Vapnik-Chervonenkis Dimension and (Pseudo-)Hyperplane Arrangements

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Abstract

An arrangement of oriented pseudohyperplanes in Euclidean d-space defines on the set X of pseudohyperplanes a set system (or range space) (X,R), $R \subseteq 2^X$ of VC-dimension d in a natural way: to every cell c in the arrangement assign the subset of pseudohyperplanes having c on their positive side, and let R be the collection of all these subsets. We investigate and characterize the range spaces corresponding to *simple* arrangements of pseudohyperplanes in this way; such range spaces are called *pseudogeometric*, and they have the property that the cardinality of R is maximum for the given VC-dimension. In general, such range spaces are called *complete*, and we show that the number of ranges $r \in R$ for which also $X - r \in R$, determines whether a complete range space is pseudogeometric. Two other characterizations go via a simple duality concept and 'small' subspaces. The correspondence to arrangements is obtained indirectly via a new characterization of uniform oriented matroids: a range space (X,R) naturally corresponds to a uniform oriented matroid of rank $|X|-d$ if and only if its VC-dimension is d, $r \in R$ implies $X - r \in R$ and $|R|$ is maximum under these conditions.
Keywords: VC-dimension, hyperplane arrangements, oriented matroids, pseudohyperplane arrangements.

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1 Introduction and statement of results

Set systems of finite VC-dimension have been investigated since the early seventies (starting with [She], [Sau], and [VC1]), and the concept has found numerous applications in statistics [Dud1, Vap, Ale, GZ, Dud2, Pol1, Tal, Pol2], combinatorics [Ass, DSW, Hau, KPW, MWW], learning theory [BEHW, Flo] and computational geometry [CM, CW, HW, Mat]. Although the VC-dimension is a purely combinatorial parameter associated with a set system, it seems that it is mainly applicable to (and naturally occurs in) geometric settings, i.e. when the set system (X, R) is obtained with X as a set of points in d-space, and with R containing the intersections of X with certain *ranges* in d-space (hyperplanes, halfspaces, balls, simplices, etc.). That is why we use the terms *range space* for (X, R), and *range* for a set in R.

The goal of this paper is to elaborate on this connection to geometry, in particular to arrangements of (oriented) hyperplanes. We will succeed in characterizing those range spaces – called *pseudogeometric* range spaces – which come from hyperplanes, but we have to respect the usual frontiers of such combinatorial characterizations (pseudolines [Lev, Grü2, GP2, Rin], circular sequences [Per, GP1], oriented matroids [BL, FL, EM, BLSWZ]): we cannot distinguish between stretchable and non-stretchable pseudoline (or pseudohyperplane) arrangements, so our analogy is actually to simple pseudohyperplane arrangements.

Intuitively speaking, arrangements of pseudohyperplanes consist of 'topological' hyperplanes with the same intersection properties as straight hyperplanes, so they differ from the usual arrangements only with respect to the geometric notion of straightness that is not 'recognized' by combinatorial structures like range spaces.

A key concept in our approach is to exploit the structure of range spaces induced by maximality conditions on the number of ranges; an interesting new insight we have to offer in this context is the fact that in order to tell whether a range space (X, R) is pseudogeometric, it suffices to count the number of ranges $r \in R$ for which the complement $X - r$ is in R; this characterization presumes that (X, R) is *complete*, i.e. $|R|$ is maximum for the VC-dimension of (X, R). This is also the basis of another characterization where we show that it suffices to consider 'small' subspaces to decide upon the pseudogeometric nature of the range space.

When we consider range spaces where $|R|$ is maximum under the additional restriction that R is *closed*, i.e. $r \in R$ implies $X - r \in R$, then this class on the one hand has a very close relation to the pseudogeometric range spaces and on the other hand is already powerful enough to encode uniform oriented matroids; these combinatorial objects are known to have topological representations as arrangements of pseudohyperplanes in projective space, and they will form the 'bridge' between pseudogeometric range spaces and the Euclidean arrangements of pseudohyperplanes.

We want to avoid to introduce arrangements of pseudohyperplanes in this paper; this, however, raises the problem of properly defining pseudogeometric range spaces. Our approach will be to extract just one intuitive property that one 'expects' these arrangements to have, and use it for the definition. Only at the end of the paper we will justify this proceeding by relating the range spaces obtained in this way to oriented matroids. This has the advantage that the paper presents itself at a completely combinatorial level; the way to get from oriented matroids to actual arrangements and vice versa is not part of it but can be found elsewhere. The reader familiar with oriented matroid terminology might discover a certain coincidence with concepts introduced here. For the benefit of the unacquainted reader, however, we will avoid to refer to this terminology and rather develop the theory from scratch using the range space language which we feel to be more appealing for a first encounter with the subject.

In the rest of this section we will formally introduce the crucial concepts and state our results. Proofs and the introduction of further (mainly technical) tools are postponed to the rest of the paper.

Range spaces, VC-dimension, and the fundamental lemma. We start by reviewing the basic definitions and facts about VC-dimension. We will use the term 'range space' rather than 'set system' or 'hypergraph', because of the motivating examples and in order to distinguish from the graphs we use as tools.

Definition 1 *A* range space *is a pair $S = (X, R)$, with X a set and $R \subseteq 2^X$. The elements in X are called* elements *of S, and the sets in R are called* ranges*. S is called* finite*, if X is finite.*

For $Y \subseteq X$, the restriction of S to Y *is defined by $S|_Y = (Y, R|_Y)$, $R|_Y := \{r \cap Y \mid r \in R\}$. We say that Y is* shattered by R *if $R|_Y = 2^Y$.*

The VC-dimension *of S, denoted by* $\dim(S)$*, is the maximum cardinality of a set $Y \subseteq X$ shattered by R; if R is empty, then we define the VC-dimension to be -1.*

For example, if X is a set of real numbers, and the set R of ranges is determined by intersecting X with intervals, then no three-element set is shattered: we can never 'cut out' the smallest and largest out of three numbers by an interval. Since any two number set can be shattered, the VC-dimension of this range space is two. Many more examples can be obtained via geometric ranges, some of which we will meet shortly.

Obviously, the number of intervals defined on n real numbers is quadratic in n. The following lemma shows that this follows also from the fact that the range space has VC-dimension two. The lemma can be seen as the fundamental lemma and the starting point of investigations of VC-dimension, and it was proved independently (and with different motivations) by Shelah [She], Sauer [Sau] (answering a question of Erdős), and Vapnik and Chervonenkis [VC1]. Although this lemma (and some notions we will use in the sequel) can be formulated for infinite range spaces as well, we will restrict our attention to the finite case, which is the one occurring in our application. Therefore, in all subsequent considerations any range space is assumed to be finite.

In the following we will use the integer function

$$\Phi_d(n) = \binom{n}{\le d} := \sum_{i=0}^{d} \binom{n}{i}$$

for $d \geq -1$ and $n \geq 0$. Φ is additive in the following sense:

Fact 2

$$\Phi_d(n) = \Phi_d(n-1) + \Phi_{d-1}(n-1), \text{ for } d \geq 0, n \geq 1.$$

Lemma 3 *Let (X, R) be a range space of VC-dimension d. Then $|R| \leq \Phi_d(|X|)$.*

To see that the bound is tight, let X be a finite set and let R be the set of all subsets of X with at most d elements. Clearly, the resulting range space has VC-dimension d, and indeed $|R|$ attains the upper bound of the lemma. The above example with intervals is another example for VC-dimension two where the upper bound in Lemma 3 is attained. An interesting implication is that for fixed d, $|R|$ is only polynomial (namely of the order $O(n^d)$) rather than exponential.

Complete range spaces and range spaces from halfspaces. This paper concentrates on range spaces for which the upper bound in Lemma 3 is attained with equality:

Definition 4 *A range space (X, R) of VC-dimension d is called* complete *if $|R|$ equals $\Phi_d(|X|)$.*

An interesting instance of a complete space can be derived from an arrangement of hyperplanes. Let X be a set of n hyperplanes in d-space and let $\mathcal{A}(X)$ denote the arrangement formed by the hyperplanes. We assume X to be in general position, i.e. any d hyperplanes meet in a unique vertex, and any $d+1$ have empty intersection. Suppose that for every hyperplane one of the two halfspaces is distinguished as positive. Then each cell (or d-face) c of $\mathcal{A}(X)$ can be labeled with a subset of X, namely the set of hyperplanes which have c in its positive halfspace (Figure 1). If R denotes the set of all cell labels, then $S = (X, R)$ is called the *description of cells* of $\mathcal{A}(X)$ [Ass, Dud1] and is complete of VC-dimension d. This follows from the well-known fact that number of cells of $\mathcal{A}(X)$ is exactly $\Phi_d(n)$ [Grü1, Zas, Ede].

A range space which stems from a set of oriented hyperplanes (or equivalently, from an arrangement of halfspaces) in this way is called *geometric*.

Pseudogeometric range spaces. A key step in many inductive proofs for arrangements of hyperplanes is to consider the arrangement obtained by removing one of the hyperplanes and the arrangement (of one dimension smaller) obtained as the intersection of one of the hyperplanes with the remaining hyperplanes. We want corresponding operations for our range spaces. For a geometric range space, removing a hyperplane just means to remove its label from every range. For the other

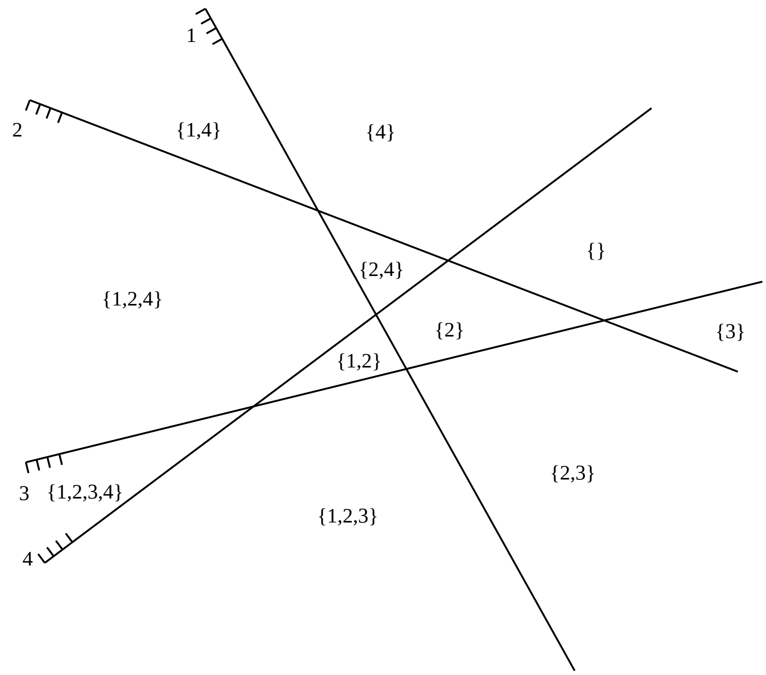

Figure 1: Description of cells of an oriented hyperplane arrangement

operation, observe that every $(d-1)$-face on a hyperplane x corresponds to two adjacent cells whose label sets differ exactly by x. That is, in the corresponding range space those adjacent cells give rise to pairs of ranges $(r, r \cup \{x\})$. This motivates the following definition for a general range space.

Definition 5 *For a range space $S = (X, R)$ and $x \in X$, we define*

$$S - \{x\} = (X - \{x\}, R - \{x\}), \text{ where } R - \{x\} := \{r - \{x\} \mid r \in R\}$$

and

$$S^{\{x\}} = (X - \{x\}, R^{\{x\}}), \text{ where } R^{\{x\}} := \{r \in R \mid x \notin r, r \cup \{x\} \in R\}.$$

Since the pairs of ranges which differ in exactly one element seem to be crucial for the structure of a range space, we look at the collection of such pairs which yields a graph on the ranges. (We denote by $A \triangle B$ the symmetric difference of sets A and B.)

Definition 6 *For a range space $S = (X, R)$, the* distance-1-graph $D^1(S)$ of S *is the undirected graph on vertex set R with edge set*

$$E := \{\{r, r'\} \subseteq R \mid |r \triangle r'| = 1\},$$

where edge $\{r, r'\}$ is labeled with the unique element in $r \triangle r'$.

Let us consider a range space obtained from a 1-dimensional arrangement of hyperplanes, i.e. a set of points on a line. Then the resulting VC-dimension is one, and it is easy to see that the distance-1-graph is simply a path (connecting the cells in the order as they appear on the line). In general, we get the following nice property, proved e.g. in [Dud3, AHW].

Lemma 7 *If* $S = (X, R)$ *is a complete range space of VC-dimension* 1*, then* $D^1(S)$ *is a tree, and each* $x \in X$ *occurs exactly once as an edge label of* $D^1(S)$*.*

So there is a natural one-to-one correspondence between trees and complete range spaces of VC-dimension one. It is quite easy to see that whenever the distance-1-graph is a path, then the range space is geometric (and vice versa). Consequently, geometric range spaces of VC-dimension one are completely characterized.

In order to carry this characterization to higher VC-dimension, we should at least require that in a geometric range space (X, R) the subspace $R^{\{x\}}$ (coming from the subarrangement on the hyperplane x) is geometric for all $x \in X$, and apply this property recursively until we reach the just settled one-dimensional case. This should also make sense if the arrangement in question actually consists of pseudohyperplanes (which coincide with hyperplanes in the one-dimensional case); based on this property we will define pseudogeometric range spaces. As mentioned above, the question whether the following definition really describes the range spaces coming from arrangements of pseudohyperplanes, will become an issue only in the last section. For the time being it suffices to have a formal definition we can work with, along with the intuition that it describes arrangements.

Definition 8 *A complete range space* $S = (X, R)$ *of VC-dimension* d *is called* pseudogeometric *if either*

(i) $d \leq 0$*, or*
(ii) $d = 1$ *and* $D^1(S)$ *is a path, or*
(iii) $d \geq 2$ *and* $S^{\{x\}}$ *is pseudogeometric for all* $x \in X$*.*

It is interesting to observe that the first example of a complete range space we had (take as ranges all sets of up to d elements) is as non-geometric as possible. For example, for $d = 1$ this gives a range space where the distance-1-graph is a star.

We will now proceed by exhibiting (probably easier to grasp) equivalent conditions for a complete range space to be pseudogeometric. While the necessity of these conditions will be quite obvious (from the geometric intuition), it is somewhat surprising that they are already sufficient.

Duality and characterization via small subspaces.

Definition 9 *For a range space* $S = (X, R)$ *the* complementary dual $-S$ *of* S *is defined as*

$$-S = (X, -R), \text{ where } -R := 2^X - R \ .$$

We will prove that the complementary dual of a complete range space of VC-dimension d with n elements is again complete of VC-dimension $n-d-1$. Similarly, we get for pseudogeometric range spaces:

Theorem 10 *A range space is pseudogeometric if and only if its complementary dual is pseudogeometric.*

In particular, this implies that if $S = (X, R)$ is pseudogeometric of VC-dimension d, and $|X| = d+2$, then $-S$ is pseudogeometric of VC-dimension 1 and so its structure is completely determined, which – vice versa – implies that the structure of S is completely determined (we will be more specific about this later). This is the range space version of the fact that – with respect to combinatorial type – there is only one simple d-dimensional arrangement of $d+2$ (pseudo-)hyperplanes.

We can also prove that for determining whether a complete range space of VC-dimension d is pseudogeometric, it suffices to look at all the $(d+2)$-element subspaces. This can be summarized to give

Theorem 11 *Let $S = (X, R)$ be complete of VC-dimension d. The following statements are equivalent:*

(i) S is pseudogeometric.
(ii) $S|_Y$ is pseudogeometric for all $Y \subseteq X$, $|Y| = d+2$.
(iii) $S|_Y$ is geometric for all $Y \subseteq X$, $|Y| = d+2$.

Characterization via cardinality of boundary. The number of unbounded cells in a simple hyperplane arrangement of n hyperplanes in d-space is $2\Phi_{d-1}(n-1)$. This can easily be seen by choosing one of the hyperplanes, call it h, and considering two hyperplanes parallel to h on either side, sufficiently far away so that all unbounded (and only unbounded) cells are intersected. In terms of the corresponding range space, the labels associated with these unbounded cells are those where also the complementary label appears.

Definition 12 *For a range space $S = (X, R)$ the* (complementary) boundary *is defined as*

$$\partial S = (X, \partial R), \text{ where } \partial R := \{r \in R \,|\, X - r \in R\}.$$

Similar as in Lemma 3 we can prove an upper bound for $|\partial R|$, namely $|\partial R| \leq 2\Phi_{d-1}(n-1)$ for a range space (X, R) with $|X| = n$ and $\dim(X, R) = d$. Again simple hyperplane arrangements give rise to range spaces which attain this bound, and actually we get:

Theorem 13 *A complete range space (X, R) of VC-dimension $d \geq 1$ is pseudogeometric if and only if $|\partial R| = 2\Phi_{d-1}(|X| - 1)$.*

Correspondence to oriented matroids. In order to relate pseudogeometric range spaces to simple arrangements of oriented pseudohyperplanes we exploit the representation theorem of Folkman& Lawrence [FL] that relates such arrangements to oriented matroids; so actually we want a correspondence between pseudogeometric spaces and oriented matroids. To this end we need to introduce a new class of range spaces, called *pseudohemispherical* range spaces. This is due to the fact that pseudogeometric spaces come from arrangements in *Euclidean* space while oriented matroids correspond to arrangements in *Projective* space; the pseudohemispherical property is the 'projective version' of the pseudogeometric one:

Definition 14 *Let $S = (X, R)$ be a range space. The* (complementary) closure *of S is the range space*

$$\overline{S} = (X, \overline{R}), \text{ where } \overline{R} := R \cup \{X - r \mid r \in R\}.$$

S is called closed*, if $S = \overline{S}$.*

Definition 15 *Let $S = (X, R)$ be a range space of VC-dimension $d \geq 1$. S is called* pseudohemispherical *if there exists a pseudogeometric space $T \neq S$ with $S = \overline{T}$. T is called an* underlying *space of S.*

In order to get an intuitive idea what this definition means, recall that the d-dimensional Projective space can be visualized as the sphere S^d with hyperplanes being great $(d-1)$-spheres, and we can get from a Euclidean hyperplane arrangement to its corresponding projective one as follows: think of E^d as the tangential hyperplane touching $S^d \subseteq E^{d+1}$ in the north pole. E^d can be mapped bijectively to the open northern hemisphere of S^d using central projection. This transformation takes a hyperplane h of E^d to a relatively open great halfsphere of dimension $d-1$. This halfsphere can be continued to a full great $(d-1)$-sphere in S^d, so an arrangement of hyperplanes in E^d induces an arrangement of great spheres in S^d, hence a Projective arrangement – the equator plays the role of the 'line at infinity' (Figure 2). Moreover, if we have positive and negative halfspaces associated with the hyperplanes, this information in an obvious way determines positive and negative hemispheres associated with the great spheres, so that we obtain an *arrangement of hemispheres* in S^d; since an antipodal cell has been generated for every cell in the underlying hyperplane arrangement, the corresponding description of cells (defined in the obvious way as for halfspace arrangements) is the closure of a geometric range space and will be called a *hemispherical* range space. Consequently, we will call the closure of a pseudogeometric range space *pseudohemispherical*.

Under the closure operation we lose information, since different pseudogeometric range spaces can have the same closure. This corresponds to the fact that depending on where the equator is chosen in an arrangement of hemispheres, the underlying Euclidean arrangement generating it changes. This, however, is hardly a nuisance – by 'fixing' the equator we get a one-to-one correspondence.

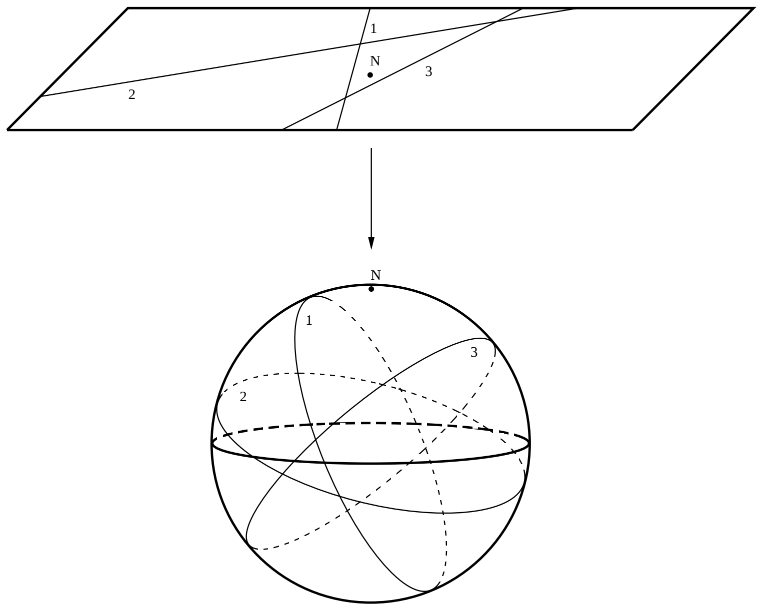

Figure 2: From halfspaces to hemispheres

Definition 16 *For a range space $S = (X, R)$ and $e \notin X$, the range space*

$$\hat{S} = (X \cup \{e\}, \hat{R}) \text{ with } \hat{R} := R \cup \{(X \cup \{e\}) - r \mid r \in R\}$$

is called the extended closure *of S.*

It is not surprising from the intuition that the extended closure of a pseudogeometric range space is pseudohemispherical as well; without proof we state

Theorem 17 *The mapping $S \mapsto \hat{S}$ forms a bijection between the pseudogeometric range spaces on X and the pseudohemispherical range spaces on $X \cup \{e\}$.*

It turns out that a pseudohemispherical space $S = (X, R)$ of VC-dimension d with $|X| = n$ attains the maximum number of ranges that a closed range space of this VC-dimension can have, namely $|R| = 2\Phi_{d-1}(n-1)$ (this is the bound of Therorem 13). Moreover, the pseudohemispherical spaces are already characterized by this property, a fact that is not apparent from their rather clumsy definition. As a consequence we obtain a new and simple characterization of the uniform oriented matroids, and this will finally give us the relation to arrangements (details are given in the last section).

Theorem 18 *For a set X of cardinality n there exists a natural (one-to-one) correspondence between the uniform oriented matroids of rank $n - d \geq 0$ on X and the closed range spaces (X, R) of VC-dimension d with $|R| = 2\Phi_{d-1}(n-1)$.*

2 Basics and Complete Range Spaces

This section will make the reader familiar with the necessary range space terminology; at the same time it presents basic properties of complete range spaces. In particular, we will introduce minors (or subspaces) of range spaces and prove the fundamental lemma of VC-dimension theory as well as the related bound on the number of ranges in the boundary of a range space. We give equivalent characterizations of completeness and discuss the structure of the distance-1-graph for complete range spaces.

Basics on range spaces.

Definition 19 *For a range space $S = (X, R)$, $Y \subseteq X$, we define*

$$\begin{aligned} S - Y &= (X - Y, R - Y), \text{ where } R - Y := \{r - Y \mid r \in R\}, \\ S^Y &= (X - Y, R^Y), \text{ where } R^Y := \{r \in R \mid r \cap Y = \emptyset, r \cup Y' \in R \ \forall\ Y' \subseteq Y\}. \end{aligned}$$

$S - Y$ and S^Y are the *minors* of S with respect to Y. $S - Y$ is said to arise from S by *deletion* of Y, while S^Y arises by *contraction* of Y. In a natural way $S - Y$ and S^Y generalize $S - \{x\}$ and $S^{\{x\}}$, as introduced in Definition 5. If S is geometric, $S - Y$ is obtained by deleting the hyperplanes in Y from the generating arrangement, while S^Y corresponds to the subarrangement induced by the remaining hyperplanes in the flat $\bigcap_{h \in Y} h$.

If Y is nonempty and $y_1, ..., y_k$ is any ordering of the elements of Y, then clearly $S - Y = (...(S - \{y_1\}) - ...) - \{y_k\}$. Via an easy induction part(i) of the following lemma also implies $S^Y = (...(S^{\{y_1\}})...)^{\{y_k\}}$.

Lemma 20 *Let $S = (X, R)$ be a range space, $x, y \in X, Y \subseteq X$.*

(i) $(R^Y)^{\{x\}} = R^{Y \cup \{x\}}$, *for* $x \notin Y$.
(ii) $|R| = |R - \{x\}| + |R^{\{x\}}|$.
(iii) $R - Y = R|_{X-Y}$.
(iv) $R^{\{x\}} - \{y\} \subseteq (R - \{y\})^{\{x\}}$.
(v) $\dim(S) = d \geq 0$ *implies* $\dim(S - \{x\}) \leq d$, $\dim(S^{\{x\}}) \leq d - 1$.

The proof requires only elementary set manipulations and is omitted for the sake of brevity. Now we are able to show the fundamental lemma of VC-dimension theory that establishes a bound of $|R| \leq \Phi_d(n)$ for any range space (X, R) of VC-dimension d with $|X| = n$ elements.

Proof of Lemma 3: We proceed by induction on d and n. The assertion is easily seen to be true for $d \leq 0$ and for $n = d \geq 0$, since in this case $|R| = 2^d = \sum_{i=0}^{d} \binom{n}{i} = \Phi_d(n)$.

Now assume $d > 0, n > d$; by hypothesis the bound holds for $S - \{x\}$ and $S^{\{x\}}$, $x \in X$. Using the preceding lemma this immediately yields

$$|R - \{x\}| \leq \Phi_d(n-1) \text{ and } |R^{\{x\}}| \leq \Phi_{d-1}(n-1),$$

so

$$|R| = |R - \{x\}| + |R^{\{x\}}| \leq \Phi_d(n-1) + \Phi_{d-1}(n-1) = \Phi_d(n).$$

$S = (X, R)$ is *maximal* if $\dim(X, R \cup \{r\}) > \dim(X, R)$ for all $r \in 2^X - R$. By the fundamental lemma every complete space is maximal, but the converse is not true in general. As a counterexample consider the range space (X, R) with

$$\begin{aligned} X &= \{1,2,3,4\}, \\ R &= \{\{1\},\{2\},\{4\},\{1,4\},\{2,4\},\{3,4\},\{1,2,3\},\{1,3,4\},\{2,3,4\},\{1,2,3,4\}\}. \end{aligned}$$

It is straightforward to check that S is maximal of VC-dimension 2 but not complete, since $|R| = 10 < \Phi_2(4) = 11$.

The fundamental lemma helps to prove another bound of a similar flavor we have already mentioned, namely the one on the maximum number of ranges in the *boundary* of a range space (Definition 12). We obtain

Theorem 21 *Let (X, R) be a range space of VC-dimension $d \geq 0$, $|X| = n > 0$. Then $|\partial R| \leq 2\Phi_{d-1}(n-1)$.*

Proof: Fix $x \in X$ and define $R' := \{r \in \partial R \mid x \notin R\}$. It is easily seen that if $Y \subseteq X - \{x\}$ is shattered by R', then $Y \cup \{x\}$ is shattered by ∂R; so $\dim(X - \{x\}, R') \leq d - 1$, which by the fundamental lemma implies $|R'| \leq \Phi_{d-1}(n-1)$. Finally observe that $|\partial R| = 2|R'|$. □

Characterizing complete range spaces. The extremal property defining complete range spaces (Definition 4) does not give immediate insights into the structure of these range spaces, so it seems appropriate to look for equivalent characterizations that reveal more of it. For example, one can show that the completeness property is inherited by the minors, a fact that is the basis of many subsequent inductive proofs. Another useful property is that completeness is maintained under duality (Definition 9). Before we give a list of equivalent statements most of which characterize completeness via certain properties of minors, let us briefly discuss the relation between the two minor operations of deleting and contracting elements (Definition 19); the point we want to stress is that although they look like very different operations at first glance, they aren't. On the contrary, they should be considered as having equal rights with respect to all concepts in this paper. The reason is that deletion and contraction change their roles under duality:

Observation 22 *Let $S = (X, R)$ be a range space, $Y \subseteq X$. Then*

(i) $-(R - Y) = (-R)^Y$.
(ii) $-(R^Y) = (-R) - Y$.

As it will turn out, we are only concerned with classes of range spaces that are closed under duality, so in any context referring to the structure of a range space the minor operations will appear in a completely symmetric way; if one of them is preferred in an argument, this is merely due to technical convenience. The symmetry already appears in the next theorem which will be the major tool to handle and manipulate complete range spaces.

Theorem 23 *Let $S = (X, R)$ be a range space, $d \geq 0$ a natural number with $|X| = n > d$. The following statements are equivalent:*

(i) S is complete of VC-dimension d.

(ii) $S - \{x\}$ and $S^{\{x\}}$ are complete of VC-dimension d and $d-1$, respectively, for all $x \in X$.

(iii) $\dim(S) = d$ and $S - \{x\}$ and $S^{\{x\}}$ are complete of VC-dimension d and $d-1$, respectively, for some $x \in X$.

(iv) $\dim(S) = d$ and $S^{\{x\}}$ is complete of VC-dimension $d-1$, for all $x \in X$.

(v) $\dim(S) = d$ and $|R^A| = 1$, for all $A \subset X$, $|A| = d$.

(vi) $-S$ is complete of VC-dimension $n-d-1$.

(vii) $\dim(-S) = n-d-1$ and $S - \{x\}$ is complete of VC-dimension d, for all $x \in X$.

(viii) $\dim(-S) = n-d-1$ and $|R|_A| = 2^{d+1} - 1$, for all $A \subseteq X$, $|A| = d+1$.

To see that the the additional dimension requirements in some of the statements are necessary in order to guarantee equivalence with (i), consider $X = \{1, 2, 3\}$ and

$$\begin{aligned} R &= \{\emptyset, \{1\}, \{2\}, \{1,2,3\}\} \text{ with } x = 1 \text{ for (iii)}, \\ R &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2,3\}\} \text{ for (iv), (v) and} \\ R &= \{\{1,2\}, \{1,3\}, \{2,3\}\} \text{ for (vii), (viii).} \end{aligned}$$

Such examples exist for arbitrary $|X|$ and d.

Proof: We proceed by showing first the equivalence of statements (i) through (v), then prove (i) $\Leftrightarrow$ (vi); together this yields the missing equivalences.

(i) $\Rightarrow$(ii) let S be complete of VC-dimension d, $x \in X$. Then

$$\Phi_d(n) = |R| = |R - \{x\}| + |R^{\{x\}}| \leq \Phi_d(n-1) + \Phi_{d-1}(n-1) = \Phi_d(n).$$

This yields $|R - \{x\}| = \Phi_d(n-1)$ and $|R^{\{x\}}| = \Phi_{d-1}(n-1)$, so $S - \{x\}$ and $S^{\{x\}}$ are complete of VC-dimension d and $d-1$, respectively, for all $x \in X$.

(ii) $\Rightarrow$ (iii),(iv) we only need to show that $\dim(S) = d$. Let $d' \geq d$ denote $\dim(S)$, and let A with $|A| = d'$ be shattered in R. If $|X| > d'$ then there is $y \in X - A$, and A is shattered also in $R - \{y\}$; since $S - \{y\}$ is of VC-dimension d we get $|A| = d$. If $|A| = |X| = d'$ then $R = 2^A$ which implies $R^{\{x\}} = 2^{A-\{x\}}$ for all $x \in X$, so $d - 1 = \dim(S^{\{x\}}) = |A| - 1$.

(iv) $\Rightarrow$ (i) we proceed by induction on n. If $n = d+1$, let Y be a set of cardinality d shattered in R. Then $R - \{x\} = R|_Y = 2^Y$ for x the unique element in $X - Y$, and observing that $2^d = \Phi_d(n-1)$ we obtain

$$|R| = |R - \{x\}| + |R^{\{x\}}| = 2^d + \Phi_{d-1}(n-1) = \Phi_d(n).$$

Now assume $n > d+1$ and choose $x \in X$. $(S - \{x\})^{\{y\}}$ is of VC-dimension at most $d-1$ for all $y \neq x$, and applying Lemma 20(iv) we get

$$\Phi_{d-1}(n-2) \geq |(R - \{x\})^{\{y\}}| \geq |R^{\{y\}} - \{x\}| = \Phi_{d-1}(n-2),$$

which holds because $S^{\{y\}} - \{x\}$ is complete of VC-dimension $d-1$ by implication (i)$\Rightarrow$(ii). But then $(S - \{x\})^{\{y\}} = S^{\{y\}} - \{x\}$, so $(S - \{x\})^{\{y\}}$ is complete of VC-dimension $d-1$. Since this holds for all y, $S - \{x\}$ is complete of dimension d by the inductive hypothesis. Finally we get

$$|R| = |R - \{x\}| + |R^{\{x\}}| = \Phi_d(n-1) + \Phi_{d-1}(n-1) = \Phi_d(n),$$

which means that S is complete. The last equation also yields implication (iii)$\Rightarrow$ (i).

(i)$\Leftrightarrow$(v) to see that '$\Rightarrow$' holds, iterate implication (i)$\Rightarrow$(iv) d times, starting from S. This shows that S^A is complete of VC-dimension 0 for all $|A| = d$, which implies $|R^A| = \Phi_0(n) = 1$. If on the other hand $|R^A| = 1$ then S^A is complete of VC-dimension 0, for all $|A| = d$. Using the fact that $\dim(S) = d$ and Lemma 20(v) we get $\dim(S^B) = d - k$ for $|B| = k$. Iterative application of (iv)$\Rightarrow$(i) then shows that S is complete.

(i)$\Leftrightarrow$ (vi) because of symmetry it suffices to show '$\Rightarrow$'; we have $2^n - \Phi_d(n) = \Phi_{n-d-1}(n)$, so it remains to show that $-S$ is of VC-dimension at most $n-d-1$. Assume on the contrary that there is $Y \subseteq X$, $|Y| = n-d$, shattered in $-R$. Then $|X - Y| = d$, and from (i)$\Rightarrow$ (iv) we get that there is a unique range $r \in R^{X-Y}$. Since $r \subseteq Y$, there is $r' \in -R$, such that $Y \cap r' = r$. This implies $r' \supseteq r$ and $r' - r$ contains no element of Y. But then r' is of the form $r' = r \cup Z, Z \subseteq X - Y$, which is a contradiction, since $r \in R^{X-Y}$ implies that all the ranges of this form are contained in R.

(vi) $\Leftrightarrow$ (vii) $\Leftrightarrow$ (viii) these equivalences are obtained by applying the 'dual' equivalences (i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) to $-S$, together with Observation 22. ☐

Corollary 24 *Let $S = (X, R)$ be complete of VC-dimension $d, |X| = n$. Then for all $x, y \in X$*

$$S^{\{x\}} - \{y\} = (S - \{y\})^{\{x\}}.$$

Proof: For $d \leq 0$ the statement is obvious, and for $n = d$ we have $R^{\{x\}} - \{y\} = (R - \{y\})^{\{x\}} = 2^{X - \{x,y\}}$. In any other case the theorem implies $|R^{\{x\}} - \{y\}| = |(R - \{y\})^{\{x\}}| = \Phi_{d-1}(n-2)$. Together with Lemma 20(iv) the claim follows. ☐

The distance-1-graph. As another, more technical tool to facilitate the subsequent considerations, we introduce the notion of 'swapping' a range space, which in case of geometric range spaces corresponds to the reorientation of hyperplanes in the generating arrangement.

Definition 25 *For $S = (X, R)$ and $D \subseteq X$, S* swapped *D is the the range space*

$$S \triangle D = (X, R \triangle D) \text{ with } R \triangle D := \{r \triangle D \mid r \in R\}.$$

Lemma 26 *For any range space $S = (X, R), D \subseteq X$ we have*

(i) $|R \triangle D| = |R|$.
(ii) $\dim(S \triangle D) = \dim(S)$.

We have already indicated that the *distance-1-graph* (Definition 6) captures crucial properties of a range space; in particular, pseudogeometric spaces are defined via a certain property of it (Definition 8). We will conclude this section by exhibiting a basic feature of the D^1-graph in the case of complete range spaces, and we use the fact that swapping does not change the D^1-graph (strictly speaking, $D^1(S)$ and $D^1(S \triangle D)$ are isomorphic with corresponding edges having the same labels); for geometric spaces this reflects the fact that reorienting some hyperplanes does not change the combinatorial structure of the arrangement. So whenever we consider some structural property of $D^1(S)$ (isomorphism type, connectivity, etc.) we are free to replace S with some swapped version $S \triangle D$, and an appropriate choice of D may result in shorter and more elegant formulations.

The key result on the D^1-graph of a complete range space is that it is connected; actually, there holds a stronger property: any two ranges are joined by a path of the shortest possible length which equals the cardinality of their symmetric difference. For a characterization of such graphs see [Djo]. First we need a lemma:

Lemma 27 *Let $S = (X, R)$ be complete of VC-dimension $d \geq 1$ and assume $X \in R$. Then, for all $r \in R, r \neq X$ there exists $x \in X$ such that $r \cup \{x\} \in R$.*

Proof: We proceed by induction on $n := |X|$. For $n = d$, any subset of X is a range so the lemma holds in this case. Now assume $n > d$ and consider $r \in R, r \neq X$. Choose $y \in X$ with $y \notin r$. If $r = X - \{y\}$ then $r \cup \{y\} \in R$. Otherwise the inductive hypothesis applies to $r \in R - \{y\}$, so there exists $z \in X - \{y\}$ with $r \in (R - \{y\})^{\{z\}} = R^{\{z\}} - \{y\}$ (Corollary 24). This is equivalent to $r \in R^{\{z\}}$ or $r \cup \{y\} \in R^{\{z\}}$, which implies $r \cup \{z\} \in R$ or $r \cup \{y\} \in R$. $\square$

Theorem 28 *Let $S = (X, R)$ be complete of VC-dimension $d \geq 1$. For any two ranges $r, r' \in R$ there is a path of length $\delta(r, r') := |r \triangle r'|$ joining r and r' in $D^1(S)$.*

Proof: By swapping assume $r' = X$ and iterate the lemma. □

In case of $\dim(S) = 1$, $D^1(S)$ is a tree on R with every element of X occurring exactly once as an edge label; this has been stated in Lemma 7, and to prove it is easy now: from the previous theorem we get that $D^1(S)$ is connected. To see that it is acyclic note that $x \in X$ occurs exactly $|R^{\{x\}}| = 1$ times as an edge label. On the other hand it is an easy observation that if $x \in X$ occurs as a label in a cycle of edges then it has to occur at least twice in this cycle. It follows that there can be no cycle.

3 Pseudogeometric Range Spaces

In this section we basically prove the characterizations of pseudogeometric spaces via duality (Therorem 10), small subspaces (Therorem 11) and cardinality of boundary (Therorem 13). The latter will be based on a version of Levi's Lemma for pseudogeometric spaces. Before this we present a characterization theorem similar to Therorem 23 for complete spaces.

Let us review the definition of pseudogeometric spaces; the following is just the non-recursive version of Definition 8:

Definition 29 *A complete range space $S = (X, R)$ of VC-dimension d is called pseudogeometric if either $d \leq 0$ or $d > 0$ and $D^1(S^Y)$ is a path for any $|Y| = d - 1$.*

Observe that for $|X| \leq d + 1$, any complete space is pseudogeometric. As in the complete case, we can come up with a list of equivalent statements characterizing the pseudogeometric property:

Theorem 30 *Let $S = (X, R)$ be a range space, $d \geq 2$ a natural number with $|X| = n > d + 2$. The following statements are equivalent:*

(i) S is pseudogeometric of VC-dimension d.

(ii) $S - \{x\}$ and $S^{\{x\}}$ are pseudogeometric of VC-dimension d and $d - 1$, respectively, for all $x \in X$.

(iii) $\dim(S) = d$ *and $S^{\{x\}}$ is pseudogeometric of VC-dimension $d-1$, for all $x \in X$.*

(iv) $\dim(S) = d$ *and S^A is pseudogeometric of VC-dimension 1, for all $A \subset X$, $|A| = d - 1$.*

(v) $-S$ is pseudogeometric of VC-dimension $n - d - 1$.

(vi) $\dim(-S) = n - d - 1$ *and $S - \{x\}$ is pseudogeometric of VC-dimension d, for all $x \in X$.*

(vii) $\dim(-S) = n - d - 1$ *and $S|_A$ is pseudogeometric of VC-dimension d, for all $A \subset X$, $|A| = d + 2$.*

Compared with the corresponding Therorem 23 for complete range spaces, we lose the characterizations via the minors S^A for $|A| = d$ and $S|_A$ for $|A| = d+1$ – they can be pseudogeometric even if S is not. However, if we consider minors on one element more, i.e. S^A for $|A| = d-1$ and $S|_A$ for $|A| = d+2$ then we can already recognize the pseudogeometric property.

An analogue of statement (iii) in Therorem 23 cannot be added here. There are cases where $S - \{x\}$ and $S^{\{x\}}$ are pseudogeometric of VC-dimension d and $d-1$, respectively, for some x, but S itself is no more than complete. To get such an example, let $S' = (X, R')$ be a pseudogeometric range space, fix $x \in X$ and define $S = (X, R)$ by $R := R'^{\{x\}} \cup \{r \cup \{x\} \mid r \in R' - R'^{\{x\}}\}$, i.e. R arises from R' by adding x to every range not in $R^{\{x\}}$. We get $R - \{x\} = R' - \{x\}$ and $R^{\{x\}} = R'^{\{x\}}$, so these minors of R will be pseudogeometric. On the other hand it is not hard to show that S is again complete, but since for $A \subset X - \{x\}$ we have

$$R^A = R^{A\cup\{x\}} \cup \{r \cup \{x\} \mid r \in R'^A - R'^{A\cup\{x\}}\},$$

by chosing $x \in X$ and $|A| = d-1$ such that $R^A \neq R'^A$ (which we can do for $|X| > d$) we see that S^A is not pseudogeometric, so S cannot be pseudogeometric by definition. Observe that we need to require $d \geq 2$ – otherwise statement (iii) only implies that S is complete; the same holds for the requirement $|X| > d+2$ in connection with statement (v).

Proof: The equivalence of (i), (iii) and (iv) just repeats the two definitions of the pseudogeometric property we had (Definition 8, Definition 29). Furthermore, (ii) implies (iii); equivalence (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii) is dual to (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv), so we are left to prove equivalence (i) $\Leftrightarrow$ (iv), where because of symmetry one implication suffices. Assume S is pseudogeometric of VC-dimension d. By Therorem 23 $-S$ is already complete of VC-dimension $n-d-1$, so by Therorem 13 (which we will prove shortly) it suffices to show that $|\partial(-R)| = 2\Phi_{n-d-2}(n-1)$, which by an easy computation follows from $|\partial(-R)| = |-R| - |R - \partial R|$. □

The characterization of the pseudogeometric property via small subspaces (Therorem 11) is now an immediate consequence of the theorem. The requirement '$\dim(-S) = n-d-1$' can be omitted since it is already imposed by the completeness of S, and the fact that for $|Y| = d+2$ any pseudogeometric range space $S|_Y$ is actually geometric follows by considering the dual range space $-(S|_Y) = (-S)^{X-Y}$ which is pseudogeometric of VC-dimension 1; its D^1-graph is a path connecting all the ranges, so any two pseudogeometric range spaces of VC-dimension 1 on Y are isomorphic, i.e. equal up to swapping and renaming of elements. Of course, this carries over to the primal setting, so any arrangement of $d+2$ hyperplanes in d-spaces has to generate an isomorphic copy of $S|_Y$, which means that this range space has to be geometric.

We have just mentioned the swap operation (Definition 25) in connection with pseudogeometric spaces, and it is quite clear that swapping does not affect the pseudogeometric property.

Lemma 31 *$S = (X, R)$ is pseudogeometric if and only if $S \triangle D$ is pseudogeometric, $D \subseteq X$.*

Levi's Lemma. After we have already used Therorem 13 (the characterization of pseudogeometric range spaces via the number of ranges in the boundary), we are now approaching its proof. It will be based on a variant of *Levi's Lemma* for pseudogeometric range spaces. The original version states that a pseudoline arrangement in the plane can be enlarged by a new pseudoline containing any two given points (which do not lie already on a common pseudoline). Although this fact is not very hard to prove, it should not be considered trivial: already in three dimensions, it is not true that every pseudoplane arrangement can be enlarged by a pseudoplane containing three given points [GP3] (recently, Richter-Gebert [RiG] has shown that there are arrangements that do not even allow a new pseudoplane containing certain two points). However, it is still true in all dimensions that any two points can be connected by a pseudoline, i.e. a curve in space which intersects (and crosses) every pseudohyperplane exactly once. In the following we define the range space analogue of such a curve:

Definition 32 *Let $S = (X, R)$ be a range space. A* segment *in S is a set of ranges which can be enumerated as $\{r_0, ..., r_k\}$ such that for $1 \leq i \leq k$, $r_{i-1} \triangle r_i = \{x_i\}$, $x_1, ..., x_k$ distinct elements from X. The segment is said to* join *r_0 and r_k. $R' \subseteq R$* admits *a segment if there exists a segment containing R'. The segment is a* line *if $k = |X|$.*

Equivalently we could say that a line is a pseudogeometric subspace (X, L), $L \subseteq R$, of VC-dimension 1. Note that Therorem 28 states that in a complete range space any two ranges admit a segment. Using this fact we obtain

Lemma 33 *For $S = (X, R)$ complete, ranges r and r' admit a line if and only if there are ranges $t, X - t \in R$ such that $r - r' \subseteq t$ and $r' - r \subseteq X - t$.*

Theorem 34 (Levi's Lemma) *If $S = (X, R)$ is pseudogeometric of VC-dimension $d \geq 1$, then any two ranges $r, r' \in R$ admit a line.*

Proof: We proceed by induction on d and $\delta(r, r') = |r \triangle r'|$.

The assertion is true for $d = 1$, since in this case R itself is a line. Furthermore, if $\delta(r, r') = 0$, i.e. $r = r'$, then the preceding lemma shows that it is sufficient to find one pair of complementary ranges $t, X - t$. Such a pair always exists, as follows by easy induction on d.

Now let $S = (X, R)$ be pseudogeometric of VC-dimension $d > 1$, $r, r' \in R$ with distance $\Delta := \delta(r, r') > 0$ and assume the theorem holds for any pseudogeometric range space of VC-dimension less than d and any pair of ranges with distance less than Δ in R.

Consider a segment joining r and r' and let u be the range followed by r' on this segment. After swapping, if necessary, we may assume $r' = u \cup \{x\}$ for some $x \in X$.

Since $\delta(r,u) = \Delta - 1$, r and u admit a line L by hypothesis, so there are ranges $t, X-t$ with

$$r - u \subseteq t, u - r \subseteq X - t.$$

If $x \in X - t$ then we obtain

$$r - r' \subseteq t, r' - r \subseteq X - t,$$

so we are done. Otherwise $x \in t$, and since $x \notin r$, by traversing L from r to t we encounter a range $s \in R^{\{x\}}$. $S^{\{x\}}$ is pseudogeometric, so by hypothesis there is a line in $R^{\{x\}}$ containing s and u, so we have $t', X - \{x\} - t' \in R^{\{x\}}$ with

$$s - u \subseteq t', u - s \subseteq X - \{x\} - t',$$

which yields

$$s - r' \subseteq t', r' - s \subseteq X - t'.$$

Now observe that $r - r' \subseteq s - r'$, $r' - r \subseteq r' - s$, which follows from the fact that s, r and $u = r' - \{x\}$ appear on the original line L in this order. Consequently, we get

$$r - r' \subseteq t', r' - r \subseteq X - t',$$

and together with the fact that $X - t'$ is a range in R, this shows that r and r' admit a line in S. ◻

For any $d \geq 2$, there are complete range spaces of VC-dimension d which are not pseudogeometric, with the property that any two ranges admit a line (let $d + 2 \leq |X| \leq 2d$ and $R = \binom{X}{\leq d}$)). For $d = 2$, however, the largest such example has 4 elements (see Therorem 36 below). The question whether this generalizes to higher VC-dimension is an interesting open problem.

Problem 3.1 *Given $d > 2$, does there exist a constant $C(d)$ such that for $S = (X, R)$ complete of VC-dimension d with $|X| > C(d)$, Levi's Lemma holds in S if and only if S is pseudogeometric? If the answer is yes, is $C(d) = 2d$?*

Here is a general characterization that might be helpful:

Lemma 35 *Let $S = (X, R)$ be complete; Levi's Lemma holds in S if and only if*

$$(\partial R) - Y = \partial(R - Y)$$

for any $Y \subseteq X$.

Observe that for any range space $(\partial R) - Y \subseteq \partial(R - Y)$ holds. In case of contraction we always have equality, i.e. $(\partial R)^Y = \partial(R^Y)$.

We will conclude the discussion by settling the 2-dimensional case:

Theorem 36 *Let $S = (X, R)$ be complete of VC-dimension 2, $|X| \geq 5$. S is pseudogeometric if and only if Levi's Lemma holds in S.*

Proof: Consider first the case $|X| = 5$, and assume Levi's Lemma holds. From $(\partial R) - Y = \partial(R - Y)$ it follows that ∂R shatters any two-element subset of X. With an easy case analysis one can check that this implies $|\partial R| = 2\Phi_1(4) = 10$, so R is pseudogeometric by Therorem 13. For $|X| > 5$ observe that if Levi's Lemma holds in S then it also holds in $S|_Y$ for any $|Y| = 5$. Consequently $S|_Y$ is pseudogeometric and from Therorem 30 we obtain that S itself has to be pseudogeometric. ◻

Characterization via cardinality of boundary. Now we can prove **Therorem 13**, which states that a complete range space $S = (X, R)$ of VC-dimension $d \geq 0$ and $|X| = n$ is pseudogeometric if and only if $|\partial R| = 2\Phi_{d-1}(n-1)$.

First suppose S is pseudogeometric; if $d = 1$, R is a line joining the only two complementary ranges of R, so $|\partial R| = 2 = 2\Phi_0(n-1)$. If $n = d$, then $|\partial R| = |R| = \Phi_d(n) = 2\Phi_{d-1}(n-1)$.

Now let $d > 1, n > d$ and inductively assume that $\partial(R - \{x\})$ and $\partial(R^{\{x\}})$ have the right cardinalities for some $x \in X$. Levi's Lemma holds in S, so we can apply Lemma 35 and obtain

$$\begin{aligned} |\partial R| &= |(\partial R) - \{x\}| + |(\partial R)^{\{x\}}| \\ &= |\partial(R - \{x\})| + |\partial(R^{\{x\}})| \\ &= 2\Phi_{d-1}(n-2) + 2\Phi_{d-2}(n-2) = 2\Phi_{d-1}(n-1). \end{aligned}$$

Now assume $|\partial R| = 2\Phi_{d-1}(n-1)$. We use induction on d to show that S is pseudogeometric: if $d = 1$, by Therorem 28 the $2 = 2\Phi_0(n-1)$ ranges in ∂R are joined by a path of length n in $D^1(S)$. Since $D^1(S)$ itself has only n edges it is identical with this path, so S is pseudogeometric.

Using Therorem 21 we get for $d > 1$ and $x \in X$

$$\begin{aligned} 2\Phi_{d-1}(n-1) &= |\partial R| \leq |(\partial R) - \{x\}| + |(\partial R)^{\{x\}}| \\ &\leq |\partial(R - \{x\})| + |\partial(R^{\{x\}})| \\ &\leq 2\Phi_{d-1}(n-2) + 2\Phi_{d-2}(n-2) = 2\Phi_{d-1}(n-1), \end{aligned}$$

which especially shows $|\partial(R^{\{x\}})| = 2\Phi_{d-2}(n-2)$, so $S^{\{x\}}$ is pseudogeometric by hypothesis. Since this holds for any $x \in X$, Therorem 30 shows that S is pseudogeometric.

4 Relation to Oriented Matroids

We have already introduced pseudohemispherical range spaces (Definition 15) which arise as the closure of pseudogeometric range spaces, and the intuition behind this definition was to have a class of range spaces generated by Projective rather than Euclidean arrangements. Therorem 17 states that both classes are in one-to-one correspondence provided we introduce a distinguished 'equator' element. This section will develop the basic properties of pseudohemispherical range spaces – the main statement will be a characterization via the number of ranges – and relate them to oriented matroids; it was first shown by Folkman & Lawrence [FL] that these combinatorial objects have natural representations as an arrangements of pseudohemispheres, and vice versa. The oriented matroid approach can handle arbitrary arrangements, while we are only talking about *simple* arrangements in this paper; so we restrict our attention to simple (or uniform) oriented matroids.

Let us start by showing that although the pseudogeometric space underlying a pseudohemispherical space is not unique, all underlying spaces have the same VC-dimension.

Lemma 37 *Let S be pseudohemispherical of VC-dimension $d \geq 1$ with underlying space T. Then T is of VC-dimension $d-1$.*

Proof: Equivalently we show that if $T = (X, R)$ $(T \neq \overline{T})$ is pseudogeometric of VC-dimension $d-1 \geq 0$, then $\dim(\overline{T}) = d$.

If $T \neq \overline{T}$, then $|X| \geq d$, so $T|_Y$ is again pseudogeometric of VC-dimension $d-1$ for $|Y| \geq d$. We obtain

$$|\overline{R|_Y}| = 2|R|_Y| - |\partial(R|_Y)| = 2\Phi_{d-1}(|Y|) - 2\Phi_{d-2}(|Y|-1) = 2\Phi_{d-1}(|Y|-1).$$

Any range space satisfies $\overline{R|_Y} = \overline{R}|_Y$, so

$$|\overline{R}|_Y| = 2\Phi_{d-1}(|Y|-1).$$

For $|Y| = d$ this number equals 2^d, so Y is shattered by $\overline{R}$, while for $|Y| \geq d+1$ the value is strictly less than $2^{|Y|}$, which implies that $\dim(\overline{T}) = d$. □

From the lemma it follows that a pseudohemispherical space of VC-dimension d has $2\Phi_{d-1}(n-1)$ ranges, and from Therorem 21 we know that this number is maximum for closed range spaces (Definition 14). Taking pattern from the complete spaces that attain the bound of Lemma 3 we define the concept of *c-completeness* ('c' stands for 'closed').

Definition 38 *$S = (X, R)$ closed of VC-dimension $d \geq 1$ with $|X| = n$ is called* c-complete *if $|R| = 2\Phi_{d-1}(n-1)$.*

Corresponding to Therorem 23 for complete spaces we obtain similar characterizations also for c-complete spaces (where only some numbers have to be adjusted):

Theorem 39 *Let $S = (X, R)$ be a closed range space, $d \geq 2$ a natural number with $|X| = n > d$. Then the following statements are equivalent:*

(i) S *is c-complete of VC-dimension* d.

(ii) $S - \{x\}$ *and* $S^{\{x\}}$ *are c-complete of VC-dimension* d *and* $d-1$, *respectively, for all* $x \in X$.

(iii) $\dim(S) = d$ *and* $S - \{x\}$ *and* $S^{\{x\}}$ *are c-complete of VC-dimension* d *and* $d-1$, *respectively, for some* $x \in X$.

(iv) $\dim(S) = d$ *and* $S^{\{x\}}$ *is c-complete of VC-dimension* $d-1$, *for all* $x \in X$.

(v) $\dim(S) = d$ *and* $|R^A| = 2$, *for all* $A \subseteq X$, $|A| = d-1$.

(vi) $-S$ *is c-complete of VC-dimension* $n-d$.

(vii) $\dim(-S) = n-d$ *and* $S - \{x\}$ *is c-complete of VC-dimension* d, *for all* $x \in X$.

(viii) $\dim(-S) = n-d$ *and* $|R|_A| = 2^{d+1} - 2$, *for all* $A \subseteq X$, $|A| = d+1$.

The proof is completely similar to the one of Therorem 23, so we do not repeat the arguments.

We also get

Theorem 40 *Let* $S = (X, R)$ *be c-complete of VC-dimension* $d \geq 2$. *For any two ranges* $r, r' \in R$ *there is a path of length* $\delta(r, r') = |r \triangle r'|$ *joining* r *and* r' *in* $D^1(S)$.

Again the proof is almost literally the same as that of Therorem 28.

Pseudohemispherical spaces are c-complete. The surprising fact is that the converse is also true:

Theorem 41 *Let* $S = (X, R)$ *be closed of VC-dimension* $d \geq 1, |X| = n$. S *is pseudohemispherical if and only if* S *is c-complete.*

Proof: We need to show that if S is c-complete space then S is pseudohemispherical, and we proceed by induction on d. If S is of VC-dimension 1 with $|R| = 2 = 2\Phi_0(n-1)$ then $S = (X, \{r, X - r\}), r \subseteq X$. Now $T = (X, \{r\})$ is of VC-dimension 0 and hence pseudogeometric with $S = \overline{T}$.

Now suppose $d > 1$, $x \in X$. $S^{\{x\}}$ is c-complete, so $S^{\{x\}}$ is pseudohemispherical of VC-dimension $d-1$ by hypothesis. Let $S' = (X - \{x\}, R')$ be a pseudogeometric space (of VC-dimension $d-2$) underlying $S^{\{x\}}$ and consider the range space

$$T = (X, R' \cup R''),$$

where $R'' := \{r \in R \mid x \in r\}$. Obviously $S = \overline{T}$, so to see that S is pseudohemispherical it remains to show that T is pseudogeometric of VC-dimension $d-1$.

The number of ranges of T is

$$|R'| + |R''| = \Phi_{d-2}(n-1) + \Phi_{d-1}(n-1) = \Phi_{d-1}(n).$$

Furthermore, T has $2|R'| = 2\Phi_{d-2}(n-1)$ ranges in the boundary. If we can show that T is of VC-dimension at most $d-1$, then T is complete and therefore pseudogeometric by Therorem 13. To this end consider $A \subseteq X$, such that A is shattered by $R' \cup R''$; we show that this implies $|A| \leq d-1$. There are two cases:

(a) $x \notin A$: For $r \in R'$ we have $r \cup \{x\} \in R''$, and since $A \cap r = A \cap (r \cup \{x\})$ we know that A is already shattered by R''. This implies that $A \cup \{x\}$ is shattered by R, so $|A \cup \{x\}| \leq d$, i.e. $|A| \leq d-1$.

(b) $x \in A$: By intersecting A with the ranges in R'' we only get subsets of A that contain x. This means, $A - \{x\}$ is shattered by R'. We get $|A - \{x\}| \leq d-2$, so $|A| \leq d-1$. □

Now that we know that c-complete and pseudohemispherical spaces are the same, we will use the latter term only in the following because it is more intuitive for our applications.

To prepare the correspondence to oriented matroids we need the notion of a *vertex* associated with a pseudohemispherical range space. Consider a simple arrangement of hemispheres labeled by X in S^{d-1}, and let A be a subset of $d-1$ hemispheres. The underlying great spheres intersect in two opposite vertices, and incident to one vertex are 2^{d-1} cells; their labels in the corresponding description of cells (X, R) of VC-dimension d are of the form $r \cup A'$ for any $A' \subseteq A$, where r is the label of the cell contained in no hemisphere from A. We have $r \in R^A$, and r can be identified with the vertex of the arrangement. Motivated by this we give

Definition 42 *Let $S = (X, R)$ be pseudohemispherical of VC-dimension $d \geq 1$, $r \in R$, $A \subseteq X$ with $|A| = d-1$. The pair (r, A) is called a* vertex *of S if $r \in R^A$.*

Therorem 39(v) shows that any $A \subseteq X$ with $|A| = d-1$ defines two vertices, just as in the case of the hemisphere arrangement. Obviously, any cell of a hemisphere arrangement is incident to a vertex, so the following theorem for pseudohemispherical spaces is not surprising.

Theorem 43 *A pseudohemispherical space $S = (X, R)$ of VC-dimension d is determined by its vertices, i.e.*

$$R = \bigcup_{|A|=d-1} \{r \cup A' \mid (r, A) \text{ vertex of } S, A' \subseteq A\}.$$

Proof: Set $R' := \bigcup_{|A|=d-1} \{r \cup A' \mid (r, A) \text{ vertex of } S, A' \subseteq A\}$. Obviously $R' \subseteq R$. On the other hand $|R'^A| \geq 2$ for any $|A| = d-1$, which implies $\dim(X, R') \geq d$, so $\dim(X, R') = d$ and $|R'^A| = 2$. Therorem 39 shows that S' is pseudohemispherical of VC-dimension d. It follows that $|R| = |R'|$ and therefore $R = R'$. □

Oriented Matroids. We start by introducing the basic oriented matroid terminology and then go straight for the proof of the correspondence between oriented matroids and pseudohemispherical spaces. We have tried to keep the section reasonably short but at the same time self-contained, i.e. readable without any previous knowledge about oriented matroids. In any case we recommend to study the standard literature on the subject for a more detailed treatment [BL, FL, EM, BLSWZ].

Let X be a finite set. A *signed vector* on X is a mapping $F : X \to \{0, +1, -1\}$. $F(x)$ will be denoted by F_x. The *support* of F is defined as the set $\underline{F} := \{x \in X \mid F_x \neq 0\}$. $F^0 := X - \underline{F}$ is called the *zeroset* of F.

In the context of signed vectors 0 is the vector satisfying $0_x = 0$ for all $x \in X$. $-F$ is defined by $(-F)_x := -(F_x)$.

A partial order on signed vectors is defined by

$$F \leq G :\Leftrightarrow F_x = 0 \text{ or } F_x = G_x \text{ for all } x \in X,$$

i.e. F results from G by switching some components to zero.

For $Y \subseteq X$, $F|_Y$ denotes the restriction of F to Y.

Definition 44 *Let X be a finite set, $\mathcal{C}$ a set of signed vectors on X, $0 \notin \mathcal{C}$. The pair $\mathcal{O} = (X, \mathcal{C})$ is called an* oriented matroid *if*

(OM1) If $F, G \in \mathcal{C}$ with $F \leq G$ then $F = G$.
(OM2) $F \in \mathcal{C}$ implies $-F \in \mathcal{C}$.
(OM3) If $F, G \in \mathcal{C}$ with $F_x = -G_x \neq 0$ and $F \neq -G$, then there exists $H \in \mathcal{C}$ with $H_x = 0$ and $H_y \in \{F_y, G_y, 0\}$ for all $y \neq x$.

$\mathcal{C}$ is the set of circuits *of the oriented matroid. We say that $\mathcal{O}$ is* uniform of rank *d if exactly all subsets of X with cardinality $d+1$ occur as support sets of signed vectors in $\mathcal{C}$.*

Apart from minor differences in notation this is the definition of Folkman & Lawrence ([FL]) as well as Bland & Las Vergnas ([BL]). In general the rank of an oriented matroid is defined via the rank of its underlying ordinary matroid. This is discussed in [EM, BLSWZ]. See also the original article about matroids by Whitney ([Whi]).

What follows are the two theorems which construct a pseudohemispherical range space from an oriented matroid and vice versa. It will be obvious that both constructions are inverse to each other. One remark to the reader familiar with oriented matroids: it would be quite intuitive to relate the ranges of the pseudohemispherical space to the topes of the oriented matroid; this, however, would require to work from an axiomatization of the topes or the whole face lattice of the oriented matroid. For the benefit of the unacquainted reader we have chosen a dual approach and relate the vertices of the range space to the vertices of the oriented matroid, i.e. the circuits of the dual oriented matroid. This enables us to work with the standard axiomatization from [BL, FL]. After our proof we cite two results from the literature that yield direct correspondences to the topes.

Theorem 45 *Let* $\mathcal{O} = (X, \mathcal{C})$ *with* $|X| = n$ *be an oriented matroid, uniform of rank* $d \geq 0$. *For* $F \in \mathcal{C}$ *define* $r_F := \{x \in X \mid F_x = +1\}$. *Then the range space* $S = (X, R)$ *with*

$$R := \bigcup_{F \in \mathcal{C}} \{r_F \cup F' \mid F' \subseteq F^0\}$$

is pseudohemispherical of VC-dimension $n - d$, *and the pairs* (r_F, F^0) *are exactly the vertices of* S.

Proof: If $d = 0$ then $\mathcal{C}$ consists of $2n$ circuits, two for each singleton support set, and it is easy to see that $R = 2^X$ in this case, which shows that S is pseudohemispherical of VC-dimension $n - d = n$ with vertices as required.

Now assume $d > 0$. We start by showing $\dim(S) = n - d$. First note that $r_F, r_{-F} \in R^{F^0}$ for all $F \in \mathcal{C}$, so $\dim(S^{F^0}) \geq 1$. Using Lemma 20(v) and the fact that $|F^0| = n - d - 1$ we get $\dim(S) \geq n - d$.

To see that $\dim(S) \leq n - d$ consider $Y \subseteq X, |Y| = n - d + 1$. We show that Y is not shattered by R. Note that $|\underline{F} \cap Y| \geq 2$ for all $F \in \mathcal{C}$. The crucial property is that if for $Y' \subseteq Y$ we have

$$(r_F \cup F') \cap Y = Y' \text{ for some } F \in \mathcal{C}, F' \subseteq F^0, \tag{1}$$

then we can assume without loss of generality that $|\underline{F} \cap Y| = 2$. To prove this choose F satisfying (1) for some $F' \subseteq F^0$ with $|\underline{F} \cap Y|$ minimal, and suppose $|\underline{F} \cap Y| > 2$. For $x, y \in \underline{F} \cap Y$ simplicity implies that there exists a circuit G with $\underline{G} = (X - Y) \cup \{x, y\}$, so $\underline{G} \cap Y = \{x, y\}$. If $G|_Y \leq F|_Y$ (analogously if $(-G)|_Y \leq F|_Y$) then G satisfies (1) for G' resulting from F' by adding all $z \in Y$ for which $F_z = +1$ and $G_z = 0$. Since this contradicts the minimality of $|\underline{F} \cap Y|$, we can assume $F_x = -G_x$, $F_y = G_y$. Applying (OM3) to F and G yields a circuit H with $H_x = 0$ and $H_Y \leq F_Y$, so H satisfies (1), again contradicting the choice of F. It follows that F can always be chosen in such a way that $|\underline{F} \cap Y| = 2$. This also implies $|F^0 \cap Y| = n - d - 1$, so $F^0 \subseteq Y$. Consequently, to show that Y is not shattered we have to make sure that there is a subset Y' which is not of the form

$$Y' = (r_F \cap Y) \cup F' \text{ with } |\underline{F} \cap Y| = 2, F' \subseteq F^0. \tag{2}$$

To construct Y' we proceed as follows: for $y, y' \in Y, y \neq y'$ choose $F \in \mathcal{C}$ with $\underline{F} \cap Y = \{y, y'\}$. y and y' are called *co-oriented* if $F_y = F_{y'}$. This notion does not depend on the specific F - otherwise there were circuits F, G with $\underline{F} \cap Y = \underline{G} \cap Y = \{y, y'\}$ and $F_y = F_{y'}, G_y \neq G_{y'}$. Applying (OM3) to F and G would yield a circuit H with $|\underline{H} \cap Y| \leq 1$, a contradiction.

Fix $x \in Y$ and define

$$Y' := \{y \in Y - \{x\} \mid x \text{ and } y \text{ are co-oriented}\}.$$

Using (OM3) it is easy to check that $y, y' \in Y$ are co-oriented if and only if $y \in Y', y' \in Y - Y'$ (or $y \in Y - Y', y' \in Y'$, of course).

Let us show that Y' cannot be expressed in the form of (2), which completes the proof that Y is not shattered by R; to this end choose any circuit F with $|\underline{F} \cap Y| = 2$. Suppose $\underline{F} \cap Y = \{y, y'\}$.

If y and y' are co-oriented, then Y' distinguishes between y and y' but any set of the form $(r_F \cap Y) \cup F'$ does not (observe that $y, y' \notin F^0$). If on the other hand y and y' are not co-oriented then $(r_F \cap Y) \cup F'$ distinguishes between them while Y' does not. In either case this implies $Y' \neq (r_F \cap Y) \cup F'$.

We have shown that $\dim(S) = n - d$. Now it is very easy to see that S is pseudohemispherical: it is a simple observation that S is closed; furthermore we have already seen that $|R^A| \geq 2$ for $|A| = n - d - 1$. From $\dim(S) = n - d$ we get $\dim(S^A) \leq 1$, so $|R^A| = 2$ (S^A is closed also).

Now Therorem 39 shows that S is pseudohemispherical. Obviously the vertices of S are the pairs (r_F, F^0) with $F \in \mathcal{C}$, as required. $\square$

Now the inverse construction:

Theorem 46 *Let $S = (X, R)$ with $|X| = n$ be pseudohemispherical of VC-dimension $n - d$. For $A \subseteq X, |A| = n - d - 1$ and (r, A) a vertex of S define a signed vector $F^{(r,A)}$ by*

$$F_x^{(r,A)} := \begin{cases} 0, & \textit{if } x \in A \\ +1, & \textit{if } x \in r \\ -1, & \textit{otherwise} \end{cases}$$

Then the pair $\mathcal{O} = (X, \mathcal{C})$ with

$$\mathcal{C} := \{F^{(r,A)} \mid (r, A) \textit{ vertex of} S\}$$

is an oriented matroid, uniform of rank d.

Proof: First note that if $\mathcal{O}$ is an oriented matroid then it is uniform of rank d by definition. So it remains to show that $\mathcal{O}$ satisfies properties (OM1), (OM2) and (OM3).

(OM1) is obvious by definition. To see that (OM2) holds observe that $F^{(r,A)} = -F^{(r',A)}$ for r, r' the two ranges in some R^A, $|A| = n - d - 1$.

To show (OM3) let $(r, A), (r', A')$ be vertices of S such that $F := F^{(r,A)}$ and $G := F^{(r',A')}$ satisfy the requirements of (OM3). This can be the case only for $n - d > 1$. Define $B := (A - A') \cap (r' - r)$, $B' := (A' - A) \cap (r - r')$ and set $s := r \cup B, s' := r' \cup B'$. Since $B \subseteq A$ and $B' \subseteq A'$, s and s' are ranges in R; moreover $s \in R^{A-B}$, $s' \in R^{A'-B'}$. For $C := (A - B) \cap (A' - B')$ we have $s, s' \in R^C$. S^C is pseudohemispherical, so there is a path of length $\delta(s, s')$ joining s and s' in $D^1(S^C)$, and the edge labels on the path are exactly the elements from $s \triangle s'$.

If we assume $F_x = +1, G_x = -1$ then $x \in r, x \notin r'$ and $x \notin A, A'$, which yields $x \in s, x \notin s'$. This means that on the path joining s, s' there is a range

$u \in (R^C)^{\{x\}} = R^{C\cup\{x\}}$. By Therorem 43 there is a vertex (v, D), $C \cup \{x\} \subseteq D$, $|D| = n - d - 1$ with $u = v \cup D'$, $D' \subseteq D - (C \cup \{x\})$. Set $H := F^{(v,D)}$. We claim that H is the circuit required by (OM3). Clearly $H_x = 0$ since $x \in D$. To show that $H_y \in \{F_y, G_y, 0\}$ for $y \neq x$ we consider the following cases (the remaining cases follow by symmetry):

(a) $F_y = G_y = 0$ means $y \in A, A'$ and $y \notin B, B'$, so $y \in C \subseteq D$; therefore $H_y = 0$.

(b) if $F_y = G_y = +1$ then $y \in r \subseteq s, y \in r' \subseteq s'$, so $y \in u$. Then either $y \in v$ which means $H_y = +1$ or $y \in D' \subseteq D$ which shows $H_y = 0$.

(c) for $F_y = G_y = -1$ we have $y \notin r, r'$ and because of $y \notin B, B'$ we get $y \notin s, s'$, so $y \notin u \supseteq v$. This gives $H_y \neq +1$.

(d) if $F_y = -G_y \neq 0$ there are no restrictions for H_y, so there is nothing to show.

(e) if $F_y = +1, G_y = 0$ then $y \in B'$, and this implies $y \in s, s'$. Now we continue as in case (b).

(f) $F_y = -1, G_y = 0$ gives $y \notin r, r', r \triangle r'$, so $y \notin s, s'$. Proceed as in case (c). ◻

The two theorems together give the main Therorem 18. By replacing the pseudohemispherical space with its dual for $d > 0$ and exploiting the representation theorem in [FL] (the reader can find other versions also in [BLSWZ, EM]) we get

Theorem 47 *For $d \geq 1$ there is a natural (one-to-one) correspondence between the pseudohemispherical range spaces of VC-dimension d on a set X and the simple arrangements of $|X|$ pseudohemispheres in S^{d-1}.*

By Therorem 17, pseudogeometric spaces correspond to oriented matroids with a distinguished element playing the role of the equator. Such objects are called *affine* oriented matroids, and as shown by Edmonds & Mandel [EM] they exactly encode Euclidean pseudohalfspace arrangements (actually they have been invented just for this purpose). Consequently we get the following theorem that finally justifies our definition of pseudogeometric range spaces.

Theorem 48 *For $d \geq 0$ there is a natural (one-to-one) correspondence between the pseudogeometric range spaces of VC-dimension d on a set X and the simple arrangements of $|X|$ oriented pseudohyperplanes in E^d.*

Related Results. We will briefly describe results related to our correspondence and figure out how our approach fits in there. We basically review two results from the literature that give alternative proofs for Therorem 46. The first one, by Lawrence [Law] is based on the notion of *lopsidedness*. To remain consistent we keep on using range space terminology.

Definition 49 *A range space $S = (X, R)$ is called* lopsided *if for any $A \subseteq X$ either*

(i) A is not shattered in R, or
(ii) $X - A$ is not shattered in $-R$.

It is quite easy to show that not both conditions can hold at the same time, so S is lopsided if for any A one of them is satisfied. As it turns out, lopsided range spaces properly generalize complete range spaces:

Lemma 50 *If $S = (X, R)$ is complete, then (X, R) is lopsided.*

A lopsided range space that is not complete is given by $X = \{1, 2, 3\}$ and

$$R = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}\}.$$

Just from the definition it follows that lopsidedness is maintained under duality, i.e. S is lopsided if and only if $-S$ is lopsided. Lawrence proved the following result that characterizes the tope sets of uniform oriented matroids (for the notion of *topes* see e.g. [BLSWZ, EM])

Theorem 51 *Let $S = (X, R)$ be a closed range space. R is isomorphic to the tope set of a uniform oriented matroid on X if and only if the range space $S' = (X - \{x\}, R')$, $R' := \{r \in R \mid x \notin r\}$ is lopsided for any $x \in X$.*

It is well-known that the topes uniquely determine the oriented matroid itself, so the range spaces having the property required in the theorem correspond to the uniform oriented matroids. This immediately proves that any pseudohemispherical range space determines an oriented matroid: if $S = (X, R)$ with $|X| = n$ is pseudohemispherical of VC-dimension d then $|R| = 2\Phi_{d-1}(n-1)$, which gives $|R'| = \Phi_{d-1}(n-1)$. S' is of VC-dimension $d-1$ and hence complete (and even pseudogeometric). Together with Lemma 50 the result follows.

The surprising fact about this theorem is that (via our correspondence) the lopsidedness of each S' already implies the pseudogeometric property.

Another concept closely related to oriented matroids is that of *acycloids* introduced by Tomizawa [Tom].

Definition 52 *Let $S = (X, R)$ be a closed range space. S is called an* acycloid *if for any two ranges $r, r' \in R$ there is a path of length $|r \triangle r'|$ joining r and r' in $D^1(S)$.*

We have seen that pseudohemispherical spaces have this property (Therorem 40), so they are acycloids. Acycloids properly generalize tope sets of oriented matroids. Fukuda (see e.g. [Han]) has given an example of an acycloid that is not the tope set of any oriented matroid. Such examples can also be found as follows: let S be complete but non-pseudogeometric, such that Levi's Lemma (Therorem 34) holds in S. Then the extended closure $\hat{S}$ (Definition 16) is an acycloid (because of Levi's Lemma) that is not matroidal (because S is not pseudogeometric). Actually, Fukuda's example can be obtained in this way with $|X| = 4$ and $S = (X, \binom{X}{\leq 2})$. The following theorem has been proved by Handa [Han]:

Theorem 53 *Let $S = (X, R)$ be an acycloid. R is isomorphic to the tope set of an oriented matroid on X if and only if S^A is an acycloid, for any $A \subseteq X$.*

Obviously this property holds for S pseudohemispherical, since S^A is again pseudohemispherical for any S^A with at least two ranges. As above, via the theorem we immediately obtain an oriented matroid from a pseudohemispherical space.

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