On-line Algorithms for q-adic Covering of the Unit Interval and for Covering a Cube by Cubes*

Marek Lassak

Institut für Informatik, FU Berlin,
D-14195, Berlin, Germany

permanent address: Instytut Matematyki i Fizyki ATR
Kaliskiego 7, 85-796 Bydgoszcz, Poland

Abstract. We present efficient algorithms for the on-line q-adic covering of the unit interval by sequences of segments. The basic method guarantees the covering provided the total length of segments in a sequence is at least $1+2\cdot\frac{1}{q}-\frac{1}{q^3}$. Other algorithms improve this estimate for $q \geq 6$. The unit d-dimensional cube can be on-line covered by arbitrary sequence of cubes of the total volume at least $2^d + \frac{5}{3} + \frac{5}{3}\cdot 2^{-d}$.

We say that a sequence $Q_1, Q_2, \ldots$ of subsets of Euclidean space E^d *permits a covering* of a set $C \subset E^d$ if there exist rigid motions $\tau_1, \tau_2, \ldots$ such that C is contained in the union of sets $\tau_1 Q_1, \tau_2 Q_2, \ldots$. Many questions appear about efficient covering algorithms. In the *on-line* version of this problem, at the beginning we are given the first set Q_1 but then we learn every succeeding set Q_i from the sequence only after the preceding set Q_{i-1} is definitely used for the covering. The reader can find more information about on-line covering algorithms in the survey articles [1] and [7]. We prove that arbitrary sequence of cubes of the total volume at least $2^d + \frac{5}{3} + \frac{5}{3}\cdot 2^{-d}$ is able to cover on-line the unit d-dimensional cube. This is very close to the best non-on-line estimate $2^d - 1$ (see [2]).

The closed interval with end-points x and y, where $x < y$, is denoted by $[x, y]$. The symbol (x, y) means the corresponding open interval.

Recall the on-line q-adic covering problem (see [6]). Let $q \geq 2$ be an integer. Find an efficient algorithm for the on-line covering of the interval $[0, 1]$ by a sequence of closed segments S_i of lengths δ_i, where $\delta_i \in \{q^{-1}, q^{-2}, \ldots\}$, and where every segment $\tau_i S_i$ is of the form $[c_i\delta_i, (c_i+1)\delta_i]$ with $c_i \in \{0, \ldots, \delta_i^{-1} - 1\}$ for $i = 1, 2, \ldots$.

We present an algorithm which is a far going modification of the algorithm from [3]. We improve the assumption about the total length of a sequence of segments which

* Research supported by Deutsche Forschungsgemeinschaft

permits the covering from a little less than $1 + 3 \cdot \frac{1}{q}$ to a little less than $1 + 2 \cdot \frac{1}{q}$. Next we propose a more sophisticated algorithm which lessens the above estimate to a little over $1 + \frac{5}{3} \cdot \frac{1}{q}$. We also show how to decrease the factor $\frac{5}{3} \approx 1.667$ arbitrarily close to $\frac{1}{2}(1 + \sqrt{5}) \approx 1.618$. A natural questions is about more efficient algorithms. It would be nice to manage with sequences of total length $1 + \frac{1}{q}$. Another question is about a non-trivial lower estimate. The only known such an estimate is $\frac{4}{3} = 1 + \frac{2}{3} \cdot \frac{1}{2}$ for $q = 2$ (see [4]).

At every moment of the covering process we take into account the greatest number $b \in [0, 1]$ such that the whole interval $[0, b]$ is covered. We call b the *current bottom.* When a segment S, say of length q^{-r}, is given to us, we find the greatest integer a such that $aq^{-r} \leq b$. If the interval $[(a + h - 1)q^{-r}, (a + h)q^{-r}]$, where $h \in \{1, 2, \ldots\}$, is a subset of $[0, 1]$, then we call it the *h-th interval.* We put S on the first not totally covered h-th interval of length q^{-r} selected in the following order: the $(q + 1)$-th interval, then the q-th interval and so on up to the 2-nd interval, next the $(q + 2)$-nd interval and the successive intervals up to the $2q$-th interval, and finally the 1-st interval. We end the covering process when the whole interval $[0, 1]$ is covered.

It is natural to call this algorithm *the* $(q + 1, \ldots, 2, q + 2, \ldots, 2q, 1)$*-algorithm.* In particular, for $q = 2$ we get the $(3, 2, 4, 1)$-algorithm which tries to put every segment by checking successively the 3-rd, the 2-nd, the 4-th and the 1-st interval of length 2^{-r}.

For the convenience of the reader, who possibly will compare the considerations, the proof of Theorem 1 is organized similarly as the proof of Theorem 1 from [3]. We use analogical notation like this in [3]. In particular, we have three analogical lemmas. Here is a lemma similar to Lemma 1 in [3]. Also the proof is similar, so we omit it.

LEMMA 1. *Let $p < 1$ be a positive multiple of q^{-w}. Assume that the interval $[0, p]$ is not completely covered yet by the $(q + 1, \ldots, 2, q + 2, \ldots, 2q, 1)$-algorithm. For $j \geq 0$ denote by ν_j the number of segments of length q^{-w-j} put to the right of p. Assume that $\nu_0 \geq q - 1, \ldots, \nu_\ell \geq q - 1$ for some $\ell \geq 0$. Then there is at most one number $z \in \{0, \ldots, l\}$ such that a segment of length q^{-w-z} used for the covering contains p. In such a case we have $\nu_j \leq q - 1$ for each $j \in \{0, \ldots, z - 1\}$, we have $q \leq \nu_z \leq 2q - 1$, we have $q - 1 \leq \nu_j \leq 2q - 2$ for every $j > z$, and the interval $[p, p + q^{-w+1}]$ is completely covered.*

For every integer $i > 1$, we denote by b_i the position of the current bottom immediately after putting the first $i - 1$ segments from our sequence. Moreover, let $b_1 = 0$.

LEMMA 2. *Assume that we apply the $(q+1, \ldots, 2, q+2, \ldots, 2q, 1)$-algorithm and that $b_i < b_{i+1} < 1$. Let $\Delta b = b_{i+1} - b_i$ and let Δl be the total length of those among the first i placed segments which have non-empty intersection with (b_i, b_{i+1}). Then*

$$\Delta l < \Big(1 + \frac{1}{q} + \frac{1}{q^2}\Big)\Delta b.$$

Proof. Let w mean the smallest positive integer such that a segment of length q^{-w} has been used for the covering of the interval (b_i, b_{i+1}). Of course,

$$q^{-w} < \Delta b \leq 2q \cdot q^{-w}.$$

We have $\Delta b = \lambda_0 q^{-w}$, where $\lambda_0 \in \{2, \ldots, 2q\}$ or

$$\Delta b = \lambda_0 q^{-w} + \lambda_k q^{-w-k} + \ldots + \lambda_m q^{-w-m},$$

where $\lambda_0 \in \{1, \ldots, 2q\}$, $1 \leq k \leq m$ and $\lambda_k, \ldots, \lambda_m \in \{0, \ldots, q-1\}$ with $\lambda_k \geq 1$, $\lambda_m \geq 1$. Clearly, if $\lambda_0 \geq 2$, then b_{i+1} is a multiple of q^{-w}. By the *last segment* we mean the segment, such that after putting it, the whole interval (b_i, b_{i+1}) is covered. Denote by q^{-t} the length of the last segment put on (b_i, b_{i+1}) and by μ_j the number of segments of length q^{-w-j}, which are different from the last segment and which are used for the covering of the interval (b_i, b_{i+1}). We have $0 \leq \mu_j \leq 2q$. Of course,

$$\Delta l = q^{-t} + \sum_{j=0}^{\infty} \mu_j q^{-w-j}.$$

In Cases 2 and 3 we will consider the smallest multiple p of q^{-w} such that the interval $[0, p]$ is not totally covered after putting all segments besides the last segment. Observe that the last segment is put such that p becomes its right end-point. Since $\lambda_k q^{-w-k} + \ldots + \lambda_m q^{-w-m} < q^{-w-k+1}$, all segments (besides the last segment) of lengths between q^{-w-k+1} and q^{-w} used for the covering of (b_i, b_{i+1}) are put to the right of p.

Figures 1-7 below show some extremal situations in the considered cases and subcases. We present the order in which the segments are put on the interval $[b_i, b_{i+1}]$ by showing them level by level. A lover level means that a segment is put later. I order to fix attention, we always take $q = 3$. The figures show only segments of length at least q^{-w-2} since shorter segments cannot be well drawn here. For clear presentation of the worst situation to the right of p, in Figures 4-7 we have $0 < p - b_i < q^{-w-2}$ despite in general $0 < p - b_i < q^{-w}$.

Case 1, when $\Delta b = s \cdot q^{-w}$ for $s \in \{2, \ldots, 2q\}$. We will show that $\Delta l < (1 + \frac{1}{q})\Delta b$ holds true in Case 1. This inequality is stronger than the inequality announced in the formulation of Lemma 2. Observe that b_i and b_{i+1} are multiples of q^{-w}.

Subcase 1.1, when $s = 2q$. We have $\Delta l < q^{-t} + (2q-1)q^{-w} + (2q-2)q^{-w-1} + (2q-2)q^{-w-2} + \ldots \leq 2q^{-w} + (2q-2)\sum_{j=w+1}^{\infty} q^{-j} = (2q-2)\frac{q}{q-1}2q^{-w} + 2q \cdot q^{-w} = (2q+2)q^{-w} = \left(1 + \frac{1}{q}\right)\Delta b$. In this evaluation we consider at most $2q-2$ segments of each of the lengths $q^{-w-1}, q^{-w-2}, \ldots$, despite it may happen that we put $2q-1$ segments of a specific length q^{-w-c}, where $c \in \{1, 2, \ldots\}$. In such a case we have at least one less (than in the above evaluation) segment of length q^{-w-c+1} and thus the estimate still holds true.

Fig. 1. *A sequence of maximum total length in Subcase 1.1.*

Subcase 1.2, when $s \in \{q+1, \ldots, 2q-1\}$. This time the last segment has length at most q^{-w-1}. We have $\Delta l < q^{-t} + (s-1)q^{-w} + (2q-2)\sum_{j=w+1}^{\infty} q^{-j} \leq q^{-w-1} + (s-1)q^{-w} + (2q-2)\frac{q}{q-1}q^{-w-1} = (s + \frac{q+1}{q})q^{-w} \leq (s + \frac{s}{q})q^{-w} \leq (1 + \frac{1}{q})\Delta b$. Providing this calculation we take into account a similar remark about the coefficients $2q-2$ like in the preceding subcase.

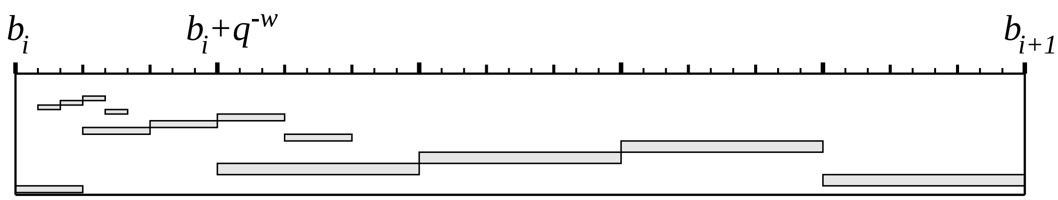

Fig. 2. *A sequence of maximum total length in Subcase 1.2.*

Subcase 1.3, when $s \in \{2, \ldots, q\}$. The situation of this subcase appears when a few segments of length q^{-w} are placed without causing an immediate increase of the current bottom, and later the current bottom grows close to those segments thanks to placing sufficiently many shorter segments. Again the last segment has length at most q^{-w-1} but less segments of length q^{-w-1} can be put to the left of $b_i + q^{-w}$. We obtain $\Delta l < q^{-t} + (s-1)q^{-w} + (q-1)q^{-w-1} + (2q-2)\sum_{j=w+2}^{\infty} q^{-j} \leq q^{-w-1} + s \cdot q^{-w} - q^{-w-1} + (2q-2)\frac{q}{q-1}q^{-w-2} = (s + \frac{2}{q})q^{-w} \leq (s + \frac{s}{q})q^{-w} \leq (1 + \frac{1}{q})\Delta b$. And again we have in mind a similar remark about the coefficients $2q-2$ like in Subcase 1.1.

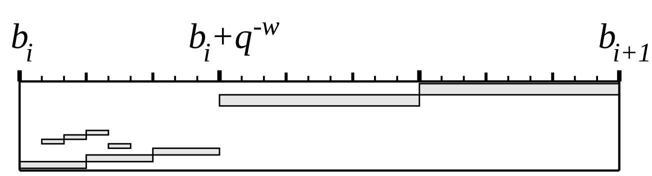

Fig. 3. *A sequence of maximum total length in Subcase 1.3.*

Case 2, when $q \cdot q^{-w} < \Delta b < 2q \cdot q^{-w}$ and when Δb is not a multiple of q^{-w}. Of course, $q \leq \lambda_0 \leq 2q-1$ and $\mu_0 \geq q-1$.

Subcase 2.1, when $\mu_1 \geq q-1, \ldots, \mu_{k-1} \geq q-1$. Assume first that there is $z \in \{1, \ldots, k-1\}$ such that a placed segment of length q^{-w-z}, different from the last segment, contains p. Lemma 1 implies that the sum of length of segments (different from the last segment) of lengths between q^{-w-k+1} and q^{-w} put on (b_i, b_{i+1}) is at most $(\lambda_0-1)q^{-w} + (2q-2)\sum_{j=w+1}^{w+z-1} q^{-j} + (2q-1)q^{-w-z} + (q-1)\sum_{j=w+z+1}^{w+k-1} q^{-j} = (\lambda_0+1)q^{-w} - q^{-w-k+1}$ (we take $z = 1$ in Fig. 4). We applied Lemma 1 since segments of length at most q^{-w-k+1} are placed to the right of p. It may also happen that $\mu_0 = \lambda_0$ and that the interval $[p, p+q^{-w}]$ is covered by a segment of length q^{-w} put "a long time before" the current bottom has arrived up to our present b_i (thus $[p+q^{-w}, p+2q^{-w}]$ is covered later by a segment of length q^{-w} than $[p, p+q^{-w}]$). Then the total length is at most $\lambda_0 q^{-w} + (q-1)q^{-w-1}$.

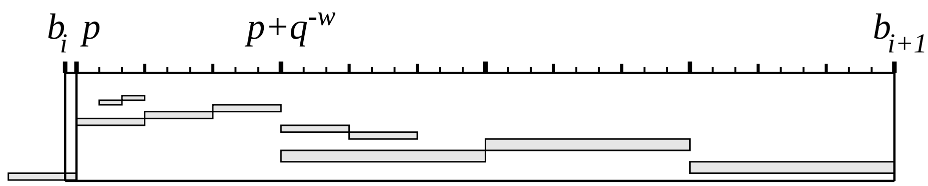

Fig. 4. *A sequence of maximum total length in the first part of Subcase 2.1.*

Now assume that p is not in the segments of lengths $q^{-w-1}, \ldots, q^{-w-k+1}$ different from the last segment used for the covering. The sum of the lengths of the considered segments is at most $\lambda_0 q^{-w} + (q-1)\sum_{j=w+1}^{w+k-1} q^{-j} = (\lambda_0+1)q^{-w} - q^{-w-k+1}$.

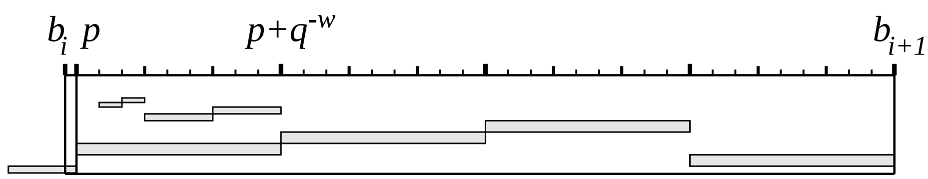

Fig. 5. *A sequence of maximum total length in the second part of Subcase 2.1.*

We obtain that always the sum of lengths of the segments which are different from the last segment put on (b_i, b_{i+1}), and whose lengths are between q^{-w-k+1} and q^{-w}, is at most

$$(\lambda_0+1)q^{-w} - q^{-w-k+1}. \tag{1}$$

Now we estimate the total length of segments of length at most q^{-w-k} different from the last segment put on (b_i, b_{i+1}). The total length of those segments is less than $\sum_{j=w+k}^{w+m-1}(\lambda_{j-w} + q - 1)q^{-j} + (\lambda_m + q - 2)q^{-w-m} + \sum_{j=w+m+1}^{\infty}(2q-2)q^{-j} = \sum_{j=w+k}^{w+m} \lambda_{j-w}q^{-j} + (q-1)\sum_{j=w+k}^{w+m-1} q^{-j} + (q-2)q^{-w-m} + (2q-2)\sum_{j=w+m+1}^{\infty} q^{-j}$, which is less than

$$\sum_{j=w+k}^{w+m} \lambda_{j-w}q^{-j} + q^{-w-k+1}. \tag{2}$$

In the above calculation we see components $(\lambda_j + q - 1)q^{-j}$ despite sometimes up to $2q-1$ segments of length q^{-j} can be put on (b_i, b_{i+1}) during the covering process. But then the estimate (2) holds true as well. Just if between $\lambda_j + q$ and $2q-1$ segments of a specific length q^{-j}, where $j \in \{w+k+1, \ldots, w+m-1\}$, are used for the covering, then one less segment of length q^{-j+1} can be put there because of lack of space. In such a case the total length is even smaller than (2). The reason is that in the calculation we add here up to $q-1$ segments of length q^{-j} and that we subtract one segment of length q^{-j+1}.

From (1) and (2) we conclude that

$$\Delta l < q^{-t} + (\lambda_0 + 1)q^{-w} + \sum_{j=w+k}^{w+m} \lambda_{j-w}q^{-j}.$$

If $\lambda_0 < 2q-1$, then $t \geq w+1$. Thus $\Delta l < \left(\lambda_0 + 1 + \frac{1}{q}\right)q^{-w} + \sum_{j=w+k}^{w+m} \lambda_{j-w}q^{-j}$. This and $q \leq \lambda_0$ imply that $\Delta l < (1 + \frac{1}{q} + \frac{1}{q^2})\Delta b$.

If $\lambda_0 = 2q-1$, then $\Delta l < (2q+1)q^{-w} + \sum_{j=w+k}^{w+m} \lambda_{j-w}q^{-j} < (1 + \frac{1}{q} + \frac{1}{q^2})\Delta b$.

Subcase 2.2, when at least one of the numbers $\mu_1, \ldots, \mu_{k-1}$ is smaller than $q-1$. Let y denote the smallest number from $\{1, \ldots, k-1\}$ such that $\mu_y < q-1$. The present evaluation differs from this in Subcase 2.1 only by a different proof of (1). Now, the total length of segments of lengths between q^{-w-k+1} and q^{-w} used for the covering of the interval (b_i, b_{i+1}) is at most $\lambda_0 q^{-w} + (q-1)\sum_{j=w+1}^{w+y-1} q^{-j} + (q-2)q^{-w-y} + (2q-2)\sum_{j=w+y+1}^{w+z-1} q^{-j} + (2q-1)q^{-w-z} + (q-1)\sum_{j=w+z+1}^{w+k-1} q^{-j}$, where z is defined at the beginning of Case 2.1 (in Fig. 6 we take $y=1$ and $z=2$). Instead of the last three components we may also have $(2q-2)\sum_{j=w+y+1}^{w+k-2} q^{-j} + (2q-1)q^{-w-k+1}$. The components $\lambda_0 q^{-w} + (q-1)\sum_{j=w+1}^{w+y-1} q^{-j}$ stand for the worst possible case and in the remaining cases we take an expression of the form $(\lambda_0 - 1)q^{-w} + (2q-2)\sum_{j=w+1}^{w+v-1} q^{-j} + (2q-1)q^{-w-v} + (q-1)\sum_{j=w+v+1}^{w+y-1} q^{-j}$. In all the variants, the total length of segments is at most $(\lambda_0 + 1)q^{-w} - q^{-w-k+1}$.

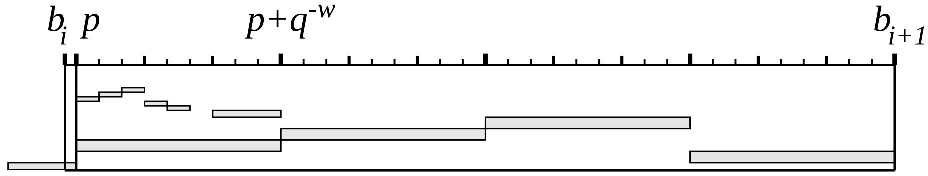

Fig. 6. *A sequence of maximum total length in Subcase 2.2.*

Case 3, when $q^{-w} < \Delta b < q \cdot q^{-w}$ and when Δb is not a multiple of q^{-w}. Similarly like in Subcase 1.3, the situation is a result of placing a number of segments of length q^{-w} with later growing of the current bottom close to those earlier put segments of length q^{-w}. Of course, $\lambda_0 \leq q-1$. From the description of our method we see that the last segment cannot be of length q^{-w}, this is $t \geq w+1$. We have

$$\big(1+\frac{1}{q}+\frac{1}{q^2}\big)\Delta b \geq \lambda_0 q^{-w} + \lambda_0 q^{-w-1} + \lambda_0 q^{-w-2} + \sum_{j=w+k}^{w+m} \lambda_{j-w} q^{-j}. \tag{3}$$

Subcase 3.1, when the interval $[p, p+q^{-w}]$ is not covered by a segment of length q^{-w}. Since at least one segment of length q^{-w} is put on $[b_1, b_{i+1}]$, we have $\lambda_0 \geq 2$.

We evaluate the sum of lengths of the segments put on (b_i, b_{i+1}) whose lengths are between q^{-w-k+1} and q^{-w} like in Case 2, but now one less segment of length q^{-w} and one more segments of length q^{-w-1} should be taken into account (of course, $\lambda_0 \geq q$ in Case 2 and now $\lambda_0 < q$). Thus this sum is not greater than

$$\big(\lambda_0 + \frac{1}{q}\big)q^{-w} - q^{-w-k+1}. \tag{4}$$

Now (4) substitutes (1) from Case 2 and the value of (2) remains unchanged. Considering the sum of (4), (2) and of the length q^{-t} of the last segment we obtain

$$\Delta l < q^{-t} + \lambda_0 q^{-w} + \sum_{j=w+k}^{w+m} \lambda_{j-w} q^{-j} + q^{-w-1}. \tag{5}$$

Since the last segment is of length at most q^{-w-1}, from (3), (5) and from $\lambda_0 \geq 2$ we obtain $\Delta l < (1+\frac{1}{q}+\frac{1}{q^2})\Delta b$.

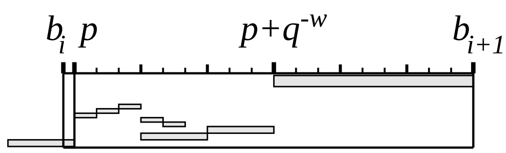

Fig. 7. *A sequence of maximum total length in Subcase 3.1.*

Subcase 3.2, when $[p, p+q^{-w}]$ is covered by a segment of length q^{-w}. We show that

$$\Delta l < q^{-t} + \lambda_0 q^{-w} + \sum_{j=w+k}^{w+m} \lambda_{j-w} q^{-j} + q^{-w-k}. \tag{6}$$

If $b_{i+1} = p + \lambda_0 q^{-w}$, we show (6) analogically like (5). Remember that (5) is the sum of (2), (4) and of q^{-t}. The difference is that now we can put at most λ_k segments of length q^{-w-k}. This lessens (2) by $(q-1)q^{-w-k} = q^{-w-k+1} - q^{-w-k}$ and thus leads to (6).

If $b_{i+1} \neq p + \lambda_0 q^{-w}$, we have $\lambda_0 = 1$ and $\lambda_1 \in \{1, \ldots, q-1\}$. Moreover, $b_{i+1} = p + q^{-w} + uq^{-w-1}$, where $u \in \{1, \ldots, \lambda_1\}$. So the only difference is that u segments of length q^{-w-1} are put to the right of $p + q^{-w-1}$ instead of to the left of p. Consequently, (6) also holds in this special situation.

From (3) and (6) we see that if $\lambda_0 \geq 2$ or if $k \geq 2$, then $\Delta l < (1 + \frac{1}{q} + \frac{1}{q^2})\Delta b$.

It remains to consider the situation when $\lambda_0 = 1$ and $k = 1$. Observe that $t \geq w+2$. Thus $\Delta l \leq q^{-w-2} + q^{-w} + \sum_{j=w+k}^{w+m} \lambda_{j-w} q^{-j} + q^{-w-1}$. Thanks to (3) we obtain $\Delta l < (1 + \frac{1}{q} + \frac{1}{q^2})\Delta b$. ■

LEMMA 3. *Consider the $(q+1, \ldots, 2, q+2, \ldots, 2q, 1)$-algorithm. Let $\Delta b = b_{i+1} - b_i$, where $b_{i+1} = 1$. Let w be the integer for which $q^{-w} < \Delta b \leq q^{-w+1}$. The total length Δl of those among the first $i-1$ segments which have non-empty intersection with $(b_i, 1)$ is less than $\Delta b + q^{-w}$.*

Proof. We consider two cases.

Case 1, when $\Delta b = s \cdot q^{-w}$ for $s \in \{2, \ldots, q\}$. We obtain $\Delta l < (s-1)q^{-w} + (2q-2)\sum_{j=w+1}^{\infty} q^{-j} \leq (s-1)q^{-w} + (2q-2)\frac{q}{q-1}q^{-w-1} = (s+1)q^{-w} \leq \Delta b + q^{-w}$. We take into account a remark about the coefficients $2q-2$ as in Case 1 of the proof of Lemma 2.

Case 2, when Δb is not a multiple of q^{-w}. We have $\Delta b = \lambda_0 q^{-w} + \lambda_k q^{-w-k} + \ldots + \lambda_m q^{-w-m}$, where $\lambda_0 \in \{1, \ldots, q-1\}$, $\lambda_k > 0$ and $\lambda_m > 0$. We provide a similar consideration like at the beginning of Case 3 in the proof of Lemma 2. The difference is that this time we can put a segment of length q^{-w} on the interval $[p, p+q^{-w}]$ provided one less segment of length q^{-w-1} has been put there. Also we do not count the last segment whose length is denoted by q^{-t} in (5). In analogy to (5), we obtain $\Delta l < (\lambda_0 + 1)q^{-w} + \sum_{j=w+k}^{w+m} \lambda_{j-w} q^{-j}$. Thus $\Delta l < \Delta b + q^{-w}$. ■

THEOREM 1. *Let $q \geq 2$ be an integer. Every sequence of segments whose lengths are from the set $\{q^{-1}, q^{-2}, \ldots\}$ and whose total length is at least*

$$1 + \frac{2}{q} - \frac{1}{q^3}$$

permits on-line covering of the interval $[0,1]$ *by the* $(q+1,\dots,2,q+2,\dots,2q,1)$*-algorithm.*

Proof. It is sufficient to show that if a sequence of segments of lengths from the set $\{q^{-1},q^{-2},\dots\}$ does not cover the interval $[0,1]$ by the algorithm, then the total length of the segments in the sequence is less than $1+\frac{2}{q}-\frac{1}{q^3}$. Observe that all segments from such a sequence are used during the covering process.

Case 1, when $b_i=0$ *during the whole covering process.* We apply Lemma 3 with $\Delta b=1$ and $w=1$. We conclude that the total length of segments put during the covering process is less than $1+\frac{1}{q}$. This is less than $1+\frac{2}{q}-\frac{1}{q^3}$ for every $q\geq 2$.

Case 2, when $\lim_{i\to\infty} b_i=1$. From Lemma 2 we see that the total length of segments used for the covering which have non-empty intersection with $[0,b_i]$ is less than $(1+\frac{1}{q}+\frac{1}{q^2})b_i$. Thus the total length of used segments is less than $1+\frac{1}{q}+\frac{1}{q^2}$ which is less than $1+\frac{2}{q}-\frac{1}{q^3}$.

Case 3, when $0<b'<1$, *where* b' *is either* $\lim_{i\to\infty} b_i$, *or* $b'=b_i$ *and* $b_{i+1}=1$. Consider the smallest integer w for which $q^{-w}<1-b'$. From Lemmas 2 and 3 we see that the total length of segments used in the covering process is less than $(1+\frac{1}{q}+\frac{1}{q^2})b'+(1-b')+q^{-w}=1+(\frac{1}{q}+\frac{1}{q^2})b'+q^{-w}\leq 1+(\frac{1}{q}+\frac{1}{q^2})(1-q^{-w})+q^{-w}=1+\frac{1}{q}+\frac{1}{q^2}+(1-\frac{1}{q}-\frac{1}{q^2})q^{-w}$. Thus it less than $1+\frac{1}{q}+\frac{1}{q^2}+(1-\frac{1}{q}-\frac{1}{q^2})q^{-1}=1+\frac{2}{q}-\frac{1}{q^3}$. ■

PROPOSITION. *Let* $q\geq 2$ *be an integer. Assume that an on-line* q*-adic covering of the interval* $[0,1]$ *is provided by the* $(q+1,\dots,2,q+2,\dots,2q,1)$*-algorithm up to the total covering of this interval. Then the total length of the used segments is less than*

$$1+\frac{3}{q}-\frac{1}{q^3}.$$

Proof. Assume that $b_i<1$ and $b_{i+1}=1$. Let w be smallest integer w such that $q^{-w}<1-b_i$. Of course, the segment which finally makes the whole interval $[0,1]$ totally covered has length at most q^{-1}. This observation and Lemmas 2 and 3 imply that the total length of segments used in the covering process is less than $(1+\frac{1}{q}+\frac{1}{q^2})b'+(1-b')+q^{-w}+q^{-1}$. This number is smaller than $1+\frac{3}{q}-\frac{1}{q^3}$ (see the calculation in Case 3 of the proof of Theorem 1). ■

If no segment is put yet on a q-adic interval A up to a moment of a covering process, we call A *empty* at this moment (despite possibly it end-points are covered). If all points of A are covered, we call A *totally covered* at this moment. If A is not empty and not totally covered, we call it *partially covered* at this moment.

LEMMA 4. *Assume that a process of the covering of the interval* $[0,1]$ *by segments according to the* $(q+1,\ldots,2,q+2,\ldots,2q,1)$*-algorithm is not finished yet and that the current bottom has arrived at least to a point* $(h-1)q^{-1}$*, where* $h \in \{2,\ldots,q-1\}$*. Then during the covering process there is a moment at which* (i) *exactly* $h-1$ *or* h *from the* q*-adic intervals of length* q^{-1} *are totally covered by segments of length at most* q^{-2} *and no* q*-adic interval of length* q^{-1} *is partially covered, or there is a moment at which* (ii) *exactly* $h-1$ *from the* q*-adic intervals of length* q^{-1} *are totally covered by segments of length at most* q^{-2} *and one or two* q*-adic intervals of length* q^{-1} *are partially covered (if two, then the second is covered only by segments of length* q^{-2}*).*

Proof. We look at the first moment (if any) before the end of the covering process, when the current bottom attains a value b at least $(h-1)q^{-1}$.

In order to fix our attention, we start from taking into consideration a covering process during which only segments of length at most q^{-2} are given to us.

First assume that $(h-1)q^{-1} \le b \le hq^{-1}$. Of course, the interval $[(h-1)q^{-1}, hq^{-1}]$ is not totally covered before the current bottom attains b. Thus from the description of the $(q+1,\ldots,2,q+2,\ldots,2q,1)$-algorithm we conclude that no segment of length at most q^{-2} is put to the right of hq^{-1} (if the current bottom is below $(h-1)q^{-1}$, then a segment can be placed to the right of hq^{-1} only if the interval $[(h-1)q^{-1}, hq^{-1}]$ is totally covered). We conclude that if $b < hq^{-1}$, then the first $h-1$ from the q-adic intervals of length q^{-1} are totally covered, the interval $[(h-1)q^{-1}, hq^{-1}]$ is empty or partially covered, and the remaining q-adic intervals of length q^{-1} are empty. We have (i), or we have (ii) with one partially covered interval of length q^{-w}. Of course, if $b = hq^{-1}$, then (i) holds true.

Now assume that $b > hq^{-1}$. As a result of placing one segment, the current bottom changes from a value $b^* < (h-1)q^{-1}$ to $b > hq^{-1}$. According to the description of our algorithm, this is possible only if the interval $[(h-1)q^{-1}, hq^{-1}]$ is totally covered. Thus, at the moment when the current bottom is at b^*, we have exactly $h-1$ intervals of length q^{-1} totally covered (by segments of length at most q^{-2}) and two such intervals partially covered. The second interval is covered only by segments of length q^{-2}. Hence (ii) is fulfilled.

If also segments of length q^{-1} are given to us, they are put successively from the right to the left on the interval $[0,1]$. It is clear that if they are put to the right of $(h+1)q^{-1}$, they do not influence on placing segments of lengths at most q^{-2} considered earlier. Observe that if a segment of length q^{-1} is put on the interval $[hq^{-1}, (h+1)q^{-1}]$ before the current bottom arrives to b, then the current bottom has no chance to attain $(h-1)q^{-1}$ before the end of the covering process and thus this situation cannot happen in our lemma. ∎

Here is the *two-stage* $(q+1,\dots,2,q+2,\dots,2q,1)$*-algorithm*. Let $h \in \{2,\dots,q-1\}$. In the first stage of the covering process we apply the $(q+1,\dots,2,q+2,\dots,2q,1)$-algorithm. If we reach the first moment described in Lemma 4, we pass immediately to the second stage. At the beginning of the second stage, applying the $(q+1,\dots,2,q+2,\dots,2q,1)$-algorithm, we put all segments of length at most q^{-2} only on the first not totally covered q-adic interval of length q^{-1} considered now as the only interval for covering by segments of length at most q^{-2}. When this interval becomes totally covered, by the $(q+1,\dots,2,q+2,\dots,2q,1)$-algorithm we put all segments of length at most q^{-2} on the next not totally covered q-adic interval of length q^{-1} considered now as the only interval for our covering process. We proceed analogically taking succeeding intervals of length q^{-1}. If in meantime we obtain segments of length q^{-1}, we put them on q-adic intervals of length q^{-1} starting from $[(q-1)q^{-1},1]$ and then proceeding one by one to the left.

Observe that the idea of the improvement in this algorithm is in avoiding the situation which may happen, if the original $(q+1,\dots,2,q+2,\dots,2q,1)$-algorithm is applied, when a segment of length q^{-1} is put on an "almost totally covered" q-adic interval of length q^{-1} and when simultaneously not many empty q-adic intervals of length q^{-1} are covered by segments of length q^{-1} during the covering process. The price payed for the introduced improvement is a weaker effectiveness in the second stage of our algorithm (just Proposition is applied instead of Lemma 2). A calculation shows that $h = \lceil \frac{2}{3}q \rceil$ optimizes the choice of the moment at which we decide to pass to the second stage.

THEOREM 2. *Let $q \geq 3$ be an integer. Every sequence of segments whose lengths are from the set $\{q^{-1}, q^{-2}, \dots\}$ and whose total length is at least*

$$1 + \frac{5}{3} \cdot \frac{1}{q} + \frac{5}{3} \cdot \frac{1}{q^2}$$

permits an on-line covering of the interval $[0,1]$ by the two-stage $(q+1,\dots,2,q+2,\dots,2q,1)$-algorithm with $h = \lceil \frac{2}{3}q \rceil$.

Proof. Since $q \geq 3$, the requirement $2 \leq h \leq q-1$ of Lemma 4 and of the description of our algorithm is fulfilled. We present $h = \lceil \frac{2}{3}q \rceil$ in the form $\frac{2}{3}q$ provided $q = 3c$, where c is a positive integer, in the form $h = \frac{2}{3}q + \frac{1}{3}$ for $q = 3c+1$, and in the form $\frac{2}{3}q + \frac{2}{3}$ for $q = 3c+2$.

Case 1, when the current bottom is below $(h-1)q^{-1}$ always before the end of the covering process. We will show that each sequence of segments of the total length at least

$$1 + \left(1 + \frac{h}{q}\right)\frac{1}{q} + \frac{h}{q} \cdot \frac{1}{q^2} \tag{7}$$

permits the covering of the interval $[0,1]$. Assume the opposite. Then there is a sequence of segments of the total length at least (7) which does not cover $[0,1]$ by our algorithm. Let b' denote the supremum of the values different from 1 accepted by the current bottom during the covering process. From Lemma 2 we conclude that the total length of segments which have non-empty intersection with the interval $(0,b')$ is less than $(1+\frac{1}{q}+\frac{1}{q^2})b'$. From Lemma 3 we see that the total length of segments (different from the segment finishing the process) which have non-empty intersection with the interval $(b',1)$ is less than $1-b'+q^{-1}$. Providing an evaluation like in Case 3 of the proof of Theorem 1 and taking into account the inequality $b'<\frac{h}{q}$ we see that the total length of segments in our sequence is smaller than (7). This contradiction confirms that every sequence of segments of the total length at least (7) permits the covering of the interval $[0,1]$ in Case 1. Substituting $h=\frac{2}{3}q$ in (7), we obtain the estimate $1+\frac{5}{3}\cdot\frac{1}{q}+\frac{2}{3}\cdot\frac{1}{q^2}$. Analogically, for $h=\frac{2}{3}q+\frac{1}{3}$ we obtain $1+\frac{5}{3}\cdot\frac{1}{q}+\frac{1}{q^2}+\frac{1}{3}\cdot\frac{1}{q^3}$, and for $h=\frac{2}{3}q+\frac{2}{3}$ we obtain $1+\frac{5}{3}\cdot\frac{1}{q}+\frac{4}{3}\cdot\frac{1}{q^2}+\frac{2}{3}\cdot\frac{1}{q^3}$.

Case 2, when the current bottom attains at least $(h-1)q^{-1}$ before the end of the covering process. According to Lemma 4 and to the description of the algorithm, when we pass to the second stage, either (i) or (ii) holds true. We will assume (ii) with the exception of one sentence at the end of Subcase 2.1 where we take care about the possibility (i).

Assume that we have two partially covered intervals (if we have only one, then we can take the first empty q-adic interval of length q^{-1} in the part of the second partially covered interval). Denote by T the more right of our two partially covered intervals.

Subcase 2.1, when T is not covered by a segment of length q^{-1} during the covering process. We apply Lemma 2. We also apply Proposition and Theorem 1 but for the q times lessened image of the original situation. They are just applied for the process of the covering of separate q-adic intervals of length q^{-1} by q-adic segments of length at most q^{-2}. This explains the factors $\frac{1}{q}$ in the following estimate: $(1+\frac{1}{q}+\frac{1}{q^2})\frac{h-1}{q}+(q-h)(1+\frac{3}{q}-\frac{1}{q^3})\frac{1}{q}+(1+\frac{2}{q}-\frac{1}{q^3})\frac{1}{q}$. Consequently, the interval $[0,1]$ can be covered if the total length of segments in a sequence is at least

$$1+\big(3-2\cdot\frac{h}{q}\big)\cdot\frac{1}{q}+\big(1+\frac{h}{q}\big)\cdot\frac{1}{q^2}+\big(-2+\frac{h}{q}\big)\cdot\frac{1}{q^3}-\frac{1}{q^4}. \tag{8}$$

Substituting $h=\frac{2}{3}q$ in (8), we obtain the estimate $1+\frac{5}{3}\cdot\frac{1}{q}+\frac{5}{3}\cdot\frac{1}{q^2}-\frac{4}{3}\cdot\frac{1}{q^3}-\frac{1}{q^4}$. Similarly, for $h=\frac{2}{3}q+\frac{1}{3}$, we get the estimate $1+\frac{5}{3}\cdot\frac{1}{q}+\frac{1}{q^2}-\frac{1}{q^3}-\frac{2}{3}\cdot\frac{1}{q^4}$, and for $h=\frac{2}{3}q+\frac{2}{3}$ we obtain $1+\frac{5}{3}\cdot\frac{1}{q}+\frac{1}{3}\cdot\frac{1}{q^2}-\frac{2}{3}\cdot\frac{1}{q^3}-\frac{1}{3}\cdot\frac{1}{q^4}$.

If (i) holds true with h totally covered intervals, then in place of (8) we have $(1+\frac{1}{q}+\frac{1}{q^2})\frac{h}{q}+(q-h-1)(1+\frac{3}{q}-\frac{1}{q^3})\frac{1}{q}+(1+\frac{2}{q}-\frac{1}{q^3})\frac{1}{q}$ which is smaller by $2\cdot\frac{1}{q^2}-\frac{1}{q^3}-\frac{1}{q^4}$ than (8), and in the case of $h-1$ totally covered intervals in (i) we get even a smaller value.

Subcase 2.2, when T is covered by a segment of length q^{-1} during the covering process. From Lemma 4, from the description of the two-stage algorithm and from the assumption of our subcase we see that before a segment of length q^{-1} covers T, only segments of length q^{-2} are put on T. Of course, the number of them is at most $q-1$. We take this into account when provide a calculation similar to that from Subcase 1.1. We conclude that the interval $[0,1]$ can be covered if the total length of segments in a sequence is at least $(1+\frac{1}{q}+\frac{1}{q^2})\frac{h-1}{q}+(q-1)\frac{1}{q^2}+(q-h)\frac{1}{q}+(1+\frac{2}{q}-\frac{1}{q^3})\frac{1}{q} = 1+\left(1+\frac{h}{q}\right)\cdot\frac{1}{q}+\frac{h}{q}\cdot\frac{1}{q^2}-\frac{1}{q^3}-\frac{1}{q^4}$. Since this value is smaller than (7), we can disregard Subcase 2.2 in further calculations.

Comparing (7) and (8) (or rather the three pairs of corresponding particular estimates resulting from (7) and (8)) we see that if q has the form $3c$, then we obtain the estimate

$$1+\frac{5}{3}\cdot\frac{1}{q}+\frac{5}{3}\cdot\frac{1}{q^2}-\frac{4}{3}\cdot\frac{1}{q^3}-\frac{1}{q^4}. \tag{9}$$

Analogically, if q has the form $3c+1$, we get the estimate

$$1+\frac{5}{3}\cdot\frac{1}{q}+\frac{1}{q^2}+\frac{1}{q^3} \tag{10}$$

and if q has the form $3c+2$, we obtain

$$1+\frac{5}{3}\cdot\frac{1}{q}+\frac{4}{3}\cdot\frac{1}{q^2}+\frac{2}{3}\cdot\frac{1}{q^3}. \tag{11}$$

Of course, (9)-(11) are smaller than $1+\frac{5}{3}\cdot\frac{1}{q}+\frac{5}{3}\cdot\frac{1}{q^2}$ for every $q \geq 3$. ∎

The formulas (9)-(11) are more precise than the simple formula in Theorem 2. They give a better estimate than Theorem 1 for $q \geq 6$.

We can improve the two-stage $(q+1,\ldots,2,q+2,\ldots,2q,1)$-algorithm by applying the two-stage approach additionally for the covering of some q-adic intervals of length q^{-2}. We apply our two-stage algorithm with an $h=h_1 \in \{2,\ldots,q-1\}$. The difference is that in the second stage, for the covering of the q-adic intervals of length q^{-2} we apply the q-times lessened variant of the two-stage $(q+1,\ldots,2,q+2,\ldots,2q,1)$-algorithm (with $h_2 = \lceil\frac{2}{3}q\rceil$) instead of the $(q+1,\ldots,2,q+2,\ldots,2q,1)$-algorithm. Just an interval of length q^{-2} is q times shorter than the interval of length q^{-1}, and we are putting q times shorter segments (now they are of lengths $q^{-2},q^{-3},\ldots$ instead of lengths $q^{-1},q^{-2},\ldots$).

Let us estimate the effectiveness of the above algorithm in analogical way as in the proof of Theorem 2. Again we have two cases.

The first case is when the current bottom is below $(h-1)q^{-1}$ always before the end of the covering process. We repeat the considerations of Case 1 of Theorem 2. We conclude that every sequence of segments of the total length at least (7) permits the covering.

The second case is when the current bottom attains at least $(h-1)q^{-1}$ before the end of the covering process. Again we apply Lemma 4 and we consider two subcases analogical to Subcases 2.1 and 2.2 of the proof of Theorem 2. In the first subcase we apply Lemma 2, Theorem 2 and a modification of Proposition related to Theorem 2 (instead to Theorem 1). We provide an analogical calculation like in Subcase 2.1 of the proof of Theorem 2: $(1+\frac{1}{q}+\frac{1}{q^2})\frac{h-1}{q}+(q-h)(1+\frac{8}{3}\cdot\frac{1}{q}+\frac{5}{3}\cdot\frac{1}{q^2})\frac{1}{q}+(1+\frac{5}{3}\cdot\frac{1}{q}+\frac{5}{3}\cdot\frac{1}{q^2})\frac{1}{q}$. We see that the interval $[0,1]$ can be covered if the total length of segments in a sequence is at least

$$1+\big(\frac{8}{3}-\frac{5}{3}\cdot\frac{h}{q}\big)\cdot\frac{1}{q}+\big(\frac{7}{3}-\frac{2}{3}\cdot\frac{h}{q}\big)\cdot\frac{1}{q^2}+\frac{2}{3}\cdot\frac{1}{q^3}. \tag{12}$$

In the second subcase we again obtain a slightly better estimate than in the first case.

For every specific $q\geq 3$ we are looking for the best choice of h_1 in the part of h such that the greater from the values (7) and (12) is minimized. When we substitute $h_1=\lceil\frac{5}{8}q\rceil$ for h in (7) and in (12), then they both become at most $1+\frac{13}{8}\cdot\frac{1}{q}$ plus a constant times $\frac{1}{q^2}$. We see that the component $\frac{5}{3}\cdot\frac{1}{q}$ taking place in Theorem 2 is now lessened to $\frac{13}{8}\cdot\frac{1}{q}$.

We can still improve the algorithm by applying the two-stage approach to shorter q-adic intervals. We omit here a calculation which shows that a proper application of this method lessens the crucial component to $\frac{34}{21}\cdot\frac{1}{q}$ when also q-adic intervals of length q^{-3} are covered in two stages, and to $\frac{89}{55}\cdot\frac{1}{q}$ when additionally the q-adic intervals of length q^{-4} are covered in two stages. An evaluation shows that the sequence $2,\frac{5}{3},\frac{13}{8},\frac{34}{21},\frac{89}{55},\dots$ of our factors tends to $\frac{1}{2}(1+\sqrt{5})=1.61803\dots$.

Each on-line 2^d-adic algorithm which permits a covering of the unit interval by sequences of segments of total length l induces an on-line algorithm which permits a covering of the unit cube of E^d by every sequence of cubes of total volume $2^d l$. This construction invented in [5] is described in Part 3 of [3] and in Part 6.2 of [7]. Thus Theorem 2 implies the following result.

THEOREM 3. *Every sequence of cubes of sides at most 1 in E^d whose total volume is at least*

$$2^d+\frac{5}{3}+\frac{5}{3}\cdot 2^{-d}$$

permits an on-line covering of the unit cube of E^d.

We see that the assumption about the total volume of a sequence of cubes is improved from almost 2^d+3 in [3] to slightly over $2^d+\frac{5}{3}$. Despite of the on-line restriction, this value is very close to the best possible non-on-line estimate 2^d-1 (see [2]). In particular, the estimate for the three dimensional case is lessened from $10.657\dots$ to $9.875\dots$. But if we apply (11), which is more precise for $q = 8$ than the estimate in the formulation of Theorem 2, we get a further improvement up to $9.843\ldots$.

References

1. J. Csirik and G.J. Woeginger, On-line packing and covering problems, *Lect. Notes Comput. Sc.* **1442** (1998), pp. 147-177.
2. H. Groemer, Covering and packing by sequences of convex sets, *Discrete Geometry and Convexity, Annals of the New York Academy of Sciences*, Vol. **440** (1985), 262–278.
3. J. Januszewski, M. Lassak, G. Rote and G. Woeginger, On-line q-adic covering by the method of the n-th segment and its application to on-line covering by cubes, *Beitr. Algebra Geom.* **37** (1996), 51-65.
4. J. Januszewski, M. Lassak, G. Rote and G. Woeginger, Solution of Problem 74, *Math. Semesterber.* **43** (1996), 94-100.
5. W. Kuperberg, On-line covering a cube by a sequence of cubes, *Discrete Comput. Geom.* **12** (1994), 83-90.
6. W. Kuperberg, Problem 74: Ein Intervallüberdeckungsspiel, *Math. Semesterber.* **41** (1994), 207–210.
7. M. Lassak, A survey of algorithms for on-line packing and covering by sequences of convex bodies, *Bolyai Society Mathematical Studies* **6** (1997), pp. 129-157.