Probability that $n$ random points are in convex position\textcopyright

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Abstract

We show that $n$ random points chosen independently and uniformly from a parallelogram are in convex position with probability

$$\left( \frac{\binom{2n-2}{n-1}}{n!} \right)^2.$$
1 The result

A finite set of points in the plane is called convex if its points are vertices of a convex polygon. In this paper we show the following result.

Theorem 1 The set $A$ of $n$ random points chosen independently and uniformly from a parallelogram $S$ is convex with probability

$$\left( \frac{(2n-2)}{n!} \right)^2.$$

A large part of studies in stochastic geometry deals with the convex hull $C$ of a set of $n$ points placed independently and uniformly in a fixed convex body $K$ in $\mathbb{R}^d$. Typical questions are: How many vertices does $C$ have? What is the volume of $C$? What is the surface area of $C$? See [WW] for a survey. In this paper we settle one very special case – the probability that $C$ has $n$ vertices in the case $K$ is a parallelogram. It is interesting that our approach is purely combinatorial, with no use of integration. We think that our method based on an approximation of the uniform distribution in a square by a large grid might have other applications. However, it is already not clear how to apply our method for $K$ a triangle or in three dimensions.

In this section we prove Theorem 1, and in the next section we mention some applications of Theorem 1.

Proof of Theorem 1. Let $n > 2$ be a fixed integer. Since a proper affine transformation transfers the uniform distribution on $S$ onto the uniform distribution on a square, we may and shall assume that $S$ is a square. We shall approximate the square $S$ by a grid whose size tends to infinity.

Let $m$ be a positive integer (denoting the size of the grid). Partition the (axis-parallel) square $S$ by $m - 1$ horizontal and by $m - 1$ vertical lines into $m^2$ squares $S_1, \ldots, S_{m^2}$ of equal size. The centers of the squares $S_1, \ldots, S_{m^2}$ form a square grid $m \times m$. Every point of $A$ lies in each of the squares $S_1, \ldots, S_{m^2}$ with the same probability $1/m^2$. Move every point of $A$ to the center of the square $S_i$ in which it lies, and denote the obtained multiset by $A(m)$. It is not difficult to see that

$$\text{Prob}(A \text{ is convex}) = \lim_{m \to \infty} \text{Prob}(A(m) \text{ is convex}).$$

Thus,

$$\text{Prob}(A \text{ is convex}) = \lim_{m \to \infty} \text{Prob}(R_m \text{ is convex}),$$

where, for every $m \geq 1$, $R_m$ is a multiset of $n$ points chosen randomly and independently from the square grid $G_m = \{(i,j) : i, j = 1, 2, \ldots, m\}$ (each point of $G_m$ is always taken with the same probability $1/m^2$).
Let $\mathcal{M}(G_m)$ be the set of all multisets of size $n$ with elements from $G_m$, and let $\mathcal{C}(G_m)$ be the set of all convex $n$-element subsets of $G_m$. It is easy to see that

$$\text{Prob}(A \text{ is convex}) = \lim_{m \to \infty} \text{Prob}(R_m \text{ is convex}) = \lim_{m \to \infty} \frac{|\mathcal{C}(G_m)|}{|\mathcal{M}(G_m)|} = \lim_{m \to \infty} \frac{|\mathcal{C}(G_m)|}{\binom{m^2}{n}}.$$ 

In the sequel we shall estimate the size of $\mathcal{C}(G_m)$.

Every convex set $R \in \mathcal{C}(G_m)$ is uniquely defined by the smallest axis-parallel rectangle $Q(R)$ containing $R$ and by the set $V(R)$ of the $n$ integer vectors forming the boundary of the convex hull of $R$ oriented in counterclockwise order.

Let $X(R)$ and $Y(R)$ be the multisets of the first and of the second coordinates of vectors in $V(R)$, respectively. Formally,

$$X(R) = \bigcup_{(x,y) \in V(R)} \{x\}, \quad Y(R) = \bigcup_{(x,y) \in V(R)} \{y\}.$$ 

Let $\mathcal{C}'(G_m)$ be the set of all convex sets $R \in \mathcal{C}(G_m)$ such that $0 \notin X(R) \cup Y(R)$ and that the directions of the $n^2$ vectors $(x,y)$ formed by all the $n^2$ pairs $x \in X(R), y \in Y(R)$ are distinct. Thus, in particular, the multisets $X(R)$ and $Y(R)$ are sets for any $R \in \mathcal{C}'(G_m)$. It is not difficult to see that

$$\lim_{m \to \infty} \frac{|\mathcal{C}'(G_m)|}{|\mathcal{C}(G_m)|} = 1.$$ 

Therefore,

$$\text{Prob}(A \text{ is convex}) = \lim_{m \to \infty} \frac{|\mathcal{C}(G_m)|}{\binom{m^2}{n}} = \lim_{m \to \infty} \frac{|\mathcal{C}'(G_m)|}{\binom{m^2}{n}}.$$ 

In the estimation of the size of $\mathcal{C}'(G_m)$ we use an auxiliary set $\mathcal{S}$ defined by

$$\mathcal{S} = \{(X(R), Y(R), Q(R)) : R \in \mathcal{C}'(G_m)\}.$$ 

The following construction shows that, for every $(X, Y, Q) \in \mathcal{S}$, there are exactly $n!$ sets $R \in \mathcal{C}'(G_m)$ with $(X(R), Y(R), Q(R)) = (X, Y, Q)$.

Take any of the $n!$ one-to-one correspondences $f : X \to Y$ between $X$ and $Y$, and define a set $V$ of $n$ vectors by $V = \{(x, f(x)) : x \in X\}$. Due to the definitions of $\mathcal{C}'(G_m)$ and $\mathcal{S}$, vectors in $V$ have distinct directions and, consequently, form the (counterclockwise oriented) boundary of the convex hull of a unique set $R \in \mathcal{C}'(G_m)$ fitting into the rectangle $Q$.

Thus,

$$|\mathcal{C}'(G_m)| = n! \cdot |\mathcal{S}|$$

and

$$\text{Prob}(A \text{ is convex}) = \lim_{m \to \infty} \frac{|\mathcal{C}'(G_m)|}{\binom{m^2}{n}} = \lim_{m \to \infty} \frac{n! \cdot |\mathcal{S}|}{\binom{m^2}{n}}.$$ 

It remains to estimate the size of the set $\mathcal{S}$ which is done in the sequel technical part of the proof.
For \((X, Y, Q) \in \mathcal{S}\), partition each of the two sets \(X\) and \(Y\) into two subsets containing elements with the same sign:

\[
X^+ = \{x \in X : x > 0\}, \quad X^- = \{x \in X : x < 0\},
\]

\[
Y^+ = \{y \in Y : y > 0\}, \quad Y^- = \{y \in Y : y < 0\}.
\]

Suppose that each of the sets \(X^+, X^-, Y^+, Y^-\) is ordered in an arbitrary way. Denote \(s = |X^+|\) and \(t = |Y^+|\). Thus,

\[
X^+ = \{x_1, \ldots, x_s\}, \quad X^- = \{x_{s+1}, \ldots, x_n\},
\]

\[
Y^+ = \{y_1, \ldots, y_t\}, \quad Y^- = \{y_{t+1}, \ldots, y_n\}.
\]

For every \((X, Y, Q) \in \mathcal{S}\), where \(Q = \{(x, y) : a_1 \leq x \leq a_2, \ b_1 \leq y \leq b_2\}\), the orders on the sets \(X^+, X^-, Y^+, Y^-\) uniquely determine four sets \(D^-, E^-, D^+, E^+\) of integers from the set \(\{1, 2, \ldots, m\}\) in the following way:

\[
D^+ = \{a_1 + \sum_{i=1}^k x_i : k = 0, 1, \ldots, s\}, \quad D^- = \{a_2 + \sum_{i=s+1}^k x_i : k = s, s+1, \ldots, n\},
\]

\[
E^+ = \{b_1 + \sum_{i=1}^k y_i : k = 0, 1, \ldots, t\}, \quad E^- = \{b_2 + \sum_{i=t+1}^k x_i : k = t, t+1, \ldots, n\}.
\]

Note that the sets \(D^-, E^-, D^+, E^+\) satisfy the following conditions:

\[
|D^+| + |D^-| = n + 2, \quad a_1 = \min D^+ = \min D^-, \quad a_2 = \max D^+ = \max D^-; \quad (1)
\]

\[
|E^+| + |E^-| = n + 2, \quad b_1 = \min E^+ = \min E^-, \quad b_2 = \max E^+ = \max E^-; \quad (2)
\]

For any \((X, Y, Q) \in \mathcal{S}\), we obtain \(|X^+|!|X^-|!|Y^+|!|Y^-|!\) different 4-tuples of sets \(D^-, E^-, D^+, E^+\) corresponding to different orders on the sets \(X^+, X^-, Y^+, Y^-\). Denote the set of all these 4-tuples \((D^-, E^-, D^+, E^+)\) by \(\mathcal{F}(X, Y, Q)\). Thus,

\[
|\mathcal{F}(X, Y, Q)| = |X^+|!|X^-|!|Y^+|!|Y^-|!
\]

\[
= (|D^+| - 1)!(|D^-| - 1)!(|E^+| - 1)!(|E^-| - 1)!
\]

where \((D^-, E^-, D^+, E^+)\) is an arbitrary 4-tuple in \(\mathcal{F}(X, Y, Q)\). For \(0 \leq i \leq n - 2\) and \(0 \leq j \leq n - 2\), we say that a 4-tuple \((D^-, E^-, D^+, E^+)\) of sets of integers has property \(\mathcal{P}_{i,j}\) if

\[
\mathcal{P}_{i,j}: \quad |D^+| = i + 2, \quad |E^+| = j + 2, \quad \text{and the sets } D^-, E^-, D^+, E^+ \text{ satisfy } (1) \text{ and } (2)
\]

for some \(1 \leq a_1 < a_2 \leq m\) and \(1 \leq b_1 < b_2 \leq m\).

There are \(\binom{n-2}{i}\binom{m}{a_1}\binom{n-2}{j}\binom{m}{b_1}\) 4-tuples \((D^-, E^-, D^+, E^+)\) with \(\mathcal{P}_{i,j}\) and \(|D^+ \cap D^-| = |E^+ \cap E^-| = 2\). It follows that there are \((1 + o(1)) \cdot \binom{n-2}{i}\binom{m}{a_1} \cdot \binom{n-2}{j}\binom{m}{b_1}\) 4-tuples \((D^-, E^-, D^+, E^+)\) with \(\mathcal{P}_{i,j}\). (Throughout the proof, \(o(1)\) denotes functions of \(m\).)
which tend to 0 as $m$ tends to infinity.) Most of them (i.e., a $(1 - o(1))$-fraction of them) lie in the disjoint union

$$
\bigcup_{(X,Y,Q) \in S} \mathcal{F}(X,Y,Q).
$$

Thus,

$$
|S| = \sum_{(X,Y,Q) \in S} 1 = \sum_{(X,Y,Q) \in S} \frac{\mathcal{F}(X,Y,Q)}{|X| |Y| |X+Y| |X-Y|} = 
$$

$$
= \sum_{i=0}^{n-2} \sum_{j=0}^{n-2} (1 - o(1)) \cdot (1 + o(1)) \left( \binom{n-2}{i} \binom{m}{i} \binom{n-2}{j} \binom{m}{j} \right) 
$$

$$
= (1 + o(1)) \left( \binom{m}{n} \right)^2 \sum_{i=0}^{n-2} \frac{\binom{n-2}{i} \cdot \binom{n-2}{n-i-1} \cdot \binom{m}{i} \cdot \binom{m}{n-i}}{(n!)^2} 
$$

$$
= (1 + o(1)) \left( \binom{m}{n} \right)^2 \frac{1}{(n!)^2} \left( \binom{2n-2}{n-1} \right)^2.
$$

Hence,

$$
\text{Prob}(A \text{ is convex}) = \lim_{m \to \infty} \frac{n! \cdot |S|}{\binom{m}{n}^2} = \lim_{m \to \infty} \frac{(1 + o(1)) \left( \binom{m}{n} \right)^2 \frac{1}{(n!)^2} \left( \binom{2n-2}{n-1} \right)^2}{\binom{m}{n}^2} = \left( \frac{\binom{2n-2}{n-1}}{n!} \right)^2.
$$

\[\square\]

2 Applications and related results

In this section we sketch some applications of our result.

1. Replacing the parallelogram by a convex body. It is known that, for every bounded convex body $K$ in the plane, there are two parallelograms, one containing $K$ and one contained in $K$, whose areas differ from the area of $K$ at most by a constant factor (e.g., see [Ba] for analogous results). Using this result and Theorem 1, it is not difficult to show that there are two positive constants $c_1$ and $c_2$ such that the set of $n$ points chosen independently and uniformly from an arbitrary convex body is convex with probability at least $\left( \frac{1}{2} \right)^n$ and at most $\left( \frac{1}{3} \right)^n$. 
2. The expected area of a random triangle. It is not difficult to show that

$$\text{Prob}(A \text{ is convex}) + 4 \cdot E[\text{Area of } T] = 1,$$

where $A$ is a set of four random points selected independently and uniformly from a convex body $S$ of area 1, and $T$ is a triangle with random vertices selected also independently and uniformly from $S$. If $S$ is a parallelogram, Theorem 1 yields that the expected area of $T$ is

$$\frac{1 - (5/6)^2}{4} = \frac{11}{144},$$

which was also shown in [He] by a different method.

3. Convex subsets of a random set. The author originally considered Theorem 1 in connection with the following result.

**Theorem 2** Let $A$ be a set of $n$ random points chosen independently and uniformly from a parallelogram. Let $c(A)$ be the largest convex subset of $A$. Set $h = 2^{4/3}e \approx 6.85$. Then $c(A) \geq \lambda n^{1/3}$ with probability smaller than $(\frac{h}{\lambda})^{3\lambda n^{1/3}}$, for any $\lambda \geq h$.

**Proof.** Let $\lambda \geq h$. For simplicity, assume that $\lambda n^{1/3}$ is an integer. The set $A$ contains \( \binom{n}{\lambda n^{1/3}} \) subsets of size $\lambda n^{1/3}$. According to Theorem 1, each of them is convex with probability

$$\frac{\left(\frac{2\lambda n^{1/3} - 2}{\lambda n^{1/3} - 1}\right)^2}{\left(\frac{\lambda n^{1/3}}{e}\right)!}.$$

It follows that the expected number of convex independent subsets of $A$ of size $\lambda n^{1/3}$ is at most

$$\left(\frac{n}{\lambda n^{1/3}}\right) \cdot \frac{\left(\frac{2\lambda n^{1/3} - 2}{\lambda n^{1/3} - 1}\right)^2}{\left(\frac{\lambda n^{1/3}}{e}\right)!} \cdot \left(\frac{4\lambda n^{1/3}}{\lambda n^{1/3} - e}\right) \cdot \left(\frac{\lambda n^{1/3}}{e}\right)^{\lambda n^{1/3}} = \left(\frac{4^2 e^3}{\lambda^3}\right)^{\lambda n^{1/3}} = \left(\frac{h}{\lambda}\right)^{3\lambda n^{1/3}}.$$

Consequently, $A$ contains a convex independent subset of size $\geq \lambda n^{1/3}$ with probability smaller than $(\frac{h}{\lambda})^{3\lambda n^{1/3}}$. 

One application of Theorem 2 on so-called dense sets may be found in the author’s PhD thesis [Va].

By a more careful handling with the result of Theorem 1, one can prove that, for any $\varepsilon > 0$ and any sufficiently large $n \geq n(\varepsilon)$,

$$(h/2 - \varepsilon)n^{1/3} \leq c(A) \leq hn^{1/3}$$
holds with a high probability.

4. Construction of random convex sets. Emo Welzl pointed out that the above proof of Theorem 1 yields a fast way how to construct a random convex set of size \( n \) in a square. Let \( M_n \) be the set of all \( n \)-element subsets of a square \( S \), and let \( \mu \) be the probabilistic measure on \( M_n \) corresponding to a choice of \( n \) points selected independently and uniformly from the square \( S \). Let \( C_n \) be the set of all convex \( n \)-element subsets of \( S \). Theorem 1 gives \( \mu(C_n) = \left(\frac{(2^n-2)}{n!}\right)^2 \). The measure \( \mu' = \mu/(\mu(C_n))|_{C_n} \) is a probabilistic measure on \( C_n \). With respect to \( \mu' \), a random convex set \( A \in C_n \) can be constructed in a straightforward way by repeated choosings of an \( n \)-point random subset \( A \) with respect to \( \mu \), until the set \( A \) is convex. However, this procedure has the expected running time at least \( \Omega(n \log n + n \cdot R + P(n)) \), where \( R \) is as above and \( P(n) \) is the time required for constructing a random permutation of the set \( \{1, 2, \ldots, n\} \). Of course, the argument also applies for any parallelogram.

5. The limit shape of a random convex set. Scale and shift the square grid \( n \times n \) so that it fits into the square \( S = \{(x, y) : -1 \leq x, y \leq 1 \} \), and consider the set \( K(n) \) of all its convex subsets. Bárány [Bá] proved that for every \( \varepsilon > 0 \) there exists \( n_0 \) such that for every \( n \geq n_0 \) the following holds: if we randomly choose an element \( A \) of \( K(n) \), each with the same probability, then the Hausdorff distance between the boundary of the convex hull of \( A \) and the curve \( \{(x, y) : \sqrt{1 - |x|} + \sqrt{1 - |y|} = 1\} \) is smaller than \( \varepsilon \) with a high probability. (Hausdorff distance between two sets is the maximum distance of a point in any of the two sets to the other set.)

It is interesting that random convex sets have the same limit shape. Consider the square \( S = \{(x, y) : -1 \leq x, y \leq 1 \} \) again, and define \( C_n \) and \( \mu' \) as in the above paragraph "Construction of random convex sets". With a help of the above proof of Theorem 1, it can be shown that for every \( \varepsilon > 0 \) there exists \( n_0 \) such that for every \( n \geq n_0 \) the following holds: if we randomly choose an element \( A \) of \( C_n \) with respect to the measure \( \mu' \), then the Hausdorff distance between the boundary of the convex hull of \( A \) and the curve \( \{(x, y) : \sqrt{1 - |x|} + \sqrt{1 - |y|} = 1\} \) is smaller than \( \varepsilon \) with a high probability.

Let us note that the limit shape curve of the boundary of the convex hull of \( n \) random points chosen independently and uniformly inside any planar convex body \( K \) is (obviously) the perimeter of \( K \).

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References


