On the number of maximum-area triangles in a planar pointset

Peter Braß[1]*, Konrad Swanepoel[2]

[1] Institute for Computer Science, FU Berlin,
Takustrasse 9, D-14195 Berlin, Germany
brass@inf.fu-berlin.de

[2] Department of Mathematics and Applied Mathematics, University of Pretoria,
0002 Pretoria, South Africa
konrad@math.up.ac.za

Abstract: In this note we prove that in a set of n points in the plane, not all on a line, the maximum area of a triangle is reached by at most n of the $\binom{n}{3}$ triangles determined by these points, and this number of maximum-area triangles is reached by several constructions. This answers a question of Erdős and Purdy of 1971.

1. Introduction

It is a classical problem of combinatorial geometry, raised first by Erdős in 1946 and still far from a solution, to bound the maximum number of occurences of the same distance among n points in the plane. Numerous variants of this were considered, such as special distances (largest, smallest,...), special pointsets (convex position, general position,...), other metrics and higher dimensions. As a related problem Erdős and Purdy studied in a sequence of papers [4,12,5,6,7] the maximum number of occurences of the same area among the triangles determined by n points in the plane, and as a common generalization the number of occurences of the same k-dimensional measure among the k-dimensional simplices determined by n points in d-dimensional space (for $k = 1$ this is the distances problem, and for $k = d = 2$ the area problem). Here the low-dimensional problems are especially interesting, since for higher dimensions ($d \geq 4$) constructions of Lenz type give large enough lower bounds that relatively weak combinatorial structures are sufficient to obtain good upper bounds; but in the plane (and 3-dimensional space) such constructions are not possible, and lattice sections seem to play a key role. For the unit-distance as well as the unit-area problem, sections of a square or triangular lattice are the asymptotically best known constructions, giving $\Omega\left(ne^{c\frac{\log n}{\log\log n}}\right)$ unit distances and $\Omega(n^2 \log\log n)$ unit-area triangles among n points in the plane. The corresponding best known upper bounds are $O(n^{\frac{4}{3}})$ and $O(n^{\frac{7}{3}})$, respectively. Erdős and Purdy also asked for the maximum number of maximum-area and minimum-area triangles [4], since the corresponding distance problems are much simpler (at most n maximum [13] and

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$\left\lfloor 3n - \sqrt{12n-3} \right\rfloor$ minimum [9] distances among n points in the plane), but remarked that 'Unfortunately we have only trivial results': $O(n^2)$ and $\Omega(n)$ for the number of maximum-area triangles. It is the aim of this note to show that the number of maximum-area triangles is indeed similar to the number of maximum distances, and can be determined exactly. The maximum number of minimum-area triangles, however, is quite unlike its distance counterpart, since it grows at least quadratically (lattice sections, or points equidistant on two parallel lines).

2. The result

Theorem: Among the triangles determined by n noncollinear points in the plane there are at most n triangles of maximum area.

There are several constructions that reach this upper bound, e.g. the vertices of a square together with the remaining $n-4$ points distributed arbitrarily on the sides of the square, or the vertices of a regular n-gon for n not divisible by three (Figure 1).

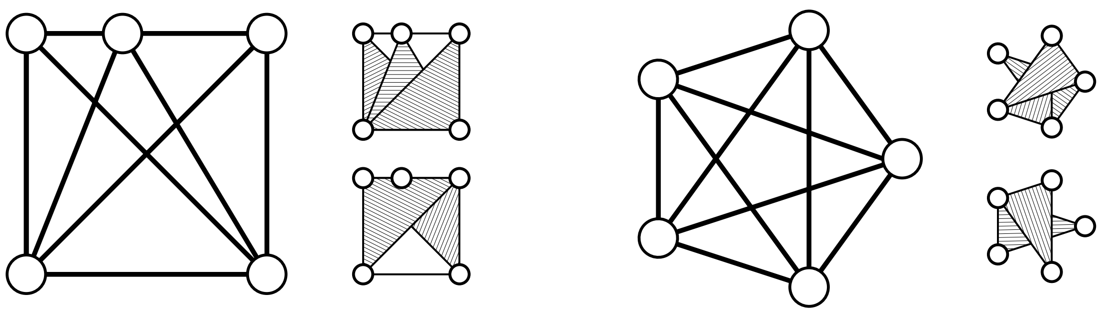

Figure 1.

To prove that a set of n noncollinear points in the plane determines at most n maximum-area triangles we note first that a point in the interior of the convex hull can never be a vertex of a maximum-area triangle. In a set with the maximum number of such triangles, each point belongs to at least one triangle, so we can assume that the points are on a convex curve. We need the following lemma, which captures the geometric content of the problem.

Lemma: If Δ, Δ^* are two maximum-area triangles determined by a pointset, then each edge of Δ^* has a point in common with some edge of Δ.

Proof of the Theorem: Let $p_1, \ldots, p_n$ be the points in clockwise order. We construct a directed graph on $\{p_1, \ldots, p_n\}$ by taking as graph edges all edges of maximum-area triangles, oriented with the clockwise orientation of the triangle (Figure 2).

$$G = (V, E) \qquad \text{with } V = \{p_1, \ldots, p_n\}$$
$$E = \{(p_i, p_j) \mid \exists p_k \text{ such that } p_i p_j p_k \text{ is a clockwise oriented maximum-area triangle}\}$$

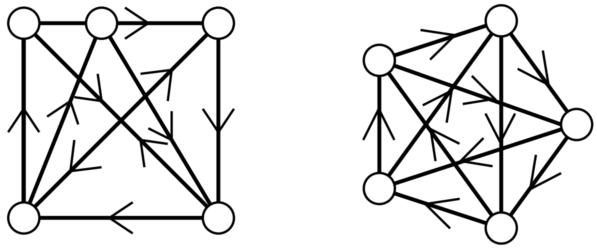

Figure 2.

We claim that this graph contains at most $2n$ directed edges. To show this, we note that the set of all oriented diagonals of a convex n-gon can be partitioned in $2n$ directed 'formal parallel classes': taking point-pairs with those indices that would give oriented parallel classes if the underlying polygon were regular, i.e. for odd n

$$\{(p_1,p_n),(p_2,p_{n-1}),\dots,(p_{\lfloor\frac{n}{2}\rfloor},p_{\lceil\frac{n}{2}\rceil})\},\quad \{(p_n,p_1),(p_{n-1},p_2),\dots,(p_{\lceil\frac{n}{2}\rceil},p_{\lfloor\frac{n}{2}\rfloor})\}$$

and their n cyclic index shifts, and for even n

$$\{(p_1,p_{n-1}),(p_2,p_{n-2}),\dots,(p_{\frac{n}{2}-1},p_{\frac{n}{2}+1})\},\quad \{(p_1,p_n),(p_2,p_{n-1}),\dots,(p_{\frac{n}{2}},p_{\frac{n}{2}+1})\}$$

and their n cyclic index shifts. Now each of these classes can contain at most one directed edge: if there were two such index-parallel edges belonging to maximum-area triangles, then we can label these edges $(p_{a_1},p_{b_1}),(p_{a_2},p_{b_2})$ so that they occur in the clockwise order $p_{a_1},p_{b_1},p_{b_2},p_{a_2}$ on the convex hull. By the definition there are points p_{c_1},p_{c_2} such that $\Delta_i = p_{a_i}p_{b_i}p_{c_i}$ are clockwise oriented maximum-area triangles, so p_{c_2} is in the clockwise order between p_{b_2} and p_{a_2}. Thus the line $p_{a_2}p_{b_2}$ separates the edge $\overline{p_{a_1}p_{b_1}}$ of Δ_1 from the triangle Δ_2, contradicting the Lemma (Figure 3, left).

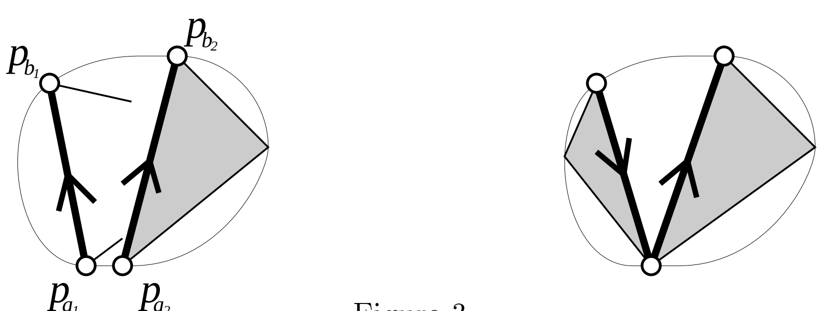

Figure 3.

We now count the triangles containing a point p_i. Let $N^{\text{out}}(p_i) = \{p_k \mid (p_i,p_k) \in E\}$, $\text{outdeg}(p_i) = |N^{\text{out}}(p_i)|$, and $N^{\text{in}}(p_i) = \{p_k \mid (p_k,p_i) \in E\}$, $\text{indeg}(p_i) = |N^{\text{out}}(p_i)|$. We claim that p_i is contained in at most $\text{outdeg}(p_i) + \text{indeg}(p_i) - 1$ triangles. To show this, we note that all points of $N^{\text{out}}(p_i)$ precede all points of $N^{\text{in}}(p_i)$ in clockwise order, for if we have an ingoing edge before an outgoing edge, we find again two triangles violating the intersection condition of the Lemma (Figure 3, right). So for each triangle there is one edge going from a point in $N^{\text{out}}(p_i)$ to a point of $N^{\text{in}}(p_i)$, and the number of triangles containing p_i is at most the number of such edges. Any two such edges must have a common point, for otherwise we again have a violation of the condition of the Lemma. Thus the edges of triangles containing p_i define a set of line-segments between points of $N^{\text{out}}(p_i)$ and $N^{\text{in}}(p_i)$ with the property that any two segments have a common point.

Since each point belongs to at least one triangle, we have $\operatorname{indeg}(p_i), \operatorname{outdeg}(p_i) \geq 1$; if $\operatorname{indeg}(p_i) = 1$ or $\operatorname{outdeg}(p_i) = 1$, our claim is trivially satisfied. So we can assume $\operatorname{outdeg}(p_i) \geq 2$. We extend the set of segments by one segment joining the first point of $N^{\mathrm{out}}(p_i)$ to the last point; again this set of segments between $\operatorname{outdeg}(v) + \operatorname{indeg}(v)$ points has the property that any two segments have a common point. But any such set consists of at most $\operatorname{outdeg}(v) + \operatorname{indeg}(v)$ segments [13,10], so there are at most $\operatorname{outdeg}(v) + \operatorname{indeg}(v) - 1$ triangles containing p_i. Therefore

$$\begin{aligned} 3 * \text{number of triangles} &= \sum_{i=1}^{n} \text{number of triangles containing } p_i \\ &\leq \sum_{i=1}^{n} (\operatorname{outdeg}(p_i) + \operatorname{indeg}(p_i) - 1) \\ &= 2|E(G)| - n \\ &\leq 3n, \end{aligned}$$

so there are at most n triangles.

Proof of the Lemma: Let Δ be a maximum-area triangle with vertices a_1, a_2, a_3. Each point of a set in which Δ is a maximum-area triangle must be in the triangle bounded by the three lines through a_i parallel to $a_{i+1}a_{i+2}$, $i = 1, 2, 3$: any point outside this region will generate together with two vertices of Δ a triangle with area larger than that of Δ. This large triangle is cut by Δ into three translates of $-\Delta$; we will denote them by $\nabla_1, \nabla_2, \nabla_3$. The vertices of Δ^* are contained in $\nabla_1 \cup \nabla_2 \cup \nabla_3$. If all vertices of Δ^* are in the same ∇_i, then Δ^* coincides with that ∇_i, since they have the same area, and the claim of the Lemma is satisfied. If each ∇_i contains one vertex of Δ^*, then the edges of Δ^* intersect Δ, and again our claim is satisfied. In the remaining case two vertices are in one ∇_i, and the third in another; we can assume $\Delta^* = pqr$ with $p, q \in \nabla_1$, $p, q \notin \Delta$, and $r \in \nabla_2$ (Figure 4). Then the line through p parallel to qr separates a_2 from qr, so the triangle a_2qr has an area larger than that of Δ^*, violating the assumption of the Lemma.

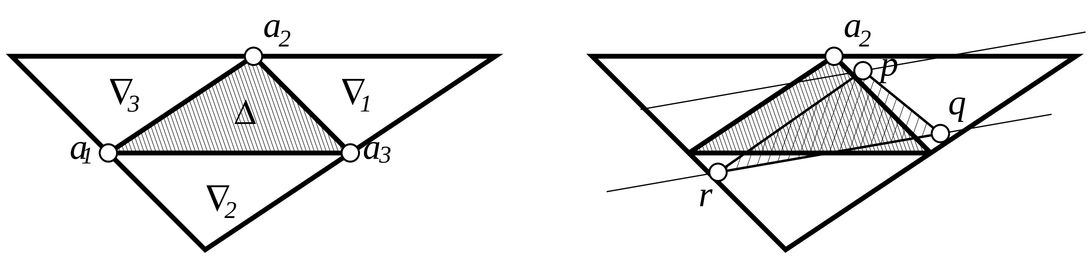

Figure 4.

3. Related Problems

Let $ua_d(n)$ denote the maximum number of unit area triangles among n points in $\mathbb{R}^d$. The best currently known bounds are

$d = 2$ $ua_2(n) = \Omega(n^2 \log\log n)$ [4] and $ua_2(n) = O(n^{\frac{7}{3}})$ [11], the lower bound by lattice sections;

$d = 3$ $ua_3(n) = O(n^{\frac{8}{3}})$ [4], no better lower bound known ($ua_3(n) \geq ua_2(n) = \Omega(n^2 \log\log n)$);

$d = 4$ no better upper or lower bounds known ($ua_4(n) \leq ua_5(n) = O(n^{3-\varepsilon})$ and $ua_4(n) \geq ua_2(n) = \Omega(n^2 \log\log n)$);

$d = 5$ $ua_5 = \Omega(n^{\frac{7}{3}})$ and $ua_5(n) = O(n^{3-\epsilon})$ for some $\epsilon > 0$ [12], the lower bound is obtained by taking $\frac{n}{2}$ points with $\Omega(n^{\frac{4}{3}})$ unit distances on a sphere in a 3-dimensional subspace [3], and $\frac{n}{2}$ points on a circle in an orthogonal plane;

$d \geq 6$ $ua_d(n) = \Theta(n^3)$ by a Lenz-type construction: $\frac{n}{3}$ points on three concentric circles in pairwise orthogonal planes [12].

Analogous to the distance problems one can also ask for the minimum number of distinct areas determined by n points in $\mathbb{R}^d$ [6]; here the conjectured value is $\left\lfloor \frac{n-1}{d} \right\rfloor$, which is reached by n points distributed equidistant on d parallel lines orthogonal to a regular $d-1$-simplex. In the plane one obtains a lower bound of $\frac{1}{2}(1-\frac{1}{3+2\sqrt{2}})n - O(1) \approx 0.4142n - O(1)$ by combining the method of [2] and the result of [14], improving previous bounds in [2] and [6].

Another generalization of the unit distances problem is the question for the maximum number of congruent triangles determined by n points in $\mathbb{R}^d$ [4]. In the plane this is the same as the unit distances problem, since each unit distance can belong to at most four triangles; the upper bounds are $O(n^{\frac{19}{9}})$ in three-dimensional space [5], $O(n^{\frac{65}{23}})$ in dimension four [1], and $\Theta(n^3)$ for dimension $d \geq 6$.

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