ORDER CONVERGENCE AND CONVERGENCE ALMOST EVERYWHERE REVISITED

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Abstract: In Analysis two modes of non-topological convergence are interesting: order convergence and convergence almost everywhere. It is proved here that oder convergence of sequences can be induced by a limit structure, even a finest one, whenever it is considered in σ-distributive lattices. Since convergence almost everywhere can be regarded as order convergence in a certain σ-distributive lattice, this result can be applied to convergence of sequences almost everywhere and thus generalizing a former result of U. Höhle [11] obtained in a more indirect way by using fuzzy topologies.
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0 Introduction

Order convergences of sequences is a well-known concept in Analysis. It has been generalized to order convergence of nets and filters in a complete lattice (even in a partially ordered set) by G. Birkhoff [6] and D. C. Kent [12] respectively. D. C. Kent[6] proved also that order convergence of filters in a complete lattice is not induced by a limit structure in general and consequently there is no topology such that convergence w. r. t. the topology means order convergence. Recently, R. Anguelov and J. H. van der Walt [1] have proved that order convergence of sequences in the vector lattice $\mathcal{C}(X)$ of all continuous real-valued maps on a topological space X is not topological but it is induced by a suitable limit structure on $\mathcal{C}(X)$.

In this paper a certain condition on a lattice, namely σ-distributivity, is studied such that order convergence of sequences is induced by a limit structure, even a finest one. It is already mentioned in the book of Luxemburg and Zaanen [15] that each vector lattice (=Riesz space) is σ-distributive.

Another non-topological mode of convergence is convergence almost everywhere which is mainly studied in measure theory and probability theory. An example that convergence almost everywhere is not topological has been given by H.-J. Kowalsky [13] and later on by E. T. Ordmann [16]. In particular H.-J. Kowalsky introduced limit structures (="Limitierungen") in order to remedy this inconvenience. Independently, limit spaces, i.e. sets endowed with a limit structure, have been introduced by H. R. Fischer [9]. They are nowadays also called convergence spaces and form a suitable framework for Functional Analysis (cf. E. Binz [4, 5], W. Gähler [10], and R. Beattie and H. P. Butzmann [3]). But they are also used in Convenient Topology (cf. e. g. [17]) together with suitable generalizations.

U. Höhle [11] pointed out that H.-J. Kowalsky used in his example everywhere defined functions whereas in probability theory almost everywhere defined functions are used. He improves this situation by introducing a suitable limit structure on the quotient set $S/\mathcal{S}$ where S is the set of all Borel measurable functions $f : [0,1] \to \mathbb{R}$ and $(f, g) \in \mathcal{S}$ iff $f = g$ λ-almost everywhere (λ denotes the Lebesque measure). This limit structure is constructed via a fuzzy topology(=many-valued topology) on $S/\mathcal{S}$.

Here a more general situation is studied, namely

1. $\mathbb{R}$ is substituted by the extended real line $\bar{\mathbb{R}}$, and
2. $([0,1], \mathcal{B}, \lambda)$, where $\mathcal{B}$ denotes the σ-algebra of Borel subsets of $[0,1]$, is substituted by a general measure space $(\Omega, \mathcal{A}, \mu)$ with $\mu \neq 0$.

Furthermore, the fact that convergence almost everywhere can be regarded as order convergence in a certain σ-distributive lattice is used explicitly. Thus, the obtained result on order convergence can be applied. In particular, Höhle's result appears as a corollary and is obtained in a direct way without using fuzzy topologies.

1 Preliminaries

1.1 Definition. Let X be a set, $F(X)$ the set of all filters on X, and $q \subset F(X) \times X$. Consider the following:

1. $(\dot{x}, x) \in q$ for each $x \in X$, where $\dot{x} = \{A \subset X : x \in A\}$,
2. $(\mathcal{F}, x) \in q$ whenever $(\mathcal{G}, x) \in q$ and $\mathcal{G} \subset \mathcal{F}$,
3. $(\mathcal{F}, x) \in q$ and $(\mathcal{G}, x) \in q$ imply $(\mathcal{F} \cap \mathcal{G}, x) \in q$,
4. For each $x \in X$, $(\mathcal{U}_q(x), x) \in q$, where $\mathcal{U}_q(x) = \bigcap_{(\mathcal{F},x)\in q} \mathcal{F}$ is the *neighborhood filter of x w. r. t. q*,
5. For each $U \in \mathcal{U}_q(x)$ there is some $V \in \mathcal{U}_q(x)$ such that $U \in \mathcal{U}_q(y)$ for each $y \in V$, and
6. $(\mathcal{F}, x) \in q$ and $(\mathcal{F}, y) \in q$ imply $x = y$.

Then q is called:

a) a *generalized convergence structure on* X iff 1. and 2. are satisfied,

b) a *limit structure on* X iff 1., 2., and 3. are satisfied,

c) a *pretopological structure (or pretopology) on* X iff 1., 2., and 4. are satisfied,

d) a *topological structure (or topology) on* X iff it is a pretopological structure such that for each $x \in X$, 5. is satisfied.

A generalized convergence structure q is called T_2 iff it satisfies 6.. If $(\mathcal{F}, x) \in q$ we say $\mathcal{F}$ *converges* to x and we write $\mathcal{F} \xrightarrow{q} x$ or shortly $\mathcal{F} \to x$. A *sequence* (x_n) in X *converges* to $x \in X$ w. r. t. a generalized convergence structure on X iff the elementary filter $\mathcal{F}_e((x_n))$ converges to x, where $\mathcal{F}_e((x_n)) = \{Y \subset X : x_n \in Y$ for all but finitely many $n \in \mathbb{N}\}$. If q_1 and q_2 are generalized convergence structures on X, then q_1 is *finer* than q_2 (or q_2 is *coarser* than q_1) iff $q_1 \subset q_2$.

1.2 Remark. There is an alternative description of pretopologies by means of closure operations in the sense of E. Čech [7] (cf. [17; 2.3.1.6.2)] for the details).

1.3 Proposition. *Let q be a pretopology on a set X. Then the convergence of sequences in X fulfills the Urysohn property, i.e. a sequence (x_n) in X converges to $x \in X$ whenever each subsequence of (x_n) has a subsequence converging to x.*

Proof. Let (x_n) be a sequence in X such that each subsequence (y_n) has a subsequence $(y_{n_a})_{a \in \mathbb{N}}$ converging to $x \in X$. Furthermore, let M be the set of all subsequences of (x_n) and $f : M \to M$ a map assiging to each subsequence (y_n) of (x_n) exactly one subsequence of (y_n) converging to x. Then

$$(*)\ \mathcal{F}_e((x_n)) = \bigcap_{(y_n) \in M} \mathcal{F}_e(f((y_n)))\ :$$

1. If $F \in \mathcal{F}_e((x_n))$, then F contains all but finitely many x_n and thus all but finitely many $f((y_n))$, i.e. F belongs to $\mathcal{F}_e(f((y_n)))$ for each $(y_n) \in M$.

2. If $F \in \bigcap_{(y_n) \in M} \mathcal{F}_e(f((y_n)))$ such that $F \notin \mathcal{F}_e((x_n))$ then, for each $n \in \mathbb{N}$, $F \not\supset E_n = \{x_m | m \geq n\}$, i.e. $(X \backslash F) \cap E_n \neq \emptyset$ for each $n \in \mathbb{N}$. Choose $z_n \in E_n \cap (X \backslash F)$ for each $n \in \mathbb{N}$. Hence (z_n) is a sequence consisting of terms of (x_n) which do not belong to F. In particular, the set $I = \{i \in \mathbb{N} : x_i = z_n$ for some $n \in \mathbb{N}\}$ is infinite, namely if I were finite with maximum m, then $z_{m+1} \in E_{m+1}$, i.e. $z_{m+1} = x_k$ with $k \geq m+1$ and $k \in I$, which would imply that m is not the maximum of I – a contradition. Let j_1 be the least $j \in \mathbb{N}$ belonging to I, j_2 the least $j \in \mathbb{N} \backslash \{j_1\}$ belonging to I, and so on. Then (j_n) is a strictly increasing sequence and $(y_n) = (x_{j_n})_{n \in \mathbb{N}}$ is a subseqence of (x_n) consisting of terms which do not belong to F. Therefore, $f((y_n))$ is a subsequence of

(y_n) converging to x whose terms do not belong to F, i.e. $F \notin \mathcal{F}_e(f((y_n)))$, which is imposible. Consequently, each $F \in \bigcap\limits_{(y_n)\in M} \mathcal{F}_e(f((y_n)))$ belongs to $\mathcal{F}_e((x_n))$.

Now $\mathcal{F}_e((x_n)) = \bigcap\limits_{(y_n)\in M} \mathcal{F}_e(f((y_n))) \supset \bigcap\limits_{(\mathcal{H},x)\in q} \mathcal{H} = \mathcal{U}_q(x)$ since each $\mathcal{F}_e(f((y_n)))$ converges to x, i.e. $(\mathcal{F}_e(f((y_n))), x) \in q$. This implies $(\mathcal{F}_e((x_n)), x) \in q$, i.e. (x_n) converges to x in X, since $(\mathcal{U}_q(x), x) \in q$.

1.4 Definition. A lattice L is called *σ-distributive (fully distributive)* provided that the following are satisfied:

1. If $A \subset L$ is countable (arbitrary) such that $\bigvee A$ exists in L, then for $x \in L$, $x \wedge \bigvee A = \bigvee\limits_{a\in A} x \wedge a$.
2. If $B \subset L$ is countable (arbitrary) such that $\bigwedge B$ exists in L, $x \vee \bigwedge B = \bigwedge\limits_{b\in B} x \vee b$.

1.5 Remark. Even if L is a complete lattice, 1. does not imply 2. (cf. [14; 1.1.30] for an example). In particular, if L is a vector lattice (=Riesz space), then L is fully distributive (cf. [14; 12.2]) and thus σ-distributive. Furthermore, every complete chain is completely distributive and hence infinitely distributive (= complete and fully distributive).

1.6 Corollary. *Let L be a σ-distributive (fully distributive) lattice. If $A, B \subset L$ are countable (arbitrary) such that $\bigvee A$ and $\bigvee B$ or $\bigwedge A$ and $\bigwedge B$ exist, then*

1. $\bigvee A \wedge \bigvee B = \bigvee\limits_{a\in A,\, b\in B} a \wedge b$

or

2. $\bigwedge A \vee \bigwedge B = \bigwedge\limits_{a\in A,\, b\in B} a \vee b$

Proof. In order to prove 1. or 2., apply 1. or 2. in 1.4.

1.7 Proposition. *Let L be a σ-distributive lattice.*

1. *If $(\alpha_n^1)_{n\in\mathbb{N}}$ and $(\alpha_n^2)_{n\in\mathbb{N}}$ are increasing sequences in L such that $\bigvee\limits_n \alpha_n^1 = x$ and $\bigvee\limits_n \alpha_n^2 = y$ exist in L, then $(\alpha_n^1 \wedge \alpha_n^2)_{n\in\mathbb{N}}$ is increasing and $\bigvee\limits_n \alpha_n^1 \wedge \alpha_n^2 = x \wedge y$.*
2. *If $(\beta_n^1)_{n\in\mathbb{N}}$ and $(\beta_n^2)_{n\in\mathbb{N}}$ are decreasing sequences in L such that $\bigwedge\limits_n \beta_n^1 = x$ and $\bigwedge\limits_n \beta_n^2 = y$ exist in L, then $(\beta_n^1 \vee \beta_n^2)_{n\in\mathbb{N}}$ is decreasing and $\bigwedge\limits_n \beta_n^1 \vee \beta_n^2 = x \vee y$.*

Proof. 1. By 1.6.1., $x \wedge y = \bigvee_{n,m} \alpha_n^1 \wedge \alpha_m^2$ and since (α_n^1), (α_n^2) are increasing, $(\alpha_n^1 \wedge \alpha_n^2)$ is increasing and $x \wedge y = \bigvee_n \alpha_n^1 \wedge \alpha_n^2$:

(a) $\bigvee_n \alpha_n^1 \wedge \alpha_n^2 \leq \bigvee_{n,m} \alpha_n^1 \wedge \alpha_m^2$.

(b) $n = m$**:** $\alpha_n^1 \wedge \alpha_m^2 = \alpha_n^1 \wedge \alpha_n^2 \leq \bigvee_n \alpha_n^1 \wedge \alpha_n^2$

$n < m$**:** $\alpha_n^1 \wedge \alpha_m^2 \leq \alpha_m^1 \wedge \alpha_m^2 \leq \bigvee_n \alpha_n^1 \wedge \alpha_n^2$

$n > m$**:** $\alpha_n^1 \wedge \alpha_m^2 \leq \alpha_n^1 \wedge \alpha_n^2 \leq \bigvee_n \alpha_n^1 \wedge \alpha_n^2$

(c) By (b) $\bigvee_n \alpha_n^1 \wedge \alpha_n^2 \geq \bigvee_{n,m} \alpha_n^1 \wedge \alpha_m^2$.

2. By 1.6.2. and the fact that (β_n^1) and (β_n^2) are decreasing, the assertion is proved.

2 Order convergence

It is well-known that a convergent sequence of real numbers is bounded. Thus, we may restrict our interest to bounded sequences, i.e. to sequences which are defined on a closed interval $[a, b]$ of the real line $\mathbb{R}$, whenever convergence will be considered. But then convergence can be defined by means of the partial order (induced by the natural partial order on $\mathbb{R}$) on the closed interval $[a, b]$. In particular, a sequence (x_n) of elements of $[a, b]$ converges to $x_0 \in [a, b]$ iff $\overline{\lim}(x_n) = \underline{\lim}(x_n) = x_0$, where $\overline{\lim}(x_n)$ (resp. $\underline{\lim}(x_n)$) denotes the limes superior (resp. limes inferior) of (x_n). Using this idea, order convergence of filters (resp. nets) in complete lattices can be introduced, which has been done by D. C. Kent [12] (resp. G. Birkhoff [6]).

2.1 Definition. Let L be a complete lattice, $\mathcal{F}$ a filter on L and $x_0 \in L$. $\mathcal{F}$ *order converges* to x_0 iff $\overline{\lim}\mathcal{F} = \underline{\lim}\mathcal{F} = x_0$ where $\overline{\lim}\mathcal{F} = \bigwedge_{F \in \mathcal{F}} \bigvee F$ and $\underline{\lim}\mathcal{F} = \bigvee_{F \in \mathcal{F}} \bigwedge F$.

2.2 Proposition. *Let L be a complete lattice and $\mathcal{F} \in F(L)$. If $\mathcal{B}$ is a base of $\mathcal{F}$, then the following hold:*

1. $\overline{\lim}\mathcal{F} = \bigwedge_{B \in \mathcal{B}} \bigvee B$

2. $\underline{\lim}\mathcal{F} = \bigvee_{B \in \mathcal{B}} \bigwedge B$

In particular,

$1'$. $\overline{\lim}\mathcal{F} = \inf\{y \in L :$ *there is some* $B \in \mathcal{B}$ *such that* $y \geq x$ *for each* $x \in B\}$

$2'$. $\underline{\lim}\mathcal{F} = \sup\{y \in L :$ *there is some* $B \in \mathcal{B}$ *such that* $y \leq x$ *for each* $x \in B\}$

Proof. 1. For each $F \in \mathcal{F}$ there is some $B' \in \mathcal{B}$ such that $B' \subset F$. Consequently, $\bigwedge_{B\in\mathcal{B}} \bigvee B \leq \bigvee B' \leq \bigvee F$ and hence, $\bigwedge_{B\in\mathcal{B}} \bigvee B \leq \bigwedge_{F\in\mathcal{F}} \bigvee F$. Since each $B \in \mathcal{B}$ belongs to F, $\bigwedge_{F\in\mathcal{F}} \bigvee F \leq \bigwedge_{B\in\mathcal{B}} \bigvee B$,

2. is proved analogously to 1..

1′. and 2′. are obvious.

2.3 Proposition. *Let L be a complete lattice. Then $\underline{\lim}\mathcal{F} \leq \overline{\lim}\mathcal{F}$ for each $\mathcal{F} \in F(L)$.*

Proof. Let $y \in S_u = \{y \in L :$ there is some $F \in \mathcal{F}$ such that $y \leq x$ for each $x \in F\}$ and $z \in S_0 = \{z \in L :$ there is some $F \in \mathcal{F}$ such that $z \geq x$ for each $x \in F\}$. Then there are $F_1, F_2 \in \mathcal{F}$ such that $y \leq x$ for each $x \in F_1$ and $z \geq x$ for each $x \in F_2$. Since $F_1 \cap F_2 \neq \emptyset$, there exists some $a \in F_1 \cap F_2$ with $y \leq a \leq z$. Therefore, $y \leq \overline{\lim}\mathcal{F} = \inf S_0$ for each $y \in S_u$. But then $\underline{\lim}\mathcal{F} = \sup S_u \leq \overline{\lim}\mathcal{F}$.

2.4 Definition. Let L be a complete lattice, $(x_\alpha)_{\alpha\in I}$ a net in L, and $x_0 \in L$. $(x_\alpha)_{\alpha\in I}$ *order converges* to x_0 iff the filter $\mathcal{F}$ generated by $(x_\alpha)_{\alpha\in I}$ order converges to x_0. Instead of $\underline{\lim}\mathcal{F}$ (resp. $\overline{\lim}\mathcal{F}$) one writes $\underline{\lim}(x_\alpha)$ (resp. $\overline{\lim}(x_\alpha)$).

2.5 Remark. The filter generated by $(x_\alpha)_{\alpha\in I}$ has the base $\mathcal{B} = \{E_\beta : \beta \in I\}$ where $E_\beta = \{x_\alpha : \alpha \geq \beta\}$. If $I = \mathbb{N}$, i.e. $(x_n)_{n\in\mathbb{N}}$ is a sequence in L, the filter generated by (x_n) is the elementary filter of (x_n).

2.6 Corollary. *If (x_α) is a net in a complete lattice, then*

$$\underline{\lim}(x_\alpha) = \bigvee_\beta \bigwedge E_\beta$$

$$\overline{\lim}(x_\alpha) = \bigwedge_\beta \bigvee E_\beta$$

(This is G. Birkhoff's original definition).

2.7 Corollary. ([6; p.244]) *Let L be a complete lattice. A net (x_α) in L order converges to $x_0 \in L$ iff there are nets $t_\alpha \uparrow x_0$ and $u_\alpha \downarrow x_0$ such that $t_\alpha \leq x_0 \leq u_\alpha$, where $t_\alpha \uparrow x_0$ means that the map $\alpha \mapsto t_\alpha$ is isotone and $\sup t_\alpha = x_0$, and $u_\alpha \downarrow x_0$ is defined dually.*

This leads to the following generalization of order convergence to arbitrary partially ordered sets.

2.8 Definition. Let P be a partially ordered set. A sequence (x_n) in P *order converges* to $x_0 \in P$ iff there is an increasing sequence (α_n) with $\sup\{\alpha_n : n \in \mathbb{N}\} = x_0$ and a decreasing sequence (β_n) with $\inf\{\beta_n : n \in \mathbb{N}\} = x_0$ such that $\alpha_n \leq x_n \leq \beta_n$ for each $n \in \mathbb{N}$.

2.9 Remark. An analogous definition can be given for order convergence of nets in a partially ordered set.

2.10 Corollary. ([6; p. 245]). *Each partially ordered set P is a Fréchet L-space with respect to order convergence of sequences, i.e. the following are satisfied:*

1. *If $x_n = x_0$ for each $n \in \mathbb{N}$, then (x_n) order converges to x_0.*
2. *If (x_n) order converges to $x_0 \in P$, then each subsequence $(x_{n_k})_{k\in\mathbb{N}}$ order converges to x_0.*
3. *Order convergence of sequences is unique, i.e. if (x_n) order converges to $x, y \in P$, then $x = y$.*

2.11 Proposition. *Let L be a σ- lattice and (x_n) a sequence in L. Then $\underline{\lim}(x_n) = \bigvee\limits_{n}\bigwedge\limits_{m\geq n} x_m$ and $\overline{\lim}(x_n) = \bigwedge\limits_{n}\bigvee\limits_{m\geq n} x_m$ exist and the following are valid:*

1. (a) $\overline{\lim}(x_n) = \bigwedge\{y \in L :$ there is some $m \in \mathbb{N}$ such then $y \geq x_n$ for each $n \geq m\}$.
   (b) $\underline{\lim}(x_n) = \bigvee\{y \in L :$ there is some $m \in \mathbb{N}$ such then $y \leq x_n$ for each $n \geq m\}$.
2. *(x_n) order converges to $x \in L$ in the sense of 2.8 iff $\overline{\lim}(x_n) = \underline{\lim}(x_n) = x$.*

Proof. 1. is obvious.

2. α) "$\Leftarrow$". Let $\overline{\lim}(x_n) = \underline{\lim}(x_n) = x$. Then $(t_n = \bigvee\limits_{m\geq n} x_m)_{n\in\mathbb{N}}$ is a decreasing sequence with $\sup\limits_n t_n = x$ and $(s_n = \bigwedge\limits_{m\geq n} x_m)_{n\in\mathbb{N}}$ is an increasing sequence with $\sup\limits_n s_n = x$. Furthermore, $s_n \leq x_n \leq t_n$ for each $n \in \mathbb{N}$. Thus, (x_n) order converges to x in the sense of 2.8.

   β) "$\Rightarrow$". By assumption, there exists an increasing sequence (s_n) and a decreasing sequence (t_n) in L with $s_n \leq x_n \leq t_n$ for each $n \in \mathbb{N}$ such that $\sup\limits_n s_n = x$ and $\inf\limits_n t_n = x$. By [18, 8.36], $\underline{\lim}(s_n) = \overline{\lim}(s_n) = \sup\limits_n s_n = x$ and $\underline{\lim}(t_n) = \overline{\lim}(t_n) = \inf\limits_n t_n = x$. Thus, $x = \underline{\lim}(s_n) \leq \underline{\lim}(x_n) \leq \overline{\lim}(x_n) \leq \overline{\lim}(t_n) = x$. Consequently, $\overline{\lim}(x_n) = \underline{\lim}(x_n) = x$.

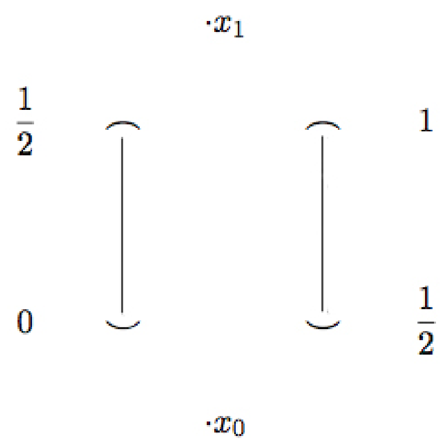
2.12 Proposition. *Let L be a complete lattice and define $q \subset F(L) \times L$ by $(\mathcal{F}, x) \in q$ iff $\mathcal{F}$ order converges to x. Then q is a T_2 generalized convergence structure on L.*

Proof. 1. $\overline{\lim}\dot{x} = \inf\{y \in L : y \geq x\} = x$ and $\underline{\lim}\dot{x} = \sup\{y \in L : y \leq x\} = x$ imply that $\dot{x}$ order converges to x for each $x \in L$.

2. Let $\mathcal{F} \in F(L)$ order converge to $x \in L$ and let $\mathcal{G} \in F(L)$ such that $\mathcal{G} \supset \mathcal{F}$. Obviously, $\overline{\lim}\mathcal{F} \geq \overline{\lim}\mathcal{G}$ and $\underline{\lim}\mathcal{F} \leq \underline{\lim}\mathcal{G}$. Using 2.3 we obtain $\underline{\lim}\mathcal{F} \leq \underline{\lim}\mathcal{G} \leq \overline{\lim}\mathcal{G} \leq \overline{\lim}\mathcal{F}$. Since $\underline{\lim}\mathcal{F} = \overline{\lim}\mathcal{F} = x$, $\underline{\lim}\mathcal{G} = \overline{\lim}\mathcal{G} = x$, i.e. $\mathcal{G}$ order converges to x.

3. The uniqueness of order convergence of filters is obvious.

2.13 Remarks. 1. The T_2 generalized convergence structure q on L in 2.12 is generally not a limit structure which D.C. Kent [12] has been demonstrated by means of the following example: Let L be the (set) union of two replicas of the open interval $(0, 1)$ together with the addition of a least element x_0 and a greatest element x_1 as follows:



Then L is a complete lattice and q is no limit structure: Let $\mathcal{F}_1$ be the elementary filter of $(\frac{1}{2} - \frac{1}{n})_{n>2}$ and $\mathcal{F}_2$ the elementary filter of $(1 - \frac{1}{n})_{n>2}$. Then $\mathcal{F}_1 \xrightarrow{q} x_1$ and $\mathcal{F}_2 \xrightarrow{q} x_1$ but $\mathcal{F}_1 \cap \mathcal{F}_2 \not\rightarrow x_1$ since $\overline{\lim}\mathcal{F}_1 \cap \mathcal{F}_2 = x_1$ and $\underline{\lim}\mathcal{F}_1 \cap \mathcal{F}_2 = x_0$.

2. Next, let us restrict our interest to order convergence of sequences. R. Anguelov and J. H. van der Walt [1] have proved that order convergence of sequences in the lattice $\mathcal{C}(\mathbb{R})$ of all real-valued continous functions on the usual topological space of real numbers does not fulfill the Urysohn property (cf. their example 20 on page 437). Thus, by 1.3, *order convergence of sequences is generally not induced by a pretopology (or topology).* Under certain conditions on a lattice L there exists a limit structure which describes the order convergence of sequences. This will be realized in the following.

2.14 Theorem. *Let L be a σ-distributive lattice and define $q \subset F(L) \times L$ by*

$$(\mathcal{F}, x) \in q \Leftrightarrow \begin{cases} 1. \text{ For each } n \in \mathbb{N} \text{ there is some closed interval } [\alpha_n, \beta_n] \subset L \\ 2. \ [\alpha_{n+1}, \beta_{n+1}] \subset [\alpha_n, \beta_n] \text{ for each } n \in \mathbb{N} \\ 3. \ \sup_n \alpha_n = \inf_n \beta_n = x \\ 4. \ \mathcal{G} \subset \mathcal{F} \text{ where } \mathcal{G} \text{ is the filter generated by the filter base} \\ \quad \{[\alpha_n, \beta_n] : n \in \mathbb{N}\} \end{cases}$$

Then q is a limit structure on L such that for each sequence (x_n) in L the following is valid:
(x_n) order converges to x in the sense of 2.8 iff $(\mathcal{F}_e((x_n)), x) \in q$, i.e. q induces the order convergence of sequences.

Proof. 1. q is a limit structure:

(a) $\dot{x} \xrightarrow{q} x$ is trivial for each $x \in L$ since $\{x\}$is a closed interval.

(b) $\mathcal{F} \xrightarrow{q} x$ and $\mathcal{G} \in F(L)$ with $\mathcal{G} \supset \mathcal{F}$ imply obviously $\mathcal{G} \xrightarrow{q} x$.

(c) Let $\mathcal{F} \xrightarrow{q} x$ and $\mathcal{G} \xrightarrow{q} x$. By assumption, there are $[\alpha_n, \beta_n] \subset L$ with $[\alpha_{n+1}, \beta_{n+1}] \subset [\alpha_n, \beta_n]$ for each $n \in \mathbb{N}$ such that $\sup_n \alpha_n = \inf_n \beta_n = x$ as well as $[\alpha'_n, \beta'_n] \subset L$ with $[\alpha'_{n+1}, \beta'_{n+1}] \subset [\alpha'_n, \beta'_n]$ such that $\sup_n \alpha'_n = \inf_n \beta'_n = x$, where $\mathcal{F}$ and $\mathcal{G}$ are generated by $\{[\alpha_n, \beta_n] : n \in \mathbb{N}\}$ and $\{[\alpha'_n, \beta'_n] : n \in \mathbb{N}\}$ respectively. Then

$$(1) \quad [\alpha_n \wedge \alpha'_n, \beta_n \wedge \beta'_n] \supset [\alpha_n, \beta_n] \cup [\alpha'_n, \beta'_n] \text{ for each } n \in \mathbb{N}$$

By 1.7, $(\alpha_n \wedge \alpha'_n)_{n \in \mathbb{N}}$ is increasing and $(\beta_n \vee \beta'_n)_{n \in \mathbb{N}}$ is decreasing such that $\sup_n(\alpha_n \wedge \alpha'_n) = \inf_n(\beta_n \vee \beta'_n) = x$. Obviously, for each $n \in \mathbb{N}$, $[\alpha_{n+1} \wedge \alpha'_{n+1}, \beta_{n+1} \vee \beta'_{n+1}] \subset [\alpha_n \wedge \alpha'_n, \beta_n \vee \beta'_n]$. Furthermore (cf. (1)), $\mathcal{H} \subset \mathcal{F} \cap \mathcal{G}$, where $\mathcal{H}$ is the filter generated by $\{[\alpha_n \wedge \alpha'_n, \beta_n \vee \beta'_n] : n \in \mathbb{N}\}$, i.e. $\mathcal{F} \cap \mathcal{G} \xrightarrow{q} x$.

2. (a) Let (x_n) be a sequence in L order converging to $x \in L$. Then there are sequences $(\alpha_n), (\beta_n)$ in L with $\alpha_n \leq a_{n+1}$ and $\beta_n \geq \beta_{n+1}$ for each $n \in \mathbb{N}$ such that $\alpha_n \leq x_n \leq \beta_n$, i.e. $x_n \in [\alpha_n, \beta_n]$, and $\sup_n \alpha_n = \inf_n \beta_n = x$. Then the filter generated by $\{[\alpha_n, \beta_n] : n \in \mathbb{N}\}$ is contained in $\mathcal{F}_e((x_n))$, i.e. $\mathcal{F}_e((x_n)) \xrightarrow{q} x$ since each $[\alpha_n, \beta_n]$ contains all but finitely many x_n.

(b) Let (x_n) be a sequence in L such that $\mathcal{F}_e((x_n)) \xrightarrow{q} x$. Then there exists $[\alpha_n, \beta_n] \subset L$ for each $n \in \mathbb{N}$ with $\alpha_n \leq \alpha_{n+1}$ and $\beta_n \geq \beta_{n+1}$ as well as $\sup_n \alpha_n = \inf_n \beta_n = x$. Additionally, $\{[\alpha_n, \beta_n] : n \in \mathbb{N}\} \subset \mathcal{F}_e((x_n))$: Inductively, an increasing sequence $(k_n)_{n \in \mathbb{N}}$ of natural numbers is constructed such that

$$(2) \quad \alpha_n \leq x_m \leq \beta_n \text{ for each } m \geq k_n.$$

Then two sequences $(a_m)_{m\in\mathbb{N}}$ and $(b_m)_{m\in\mathbb{N}}$ are defined as follows:

$$\begin{aligned} a_m &= \inf\{x_1,\ldots,x_{k_1-1},\alpha_1\},\ m=1,2,\ldots,k_1-1 \\ a_m &= \alpha_n,\ m=k_n,k_n+1,\ldots,k_{n+1}-1,\ n=1,2,\ldots \\ b_m &= \sup\{x_1,\ldots,x_{k_1-1},\beta_1\},\ m=1,2,\ldots,k_1-1 \\ b_m &= \beta_n,\ m=k_n,k_n+1,\ldots,k_{n+1}-1,\ n=1,2,\ldots \end{aligned}$$

Obviously, $(a_m)_{n\in\mathbb{N}}$ is increasing and $(b_m)_{m\in\mathbb{N}}$ is decreasing. By (2), $a_m \leq x_m \leq b_m$, $m\in\mathbb{N}$. Furthermore, $\sup\limits_m a_m = \sup\limits_n \alpha_n = x$ and $\inf\limits_m b_m = \inf\limits_n \beta_n = x$. Thus, (x_n) order converges to x.

2.15 Corollary. *Let L be a σ-distributive lattice. Then the limit structure q on L in 2.14 is* T_2.

Proof. Let $\mathcal{F} \xrightarrow{q} x, y$. Then, for each $n\in\mathbb{N}$, there are intervals $[\alpha_n^1,\beta_n^1]$ and $[\alpha_n^2,\beta_n^2]$ in L with $[\alpha_{n+1}^1,\beta_{n+1}^1]\subset[\alpha_n^1,\beta_n^1]$ and $[\alpha_{n+1}^2,\beta_{n+1}^2]\subset[\alpha_n^2,\beta_n^2]$ such that $\sup\limits_n \alpha_n^1 = \inf\limits_n \beta_n^1 = x$ and $\sup\limits_n \alpha_n^2 = \inf\limits_n \beta_n^2 = y$ as well as $\{[\alpha_n^1,\beta_n^1] : n\in\mathbb{N}\}\subset\mathcal{F}$ and $\{[\alpha_n^2,\beta_n^2] : n\in\mathbb{N}\}\subset\mathcal{F}$.
Thus, for each $n\in\mathbb{N}$, $[\alpha_n^1,\beta_n^1]\cap[\alpha_n^2,\beta_n^2]\neq\emptyset$. Choose exactly one x_n form each $[\alpha_n^1,\beta_n^1]\cap[\alpha_n^2,\beta_n^2]$. Hence, (x_n) is a sequence in L order converging to x and y. This implies $x=y$ (cf. 2.10.3.).

2.16 Remarks. 1. *If X is a lattice, then $q\subset F(X)\times X$ as defined in 2.14 is a generalized convergence structure, which induces the order convergence of sequences. Furthermore, q is* T_2. (cf. the corresponding proofs of 2.14 and 2.15).

2. By [15; theorem 12.2], each vector lattice (=Riesz space) is σ-distributive. In particular, the set $\mathcal{C}(X)$ of all continuous real-valued maps on a topological space X endowed with the pointwise order is a vector lattice. Thus, *the above results on $\sigma-$distributive lattices are valid for $\mathcal{C}(X)$* (cf. also [1]). But also the following in this section is valid.

2.17 Proposition. *Let L be a σ-distributive lattice. Then the order convergence of sequences fulfills the following conditions (and additionally the conditions 1., 2. and 3. in 2.10):*

4. *If ζ,η are sequences in L such that ζ order converges to $x\in L$, and $\mathcal{F}_e(\zeta)=\mathcal{F}_e(\eta)$, then η order converges to x.*

5. *If ζ,η are sequences in L order converging to $x\in L$, then $\zeta*\eta$ order converges to x, where $\zeta*\eta$ denotes the simple mixture of ζ and η defined by $\zeta*\eta(2n-1)=\zeta(n)$ and $\zeta*\eta(2n)=\eta(n)$ for each $n\in\mathbb{N}$*

Proof. Since, L is σ-distributive, there is a limit structure q on L inducing the order convergence of sequences (cf. 2.14). Thus, 4. is obvious and 5. follows from 3. in 1.1..

2.18 Corollary. *Let L be a σ-distributive lattice. Then there is a finest limit structure q_0 on L inducing the order convergence of sequences and being defined by*

$$(\mathcal{F}, x) \in q_0 \Leftrightarrow \quad \text{There is a sequence } \zeta \text{ in } L \text{ order converging to } x \text{ such that } \mathcal{F}_e(\zeta) \subset \mathcal{F}.$$

In particular, q_0 is T_2.

Proof. 1. q_0 is a limit structure (cf. [3; proof of 1.7.6]).

2. Let q' be a limit structure on L inducing the order convergence of sequences. If $(\mathcal{F}, x) \in q_0$ there is a sequence ζ in L order converging to x such that $\mathcal{F}_e(\zeta) \subset \mathcal{F}$. Since $\mathcal{F}_e(\zeta)$ converges to x w.r.t. q', $\mathcal{F}$ converges to x w.r.t. q', i.e. $q_0 \subset q'$.

3. By 2.14 and 2.15, q is a T_2 limit structure and q induces the order convergence of sequences. Thus, by 2., q_0 is finer than q which implies that q_0 is also T_2.

3 Convergence almost everywhere

In this section a measure μ on a σ-algebra $(\Omega, \mathcal{A})$ is assumed to be non-trivial, i.e. $\mu \neq 0$ (or equivalently: $\mu(\Omega) \neq 0$).

3.1 Definition. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $\bar{\mathbb{R}}$ the extended real line, i.e. $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.

1. A function $f : \Omega \to \bar{\mathbb{R}}$ is called $(\mathcal{A}-)$*measurable* provided that it is $\mathcal{A}-\mathcal{B}^1-$measurable, where $\mathcal{B}^1$ is the σ-algebra on $\bar{\mathbb{R}}$ whose trace on $\mathbb{R}$ is the set $\mathcal{B}$ of all Borel sets in $\mathbb{R}$ (i.e. $\mathcal{B}^1$ consists of B, $B \cup \{+\infty\}$, $B \cup \{-\infty\}$, and $B \cup \{-\infty, +\infty\}$ for all $B \in \mathcal{B}$).

2. Let $N \subset \Omega$ be a $(\mu-)$null set (i.e. $N \in \mathcal{A}$ and $\mu(N) = 0$) and $M = \Omega \backslash N$. A map $f : M \to \bar{\mathbb{R}}$ which is $(M \cap \mathcal{A}-)$measurable, is called *a $(\mu-)$almost everywhere defined $(\mathcal{A}-)$measurable function* (note: $M \cap \mathcal{A} = \{M \cap A : A \in \mathcal{A}\}$.

3. If $f, g : \Omega \to \bar{\mathbb{R}}$ are $(\mathcal{A})$-measurable functions, then f is said to be *equivalent* to g iff $f = g$ (μ-)almost everywhere, i.e. $f(x) = g(x)$ for each $x \in \Omega \backslash N$ where N is a (μ-)null set. The equivalence class belonging to f is denoted by $[f]$.

4. (a) If $f : M \to \bar{\mathbb{R}}$ is a μ-almost everywhere defined $\mathcal{A}$-measurable function, then f° denotes its trivial extension,
   i.e. $f^\circ(x) = \begin{cases} f(x) & \text{for each } x \in M \ , \\ 0 & \text{for each } x \in \Omega \backslash M \ . \end{cases}$

(b) Let $f : A \to \bar{\mathbb{R}}$ and $g : B \to \bar{\mathbb{R}}$ be μ-almost everywhere defined $\mathcal{A}$-measurable functions. Then f is called *equivalent* to g iff $[f^\circ] = [g^\circ]$, i.e. iff there is a μ-null set N such that for each $x \in M = A \cap B \cap (\Omega \backslash N)$, $f(x) = g(x)$ (obviously M is the complement of a μ-null set, i.e. $f = g$ μ-almost everywhere). The corresponding equivalence class of f is denoted by $\tilde{f}$.

3.2 Corollary. a) *Let $X = \{[f^\circ] : f$ μ-almost everywhere defined $\mathcal{A}$-measurable function$\}$ and $Z = \{\tilde{f} : f$ $\mu-$almost everywhere defined $\mathcal{A}$-measurable function$\}$. Then $H : Z \to X$ defined by $H(\tilde{f}) = [f^\circ]$ is bijective.*

b) *Let $Y = \{[f] : f : \Omega \to \bar{\mathbb{R}}$ $\mathcal{A}$-measurable function$\}$. Then $X = Y$.*

3.3 Definition. A *partial order* $\leq$ *on* X is defined as follows:

$$[f] \leq [g] \Leftrightarrow \quad f \leq g \ \mu\text{-almost everywhere} \quad (\text{i.e. } f(x) \leq g(x) \text{ for each } x \in \Omega \backslash N, \text{ where } N \text{ is a } \mu\text{-null set}).$$

3.4 Remark. 3.3 is independent of the choice of the representatives.

3.5 Proposition. *Let $([f_n])_{n\in\mathbb{N}}$ be a sequence in X. Then $\sup_n [f_n]$ and $\inf_n [f_n]$ exist and the following are valid:*

1. $\sup_n [f_n] = [\sup_n f_n]$,
2. $\inf_n [f_n] = [\inf_n f_n]$.

In particular, $(X, \leq)$ is a σ-lattice which is σ-distributive.

3.6 Corollary. *Let $([f_n])_{n\in\mathbb{N}}$ be a sequence in X. Then $\overline{\lim}[f_n]$ and $\underline{\lim}[f_n]$ exist and the following are satisfied:*

1. $\overline{\lim}[f_n] = [\overline{\lim} f_n]$,
2. $\underline{\lim}[f_n] = [\underline{\lim}[f_n]$.

Proof of 3.5. By [2; 9.5], $\sup_n f_n$ and $\inf_n f_n$ are $\mathcal{A}$-measurable functions from Ω to $\bar{\mathbb{R}}$.

1. (a) Let $g_n \in [f_n]$, i.e. $[g_n] = [f_n]$ for each $n \in \mathbb{N}$. Then $g_n = f_n$ almost everywhere for each $n \in \mathbb{N}$, which implies that there is some μ-null set N such that for each $x \in \Omega \backslash N$, $g_n(x) = f_n(x)$ for all $n \in \mathbb{N}$, i.e. $\sup_n f_n = \sup_n g_n$ almost everywhere. Thus, $[\sup_n f_n] = [\sup_n g_n]$.

(b) α) $[\sup_n f_n]$ is an upper-bound of $\{[f_n] : n \in \mathbb{N}\}$ ($f_n(x) \leq \sup_n f_n(x)$ for each $x \in \Omega$ and each $n \in \mathbb{N}$, i.e. $f_n \leq \sup_n f_n$ even everywhere).

β) Let $[f_n] \leq [h]$ for each $n \in \mathbb{N}$. Then $f_n \leq h$ almost everywhere for each $n \in \mathbb{N}$ which implies that there is a μ-null set N such that for each $x \in \Omega \backslash N$, $f_n(x) \leq h(x)$ for all $n \in \mathbb{N}$. Thus, $\sup_n f_n \leq h$ almost everywhere, i.e. $[\sup_n f_n] \leq [h]$.

2. is proved analogously.

3. The σ- distributivity of $(X, \leq)$ follows from the rules for equivalence classes and the fact that the $\mathcal{A}$-measurable functions on Ω form a σ-distributive σ-lattice (note: $\bar{\mathbb{R}}$ is a complete chain and thus completely distributive).

Proof of 3.6. By [2; 9.5], $\overline{\lim} f_n$ and $\underline{\lim} f_n$ are $\mathcal{A}$-measurable.

a) Let $[f_n] = [g_n]$ for each $n \in \mathbb{N}$. Then there is a μ-null set N such that for each $x \in \Omega \backslash N$ $f_n(x) = g_n(x)$ for all $n \in \mathbb{N}$ which implies $\overline{\lim} f_n = \overline{\lim} g_n$ almost everywhere, i.e. $[\overline{\lim} f_n] = [\overline{\lim} g_n]$.

b) Applying 3.5 one obtains $\overline{\lim}[f_n] = \bigwedge_m \bigvee \{[f_n] : n \geq m\} =$

$\bigwedge_m \{[\bigvee f_n] : n \geq m\} = [\bigwedge_m \bigvee_{n \geq m} f_n] = [\overline{\lim} f_n]$

Thus, 1. is proved. The proof of 2. is similar.

3.7 Definition. A sequence $([f_n])_{n \in \mathbb{N}}$ in X is called *convergent to* $[f] \in X$ *(μ-)almost everywhere* iff $(f_n)_{n \in \mathbb{N}}$ converges to f (μ-)almost everywhere.

3.8 Remarks. 1. The above definition is independent of the choice of the representatives.

2. *There is – in general – no pretopology (and thus no topology) on X inducing the convergence of sequences (μ-)almost everywhere* as the following example shows:

3.9 Example. Let $\Omega = [0, 1]$ (= closed unit interval), $\mathcal{A}$ the σ-algebra of Borel sets, and λ the Lebesgue measure. Further, for each $S \subset [0, 1]$ let χ_S be the characteristic function of S and let $(f_n)_{n \in \mathbb{N}}$ be a sequence of $\mathcal{A}$-measurable real-valued functions on $[0, 1]$ whose first term is $\chi_{[0,1]}$, whose next two terms are $\chi_{[0,\frac{1}{2}]}$ and $\chi_{[\frac{1}{2},1]}$, whose next three terms are $\chi_{[0,\frac{1}{3}]}$, $\chi_{[\frac{1}{3},\frac{2}{3}]}$, and $\chi_{[\frac{2}{3},1]}$, whose next four terms are $\chi_{[0,\frac{1}{4}]}$, $\chi_{[\frac{1}{4},\frac{2}{4}]}$, $\chi_{[\frac{2}{4},\frac{3}{4}]}$, and $\chi_{[\frac{3}{4},1]}$, and so on. Obviously, (f_n) converges to the zero function 0 in measure (i.e. for each $\varepsilon > 0$, $\lim \mu(\{x \in [0, 1] : |f_n(x)| > \varepsilon\} = 0)$ but (f_n) does not converge pointwise to the zero function since, for each $x \in [0, 1]$, the sequence (f_n) contains infinitely many ones and infinitely many zeros. Thus, (f_n) does not converges to the zero function 0 λ-almost everywhere. Consequently, $([f_n])$ does not converge to $[0]$ λ-almost everywhere. If there is a pretopology on X inducing the convergence of sequences λ-almost everywhere, then there exists a neighborhood N of $[0]$ such that infinitely many terms of $([f_n])$ are outside of N. Hence, there is a subsequence $([f_{n_l}])_{l\in\mathbb{N}}$ of $([f_n])$ whose terms belong to the complement of N. Furthermore, $(f_{n_l})_{l\in\mathbb{N}}$ converges to 0 in measure. By [8; 3.13], there is a subsequence of $(f_{n_l})_{l\in\mathbb{N}}$ which converges to 0 λ-almost everywhere and whose corresponding subsequence of equivalence classes converges to $[0]$ λ-almost everywhere.Then all but finitely many terms of this subsequence belong to N in contrast of the fact that all terms of $([f_{n_l}])$ are outside of N. Hence, such a pretopology cannot exist.

3.10 Theorem. *Let X be the σ-lattice described under 3.2-3.6. The order convergence of sequences in X is exactly the convergence almost everywhere. It is induced by a limit structure on X, in particular by a finest one.*

Proof.

1. Let $([f_n])$ be a sequence in X order converging to $[f] \in X$. Thus, $\overline{\lim}([f_n]) = [\overline{\lim}(f_n)] = [f]$ and $\underline{\lim}([f_n]) = [\underline{\lim}(f_n)] = [f]$, i.e. $\overline{\lim}(f_n) = f$ μ-almost everywhere and $\underline{\lim}(f_n) = f$ μ-almost everywhere. Then (f_n) converges to f μ-almost everywhere, i.e. $([f_n])$ converges to $[f]$ μ-almost everywhere.

2. Let $([f_n])$ be a sequence in X converging to $[f] \in X$ μ-almost everywhere. Then $[f] = [\underline{\lim}(f_n)] = \underline{\lim}([f_n])$ and $[f] = [\overline{\lim}(f_n)] = \overline{\lim}([f_n])$, i.e. $([f_n])$ order converges to $[f]$ in $(X, \leq)$.

3. Since X is σ-distributive there is a limit structure on X inducing the order convergence of sequences (cf. 2.14), in particular a finest one (cf. 2.18).

3.11 Remark. A pair (S, q_S) is called a *limit space* iff X is a set and q_S a limit structure on S. A map $f : (S, q_S) \to (S', q_{S'})$ between limit spaces is said to be *continuous* iff $(f(\mathcal{F}), f(x)) \in q_{S'}$ for each $(\mathcal{F}, x) \in q_S$. The category **Lim** of limit spaces (and continuous maps) is a topological construct (cf. [17]). In particular, if S is a set, $((S_i, q_{S_i}))_{i\in I}$ a family of limit spaces, and $(f_i : S \to S_i)_{i\in I}$ a family of maps, then $q_S = \{(\mathcal{F}, x) \in F(S) \times S : (f_i(\mathcal{F}), f_i(x)) \in q_{S_i}$ for each $i \in I\}$ is the initial limit structure such that each $f_i : (S, q_S) \to (S_i, q_{S_i})$ is continuous. This one is needed for the following corollaries.

3.12 Definition. A sequence $(\tilde{f}_n)_{n\in\mathbb{N}}$ in $Z = \{\tilde{f} : f$ μ-almost everywhere defined $\mathcal{A}$-measurable function $\}$ is said to *converge* to $\tilde{f} \in Z$ (μ-)*almost everywhere* iff the sequence $(f_n)_{n\in\mathbb{N}}$ convergence to f (μ-)almost everywhere, i.e. iff the set of all points $x \in \Omega$, where all $f_n(x)$ are defined and the sequence $(f_n(x))$ converges to $f(x)$, is the complement of a (μ)-null set.

3.13 Remark. The above definition is independent of the choice of the representatives.

3.14 Corollary. *Let q be a limit structure on X inducing the convergence of sequences almost everywhere (cf. 2.14 and 2.16). Then the initial limit structure $\tilde{q}$ on Z w.r.t. the bijective map $H : Z \to X$ defined by $H(\tilde{f}) = [f^\circ]$ induces the convergence of sequences in Z almost everywhere. In particular, $H : (Z, \tilde{q}) \to (X, q)$ is an isomorphism (in* **Lim**).

Proof. 1. By definition, $(\mathcal{F}, \tilde{f}) \in \tilde{q}$ iff $(H(\mathcal{F}), H(\tilde{f})) \in q$. Thus, a sequence $(\tilde{f_n})$ in Z converges to $\tilde{f} \in Z$ w.r.t. $\tilde{q}$ iff $([f_n^\circ])$ converges to $[f^\circ]$ w.r.t. q, i.e. iff $(f_n)_{n\in\mathbb{N}}$ converges to f almost everywhere. But this means that $(\tilde{f_n})$ converges to $\tilde{f}$ almost everywhere.

2. H is an isomorphism since that it is a surjective embedding.

3.15 Remark. 3.14 means that the almost everywhere convergence of sequences of almost everywhere defined $\mathcal{A}$-measurable functions is included in our considerations. If we restrict our interest to equivalence classes of real-valued $\mathcal{A}$-measurable functions, we obtain the following corollary, which corresponds to the case described by U. Höhle [11] for $\Omega = [0, 1]$, $\mathcal{A} = [0, 1] \cap \mathcal{B}$, and $\mu = \lambda$ in a different way.

3.16 Corollary. *Let $X' = \{f' : f : \Omega \to \mathbb{R}\ \mathcal{A}\text{-measurable}\}$ where $f : \Omega \to \mathbb{R}$ is called $\mathcal{A}$-measurable iff it is $\mathcal{A}$-$\mathcal{B}$-measurable and f' is the equivalence class of the $\mathcal{A}$-measurable real-valued function f w.r.t. the equivalence relation ρ on $\{f : f : \Omega \to \mathbb{R}\ \mathcal{A}\text{-measurable}\}$ defined by $f\rho g$ iff $f = g$ μ-almost everywhere. If $i : \mathbb{R} \to \bar{\mathbb{R}}$ denotes the inclusion map, then $k : X' \to X$ defined by $k(f') = [i \circ f] \in X$ for each $f' \in X'$ is well-defined and injective. Further, let q be a limit structure on X inducing the convergence of sequences almost everywhere and denote the initial limit structure on X' w.r.t. k by q', then a sequence (f_n') in X converges to $f' \in X'$ w.r.t. q' iff $([i \circ f_n])$ converges to $[i \circ f]$ w.r.t. q, i.e. iff (f_n) converges to f μ-almost everywhere. In particular, $k : (X', q') \to (X, q)$ is an embedding (in* **Lim**).

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