Adaptive Modelling of Coupled Hydrological Processes with Application in Water Management

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Abstract This paper presents recent results of a network project aiming at the modelling and simulation of coupled surface and subsurface flows. In particular, a discontinuous Galerkin method for the shallow water equations has been developed which includes a special treatment of wetting and drying. A robust solver for saturated–unsaturated groundwater flow in homogeneous soil is at hand, which, by domain decomposition techniques, can be reused as a subdomain solver for flow in heterogeneous soil. Coupling of surface and subsurface processes is implemented based on a heterogeneous nonlinear Dirichlet–Neumann method, using the `dune-grid-glue` module in the numerics software DUNE.

1 Introduction

Reliable flood prediction and the design of efficient flood protection measures are tasks that engineering companies for water management have to face. Since flood control basins and water retention walls on rivers have effects on the groundwater table, sound modelling and simulation of coupled ground- and surface water processes is required. Challenges include the temporal and spatial scale differences between ground- and surface water, and the anisotropy of the groundwater domain.

As the surface water model we consider the shallow water equations, given by hyperbolic conservation laws for the water height h and the discharge $\mathbf{q}$

$$\partial_t h + \operatorname{div} \mathbf{q} = S_h\,, \qquad \partial_t \mathbf{q} + \operatorname{div}(\mathbf{q}^2/h + 0.5\,g h^2) = -gh\nabla b\,. \tag{1}$$

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Here, S_h denotes a source term, g the gravity constant and the graph Σ of the function b on the domain $\Omega_s \subset \mathbb{R}^2$ coincides with the topography. The latter is identical to the top boundary of the domain $\Omega_p \subset \mathbb{R}^3$ that represents the porous medium.

Saturated–unsaturated groundwater flow in Ω_p is modelled by the Richards equation

$$n\,\theta(p)_t + \operatorname{div}\mathbf{v}(p) = 0\,, \qquad \mathbf{v}(p) = -K_h kr(\theta(p))\nabla(p-z) \tag{2}$$

in case of homogeneous equations of state for the saturation θ and the relative permeability kr. It combines mass conservation on the left with a nonlinear version of Darcy's law on the right. The water or capillary pressure head is denoted by p and z is the downward pointing component of $x \in \Omega_p$. The porosity $n \in [0,1]$ and the hydraulic conductivity $K_h > 0$ may depend on x. The functions $p \mapsto \theta(p) \in [0,1]$ and $\theta \mapsto kr(\theta) \in [0,1]$ are increasing with $kr(1) = 1$.

Conditions for coupling Equations (1) and (2) across the surface Σ are given by mass conservation and continuity of the pressure

$$S_h = \mathbf{v}\cdot\mathbf{n} \quad \text{and} \quad p = h \quad \text{on } \Sigma \tag{3}$$

where $\mathbf{n}$ is the outward normal of Ω_p on Σ.

All algorithms described in this paper have been implemented in the software framework DUNE [1].

2 DG method for the shallow water equations

The shallow water equations (1) are a special case of the general evolution problem

$$\partial_t \mathbf{u}(t,\cdot) = \mathscr{L}[\mathbf{u}(t,\cdot)](\cdot) \qquad \text{in } ([0,T)\times\Omega_s) \subset (\mathbb{R}\times\mathbb{R}^d)\,,\ d \in \{1,2,3\}\,,$$

with the spatial operator $\mathscr{L}[\mathbf{w}] = S(\mathbf{w}) - \operatorname{div} F(\mathbf{w})$, where $\mathbf{w} : \Omega_s \to \Psi \subseteq \mathbb{R}^r$ belongs to some suitable function space V, Ψ is the set of states for a given problem, $S(\mathbf{w})$ is a source term function and $F(\mathbf{w})$ is the analytical flux function. For a tessellation $\mathscr{T}_h$ of Ω_s we consider the discrete space $V_h := \{\varphi \in L^2(\Omega_s) |\ \varphi_{|T} \in \mathbb{P}^p(T)\ \ \forall T \in \mathscr{T}_h\}$ and, for $\varphi \in V_h$, define the discrete operator $\mathscr{L}_h : V_h \to V_h'$ by

$$\int_{\Omega_s} \mathscr{L}_h[u_h]\varphi := \sum_{T\in\mathscr{T}_h} \left(\int_T S(u_h)\varphi + \int_T F(u_h)\cdot\nabla\varphi - \int_{\partial T} \varphi\, G(u_h^+, u_h^-, \ldots)\cdot\mathbf{n}_T \right).$$

$\mathbf{n}_T$ denotes the outward normal of T, and u_h^+ and u_h^- are the values of the function u_h on both sides of ∂T. $G(u,v,\mathbf{x}) : V\times V\times\mathbb{R}^d \to \mathbb{R}^d$ is a numerical flux function. The discontinuous Galerkin (DG) method is determined by the functions S and F, the numerical flux G (we use the Local-Lax–Friedrichs flux) and the space V_h. This space discretization leads to a system of ODEs $\frac{d}{dt}u_h(t) = \widetilde{\mathscr{L}_h}[u_h(t)]$, solved with an explicit Runge–Kutta (RK) method of order $p+1$, cf. [8]. Since the RK–DG method is not stable for nonlinear problems where strong shocks might appear, a stabilized

version $\widetilde{\mathscr{L}_h}$ of $\mathscr{L}_h$ is used. We apply the stabilization mechanism of [7], extended to a specific criterion to guarantee, in particular, the conservation of nonnegative water levels.

For a correct handling of steady states, [8] suggests a well-balancing method based on a reformulation of the topography source term in the balance law (1) for the discharge $\mathbf{q}$. We rewrite this term as $S_{\mathbf{q}} = -\frac{1}{2}gh\nabla b - \frac{1}{2}\operatorname{div}(ghb) + \frac{1}{2}gb\nabla h$, move the divergence term to the left hand side, and treat the topography b as an additional unknown. Then, under suitable assumptions, it is straightforward to derive well-balanced schemes for a wide range of standard numerical fluxes to solve this new representation of the shallow water equations with DG.

The wetting–drying treatment is based on the thin water layer approach of [6]. Simplifying the stability condition in [6], we use the "reflection numerical flux" to prohibit mass transfer through an element boundary as soon as we detect an emptying of this element. After the positivity of the mean water depth in each element is guaranteed, the so-called *positive depth operator* [6] ensures the positivity of water depth node-wise. See [8] for details of our implementation.

3 Efficient solver for the Richards equation in heterogeneous soil

Our solution technique for the Richards equation is based on Kirchhoff transformation and convex minimization for homogeneous state equations and, in addition, on domain decomposition methods for layered heterogeneous soil. For simplicity, we assume $n = K_h = 1$ here. Then, a time discretization of (2) which treats the main part implicitly and the convective part (in z-direction) explicitly leads to the spatial problem

$$\theta(p) - \operatorname{div}(kr(\theta(p))\nabla p) = f \tag{4}$$

with some function f. By the *Kirchhoff transformation* $\kappa : p \mapsto u := \int_0^p kr(\theta(q))\,dq$ and the definition $M(u) := \theta(\kappa^{-1}(u))$ we can rewrite Equation (4) as

$$M(u) - \Delta u = f\,. \tag{5}$$

We can endow (5) with quite general boundary conditions including *outflow conditions of Signorini-type*. Then, since $M : \mathbb{R} \to \mathbb{R}$ is an increasing function, the weak form of Problem (5) allows an equivalent formulation as a uniquely solvable *convex minimization problem* on a convex subset of $H^1(\Omega_p)$. Using linear finite elements, we construct a convergent discretization of (5), which is meaningful also in the physical variable p. It can be solved efficiently and robustly by *monotone multigrid methods* [3].

In layered heterogeneous soils, different functions $\theta_i(\cdot)$ and $kr_i(\cdot)$, $i = 1,\ldots,m$, belonging to the layers Ω_i of Ω_p occur. Application of different Kirchhoff transformations κ_i in the layers and the assumption of pressure continuity as well as mass conservation across the interfaces between the layers lead to local convex minimization problems that are coupled by *nonlinear transmission conditions.* In particular,

for $m = 2$ we obtain the transmission problem

$$M_i(u_i) - \Delta u_i = f \quad \text{on } \Omega_i, \qquad \kappa_1^{-1} u_1 = \kappa_2^{-1} u_2, \quad \mathbf{v}_1 \cdot \mathbf{n}_1 = \mathbf{v}_2 \cdot \mathbf{n}_1 \quad \text{on } \Gamma, \quad (6)$$

where $\Gamma := \overline{\Omega}_1 \cap \overline{\Omega}_2$ is the interface between the layers and $\mathbf{n}_1$ is the outward normal of $\partial\Omega_1$ on Γ. The coupled problem (6) can be solved iteratively by nonlinear *Dirichlet–Neumann* or *Robin methods*. For analytical and numerical results on such methods we refer to [4].

4 Coupling strategy: algorithm and modular implementation

We obtain a spatial coupled problem by implicitly discretizing the time dependent problem (1)–(3). Typically, the time scale for (1) is considerably smaller than the time scale for (2). Therefore, our time discretization of (1)–(3) attributes a certain number N of equidistant time steps to (1) within an interval between two time steps for (2). Correspondingly, we assign $1/N$ of the source term S_h in (3) from one time step of (2) equally to the sub time steps of (1).

In order to solve the resulting discretization of (1)–(3), we apply a Dirichlet–Neumann-type iteration technique. Given solutions p_0 and $\mathbf{u}_0$ of the Richards equation (RE) and the shallow water equations (SWE) for a time step t_k, using the water flux $F_\Sigma = \mathbf{v}_0 \cdot \mathbf{n}$ as the source term, we solve the SWE at sub time steps $t_k < t_k^i \leq t_{k+1}$. Thus we obtain a Dirichlet value $p = h_1$ on Σ for the RE at time step t_{k+1}. The solution of the RE at t_{k+1} yields a water flux $F_1 = \mathbf{v}_1 \cdot \mathbf{n}$. Setting $F_\Sigma = F_1$, the SWE are solved again in $t_k^i \leq t_{k+1}$ and we obtain a new Dirichlet value $p = h_2$ for the RE at the time step t_{k+1}. We repeat this iteration until the error $\|h_{j+1} - h_j\|_\infty$ is small enough.

Implementing the treatment of (1)–(3) is challenging since the surface and the subsurface solvers exist as completely separate codes on top of the DUNE libraries. Coupling is obtained via the DUNE module `dune-grid-glue`, which offers abstract interfaces for the geometric coupling of finite element grids. Its design follows the concept of intersections, used in the DUNE grid interface [1], and it supports most domain decomposition techniques in use today, including overlapping and nonoverlapping couplings between non-matching grids. In our case, by injection in $\mathbb{R}^3$ and projection onto Σ, the surface water domain $\Omega_s \subset \mathbb{R}^2$ is coupled to the top boundary Σ of the porous media domain Ω_p. The corresponding image $\mathscr{T}(\Omega_s)$ of the two-dimensional (surface) grid on Ω_s and the restriction $\mathscr{T}(\Omega_p|_\Sigma)$ of the porous media (subsurface) grid onto Σ are in general nonmatching. The `dune-grid-glue` module efficiently computes all intersections between elements of $\mathscr{T}(\Omega_s)$ and $\mathscr{T}(\Omega_p|_\Sigma)$. These so-called *remote intersections* [2] encapsulate the relationships between elements in the generating grids $\mathscr{T}(\Omega_s)$ and $\mathscr{T}(\Omega_p|_\Sigma)$ and constitute a new partition $\mathscr{T}(\Sigma)$ of Σ.

For high performance parallel computations, the concept of intersections is extended and provides parallel communication methods to exchange data associated

with intersections between elements on remote processes [9]. Using parallel index-sets [5], each remote intersection $\mathscr{I}$ provides information about relations with remote processes. The index-sets provide means of communication to exchange data between different processes. The communication interfaces are designed similar to those of a single DUNE grid. The user has to provide `DataHandle` objects defining `gather` and `scatter` methods to read and write data associated with an intersection $\mathscr{I}$.

For the Dirichlet–Neumann coupling, the fluxes F_Σ are computed at quadrature points on $\mathscr{I}$ from the solution of (2). Communicating the fluxes, the source term S_h can be computed on the surface grid. To evaluate Dirichlet values $p = h_j$ for the Richards equation, a representation of h_j on the boundary of the subsurface grid is required. On the surface grid the solution is projected onto a discontinuous function h_{disc} on $\mathscr{T}(\Sigma)$. The coefficients of h_{disc} for each intersection $\mathscr{I}$ are communicated and h_{disc} can be evaluated on the subsurface grid.

5 Benchmark problems and numerical experiments

We close by giving some numerical results obtained by the application of the Richards and shallow water solver to realistic data (grid and soil parameters) which was provided by the engineering company WALD+CORBE. The geometry is a real piece of landscape (size: 2525 m $\times$ 2415 m; 5 m $\times$ 5 m resolution of the digital elevation model). Measured groundwater levels were available (see Figure 1). The observed region consists of three separate geologic formations with different hydraulic

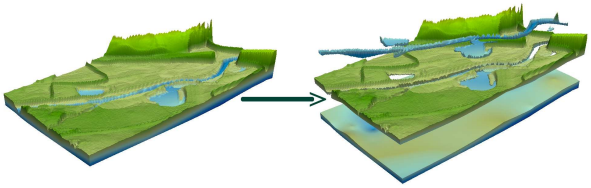

Fig. 1 Initial groundwater distribution for computation a) and decomposition of the domain in its 3 layers (overlying strata, aquifer and riverbed). Scaling in direction of gravity by a factor of 10.

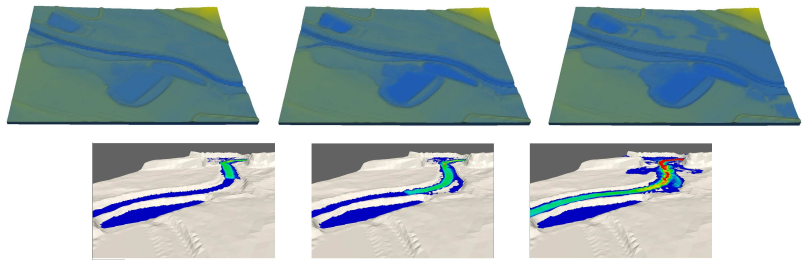

Fig. 2 Simulations of flood events: a) Richards equation – visualization of capillary pressure. b) Shallow water equations – visualization of discharge with wetting and drying effect.

conductivities: a hardly permeable top layer (alluvial clay), the aquifer (quaternary gravel sand) and the riverbed (river gravel). The decomposition is illustrated in Figure 1. Numerical experiments showed that we need a resolution of about 40 nodes in gravity direction in order to capture the infiltration adequately. Since the geological layers are very thin (0.5 m–15 m) in comparison to width and length of the domain, we developed a nonlinear line smoother for the multigrid method to treat the anisotropy. The numerical experiments simulate flood events in the considered domain (Figures 2 and 3), which contains a fictional polder and a retention basin to be built for flood protection. The results qualitatively match the behaviour expected for flood scenarios under the given circumstances.

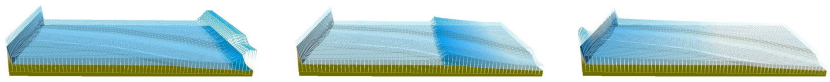

Fig. 3 Flooding of a small subdomain, with reflexion by an artificial obstacle. Here, we used the coupling of the solvers for the Richards and the shallow water equation. After several flood waves, a slight infiltration can be observed (processes take place at different time scales).

Acknowledgements This work was supported by the BMBF–Programm “Mathematik für Innovationen in Industrie und Dienstleistungen”, Förderkennzeichen (contract numbers) 03OMPAF1, 03BAPAF2, 03DEPAF3, 03KOPAF4.

References

1. Bastian, Blatt, Dedner, Engwer, Klöfkorn, Kornhuber, Ohlberger, Sander: A generic grid interface for parallel and adaptive scientific computing. Part II: Implementation and tests in DUNE. Computing **82**(2–3), 121–138 (2008)
2. Bastian, Buse, Sander: Infrastructure for the coupling of Dune grids. In: Proc. ENUMATH, vol. 9 (2010)
3. Berninger, Kornhuber, Sander: Fast and robust numerical solution of the Richards equation in homogeneous soil. FU Berlin, Math., Prepr. A /01/2010. Submitted to SIAM J. Numer. Anal.
4. Berninger, Kornhuber, Sander: Convergence behaviour of Dirichlet–Neumann and Robin methods for a nonlinear transmission problem. In: Proc. DD19, LNCSE. Springer (2010). To appear.
5. Blatt, Bastian: C++ components describing parallel domain decomposition and communication. IJPEDS **24**(6), 467–477 (2009)
6. Bunya, Kubatko, Westerink, Dawson: A wetting and drying treatment for the Runge–Kutta discontinuous Galerkin solution to the shallow water equations. Comput. Meth. Appl. Mech. Eng. **198**, 1548–1562 (2009)
7. Dedner, Klöfkorn: A generic stabilization approach for higher order Discontinuous Galerkin methods for convection dominated problems. Uni Freiburg, Preprint 8. Submitted to SISC
8. Dedner, Kröner, Shokina: Adaptive modelling of two-dimensional shallow water flows with wetting and drying. In: Comput. Science and High Perf. Comput. IV. Springer. To be published.
9. Engwer, Sander: A framework for the parallel coupling of Dune grids In preparation.