Some remarks on stronger versions of the Boundary Problem for Banach spaces

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Abstract

Let X be a real Banach space. A subset B of the dual unit sphere of X is said to be a boundary for X, if every element of X attains its norm on some functional in B. The well-known Boundary Problem originally posed by Godefroy asks whether a bounded subset of X which is compact in the topology of pointwise convergence on B is already weakly compact. This problem was recently solved by Pfitzner in the positive. In this note we collect some stronger versions of the solution to the Boundary Problem, most of which are restricted to special types of Banach spaces. We shall use the results and techniques of Pfitzner, Cascales et al., Moors and others.

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1 Introduction

First we fix some notation: Throughout this paper X denotes a real Banach space, X^* its dual, B_X its closed unit ball and S_X its unit sphere. For a subset B of X^* we denote by σ_B the topology on X of pointwise convergence on B. If $A \subseteq X$, then $\operatorname{co} A$ stands for the convex hull of A and $\overline{A}^\tau$ for the closure of A in any topology τ on X, except for the norm closure, which we simply denote by $\overline{A}$. Also, we denote by $\operatorname{ex} C$ the set of extreme points of a convex subset C of X.

Now recall that a subset B of S_{X^*} is called a boundary for X, if for every $x \in X$ there is some $b \in B$ such that $b(x) = \|x\|$. It easily follows from the Krein-Milman theorem that $\operatorname{ex} B_{X^*}$ is always a boundary for X. In 1980 Bourgain and Talagrand proved in [4] that a bounded subset A of X is weakly compact if it is merely compact in the topology σ_E, where $E = \operatorname{ex} B_{X^*}$. In [14] Godefroy asked whether the same statement holds for an arbitrary boundary B, a question which has become known as the Boundary Problem.

Long since only partial positive answers were known, for example if $X = C(K)$ for some compact Hausdorff space K (cf. [5, Proposition 3]) or $X = \ell^1(I)$ for some set I (cf. [9, Theorem 4.9]). In [24, Theorem 1.1] the positive answer for L_1-preduals is contained. Moreover, the answer is positive if the set A is additionally assumed to be convex (cf. [15, p.44]). It was only in 2008 that the positive answer to the Boundary Problem was found in full generality by Pfitzner in [19]. An extended version [20] of this work is going to appear in Inventiones Mathematicae.

An important tool in the study of the Boundary Problem is the so called Simons' equality:

Theorem 1.1 (Simons, cf. [23])**.** *If B is a boundary for X, then*

$$\sup_{x^*\in B}\limsup_{n\to\infty} x^*(x_n) = \sup_{x^*\in B_{X^*}}\limsup_{n\to\infty} x^*(x_n) \tag{1}$$

holds for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X.

In particular, it follows from Theorem 1.1 that the well-known Rainwater's theorem for the extreme points of the dual unit ball (cf. [21]) holds true for an arbitrary boundary:

Corollary 1.2 (Simons, cf. [22] or [23])**.** *If B is a boundary for X, then a bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X is weakly convergent to $x \in X$ iff it is convergent to x under every functional in B.*

Pfitzner's proof also uses Simons' equality, as well as a quantitative version of Rosenthal's ℓ^1-theorem due to Behrends (cf. [3]) and an ingenious variant of Hagler-Johnson's construction.

Next we recall the following known characterization of weak compactness (compare [16, p.145-149], [12, Theorem 5.5 and Exercise 5.19] as well as the proof of [10, Theorem V.6.2]). It is a strengthening of the usual Eberlein-Šmulian theorem.

Theorem 1.3. *Let A be a bounded subset of X. Then the following assertions are equivalent:*

(i) *A is weakly relatively compact.*

(ii) *For every sequence $(x_n)_{n\in\mathbb{N}}$ in A we have that*

$$\bigcap_{k=1}^{\infty} \overline{\mathrm{co}}\,\{x_n : n \geq k\} \neq \emptyset.$$

(iii) *For every sequence $(x_n)_{n\in\mathbb{N}}$ in A there is some $x \in X$ such that*

$$x^*(x) \leq \limsup_{n\to\infty} x^*(x_n) \quad \forall x^* \in X^*.$$

In [17] Moors proved a statement stronger than the equivalence of (i) and (ii), which also sharpens the result from [4]:

Theorem 1.4 (Moors, cf. [17]). *A bounded subset A of X is weakly relatively compact iff for every sequence $(x_n)_{n\in\mathbb{N}}$ in A we have that*

$$\bigcap_{k=1}^{\infty} \overline{\text{co}}^{\sigma_E} \{x_n : n \geq k\} \neq \emptyset,$$

where $E = \text{ex}\, B_{X^*}$. In particular, A is weakly relatively compact if it is merely relatively countably compact in the topology σ_E.

In fact, Moors gets this theorem as a corollary to the following one:

Theorem 1.5 (Moors, cf. [17]). *Let A be an infinite bounded subset of X. Then there exists a countably infinite set $F \subseteq A$ with $\overline{\text{co}}^{\sigma_E} F = \overline{\text{co}} F$, where $E = \text{ex}\, B_{X^*}$. In particular, for each bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with $\overline{\text{co}}^{\sigma_E} \{x_{n_k} : k \in \mathbb{N}\} \subseteq \overline{\text{co}}\, \{x_n : n \in \mathbb{N}\}$.*

The object of this paper is to give some results related to Theorem 1.4 in the more general context of boundaries. In particular, we shall see, by a very slight modification of the construction from [19], that a 'non-relative' version of 1.4 holds for any boundary B of X, see Theorem 2.18.

Since we will also deal with some quantitative versions of Theorem 1.4, it is necessary to introduce a bit more of terminology, which stems from [11]: Given $\varepsilon \geq 0$, a bounded subset A of X is said to be ε-weakly relatively compact (in short ε-WRC) provided that $\text{dist}(x^{**}, X) \leq \varepsilon$ for every element $x^{**} \in \overline{A}^{w^*}$, where w^* refers to the weak*-topology of X^{**}. For $\varepsilon = 0$ this is equivalent to the classical case of weak relative compactness.

The authors of [11] used this notion to give a quantitative version of the well known theorem of Krein (cf. [11, Theorem 2]). In their proof they made use of double limit techniques in the spirit of Grothendieck. More precisely, they worked with the following definition: Let bounded subsets A of X, M of X^* and $\varepsilon \geq 0$ be given. Then A is said to have ε-interchangeable double limits with M if for any two sequences $(x_n)_{n\in\mathbb{N}}$ in A and $(x_m^*)_{m\in\mathbb{N}}$ in M we have

$$\left| \lim_{n\to\infty} \lim_{m\to\infty} x_m^*(x_n) - \lim_{m\to\infty} \lim_{n\to\infty} x_m^*(x_n) \right| \leq \varepsilon,$$

provided that all the limits involved exist. In this case we write $A \S \varepsilon \S M$. The connection to ε-WRC sets is given by the following proposition:

Proposition 1.6 (Fabian et al., cf. [11]). *Let $A \subseteq X$ be bounded and $\varepsilon \geq 0$. Then the following hold:*

(i) *If A is ε-WRC, then $A \S 2\varepsilon \S B_{X^*}$.*

(ii) *If $A \S \varepsilon \S B_{X^*}$, then A is ε-WRC.*

In case $\varepsilon = 0$ this is the classical Grothendieck double limit criterion. For various other quantitative results on weak compactness we refer the interested reader to [2], [6], [7] and [11].

We are now ready to formulate and prove our results. However, it should be added that all of them can easily be derived from already known results and techniques.

2 Results and proofs

We begin with a quantitative version of Theorem 1.3. First we prove an easy lemma that generalizes the equivalence of (ii) and (iii) in said theorem (the proof is practically the same).

Lemma 2.1. *Let B be a subset of B_{X^*} that separates the points of X and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X as well as $x \in X$ and $\varepsilon \geq 0$. Then the following assertions are equivalent:*

(i) $x \in \bigcap_{k=1}^{\infty} \overline{\mathrm{co}}^{\sigma_B}\left(\{x_n : n \geq k\} + \varepsilon B_X\right)$.

(ii) $x^*(x) \leq \limsup_{n\to\infty} x^*(x_n) + \varepsilon \quad \forall x^* \in B_{X^*} \cap \operatorname{span} B$.

Proof. First we assume (i). It then directly follows that

$$x^*(x) \in \overline{\mathrm{co}}\left(\{x^*(x_n) : n \geq k\} + [-\varepsilon, \varepsilon]\right) \;\forall k \in \mathbb{N}\; \forall x^* \in B_{X^*} \cap \operatorname{span} B.$$

Thus we also have $x^*(x) \leq \sup_{n\geq k} x^*(x_n) + \varepsilon$ for all $k \in \mathbb{N}$ and all $x^* \in B_{X^*} \cap \operatorname{span} B$ and the assertion (ii) follows.

Now we assume that (ii) holds and take $k \in \mathbb{N}$ arbitrary. Suppose that

$$x \notin \overline{\mathrm{co}}^{\sigma_B}\left(\{x_n : n \geq k\} + \varepsilon B_X\right).$$

Then by the separartion theorem we could find a functional $x^* \in (X_{\sigma_B})^* = \operatorname{span} B$ with $\|x^*\| = 1$ and a number $\alpha \in \mathbb{R}$ such that

$$x^*(y) \leq \alpha < x^*(x) \quad \forall y \in \overline{\mathrm{co}}^{\sigma_B}\left(\{x_n : n \geq k\} + \varepsilon B_X\right).$$

It follows that

$$\limsup_{n\to\infty} x^*(x_n) + \varepsilon \leq \alpha < x^*(x),$$

a contradiction which ends the proof. □

Now we can give a quantitative version of the first equivalence in Theorem 1.3.

Theorem 2.2. *Let $A \subseteq X$ be bounded and $\varepsilon \geq 0$. If for each sequence $(x_n)_{n\in\mathbb{N}}$ in A we have*

$$\bigcap_{k=1}^{\infty} \overline{\mathrm{co}}\left(\{x_n : n \geq k\} + \varepsilon B_X\right) \neq \emptyset, \tag{2}$$

then A is 2ε-WRC. If A is ε-WRC, then (2) holds for every sequence in A.

Proof. First we assume that (2) holds for every sequence in A. Let $(x_n)_{n\in\mathbb{N}}$ and $(x_m^*)_{m\in\mathbb{N}}$ be sequences in A and B_{X^*}, respectively, such that the limits

$$\lim_{n\to\infty}\lim_{m\to\infty} x_m^*(x_n) \text{ and } \lim_{m\to\infty}\lim_{n\to\infty} x_m^*(x_n)$$

exist. By assumption, we can pick an element

$$x \in \bigcap_{k=1}^{\infty} \overline{\mathrm{co}}\left(\{x_n : n \geq k\} + \varepsilon B_X\right).$$

From Lemma 2.1 we conclude that

$$\liminf_{n\to\infty} x^*(x_n) - \varepsilon \leq x^*(x) \leq \limsup_{n\to\infty} x^*(x_n) + \varepsilon \quad \forall x^* \in B_{X^*}. \tag{3}$$

It follows that

$$\left| x_m^*(x) - \lim_{n\to\infty} x_m^*(x_n) \right| \leq \varepsilon \quad \forall m \in \mathbb{N}.$$

By passing to a subsequence we may assume that $\lim_{m\to\infty} x_m^*(x)$ exists. Thus we get

$$\left| \lim_{m\to\infty} x_m^*(x) - \lim_{m\to\infty}\lim_{n\to\infty} x_m^*(x_n) \right| \leq \varepsilon. \tag{4}$$

Now take a weak*-cluster point $x^* \in B_{X^*}$ of the sequence $(x_m^*)_{m\in\mathbb{N}}$. Then

$$\lim_{m\to\infty} x_m^*(x) = x^*(x) \text{ and } \lim_{n\to\infty}\lim_{m\to\infty} x_m^*(x_n) = \lim_{n\to\infty} x^*(x_n). \tag{5}$$

By (3) we have

$$\left| x^*(x) - \lim_{n\to\infty} x^*(x_n) \right| \leq \varepsilon. \tag{6}$$

From (4), (5) and (6) we get

$$\left| \lim_{m\to\infty}\lim_{n\to\infty} x_m^*(x_n) - \lim_{n\to\infty}\lim_{m\to\infty} x_m^*(x_n) \right| \leq 2\varepsilon.$$

Thus we have proved $A \S 2\varepsilon \S B_{X^*}$. Hence, by Proposition 1.6, A is 2ε-WRC.

Now assume that A is ε-WRC and take any sequence $(x_n)_{n\in\mathbb{N}}$ in A. Let $x^{**} \in \overline{A}^{w^*}$ be a weak*-cluster point of $(x_n)_{n\in\mathbb{N}}$ and fix $\delta > 0$. Since A is ε-WRC there is some $x \in X$ such that $\|x^{**} - x\| \leq \varepsilon + \delta$.

For every $x^* \in B_{X^*}$ the number $x^{**}(x^*)$ is a cluster point of the sequence $(x^*(x_n))_{n\in\mathbb{N}}$ and thus

$$x^*(x) \leq \|x - x^{**}\| \, \|x^*\| + x^{**}(x^*) \leq \varepsilon + \delta + \limsup_{n\to\infty} x^*(x_n).$$

Since $\delta > 0$ was arbitrary, we conclude that

$$x^*(x) \leq \varepsilon + \limsup_{n\to\infty} x^*(x_n) \quad \forall x^* \in B_{X^*}.$$

Lemma 2.1 now yields

$$x \in \bigcap_{k=1}^{\infty} \overline{\mathrm{co}}\left(\{x_n : n \geq k\} + \varepsilon B_X\right)$$

and the proof is finished. □

As an immediate corollary we get

Corollary 2.3. *If $A \subseteq X$ is bounded and $\varepsilon \geq 0$ such that*

$$\bigcap_{k=1}^{\infty} \left(\overline{\text{co}}\,\{x_n : n \geq k\} + \varepsilon B_X\right) \neq \emptyset$$

for every sequence $(x_n)_{n\in\mathbb{N}}$ in A, then A is 2ε-WRC.

Now we can also prove a quantitative version of Theorem 1.4:

Corollary 2.4. *Let $A \subseteq X$ be bounded, $\varepsilon \geq 0$ and $E = \operatorname{ex} B_{X^*}$. If for each sequence $(x_n)_{n\in\mathbb{N}}$ in A we have that*

$$\bigcap_{k=1}^{\infty} \left(\overline{\text{co}}^{\sigma_E}\,\{x_n : n \geq k\} + \varepsilon B_X\right) \neq \emptyset,$$

then A is 2ε-WRC.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in A. By means of Theorem 1.5 and an easy diagonal argument we can find a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $\overline{\text{co}}^{\sigma_E}\,\{x_{n_k} : k \geq l\} \subseteq \overline{\text{co}}\,\{x_n : n \geq l\}$ for all l (compare [17, Corollary 0.2]). It then follows from our assumption that

$$\bigcap_{l=1}^{\infty} \left(\overline{\text{co}}\,\{x_n : n \geq l\} + \varepsilon B_X\right) \neq \emptyset.$$

Hence, by Corollary 2.3, A is 2ε-WRC. □

Next we observe that Moors' Theorem 1.5 does not only work for the extreme points of B_{X^*} but also for any weak*-separable boundary.

Theorem 2.5. *Let B be a weak*-separable boundary for X and A a bounded infinite subset of X. Then there is a countably infinite set $F \subseteq A$ such that $\overline{\text{co}}F = \overline{\text{co}}^{\sigma_B}F$. In particular, for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with $\overline{\text{co}}^{\sigma_B}\,\{x_{n_k} : k \in \mathbb{N}\} \subseteq \overline{\text{co}}\,\{x_n : n \in \mathbb{N}\}$.*

Proof. The proof is completely analogous to that of Theorem 1.5 given in [17], in fact it is even simplier, so we shall only sketch it. Arguing by contradiction, we suppose that for each countably infinite subset F of A there is an element $z \in \overline{\text{co}}^{\sigma_B}F \setminus \overline{\text{co}}F$.

Then we can show exactly as in [17] (using the Bishop-Phelps theorem (cf. [13, Theorem 5.5]) and the Hahn-Banach separation theorem) that for every sequence $(x_n)_{n\in\mathbb{N}}$ in A for which the set $\{x_n : n \in \mathbb{N}\}$ is infinite, there is an element

$$x \in \bigcap_{k=1}^{\infty} \overline{\text{co}}^{\sigma_B}\,\{x_n : n \geq k\} \setminus \overline{\text{co}}\,\{x_n : n \in \mathbb{N}\}\,. \tag{7}$$

Next we fix a sequence $(x_n)_{n\in\mathbb{N}}$ in A whose members are distinct and a countable weak*-dense subset $\{x_m^* : m \in \mathbb{N}\}$ of B. By the usual diagonal argument we may select a subsequence (again denoted by $(x_n)_{n\in\mathbb{N}}$) such that $\lim_{n\to\infty} x_m^*(x_n)$ exists for all m.

We then choose an element x according to (7) and conclude that for each $m \in \mathbb{N}$ we have $\lim_{n\to\infty} x_m^*(x_n) = x_m^*(x)$.

Now let $x^* \in B$ be arbitrary. Again as in [17] we will show that $\lim_{n\to\infty} x^*(x_n) = x^*(x)$. Suppose that this is not the case. Then there is an $\varepsilon > 0$ such that $|x^*(x) - x^*(x_n)| > \varepsilon$ for infinitely many $n \in \mathbb{N}$. Let us assume $x^*(x_n) > \varepsilon + x^*(x)$ for infinitely many n and arrange these indices in an increasing sequence $(n_k)_{k\in\mathbb{N}}$. By (7) we can find

$$z \in \bigcap_{l=1}^{\infty} \overline{\text{co}}^{\sigma_B} \{x_{n_k} : k \geq l\} \setminus \overline{\text{co}} \{x_{n_k} : k \in \mathbb{N}\}.$$

It follows that $x_m^*(z) = \lim_{k\to\infty} x_m^*(x_{n_k}) = x_m^*(x)$ for all m and since $\{x_m^* : m \in \mathbb{N}\}$ is weak*-dense in B this implies $x^*(x) = x^*(z)$, whereas on the other hand $x^*(z) \geq \varepsilon + x^*(x)$, a contradiction.

Thus $(x_n)_{n\in\mathbb{N}}$ is σ_B-convergent to x and hence, by Corollary 1.2 it is also weakly convergent to x, which in turn implies $x \in \overline{\text{co}} \{x_n : n \in \mathbb{N}\}$, contradicting the choice of x. □

Note that the assumption of weak*-separability of B is fulfilled, in particular, if X is separable, for then the weak*-topology on B_{X^*} is metrizable. As an immediate corollary we get 2.4 for weak*-separable boundaries.

Corollary 2.6. *Let B be a boundary for X and A a bounded subset of X as well as $\varepsilon \geq 0$. If B is weak*-separable (in particular, if X is separable) and for each sequence $(x_n)_{n\in\mathbb{N}}$ in A we have*

$$\bigcap_{k=1}^{\infty} \left(\overline{\text{co}}^{\sigma_B} \{x_n : n \geq k\} + \varepsilon B_X\right) \neq \emptyset,$$

then A is 2ε-WRC.

Proof. Exactly as the proof of Corollary 2.4. □

Let us now consider the case of $C(K)$-spaces, where K is a compact Hausdorff space. In [5] Cascales and Godefroy gave a solution to the Boundary Problem for this class of spaces. In fact, they proved a stronger result, namely that the space $(C(K), \sigma_B)$ is angelic[1] for every boundary B. They used the following lemma:

[1]See [5] or [12] for the definition and background.

Lemma 2.7 (Cascales-Godefroy, cf. [5]). *Let K be a compact Hausdorff space and B a boundary for $C(K)$, as well as $x \in K$ and $(f_n)_{n\in\mathbb{N}}$ any sequence in $C(K)$. Then there exists $\mu \in B$ with $\mu(f_n) = f_n(x)$ for all $n \in \mathbb{N}$.*

From this lemma we can easily deduce the following one, which is a slight variation of [5, Proposition 2 (i)].

Lemma 2.8. *If K is a compact Hausdorff space, B a boundary for $C(K)$ and $(f_n)_{n\in\mathbb{N}}$ a sequence in $C(K)$, then*

$$\overline{\text{co}}^{\sigma_B}\{f_n : n \in \mathbb{N}\} \subseteq \overline{\text{co}}^{\tau_p}\{f_n : n \in \mathbb{N}\},$$

where τ_p denotes the topology of pointwise convergence.

Proof. Let $f \in \overline{\text{co}}^{\sigma_B}\{f_n : n \in \mathbb{N}\}$, $\varepsilon > 0$ and $x_1, \dots, x_m \in K$ be arbitrary. By Lemma 2.7 we can find functionals $\mu_1, \dots, \mu_m \in B$ such that $\mu_i(f_n - f) = f_n(x_i) - f(x_i)$ for all $n \in \mathbb{N}$ and all $i = 1, \dots, m$.

Now choose a function $g \in \text{co}\{f_n : n \in \mathbb{N}\}$ with $|\mu_i(g - f)| \leq \varepsilon$ for all $i = 1, \dots, m$. It then easily follows that $|f(x_i) - g(x_i)| \leq \varepsilon$ for $i = 1, \dots, m$ and we are done. □

As a consequence we find that the statement of Corollary 2.6 holds true for every boundary B of $C(K)$.

Corollary 2.9. *Let K be a compact Hausdorff space and B a boundary for $C(K)$. If $\varepsilon \geq 0$ and $A \subseteq C(K)$ is bounded such that for each sequence $(f_n)_{n\in\mathbb{N}}$ in A we have*

$$\bigcap_{k=1}^{\infty} \left(\overline{\text{co}}^{\sigma_B}\{f_n : n \geq k\} + \varepsilon B_{C(K)} \right) \neq \emptyset,$$

then A is 2ε-WRC.

Proof. As is well known, $E = \text{ex}\, B_{C(K)^*} = \{\alpha\delta_x : x \in K, |\alpha| = 1\}$, where δ_x denotes the Dirac measure with respect to the point x. Hence the pointwise convergence topology τ_p coincides with the topology σ_E and thus the statement immediately follows from Lemma 2.8 and Corollary 2.4. □

Using the same ideas as in [5] the authors of [9] proved that the space $(\ell^1(I), \sigma_B)$ is angelic for every boundary B of $\ell^1(I)$, where I is an arbitrary index set. The proof is based on the following lemma, analogous to 2.7. Recall that $\text{ex}\, B_{\ell^\infty(I)} = \{-1; 1\}^I$.

Lemma 2.10 (Cascales-Shvydkoy, cf. [9]). *Let I be a set and B a boundary for $\ell^1(I)$. Then for every sequence $(f_n)_{n\in\mathbb{N}}$ in $\ell^1(I)$ and every $g \in \{-1; 1\}^I$ there is some $b \in B$ such that $b(f_n) = g(f_n)$ for all $n \in \mathbb{N}$.*

From this lemma we can deduce exactly as in the case of $C(K)$ the following results.

Lemma 2.11. *Let I be a set, B a boundary for $\ell^1(I)$ and $(f_n)_{n\in\mathbb{N}}$ a sequence in $\ell^1(I)$. Then we have*

$$\overline{\mathrm{co}}^{\sigma_B}\{f_n : n\in\mathbb{N}\} \subseteq \overline{\mathrm{co}}^{\sigma_E}\{f_n : n\in\mathbb{N}\},$$

where $E = \mathrm{ex}\, B_{\ell^\infty(I)}$.

Corollary 2.12. *Let I be a set and B a boundary for $\ell^1(I)$. If $A\subseteq \ell^1(I)$ is bounded and $\varepsilon\geq 0$ such that for every sequence $(f_n)_{n\in\mathbb{N}}$ in A we have*

$$\bigcap_{k=1}^{\infty}\left(\overline{\mathrm{co}}^{\sigma_B}\{f_n : n\geq k\} + \varepsilon B_{\ell^1(I)}\right)\neq\emptyset,$$

then A is 2ε-WRC.

Next we turn to spaces not containing isomorphic copies of ℓ^1. It is known that for such spaces one has $\overline{\mathrm{co}}^{\gamma}B = B_{X^*}$ for every boundary B of X, where we denote by γ the topology on X^* of uniform convergence on bounded countable subsets of X (cf. [8, Theorem 5.4]).

We will also need two easy lemmas.

Lemma 2.13. *Let $A\subseteq X$ and $S\subseteq X^*$ be bounded as well as $\varepsilon\geq 0$ such that $A\S\varepsilon\S S$. Then we also have $A\S\varepsilon\S\overline{S}^{\gamma}$.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ and $(x_m^*)_{m\in\mathbb{N}}$ be sequences in A and $\overline{S}^{\gamma}$, respectively, such that the limits

$$\lim_{n\to\infty}\lim_{m\to\infty} x_m^*(x_n) \text{ and } \lim_{m\to\infty}\lim_{n\to\infty} x_m^*(x_n)$$

exist. For each $m\in\mathbb{N}$ we can pick a functional $\tilde{x}_m^*\in S$ with

$$|x_m^*(x_n)-\tilde{x}_m^*(x_n)|\leq\frac{1}{m}\quad\forall n\in\mathbb{N}.$$

By the usual diagonal argument, choose a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}\tilde{x}_m^*(x_{n_k})$ exists for all m. It then easily follows that

$$\lim_{m\to\infty}\lim_{n\to\infty} x_m^*(x_n) = \lim_{m\to\infty}\lim_{k\to\infty}\tilde{x}_m^*(x_{n_k}) \text{ and}$$
$$\lim_{n\to\infty}\lim_{m\to\infty} x_m^*(x_n) = \lim_{k\to\infty}\lim_{m\to\infty}\tilde{x}_m^*(x_{n_k}).$$

Since $A\S\varepsilon\S S$, we conclude that

$$\left|\lim_{n\to\infty}\lim_{m\to\infty} x_m^*(x_n) - \lim_{m\to\infty}\lim_{n\to\infty} x_m^*(x_n)\right|\leq\varepsilon,$$

finishing the proof. □

Lemma 2.14. *Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in X and $B \subseteq B_{X^*}$ such $\overline{\text{co}}^\gamma B = B_{X^*}$. Then*

$$\overline{\text{co}}^{\sigma_B} \{x_n : n \in \mathbb{N}\} = \overline{\text{co}} \{x_n : n \in \mathbb{N}\}.$$

Proof. Take $x \in \overline{\text{co}}^{\sigma_B} \{x_n : n \in \mathbb{N}\}$ and let $\varepsilon > 0$ and $x_1^*, \dots, x_k^* \in B_{X^*}$ be arbitrary. By assumption, we can find $\tilde{x}_1^*, \dots, \tilde{x}_k^* \in \text{co}\, B$ such that for $i = 1, \dots, k$ we have

$$|\tilde{x}_i^*(x_n) - x_i^*(x_n)| \leq \varepsilon \ \forall n \in \mathbb{N} \text{ and } |\tilde{x}_i^*(x) - x_i^*(x)| \leq \varepsilon.$$

It follows that

$$|\tilde{x}_i^*(y) - x_i^*(y)| \leq \varepsilon \quad \forall y \in \text{co}\,(\{x_n : n \in \mathbb{N}\} \cup \{x\}) \ \forall i = 1, \dots, k.$$

Now take some element $y \in \overline{\text{co}}^{\sigma_B} \{x_n : n \in \mathbb{N}\}$ with $|\tilde{x}_i^*(y) - \tilde{x}_i^*(x)| \leq \varepsilon$ for all $i = 1, \dots, k$.
Employing the triangle inequality we can deduce $|x_i^*(x) - x_i^*(y)| \leq 3\varepsilon$, which ends the proof. □

As an immediate consequence of Lemma 2.14, Corollary 2.3 and the aforementioned result [8, Theorem 5.4] we get the following corollary.

Corollary 2.15. *Suppose $\ell^1 \not\subseteq X$ and let B be a boundary for X. If $A \subseteq X$ is bounded and $\varepsilon \geq 0$ such that for each sequence $(x_n)_{n\in\mathbb{N}}$ in A we have*

$$\bigcap_{k=1}^{\infty} \left(\overline{\text{co}}^{\sigma_B} \{x_n : n \geq k\} + \varepsilon B_X\right) \neq \emptyset,$$

then A is 2ε-WRC.

We can further get a kind of 'boundary double limit criterion'.

Proposition 2.16. *Let B be a boundary for X as well as $\varepsilon \geq 0$ and $A \subseteq X$ be bounded such that $A \S\varepsilon\S B$. Then A is 2ε-WRC. If $\ell^1 \not\subseteq X$, then A is even ε-WRC.*

Proof. From [7, Theorem 3.3] it follows that we also have $A \S\varepsilon\S\, \text{co}\, B$. Since B is a boundary for X the Hahn-Banach separation theorem implies $B_{X^*} = \overline{\text{co}}^{w^*} B$. Therefore it follows from [2, Lemma 3] that $A \S 2\varepsilon \S B_{X^*}$. Thus by (ii) of Proposition 1.6 A is 2ε-WRC.

Moreover, if $\ell^1 \not\subseteq X$ then we even have $B_{X^*} = \overline{\text{co}}^\gamma B$ by the already cited [8, Theorem 5.4]. Hence $A \S\varepsilon\S B_{X^*}$ by Lemma 2.13, thus A is ε-WRC. □

Our final aim in this note is to prove a 'non-relative' version of Theorem 1.4 for arbitrary boundaries. To do so, we will use the techniques of Pfitzner from [19]. More precisely, we can get the following slight generalization of [19, Theorem 5]. Recall that an ℓ^1-sequence in X is simply a sequence equivalent to the canonical basis of ℓ^1.

Theorem 2.17. *Let B be a boundary for X. If $A \subseteq X$ is bounded and for every sequence $(x_n)_{n\in\mathbb{N}}$ in A we have*

$$A \cap \bigcap_{k=1}^{\infty} \overline{\mathrm{co}}^{\sigma_B} \{x_n : n \geq k\} \neq \emptyset, \tag{8}$$

then A does not contain an ℓ^1-sequence.

Proof. The proof is completely analogous to that of [19, Theorem 5], therefore we will only give a very brief sketch. We use the notation and definitions from [19]. Arguing by contradiction, we assume that there is an ℓ^1-sequence $(x_n)_{n\in\mathbb{N}}$ in A. By [19, Lemma 2] we may assume that $(x_n)_{n\in\mathbb{N}}$ is ε_J-stable and δ-stable. We take a sequence $(\alpha_k)_{k\in\mathbb{N}}$ of positive numbers decreasing to zero. By [19, Lemma 4] we can find $\varepsilon \geq \varepsilon_J(x_n) > 0$, a sequence $(b_k)_{k\in\mathbb{N}}$ in B and a tree $(\Omega_\sigma)_{\sigma\in S}$ such that for each $k \in \mathbb{N}$ and every $\sigma, \sigma' \in S_k$ with $\sigma_k = 0$ and $\sigma'_k = 1$ we have

$$b_k(x_n - x_{n'}) \geq 2\varepsilon(1 - \alpha_k) \quad \forall n \in \Omega_\sigma, n' \in \Omega_{\sigma'}.$$

It follows that the same inequality holds for every $x \in \overline{\mathrm{co}}^{\sigma_B} \{x_n : n \in \Omega_\sigma\}$ and $x' \in \overline{\mathrm{co}}^{\sigma_B} \{x_{n'} : n' \in \Omega_{\sigma'}\}$.
Now using our hypothesis we can proceed completely analogous to the proof of the second part of [19, Lemma 4] to find a sequence $(y_m)_{m\in\mathbb{N}}$ in $A \cap \bigcap_{k=1}^{\infty} \overline{\mathrm{co}}^{\sigma_B} \{x_n : n \geq k\}$ such that

$$b_k(y_m - y_{m'}) \geq 2\varepsilon(1 - \alpha_k) \quad \forall m \leq k < m'.$$

Next we take an element

$$y \in A \cap \bigcap_{k=1}^{\infty} \overline{\mathrm{co}}^{\sigma_B} \{y_m : m \geq k\}.$$

As in the proof of [19, Theorem 5] we put

$$x = \sum_{m=1}^{\infty} 2^{-m}(y_m - y)$$

and proceed again exactly as in [19] to show that $\|y_m - y\| \leq 2\varepsilon$ for all m and $\|x\| = 2\varepsilon$. Finally, taking a functional $b \in B$ with $b(x) = \|x\|$ we obtain $b(y) = 2\varepsilon + b(y)$ and with this contradiction the proof is finished. $\square$

Now we can get the final result.

Theorem 2.18. *Let B be a boundary for X and $A \subseteq X$ be bounded. Then the following assertions are equivalent:*

(i) *A is countably compact in the topology σ_B.*

(ii) *For every sequence $(x_n)_{n\in\mathbb{N}}$ in A we have*

$$A \cap \bigcap_{k=1}^{\infty} \overline{\mathrm{co}}^{\sigma_B} \{x_n : n \geq k\} \neq \emptyset.$$

(iii) *For every sequence $(x_n)_{n\in\mathbb{N}}$ in A there is some $x \in A$ with*

$$x^*(x) \leq \limsup_{n\to\infty} x^*(x_n) \quad \forall x^* \in \operatorname{span} B.$$

(iv) *A is weakly compact.*

Proof. The implications (i) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (i) are clear and the equivalence of (ii) and (iii) follows from Lemma 2.1. It only remains to prove (ii) $\Rightarrow$ (iv).

Let us assume that (ii) holds and take an arbitrary sequence $(x_n)_{n\in\mathbb{N}}$ in A. By Theorem 2.17 no subsequence of $(x_n)_{n\in\mathbb{N}}$ is an ℓ^1-sequence and thus Rosenthal's theorem (cf. [3] or [1, Theorem 10.2.1]) applies to yield a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ which is weakly Cauchy. Now choose an element

$$x \in A \cap \bigcap_{l=1}^{\infty} \overline{\mathrm{co}}^{\sigma_B} \{x_{n_k} : k \geq l\}.$$

It easily follows that $\lim_{k\to\infty} b(x_{n_k}) = b(x)$ for all $b \in B$. By Corollary 1.2 $(x_{n_k})_{k\in\mathbb{N}}$ is weakly convergent to x.
Thus we have shown that A is weakly sequentially compact. Hence it is also weakly compact, by the Eberlein-Šmulian theorem. □

Remark 1. It is proved in [11, Remark 10] that for $X = \ell^1$ the statement $B_X \S \varepsilon \S B_{X^*}$ is false for every $0 < \varepsilon < 2$. An alternative proof of this fact is given [6, Example 5.2]. It is further proved in [11, Remark 10] that every separable Banach space X which contains an isomorphic copy of ℓ^1 can be equivalently renormed such that, in this renorming, the statement $B_X \S \varepsilon \S B_{X^*}$ is false for every $0 < \varepsilon < 2$. The proof makes use of the notion of octahedral norms.

We wish to point out here that the argument from [6, Example 5.2] can be carried over to arbitrary Banach spaces containing a copy of ℓ^1, precisely: If X is a (not necessarily separable) Banach space which contains ℓ^1 than the statement $B_X \S \varepsilon \S B_{X^*}$ (in the original norm of X) is false for every $0 < \varepsilon < 2$.

To see this, take $0 < \varepsilon < 2$ arbitrary and fix $0 < \delta < 1$ such that $2(1-\delta) > \varepsilon$. Since X contains ℓ^1 we may find, with the aid of James' ℓ^1-distortion theorem (cf. [1, Theorem 10.3.1]), a sequence $(x_n)_{n\in\mathbb{N}}$ in the unit sphere of X such that $T : \ell^1 \to X$ defined by

$$Ty = \sum_{k=1}^{\infty} \alpha_k x_k \quad \forall y = (\alpha_n)_{n\in\mathbb{N}} \in \ell^1$$

is an isomorphism (onto $U = \operatorname{ran} T$) with $\|T^{-1}\| \leq (1-\delta)^{-1}$. Consequently, the adjoint $T^* : U^* \to \ell^\infty$ is as well an isomorphism with $\|(T^*)^{-1}\| \leq (1-\delta)^{-1}$. Now we can define as in [6, Example 5.2] for each $n \in \mathbb{N}$ a norm one functional $y_n^* \in \ell^\infty$ by

$$y_n^*(m) = \begin{cases} 1, & \text{if } m \leq n \\ -1, & \text{if } m > n. \end{cases}$$

Put $u_n^* = (T^*)^{-1} y_n^*$ for all $n \in \mathbb{N}$. Then $\|u_n^*\| \leq (1-\delta)^{-1}$ and hence by the Hahn-Banach extension theorem we can find $x_n^* \in B_{X^*}$ with $x_n^*|_U = (1-\delta)u_n^*$ for all $n \in \mathbb{N}$. It follows that

$$\left| \lim_{n\to\infty} \lim_{m\to\infty} x_n^*(x_m) - \lim_{m\to\infty} \lim_{n\to\infty} x_n^*(x_m) \right| = 2(1-\delta) > \varepsilon$$

and the proof is finished.

In the notation of [6] we have proved $\gamma(B_X) = 2$ for every Banach space X containing an isomorphic copy of ℓ^1, which implies that the value of B_X under all other measures of weak non-compactness considered in [6] is equal to one (again compare [6, Example 5.2]). So in a certain sense a Banach space containing ℓ^1 is 'as non-reflexive as possible'.

Remark 2. Shortly after the first version of this paper was published on the web, the author received a message from Prof. Warren B. Moors, who kindly pointed out to him that the above Lemma 2.8 probably holds true in the more general setting of L_1-preduals[2], refering to the paper [18]. Indeed, an examination of the proof of [18, Theorem 3] shows that it can be straightforwardly adopted to produce the following result: If B is a boundary for the L_1-predual X and $E = \operatorname{ex} B_{X^*}$, then

$$\overline{\text{co}}^{\sigma_B} \{x_n : n \in \mathbb{N}\} \subseteq \overline{\text{co}}^{\sigma_E} \{x_n : n \in \mathbb{N}\}$$

holds for every sequence $(x_n)_{n\in\mathbb{N}}$ in X. Hence Corollary 2.9 also carries over to arbitrary boundaries of L_1-preduals.

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[2] Recall that a Banach space X is called an L_1-predual if X^* is isometric to $L_1(\mu)$ for some suitable measure μ. It is well known that every $C(K)$-space is an L_1-predual.

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