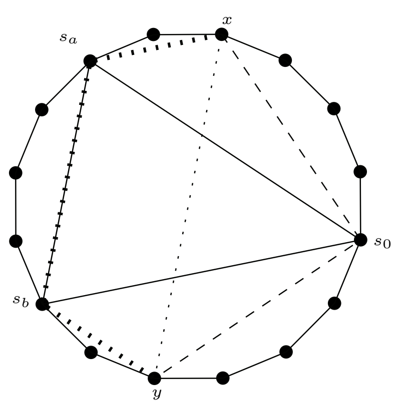

Fig. 1. We are looking at the dilation between x and y. The shortest path between x and y either includes s_0 (dashed line) or uses line segment $\overline{s_a s_b}$ (bold dotted line).

We define $\alpha := a\pi/n$ and $\beta := (n-b)\,\pi/n$. This means that α denotes half the angle between s_0 and s_a, while β represents half the angle between s_0 and s_b. By our assumptions, it follows that $\lceil n/3 \rceil \le a \le \lfloor n/2 \rfloor$ and $\lceil (n-a)/2 \rceil \le n - b \le a$, since ℓ is a longest line segment. This implies the following bounds on α and β:

$$\pi/3 \le \alpha \le \pi/2 \quad \text{and} \quad (\pi - \alpha)/2 \le \beta \le \alpha.$$

Note that these bounds imply in particular that x always lies between s_0 and s_a and that y always lies between s_0 and s_b, as shown in figure 1. This is so because from the bounds it follows that $\beta \ge \pi/4$.

Now let us compute the dilation between x and y. Clearly, the Euclidean distance between x and y is 2. The shortest path between the two points either passes s_0 or uses line segment $\overline{s_a s_b}$. In the former case, the length of the shortest path has to be at least $2\sin(\pi/4) + 2\sin(\pi/4)$, in the latter case, the length is bounded from below by $2\sin(\alpha - \pi/4) + 2\sin(\beta - \pi/4) + 2\sin(\pi - (\alpha + \beta))$, since the shortest path length can never be less than the Euclidean distance (see figure 1). Thus, we have

$$\delta_T(x, y) \ge \min\left\{\sqrt{2}, \sin(\alpha - \pi/4) + \sin(\beta - \pi/4) + \sin(\alpha + \beta)\right\}.$$

Using elementary calculus, we can show that this expression is bounded from below by $\sqrt{2 - \sqrt{3}} + \sqrt{3}/2 \approx 1.3836$. This proves the first theorem:

Theorem 1. *Let S_n be the set of vertices of a regular n-gon, and let $n \equiv 0 \pmod 4$. Then any triangulation of S_n has graph theoretic dilation at least $\sqrt{2-\sqrt{3}} + \sqrt{3}/2 \approx 1.3836$.*

As mentioned above, the assumption that n must be divisible by 4 can be removed, and we get the following general lower bound:

Theorem 2. *Let S_n be the set of vertices of a regular n-gon, and let $n \geq 74$. Then any triangulation of S_n has graph theoretic dilation at least $\sqrt{2-\sqrt{3}} + \sqrt{3}/2 \approx 1.3836$.*

Implications of the Bound Next, we are going to use the lower bound in order to establish some interesting structural properties of T. In particular, we are interested in the properties of any pair of nodes $s_a, s_b \in S_n$ such that $\delta(T) = \delta_T(s_a, s_b)$. We shall call such a pair a *maximum dilation pair*. We will show that the Euclidean distance between the vertices of a maximum dilation pair has to be at least 1.93185 if the radius of the regular n-gon is 1. If the radius of the n-gon is not 1, it is still true that the convex hull distance between the two vertices of a maximum dilation pair is at least $5n/12$.

The main idea is that the fact that between any two vertices in S_n there is a path that goes along the convex hull of S_n gives us an upper bound on the dilation between these two vertices. This upper bound can then be compared with the lower bound on $\delta(T)$ to obtain the desired property of a maximum dilation pair.

More precisely, it is easy to see that an upper bound for the dilation between any two distinct points s_a and s_b in S_n that have convex hull distance $\Delta = \Delta_{S_n}(s_a, s_b)$ is given by

$$\delta_T(s_a, s_b) \leq \frac{\Delta \sin(\pi/n)}{\sin(\Delta\pi/n)}, \tag{1}$$

since there is always a path between s_a and s_b that goes along the convex hull of S_n. On the other hand, theorem 2 tells us that for any maximum dilation pair (s_x, s_y), we have

$$\delta_T(s_x, s_y) \geq \sqrt{2-\sqrt{3}} + \sqrt{3}/2.$$

From this we can show that the convex hull distance Δ between s_x and s_y has to be more than $5n/12$: Simple calculus shows that the upper bound defined in equation (1) grows monotonically with Δ, and hence for $1 \leq \Delta \leq \frac{5}{12}n$ we have

$$\frac{\Delta \sin(\pi/n)}{\sin(\Delta\pi/n)} \leq \frac{5n \sin(\pi/n)}{12 \sin(5\pi/12)} \overset{(1)}{\leq} \frac{5\pi}{12\sin(5\pi/12)} < \sqrt{2-\sqrt{3}} + \sqrt{3}/2,$$

where (1) is due to the fact that $n \sin(\pi/n) \nearrow \pi$.

Consequently, the Euclidean distance between s_x and s_y must be larger than $2\sin(5\pi/12)$, which is about 1.93185. Let us state this result as a corollary:

Corollary 1. *Let $n \geq 74$ and S_n be the set of vertices of a regular n-gon. For any triangulation T of S_n and any maximum dilation pair $s_x, s_y \in S_n$, the convex hull distance $\Delta_{S_n}(s_a, s_b)$ between s_a and s_b is larger than $5n/12$. Furthermore, if the radius of S_n is 1, the Euclidean distance $|s_x s_y|$ is more than $\left(\sqrt{6+3\sqrt{3}} + \sqrt{2-\sqrt{3}}\right)/2 \approx 1.93185$.*

4 An Approximation Algorithm for the Regular n-gon

This section is devoted to the description of a fast approximation algorithm for the graph theoretic dilation of S_n. The underlying idea is very simple: If there are a lot of points on the convex hull of S_n, then it is likely that the dilation will not change too much if we just throw away some of these points. Taking this idea to an extreme, if we could throw away all but logarithmically many points of S_n without affecting the graph theoretic dilation of S_n too much, we could use exhaustive search in order to find a minimum dilation triangulation for the logarithmic sample. Then we add the vertices we discarded before and add edges until we obtain a triangulation T^* of S_n. This triangulation T^* will be our approximation of a triangulation that achieves the optimal graph theoretic dilation. We are going to show that the graph theoretic dilation $\delta(T^*)$ of triangulation T^* approximates $\delta(S_n)$ up to a factor of $1 + O\left(1/\sqrt{\log n}\right)$.

Description of the Algorithm We will now give a description of the algorithm. As mentioned above, the algorithm takes a logarithmic sample A of S_n for which it computes a minimum dilation triangulation T_A and then extends this triangulation to a triangulation T^* of S_n.

The first step of the algorithm is to compute the total length of the convex hull of S_n, which we will denote by l. Then the algorithm picks an arbitrary start vertex, say s_0, and proceeds counter-clockwise along the convex hull of S_n. During this process it picks the first vertex that has distance at least $d := 2l/\log n$ from s_0 along the convex hull, say s_α, and includes it in the sample. Then it picks the first vertex that has distance at least d from s_α, say s_β, and includes it in the sample, and so on. This process continues until the whole convex hull of S_n has been processed (see figure 2 for an illustration). The size of the sample A is at most $\log n/2$.

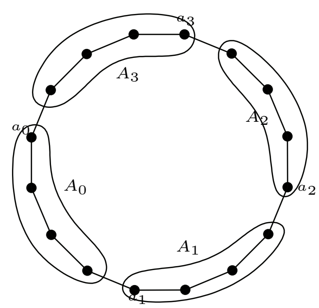

Fig. 2. Computing the logarithmic sample: Each set A_i is represented by the respective vertex a_i. In this example, we have $k = 3$.

Now the algorithm determines a triangulation T_A of A such that $\delta(A) = \delta(T_A)$. This is done by brute force by enumerating all possible triangulations of A and choosing one that achieves the minimum graph theoretic dilation. Since A is convex, there are at most $C_{(\log n)/2} = O\left(n/\log^{1.5} n\right)$ different triangulations of A, where C_m denotes the m-th Catalan number. It is possible to enumerate all these triangulations with very small overhead, and the graph theoretic dilation of a given triangulation of A can be computed in quadratic time, using the shortest path algorithm for planar graphs given in [8]. Thus, it is possible to compute T_A in time $O\left(n\sqrt{\log n}\right)$.

Finally, the algorithm proceeds to extend T_A to a triangulation T^* of S_n. To do this, we connect any point in $S_n \backslash A$ with the vertex in A that is closest to it and add edges in an arbitrary manner as long as the graph remains planar. The resulting triangulation T^* is our approximation of a minimum dilation triangulation. It is clear that the running time of our algorithm is $O\left(n\sqrt{\log n}\right)$, where $n = |S_n|$ denotes the number of points in the input.

The Dilation of the Sample In this section, we will relate the graph theoretic dilation of the sample A to the dilation between any pair of points in A when the points from $S_n \backslash A$ are added. Even though it is possible that the graph theoretic dilation of A decreases when points are added outside the convex hull of A, we are going to show that this decrease cannot be arbitrarily large.

We start by considering a shortest path P between two points a and b in an arbitrary triangulation T of S_n. The line segment $\overline{ab}$ divides S_n into two sets $Y, Z \subseteq S_n$ such that $Y \cap Z = \{a, b\}$ and Y

is the set of points to the left of $\overline{ab}$ and Z is the set of points to the right of $\overline{ab}$. Let us order the vertices in $Y = \{y_1, y_2, \ldots, y_\sigma\}$ and $Z = \{z_1, z_2, \ldots, z_\tau\}$ in increasing convex hull distance from a. Furthermore, assume that Y and Z are numbered in that order. Now, let $a = p_1 \to p_2 \to \cdots \to p_m = b$ be the sequence of nodes along P. It can easily be verified that the two sequences

$$P \cap Y = (a = y_{i_0}, y_{i_1}, \ldots, y_{i_\alpha} = b) \text{ and } P \cap Z = \left(a = z_{j_0}, z_{j_1}, \ldots, z_{i_\beta} = b\right)$$

are strictly monotonically increasing (see figure 3).

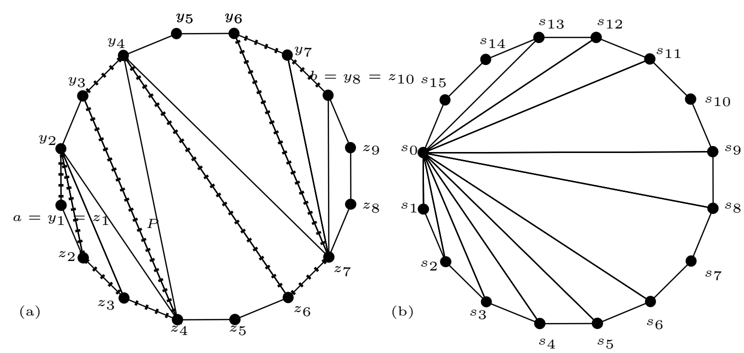

Fig. 3. In (a), we see a shortest path from a to b (bold dotted line). The sequences $P \cap Y$ and $P \cap Z$ are strictly monotonically increasing. The thin chords are the lines that define the values of the d_i. Figure (b) shows the equivalent arrangements of these chords that are used in the proof that $(d_i)_{1 \le i \le m}$ is bitonic (claim 4).

Let us look at the sequence $d_i = |p_\alpha p_\beta|$, where $\alpha = \max\{k \le i \,|p_k \in P \cap Y\}$ and $\beta = \max\{k \le i \,|p_k \in P \cap Z\}$ for $1 \le i \le m$.

Claim. The sequence $(d_i)_{1 \le i \le m}$ is *bitonic, i.e.,* it is first monotonically increasing and then monotonically decreasing.

Proof. For each d_i, let $(p_{\alpha_i}, p_{\beta_i})$ be the corresponding pair of nodes on P. It is clear that the sequence $h_i = \Delta_{S_n}(p_{\alpha_i}, a) + \Delta_{S_n}(a, p_{\beta_i})$, which counts the "hops" along the convex hull from p_{α_i} to p_{β_i} over a, is strictly increasing and that d_i is the same as $|s_0 s_{h_i}|$. Now the claim follows immediately, since every circle is unimodal (see figure 3). $\square$

Now let us consider the following set of of indices:

$$M := \left\{ 1 \le i \le m \,\middle|\, d_i \ge 2\sin\left(l/\left(2\sqrt{\log n}\right)\right) \right\}, \tag{2}$$

where l denotes the length of the convex hull of S_n. Observe that M is an interval since $(d_i)_{1 \le i \le m}$ is a bitonic sequence. Furthermore, let us call a node p_i on P a *jump node*, if either $p_i \in P \cap Y$ and $p_{i+1} \in P \cap Z$ or $p_i \in P \cap Z$ and $p_{i+1} \in P \cap Y$, *i.e.,* P changes sides between p_i and p_{i+1}. In the following claim we shall bound the number of jump nodes with indices in M.

Claim. The number of jump nodes with indices in M is $O\left(\sqrt{\log n}\right)$.

Proof. This is clear, since any jump node increases the length of P by d_i, but the length of P can be at most $\frac{l}{2}$. $\square$

After these preparations, let us now come to the heart of the argument. Let T be an arbitrary triangulation of S_n, and let a_i be the points in the sample in counterclockwise order. Let A_i be the set of points between a_i and a_{i+1}. We consider the graph $\hat{T}$ which we get when we contract all the vertices in A_i into vertex a_i for each i and then delete all the loops and multiple edges. This contraction process is performed as a series of single *contraction steps*, in which two vertices of T are contracted into one vertex. An example of one such contraction step is shown in figure 4. It is easily verified that $\hat{T}$ is a planar subdivision with vertex set A.

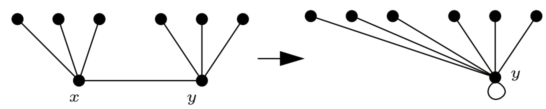

Fig. 4. An example of a contraction step. x and y are contracted into y, *i.e.*, the line segment connecting x and y becomes a loop and all the line segments that ended in x now end in y.

Let $a, b \in A$. We would like to know how the length of the shortest path $\pi_T(a, b)$ between a and b in T relates to the length of the shortest path $\pi_{\hat{T}}(a, b)$ between a and b in $\hat{T}$. Let us first consider a very special case (see Figure 5).

Claim. Let $q_1, q_2, \ldots, q_s$ be a set of points on a semicircle of radius 1 with center C in counterclockwise order and let $Q = q_1 \to q_2 \to \cdots \to q_s$ be the polygonal chain along these points. If we perturb each of the points $q_2, \ldots, q_{s-1}$ by an angle of at most ε along the semicircle in counter-clockwise direction and call the new points $q'_2, \ldots, q'_{s-1}$, then the length of polygonal chain $Q' = q_1 \to q'_2 \to \cdots \to q'_{s-1} \to q_s$ can be bounded by $|Q'| \leq |Q| + O(\varepsilon)$, where $|Q|$ and $|Q'|$ denote the length of polygonal chains Q and Q', respectively.

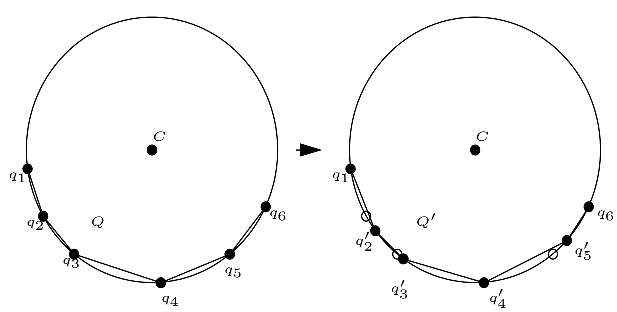

Fig. 5. Points q_2, q_3, q_4, and q_5 are perturbed to q'_2, q'_3, q'_4, and q'_5 (note that q_4 does not change). The resulting polygonal chain Q' is at most $O(\varepsilon)$ units longer than Q.

Proof. Let $l_i = \overline{q_i q_{i+1}}$ and $l'_i = \overline{q'_i q'_{i+1}}$ be the line segments of polygonal chains Q and Q' for $1 \leq i < s$ (naturally, we set $q'_1 = q_1$ and $q'_s = q_s$). Furthermore, for each line segment $l_i = \overline{q_i q_{i+1}}$ let $\alpha(l_i)$ denote the angle $\angle(q_i C q_{i+1})$. We will prove the following stronger claim:

$$|Q'| - |Q| \leq \alpha(q_1, q_s)^2 \sin(\varepsilon/2). \qquad (3)$$

Let us first consider the case $s = 3$: In this case, simple calculus yields the bound

$$|Q'| - |Q| \leq \alpha(p_1, p_3)^2 \sin(\varepsilon/2)/4. \qquad (4)$$

Now let us look at the case $s > 3$. Let β_i denote the angle $\angle(q_i C q_{i+2})$ for $1 \leq i \leq s-2$. We perturb the q_i as follows: in the first step, we fix $q_1, q_3, \ldots, q_s$ and perturb q_2 to get q_2'. In the second step, we fix $q_1, q_2', q_4, \ldots, q_s$ and perturb q_3 to get q_3'. We continue in this fashion, until all the q_i have been moved. From (4) it follows that in the i-th step the length of the polygonal chain increases by at most $\beta_i^2 \sin(\varepsilon/2)/4$, since all the points are perturbed in counter-clockwise direction and hence we have $\angle(q_i' C q_{i+2}) \leq \beta_i$. Thus, the total increase in length is at most

$$\frac{1}{4} \sin\left(\frac{\varepsilon}{2}\right) \sum_{i=1}^{s-2} \beta_i^2.$$

Now claim (3) follows from the fact that

$$\sum_{i=1}^{s-2} \beta_i \leq 2\alpha(q_1, q_s) \text{ and hence } \sum_{i=1}^{s-2} \beta_i^2 \leq 4\alpha(q_1, q_s)^2,$$

since all the β_i are positive.

Thus, the claim follows, since $\sin(\varepsilon/2) \leq \varepsilon$. □

Now let us consider a shortest path P between two points a and b in S_n. First, we look at $P \cap M$, the sub-path of P whose vertices are those whose indices lie in M, where M is the index set defined in equation (2). We have already seen that the shortest path $P \cap M$ changes sides at most $O\left(\sqrt{\log n}\right)$ times. The perturbation of each of these jump nodes can increase the length of P by at most $O(l/\log n)$, and between the jump nodes we have the situation we examined in the last claim, where we saw that the length of the shortest path between the jump nodes can increase by at most $O(l/\log n)$, since the perturbation angle is larger than the perturbation distance (see figure 6). Thus, the total length of the sub-path $P \cap M$ increases by at most $O\left(l\sqrt{\log n}/\log n\right) =$

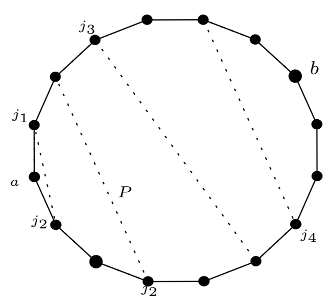

Fig. 6. A typical shortest path from a to b. Between any two sequential jump nodes, the length of the shortest path increases by at most $O(l/\log n)$.

$O\left(l/\sqrt{\log n}\right)$. We do not know what happens to the shortest path outside M, but by our choice of M we do know that the increase in length is $O\left(l/\sqrt{\log n}\right)$, since the shortest path outside M consists of two paths. each of which is restricted to a circular segment that is defined by a chord of length at most $2\sin\left(l/2\sqrt{\log n}\right)$, and the length of such a path can be at most $l/\sqrt{\log n}$, which bounds the length of the convex hull.

Therefore, we have proved that the for any $a, b \in A$ we have $\pi_{\hat{T}}(a, b) \leq \pi_T(a, b) + O\left(l/\sqrt{\log n}\right)$. Furthermore, by corollary 1 we know that any pair of vertices in A that achieves maximum dilation has a Euclidean distance that is $\Omega(l)$. Thus, we have shown that for any triangulation T of S_n in which the maximum dilation between any pair of points in A is δ, there is a triangulation $\hat{T}$ of A such that $\delta\left(\hat{T}\right) \leq \delta + O\left(1/\sqrt{\log n}\right)$. Consequently, by taking the minimum on both sides we can conclude with the following lemma:

Lemma 1. *Let S_n be the set of vertices of a regular n-gon, and let $A \subseteq S_n$ be the sample. Then we have*

$$\delta(S_n) \geq \delta(A) - O\left(1/\sqrt{\log n}\right).$$

The Approximation Factor In order to get the desired relationship between $\delta(T^*)$ and $\delta(T_A)$, we compute an upper bound on the dilation between the vertices of a maximum dilation pair $x, y \in S_n \backslash A$. Let $a, b \in A$ be the points in A that are closest to x and y, respectively. Since we know that the Euclidean distance between the vertices of a maximum dilation pair is $\Omega(l)$, we can assume that $a \neq b$ and $|ab| > 4l/\log n$. By our definition of A it follows that the distance between x and a as well as y and b along the convex hull (and hence the Euclidean distance) is at most $d = 2l/\log n$. Thus, we can upperbound the shortest path length $\pi_{T^*}(x, y)$ between x and y by $\pi_{T^*}(x, y) \leq \pi_{T_A}(a, b) + 2d$.

Thus, the dilation $\delta_{T^*}(x, y)$ between x and y in T^* is at most

$$\begin{aligned}
\delta_{T^*}(x, y) &\leq \frac{\pi_{T_A}(a, b) + 2d}{|xy|} \\
&= \frac{\pi_{T_A}(a, b)}{|ab|} + \frac{(\pi_{T_A}(a, b) + 2d)\,|ab| - \pi_{T_A}(a, b)\,|xy|}{|ab|\,|xy|} \\
&\overset{(1)}{\leq} \frac{\pi_{T_A}(a, b)}{|ab|} + \frac{(\pi_{T_A}(a, b) + 2d)\,|ab| - \pi_{T_A}(a, b)\,(|ab| - 2d)}{|ab|\,|xy|} \\
&= \frac{\pi_{T_A}(a, b)}{|ab|}\left(1 + \frac{2d\,|ab| + 2d\pi_{T_A}(a, b)}{\pi_{T_A}(a, b)\,|xy|}\right) \\
&\overset{(2)}{\leq} \frac{\pi_{T_A}(a, b)}{|ab|}\left(1 + \frac{4d}{|xy|}\right) \\
&\overset{(3)}{=} \delta(T_A)\,(1 + O(1/\log n)).
\end{aligned}$$

In this chain of inequalities, (1) is due the equation $|xy| \geq |ab| - 2d$, (2) is due to the fact that $|ab|/\pi_{T_A}(a, b) = 1/\delta_{T_A}(a, b) \leq 1$, and (3) holds because a maximum dilation pair in S_n has dilation $\Omega(l)$ and $d = 2l/\log n$. Since x and y were arbitrary, we can now conclude the following inequality

$$\delta(T^*) \leq \delta(T_A)\,(1 + O(1/\log n)). \tag{5}$$

Together with lemma 1 this inequality finally yields the desired theorem that proves the correctness of our algorithm:

Theorem 3. *Let S_n be the vertex set of a regular n-gon. Then the triangulation T^* of S_n that is computed by the algorithm described in section 4 has the property that $\delta(T^*)$ approximates $\delta(S_n)$ up to a factor of $1 + O\left(1/\sqrt{\log n}\right)$, i.e.,*

$$\delta(T^*) \leq \left(1 + O\left(1/\sqrt{\log n}\right)\right)\delta(S_n).$$

Proof. Obviously, we have $\delta(T^*) \geq \delta(T_A)$ and thus inequality (5) yields

$$\delta(T_A) \leq \delta(T^*) \leq \delta(T_A)\,(1 + O(1/\log n)).$$

Furthermore, by definition we have $\delta(S_n) \leq \delta(T^*)$ and thus by lemma 1 it follows that

$$\delta(T_A) - O\left(1/\sqrt{\log n}\right) \leq \delta(S_n) \leq \delta(T^*).$$

Therefore, we get

$$\delta(T_A)\left(1 - O\left(1/\sqrt{\log n}\right)\right) \overset{(1)}{\leq} \delta(S_n) \overset{(2)}{\leq} \delta(T_A)\left(1 + O\left(1/\sqrt{\log n}\right)\right),$$

where (1) follows from the fact that $\delta(T_A) \geq 1$ and (2) is due to the fact that $1/\log n \leq 1/\sqrt{\log n}$. Now the theorem follows. □

5 An Exclusion Region

We now leave the regular n-gon and focus our attention on the general case. When considering optimal triangulations, it is instructive to look at local properties of the edges of these triangulations, since local properties improve our understanding of the structure of optimal triangulations and sometimes lead to efficient algorithms to compute them. One important class of local properties that has been studied for minimum weight triangulations and greedy triangulations is constituted by *exclusion regions*. Exclusion regions give us a necessary condition for the inclusion of an edge into an optimal triangulation: If u and v are two points in a given planar point set S, then the edge $e := \overline{uv}$ can only be contained in an optimal triangulation of S if no other points of S lie in certain parts of the exclusion region of S. For example, Das and Joseph [2] proved that e can only be included in the minimum weight triangulation of a point set S, if at least one of the two equilateral triangles with base e and base angle $\frac{\pi}{8}$ is empty (see Figure 7). This result was improved by Drysdale *et al.* [4], who proved that the base angle can be increased to $\pi/4.6$ and that also the disk of diameter $|e|/\sqrt{2}$ centered at the midpoint of e is an exclusion region for the minimum weight triangulation. A similar result with slightly different parameters also holds for the greedy triangulation [5]. In this section, we are going to show that an analogous result applies to the minimum dilation triangulation. More specifically, we show that an edge e can only be included in the minimum dilation triangulation of S, if at least one of the two half circles with radius $\alpha|e|$ whose center is the center of e is empty (see Figure 7). Here α denotes any constant such that $0 < \alpha < 3\cos(\pi/6)/(4\pi) \approx 0.2067$.

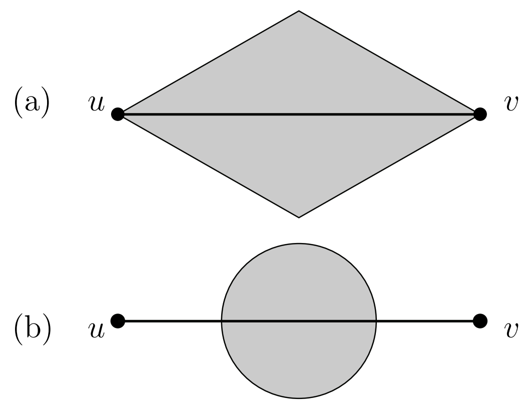

Fig. 7. (a) shows the standard exclusion region for the minimum weight triangulation. (b) shows our exclusion region for the minimum dilation triangulation.

The basic idea is very simple: Even though we do not know much about the actual minimum dilation triangulation of a planar point set S, we know that the graph theoretic dilation of the Delaunay triangulation of S is bounded by the constant $\gamma = 2\pi/(3\cos(\pi/6))$ [7]. Furthermore, it is obvious that if we have an edge e and two points that are quite close to the center of e and that lie on opposite sides of e, then the dilation between these two points is very large, because the line segment e constitutes an obstacle that any path between these two points needs to circumvent. Thus, all we need to check is that the dilation between any pair of points in the disk that lie on opposing sides of e is larger than γ, and then we know that if such a pair of points exists, then e cannot be contained in the minimum dilation triangulation of S, since the Delaunay triangulation would give us a better graph theoretic dilation than any triangulation containing e.

Thus, we assume that there exist two points $a, b \in S$ in D on opposite sides of e (see Figure 8). We need to show that $\delta_T(a, b) > \gamma$ for any triangulation T of S that contains line segment e. For this we need to know the shortest path distance between a and b in T, $\pi_T(a, b)$. Since the only thing we know about T is that T contains e, the best thing we can do is to lowerbound $\pi_T(a,b)$ by $\min(|au| + |ub|, |av| + |vb|)$.

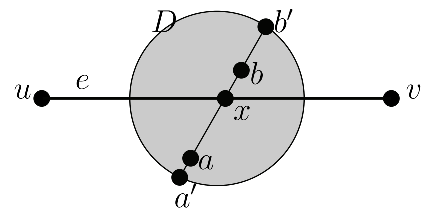

Fig. 8. The situation described in Observation 1. The dilation between a' and b' is smaller than the dilation between a and b. x is the intersection point of $\overline{ab}$ and e.

The first thing we observe is that we can assume that the two points lie on the boundary of D, since the dilation between the intersection points of the line through a and b with the boundary of D is smaller than the dilation between a and b:

Observation 1 *Let a be a point in D to the right of line segment $e = uv$, and let x be a point on e and in D. For $d > 0$, let $b(d)$ be the point to the left of line segment e on the half line ax such that $|xb(d)| = d$. Then the dilation $\delta(d)$ between a and $b(d)$ decreases as d increases.*

Proof. Due to the triangle inequality, the shortest path between a and b cannot include e, and hence $\delta(d)$ is given by

$$\delta(d) = \frac{\min\left(|ua| + |ub(d)|, |va| + |vb(d)|\right)}{|ax| + d}.$$

First, we are going to check that $\ell(d) := (|ua| + |ub(d)|)/(|ax| + d)$ is monotonically decreasing. By the law of cosines, the numerator can be written as $\text{num}(d) = |ua| + \sqrt{|ux|^2 + d^2 - 2|ux| \cos \delta}$, where δ denotes the angle between $\overline{ux}$ and $\overline{xb(d)}$. An easy calculation shows $\text{num}'(d) \leq 1$. The derivative of the denominator is 1. Therefore, by the mean value theorem, it follows that $\ell(d)$ is monotonically decreasing (note that the numerator is never smaller than the denominator), and the observation follows, since by a similar argument we can check that also $d \mapsto (|va| + |vb(d)|)/(|ax| + d)$ decreases monotonically, and hence $\delta(d)$ decreases. □

Now we are left with the task of bounding the dilation between two points on the boundary of D. First of all, it is clear that dilation $(2\alpha)^{-1}$ can be achieved when a and b are infinitesimally close to the two intersection points of D and e, respectively. We are going to show that this is already an optimal configuration. For our calculations we need a propitious parameterization. We proceed as follows: Let z be the center of D. By symmetry, we may assume that $\overline{ab}$ lies to the right of z. We describe the line segment $\overline{ab}$ by looking at the angle $\beta = \angle bza$ and the angle $x = \angle bzv - \beta/2$. The angle x describes the rotation of $\overline{ab}$ with respect to the position in which $\overline{ab}$ is orthogonal to e (see Figure 9). By our assumptions, we have $\beta \in (0, \pi]$ and $x \in (-\beta/2, \beta/2)$. Our parameterization is chosen in such a way that the following equations can be written in a symmetric manner, which simplifies some of the calculations.

The angle $\angle bza$ is at most π, and hence the shortest path between a and b passes v. Thus, the dilation between a and b is given by

$$\delta(x, \beta) := \frac{f(x) + f(-x)}{2\alpha \sin(\beta/2)},$$

where

$$f(x, \beta) = \sqrt{0.25 + \alpha^2 - \alpha \cos(\beta/2 + x)}.$$

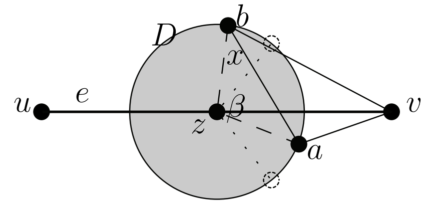

Fig. 9. Our parameterization. The angle $\angle azb$ is called β. The offset x denotes the rotation of $\overline{ab}$ with respect to the vertical position (dashed lines).

Here, $f(x)$ and $f(-x)$ denote the length of line segment $|vb|$ and $|va|$, respectively.

First, we fix $\beta \in (0, \pi]$ and optimize $x \mapsto \delta(x, \beta)$. An elementary yet tedious calculation yields the following observation:

Observation 2 *Let $\beta \in (0, \pi]$ be fixed. If we have $\cos(\beta/2) \leq 2\alpha$, the function $x \mapsto \delta(x, \beta)$ is minimal for $\cos(x) = (2\alpha)^{-1} \cos(\beta/2)$. Otherwise, $x \mapsto \delta(x, \beta)$ is minimal for $x = 0$.*

Now there are two cases to consider. If $\cos(\beta/2) \geq 2\alpha$, we need to look at $\delta(0, \beta) = f(0)/(\alpha \sin(\beta/2))$. Again, it turns out that this function is minimal if $\cos(\beta/2) = 2\alpha$, for this value of β we get that the dilation between a and b is exactly $(2\alpha)^{-1}$. What happens if $\cos(\beta/2) < 2\alpha$? In this case, we need to consider the value of $\delta(x, \beta)$, where x has the property that $\cos x = (2\alpha)^{-1} \cos(\beta/2)$. This situation is depicted in Figure 10. The condition $\cos x = (2\alpha)^{-1} \cos(\beta/2)$ means that the continuation of line segment $\overline{va}$ is orthogonal to the line through a' and z, where a', b' is the rotation of a, b along the boundary of D such that $\overline{a'b'}$ is orthogonal to $\overline{uv}$. In this configuration, the triangles $zp'b$ and zpa are congruent, and hence the length of the shortest path between a and b is $2 \cdot \frac{1}{2} \sin(\beta/2) |uv|$. Since the distance between a and b is $2\alpha \sin(\beta/2) |uv|$, we find that $\delta(x, \beta) = (2\alpha)^{-1}$.

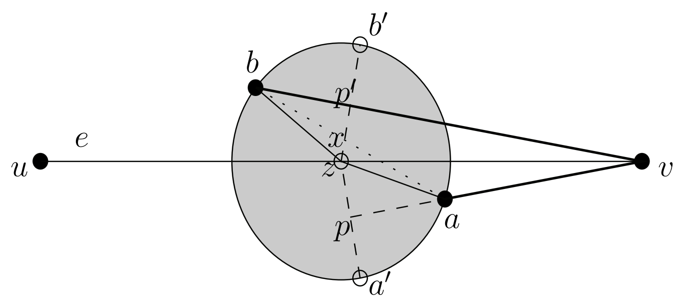

Fig. 10. On optimal configuration. The angle x is chosen in such a way that line segments $\overline{ab}$ and $\overline{zb'}$ are orthogonal. Then, line segments $\overline{p'b}$ and $\overline{pa}$ have the same length and hence the length of the shortest path between a and b is $\sin(\beta/2) |uv|$.

It follows that the dilation between a and b exhibits quite a remarkable behavior. If a and b are diametrically opposed, the minimum configuration with minimum dilation occurs when a and b are infinitesimally close to the two intersection points between D and e. As the chord $\overline{ab}$ gets shorter, the angle between e and $\overline{ab}$ in the optimal configuration becomes larger, until e and $\overline{ab}$ are orthogonal. As soon as this configuration is reached, the dilation between a and b increases as $\overline{ab}$ gets shorter.

Consequently, the minimum dilation between any points a and b in the two halves of D is $(2\alpha)^{-1}$, and by our choice of α and the upper bound on the graph theoretic dilation of the Delaunay triangulation [7], we can conclude with the following theorem:

Theorem 4. *Let $0 < \alpha < 3\cos(\pi/6)/(4\pi)$ be a constant, and let a and b be two points in the plane. Then the disk of radius $\alpha|ab|$ centered at the midpoint of $\overline{ab}$ is an exclusion region for the minimum dilation triangulation.*

Note that this exclusion region can be enlarged a little bit on the upper and lower boundary. For example, the dilation between the north- and south-pole of D is strictly less than γ. However, this would give us some curve of order 4 that is more difficult to handle than a simple circle. Moreover, the improvement would not be very substantial since the best possible radius for the north- and south-pole is $3\sqrt{3}/\sqrt{64\pi^2 - 108} \cdot |uv| \approx 0.227 \cdot |uv|$, which is only a minor improvement over our result of about $0.2067 \cdot |uv|$.

An upper bound Compared to the exclusion region for the minimum weight triangulation, our exclusion region is relatively small. Thus, it is natural to ask for upper bounds, that is, we would like to know how large the radius of the disk can be made. A simple four-point example of an upper bound is given in Figure 11. Line segments $\overline{uv}$ and $\overline{ab}$ are orthogonal, and $\overline{ab}$ is bisected by $\overline{uv}$. The triangulation shown is the Delaunay and the minimum weight triangulation of $\{a, b, u, v\}$, but it is not the minimum dilation triangulation, since the dilation between u and v is too high. The minimum dilation triangulation is the triangulation which contains the diagonal $\overline{uv}$. By setting up the equations and optimizing them using a computer algebra system like MAPLE, we find that a and b lie on a circle of radius about $0.3841 \cdot |uv|$. Hence, there is quite a substantial gap between the upper bound and our exclusion region of size $0.2067 \cdot |uv|$.

Note that the size of our exclusion region only depends on the upper bound on the graph theoretic dilation of the Delaunay triangulation. It is widely believed and suggested by experimental results that the true upper bound on the graph theoretic dilation of the Delaunay triangulation is $\frac{\pi}{2}$, which would immediately lead to an exclusion region of radius $|uv|/\pi \approx 0.318 \cdot |uv|$, which is quite close to the upper bound.

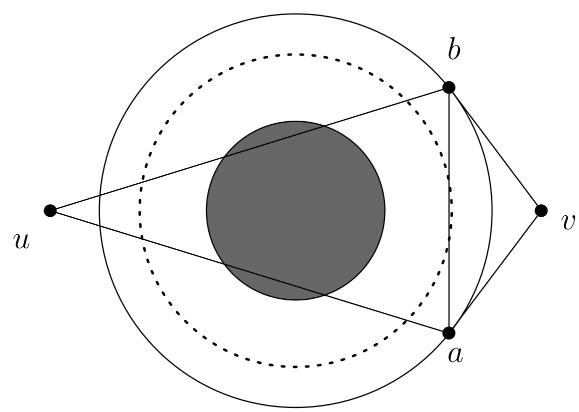

Fig. 11. A simple four-point configuration to upperbound the size of the exclusion region. The minimum dilation triangulation is the triangulation which uses the diagonal $\overline{uv}$. The shaded region show our exclusion region. The solid circle shows the upper bound. The dashed circle shows the size of the exclusion region for the case that the conjecture holds that the graph theoretic dilation of the Delaunay triangulation is bounded by $\pi/2$.

An Inclusion Region Up to now, we have discussed a necessary condition for the inclusion of an edge in the minimum dilation triangulation. We can further exploit the upper bound of [7] to obtain a sufficient condition for the inclusion of an edge. More specifically, for two points $u, v \in S$, we consider the ellipsoid E with foci u and v that is given by $E = \{x \in \mathbb{R}^2 | |ux| + |vx| \leq 2\pi / (3\cos(\pi/6)) \cdot |uv|\}$. If E is empty, then the line segment $\overline{uv}$ has to be included in the minimum dilation triangulation of S, since otherwise the dilation between u and v would be too high.

6 Conclusion

We have made some progress in the field of minimum dilation triangulations, and we have identified some useful properties of minimum dilation triangulations of the regular n-gon which we used in order to obtain an efficient approximation algorithm. In particular, the property that any maximum dilation pair must have a large Euclidean distance has proved very useful. However, the main question how to compute a minimum dilation triangulation for an *arbitrary* planar point set remains wide open. The approximation algorithm we devised can be applied to arbitrary convex sets, but in order to prove the approximation factor, we need a lower bound on the minimum Euclidean distance between the two vertices of a maximum dilation pair. If such a bound can be shown for certain types of convex planar point sets, the approximation algorithm will yield a triangulation whose graph theoretic dilation approximates the graph theoretic dilation for these point sets.

Furthermore, we have identified some local properties of the minimum dilation triangulation for general point sets. These properties can be used in order to filter out edges that are impossible for a minimum dilation triangulation of a given point set S. It can be shown that if S is uniformly distributed in a convex set C, then the expected number of edges remaining after the exclusion region test is linear. Unfortunately, it is not clear what to do after the exclusion region test, since it is not even known how to compute the minimum dilation triangulation of a convex polygon.

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