

# Generalized Guarding and Partitioning for Rectilinear Polygons

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## Abstract

A  $T_k$ -guard  $G$  in a rectilinear polygon  $P$  is a tree of diameter  $k$  completely contained in  $P$ . The guard  $G$  is said to cover a point  $x$  if  $x$  is visible (or rectangularly visible) from some point contained in  $G$ . We investigate the function  $r(n, h, k)$ , which is the largest number of  $T_k$ -guards necessary to cover any rectilinear polygon with  $h$  holes and  $n$  vertices. The aim of this paper is to prove new lower and upper bounds on parts of this function.

In particular, we show the following bounds:

1.  $r(n, 0, k) \leq \lfloor \frac{n}{k+4} \rfloor$ , with equality for even  $k$
2.  $r(n, h, 1) = \lfloor \frac{3n+4h+4}{16} \rfloor$
3.  $r(n, h, 2) \leq \lfloor \frac{n}{8} \rfloor$ .

These bounds, along with other lower bounds that we establish, suggest that the presence of holes reduces the number of guards required, if  $k > 1$ . In the course of proving the upper bounds, new results on partitioning are obtained.

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# 1 Introduction

Given two points  $x$  and  $y$  in a rectilinear polygon  $P$ , the points  $x$  and  $y$  are called *rectangularly visible*, denoted  $x \square y$ , if the smallest aligned rectangle  $R(x, y)$  spanned by  $x$  and  $y$  is contained in  $P$  [8]. This is a more restrictive notion than the usual visibility, where one only requires that the line segment  $(x, y)$  is contained in  $P$ .

In this paper we study the following (rectangular) visibility problem: Let  $P$  be a rectilinear polygon with  $h$  holes on  $n$  vertices. How can one cover  $P$  by  $T_k$ -guards? Here, a  $T_k$ -guard in  $P$  is a tree  $G$  that has graph-theoretic diameter  $k$  and is rectilinearly embedded in  $P$ . The region  $V(G)$  covered by such a guard is the set of all points rectangularly visible to  $G$ :  $V(G) = \{x \in P \mid \exists y \in G \text{ such that } x \square y\}$ . A collection  $\{G_i\}, i \in I$  of  $T_k$ -guards covers  $P$  if  $\bigcup_{i \in I} V(G_i) = P$ .

Let us define the following functions:

$$\begin{aligned} r(P, k) &= \min\{p \mid \exists \text{ a set of } p \text{ } T_k\text{-guards} \\ &\quad \text{that cover } P\} \\ r(n, h, k) &= \max\{r(P, k) \mid P \text{ is a rectilinear polygon} \\ &\quad \text{with } n \text{ vertices and } h \text{ holes}\} \end{aligned}$$

Further, let  $g(n, h, k)$  be the function analogous to  $r(n, h, k)$  defined for general polygons with the usual visibility notion. The first result concerning these functions is Chvátal's classical Art Gallery Theorem, which in our notation reads  $g(n, 0, 0) = \lfloor \frac{n}{3} \rfloor$ . After this result, many combinatorial and algorithmic variations of this problem have been studied; most of these variations can be found in [10] and [11]. For general polygons, it is known that  $g(n, 0, k) = \lfloor \frac{n}{k+3} \rfloor$  [12] and  $g(n, h, 0) = \lfloor \frac{n+h}{3} \rfloor$  [6], [2]. Throughout this paper we use the following non-standard convention:  $\lfloor \frac{n}{m} \rfloor$  is the set to be 1 for  $0 < n < m$ .

In rectilinear polygons the situation is quite different. For instance, for point guards ( $T_0$ -guards), it is known that  $r(n, h, 0) = \lfloor \frac{n}{4} \rfloor$  [7], [4]. This is unusual in that the number of holes does not affect the maximum number of guards required. However, for line guards ( $T_1$ -guards) holes make the problem harder: it is known that  $r(n, h, 1) \geq \lfloor \frac{3n+4h+4}{16} \rfloor$  [14]. This bound is tight for  $h = 0$  (i.e.,  $r(n, 0, 1) = \lfloor \frac{3n+4}{16} \rfloor$ ) [1]. So what is the correct bound for line guards, and what about general  $T_k$ -guards? This paper answers the first question and begins to address the second. We begin with some definitions and conventions.

We use the term  $(n, h)$ -*polygon* to denote a rectilinear polygon with  $h$  holes and a total of  $n$  vertices. Such a polygon is said to be in *general position* if no two reflex vertices can be joined by a horizontal or vertical line segment lying in the interior of the polygon. A short case analysis shows that by perturbing the vertices of a polygon  $P$  that is not in general position, we can obtain a polygon  $P'$  in general position such that a covering of  $P'$  by  $T_k$ -guards implies a covering of  $P$  by  $T_k$ -guards. We henceforth restrict our attention to polygons in general position.

The *rectangular decomposition* of an  $(n, h)$ -polygon  $P$  is a partition of  $P$  into rectangles by extending a horizontal chord into the polygon from every reflex vertex

Figure 1: Rectangular decomposition and R-graph

(see Figure 1). The number of rectangles in this decomposition is  $\frac{n-2}{2} + h$  (if the polygon were not in general position this number would be smaller). We define the R-graph of  $P$ , denoted  $\mathbf{R}(P)$  (or simply  $\mathbf{R}$  when  $P$  is understood), as a directed graph where each vertex corresponds to a rectangle of the rectangular decomposition of  $P$ , and an arc is directed from node  $A$  to node  $B$  iff they correspond to adjacent rectangles and the chord separating these rectangles forms an entire side of  $B$ . The direction of these arcs gives us some visibility information. R-graphs are similar to the H-graphs of O'Rourke [9]. The undirected version of  $\mathbf{R}$  is denoted  $\tilde{\mathbf{R}}$ .

For any pair of neighboring rectangles in a rectangle decomposition there is one vertical polygon edge which is a vertical boundary for both. Depending whether this edge is the left (or right) boundary of both rectangles we will call the rectangles (or their corresponding nodes in  $\mathbf{R}(P)$ ) left (or right) neighbors. The remaining terminology about rectangle decompositions should be self-explanatory (compare with Figure 1):

lower neighbor	( $B$ is a lower neighbor of $A$ ),
upper neighbor	( $C$ is an upper neighbor of $A$ ),
indegree	( $\text{indeg}(C) = 1$ ),
outdegree	( $\text{outdeg}(A) = 3$ ),
degree	( $\text{deg}(D) = \text{indeg}(D) + \text{outdeg}(D) = 3$ ).

We note that the property of being a left neighbor is symmetric, in contrast to the property of being a lower neighbor.

The rest of the paper is organized as follows. The next section provides constructions which establish a lower bound for every value of  $r(n, h, k)$ . The third section contains a proof that  $r(n, 0, k) \leq \lfloor \frac{n}{k+4} \rfloor$ , and that equality holds for even  $k$ . One feature of our proof is that it provides a procedure for partitioning a simply-connected orthogonal polygon into at most  $\lfloor \frac{n}{k+4} \rfloor$  polygons of size at most  $2k + 6$ ; this generalizes results in [9], [3] for  $k = 0$ . The fourth section shows that the lower bound

Figure 2: Lower bounds for polygons with no holes

Figure 3: Lower bounds for even  $k$ 

for line guards is tight and that  $r(n, h, 2) \leq \lfloor \frac{n}{6} \rfloor$ . The last section is a summary and discussion of future directions.

## 2 Lower bounds on $r(n, h, k)$

In this section, we establish the following lower bounds on  $r(n, h, k)$ :

$$r(n, h, k) \geq \begin{cases} \lfloor \frac{n-2h}{k+4} \rfloor & \text{even } k \\ \lfloor \frac{3n+(7-3k)h+4}{3k+13} \rfloor & k = 1, 3 \\ \lfloor \frac{3(n-2h)+4}{3k+13} \rfloor & \text{odd } k \geq 5 \end{cases}$$

These bounds are valid only for certain relationships of  $n/h$ , and  $k$ , as detailed later.

We begin with the  $\lfloor \frac{n-2h}{k+4} \rfloor$  bound for even  $k$ . This bound is valid for  $\frac{n}{h} \geq k+6$ ; this condition may be thought of as “having enough vertices per hole to make it interesting.” Note that  $\frac{n-4}{h}$  must be at least four, because each hole must have at least four vertices. Also, it is already known that  $r(n, h, 0) = \lfloor \frac{n}{4} \rfloor$  for  $k = 0$  [4], so we need only consider  $k \geq 2$ .

Figure 2 shows examples of infinite polygon classes that establish a lower bound of  $\lfloor \frac{n}{k+4} \rfloor$  for  $h = 0$ . The figure shows examples for  $k = 2$ ,  $k = 4$ , and  $k = 6$ ; these examples consist of  $\frac{n}{k+4}$  spiral arms joined in a row; one guard is needed for each arm. Examples for larger  $k$  are made by increasing the number of turns on each spiral arm (one more turn per each increase of two in  $k$ ). Examples for larger

Figure 4: The 1–pinwheel and the 3–pinwheel

$n$  are made by joining more arms to the polygon. Holes made be added to these examples in the following manner: find a spiral arm that does not contain a hole (here we use the property that  $\frac{n}{h} \geq k + 4$ ), shorten that spiral by one turn, and add a rectangle in its end. This operation increases  $n$  by two and  $h$  by one, leaving the numerator (of  $\lfloor \frac{n-2h}{k+4} \rfloor$ ) unchanged, and ensures that each arm still requires its own guard. Examples of this construction are shown in Figure 3 for  $n = 36, h = 3, k = 2$  and  $n = 34, h = 2, k = 6$ . The class of polygons thus described establishes the  $\lfloor \frac{n-2h}{k+4} \rfloor$  lower bound.

It remains to show lower bounds for odd  $k$ . Note that all both of the bounds that we wish to show (one for  $k = 1$  and  $3$ , and another for  $k \geq 5$  both simplify to  $\lfloor \frac{3n+4}{3k+13} \rfloor$  for  $h = 0$ ). We first establish this bound, and describe the general construction method for odd  $k$ .

Let the term  $t$ –*pinwheel* denote the  $(8t + 12, 0)$ –polygon formed by connecting four spiral arms of  $t$  turns in “pinwheel fashion”, as illustrated in Figure 4 for  $t = 1$  and  $t = 3$ . We will construct larger polygons from pinwheels by an operation that we call *grafting*. Grafting consists of clipping one of the spiral arms from a pinwheel, and attaching this fragment to another polygon at the first turn of one of its spiral arms (with the restriction that this spiral arm has not been grafted to before). A polygon which is formed by successively grafting only  $t$ –pinwheels to a  $t$ –pinwheel is called a  $t$ –*growth*. Figure 5 shows two 3–growths, the first the result of one grafting operation and the second the result of two.

In any  $t$ –pinwheel or  $t$ –growth, the vertices at the end of each spiral arm (one for each arm) form an independent set with respect to paths of length  $2t + 1$  inside the polygon. Thus, no  $T_{2t-1}$ –guard can see two of these vertices. To get lower bound examples for odd  $k$  and  $h = 0$ , we set  $k = 2t - 1$  ( $t = \frac{k+1}{2}$ ). Any  $(\frac{k+1}{2})$ –growth resulting from  $j$  graftings has  $3j + 4$  spiral arms (thus requiring  $3j + 4$   $T_k$ –guards) and  $n = (8t + 12) + j(6t + 10) = (4k + 16) + j(3k + 13)$  vertices. These growths thus give the desired  $\lfloor \frac{3n+4}{3k+13} \rfloor$  lower bound.

To establish the general  $\lfloor \frac{3(n-2h)+4}{3k+13} \rfloor$  bound, we start with the (holeless)  $(\frac{k+1}{2})$ –growth and add holes in the same fashion that we did for the even– $k$  examples: find an empty spiral arm, shorten it by one turn, and insert a rectangle. Once again

Figure 5: 3-growths

we have increased  $n$  by two and  $h$  by one without changing the number of guards required. An example of this construction is shown in Figure 6 for  $n = 100, h = 4, k = 5$  (requiring 10  $T_5$ -guards). This establishes the bound if the “enough vertices per hole” condition of  $\frac{n}{h} > k + 6\frac{1}{3}$  is satisfied.

Figure 6: A 3-growth with holes added

For  $k = 3$ , we wish to show a lower bound of  $\lfloor \frac{3n-2h+4}{22} \rfloor$ . We start, as expected, with 2-growths, but to add a hole we *increase* the number of turns on a spiral arm by one, and insert an L-shaped hole that sits inside this turn (see Figure 7 for an example). This process adds 8 vertices and 1 hole ( $3\Delta n - 2\Delta h = 22$ ) but the polygon

now requires one extra guard, which bears out the formula. This hole insertion may be carried out as long as  $\frac{n}{h} > 19\frac{1}{3}$ .

Figure 7: Example for  $k = 3$

For  $k = 1$ , the bound of  $\lfloor \frac{3n+4h+4}{16} \rfloor$  is established by starting with 1-growths and adding rectangular holes in the ends of empty spiral arms [14]. Each hole insertion adds 1 hole and 4 vertices, and necessitates 1 extra guard. This construction is valid for  $\frac{n}{h} > 9\frac{1}{3}$ .

### 3 Upper bound on $r(n, 0, k)$

In this section, we prove the following upper bound:

**Theorem 1**  $r(n, 0, k) \leq \lfloor \frac{n}{k+4} \rfloor$

We actually prove a stronger statement

**Theorem 2** *Any  $(n, 0)$ -polygon in general position can be partitioned into  $\lfloor \frac{n}{k+4} \rfloor$  simply-connected rectilinear polygons of at most  $2k + 6$  vertices.*

We recall once more that if  $n < k + 4$  then we have to count one for  $\lfloor \frac{n}{k+4} \rfloor$  rather than zero. The following lemma and Theorem 2 imply Theorem 1.

**Lemma 3** *Any simply-connected rectilinear polygon of at most  $2k + 6$  vertices can be covered by one  $T_k$ -guard.*

Lemma 3 can be proved easily by induction on  $k$ .

Now it is sufficient to give a proof of Theorem 2 for a polygon  $P$  with  $n \geq 2k + 8$  vertices.

We let the term *cut* denote either a chord of the horizontal or vertical rectangular decomposition of  $P$  or the L-shaped union of two line segments joining two reflex

vertices. We prove Theorem 2 inductively, using cuts to subdivide the polygon  $P$ . A cut subdivides  $P$  into two rectilinear subpolygons of  $n_1$  and  $n_2$  vertices such that  $n_1 + n_2 = n + 2$ ; we refer to such a cut as a  $(n_1, n_2)$ -cut. Such a cut will be called good if  $\lfloor \frac{n_1}{k+4} \rfloor + \lfloor \frac{n_2}{k+4} \rfloor \leq \lfloor \frac{n}{k+4} \rfloor$ , i.e. if the inductive argument can be applied.

**Lemma 4** *Let  $n, n_1, n_2$  be even numbers with  $n \geq 2k + 8$  and  $n_1 + n_2 = n + 2$ . An  $(n_1, n_2)$ -cut of an  $(n, 0)$ -polygon is good if one of the following conditions holds:*

- (i).  $n_1 \leq 2k + 6$  and  $n_2 \leq 2k + 6$
- (ii).  $n_1 \geq k + 4$  and  $n_2 \geq k + 4$  and  $n_1 \not\equiv 0$  or  $1 \pmod{k + 4}$
- (iii).  $n_1 \geq k + 4$  and  $n_2 \geq k + 4$  and  $n_2 \not\equiv 0$  or  $1 \pmod{k + 4}$
- (iv).  $n_1 \equiv n_2 \equiv 1 \pmod{k + 4}$

**Proof:** (i):  $\lfloor \frac{n_1}{k+4} \rfloor + \lfloor \frac{n_2}{k+4} \rfloor = 1 + 1 = \lfloor \frac{2k+8}{k+4} \rfloor \leq \lfloor \frac{n}{k+4} \rfloor$ .

(ii),(iii),(iv): Let  $\alpha_i$  be the residue  $n_i \pmod{k + 4}$ . Then in all cases we have  $\alpha_1 + \alpha_2 \geq 2$ . Moreover  $k + 4 \leq n_1$  and  $k + 4 \leq n_2$  holds in case (ii) and (iii) by assumption and in case (iv) because otherwise  $n_1$ , resp.  $n_2$  (as the number of vertices of  $P_1$ , resp.  $P_2$ ) would be 1. Thus we get

$$\lfloor \frac{n_1}{k+4} \rfloor + \lfloor \frac{n_2}{k+4} \rfloor = \lfloor \frac{n_1 - \alpha_1}{k+4} \rfloor + \lfloor \frac{n_2 - \alpha_2}{k+4} \rfloor \leq \lfloor \frac{n_1 + n_2 - \alpha_1 - \alpha_2}{k+4} \rfloor = \lfloor \frac{n+2 - \alpha_1 - \alpha_2}{k+4} \rfloor \leq \lfloor \frac{n}{k+4} \rfloor \quad \square$$

**Corollary** *Let  $n, n_1, n_2$  be even numbers with  $n_1 + n_2 = n + 2$ ,  $n_1 \geq k + 4$  and  $n_2 - 2 \geq k + 4$ . If an  $(n, 0)$ -polygon has an  $(n_1, n_2)$ -cut and an  $(n_1 + 2, n_2 - 2)$ -cut then at least one of them is a good cut.  $\square$*

Usually we will apply this corollary in a situation where the region between the two cuts is a rectangle. We use the term *consecutive cuts* to refer to such a pair of cuts.

**Proof of Theorem 2.** As  $P$  is an  $(n, 0)$ -polygon, the R-graph  $\mathbf{R}(P)$  is a tree with  $r = \frac{n-2}{2}$  nodes, and therefore it has a node  $R$  such that after deleting it, the size of any connected component is at most  $\frac{r}{2} = \frac{n-2}{4}$ . In terms of the polygon this means that  $\deg(R)$  horizontal cuts partition the polygon into  $\deg(R) + 1$  parts: the rectangle  $R$  and polygons  $P_1, \dots, P_{\deg(R)}$  with  $n_1, \dots, n_{\deg(R)}$  vertices such that each  $n_i$  is at most  $2 \cdot \frac{r}{2} + 2 = \frac{n+2}{2}$ . Since any cut creates two new vertices we have  $\sum_{i=1}^{\deg(R)} n_i = n + 2 \cdot \deg(R) - 4$ . Transforming this equality as follows

$$\begin{aligned} -n_i &= -n + \sum_{j \in \{1, \dots, \deg(R)\} \setminus \{i\}} n_j + 4 - 2 \cdot \deg(R) && \text{and combining it with} \\ 2n_i &\leq n + 2 && \text{we obtain} \\ n_i &\leq \sum_{j \in \{1, \dots, \deg(R)\} \setminus \{i\}} n_j + 6 - 2 \cdot \deg(R) && \text{for any } i \in \{1, \dots, \deg(R)\}. \end{aligned}$$

Now, we have the three possibilities:  $R$  has 2, 3 or 4 neighbors.

Figure 8: Illustration of Case B

**Case A:** Suppose that  $\deg(R) = 2$  and assume w.l.o.g.  $n_1 \leq n_2$ . Considering the two cuts individually we have an  $(n_1, n_2 + 2)$ -cut and an  $(n_1 + 2, n_2)$ -cut. If moreover  $n_1 \geq k + 4$  then by the corollary at least one of the cuts is good. Otherwise, if  $n_1 < k + 4$  then by the inequality derived above we get  $n_2 \leq n_1 + 6 - 2 \cdot 2 < k + 4 + 2 \leq 2k + 6$ . Thus, the  $(n_1 + 2, n_2)$ -cut will be good by Lemma 4 (i).

**Case B:** Suppose that  $\deg(R) = 3$  and assume w.l.o.g. (by symmetry) that  $P_1$  (resp.  $P_2$  and  $P_3$ ) meets  $R$  via a left upper (resp left lower and right upper) neighboring rectangle.

By the above discussion, we know that  $n_1 + n_2 + n_3 = n + 2$  and  $n_i \leq n_j + n_k$  for any permutation  $(i, j, k)$ . Clearly, we have an  $(n_1, n_2 + n_3)$ -cut, an  $(n_2, n_1 + n_3)$ -cut and an  $(n_3, n_1 + n_2)$ -cut, but, there is also a fourth  $(n_3 + 2, n_1 + n_2 - 2)$ -cut which starts vertically from  $A$  down to the horizontal edge thru  $C$  or its extension (see Figure 8 for illustration of the typical situations).

**Subcase B.1:** Suppose that  $n_3 \geq k + 4$ .

If moreover  $n_1 + n_2 - 2 \geq k + 4$  then by the corollary the third or the fourth cut will be good. Otherwise, if  $n_1 + n_2 - 2 < k + 4$  then we have  $n_3 \leq n_1 + n_2 < k + 6 \leq 2k + 6$  and hence the fourth cut is good by Lemma 4 (i).

**Subcase B.2:** Suppose that  $n_3 < k + 4$  and one of the following seven conditions holds:

- a)  $n_1 < k + 4$ ; then  $n_1 + n_3 \leq 2k + 6$  and  $n_2 \leq n_1 + n_3 \leq 2k + 6$ . Thus the first cut is good by Lemma 4 (i).
- b)  $n_2 < k + 4$ ; then analogously the second cut is good.
- c)  $n_1 \geq k + 4$  and  $n_2 \geq k + 4$  and  $n_1 \equiv 0 \pmod{k + 4}$ ; then  $(n_1 + n_3) \not\equiv 0$  or  $1 \pmod{k + 4}$  and the second cut will be good by Lemma 4 (iii).

- d)  $n_1 \geq k + 4$  and  $n_2 \geq k + 4$  and  $n_2 \equiv 0 \pmod{k + 4}$ ; then  $(n_2 + n_3) \not\equiv 0$  or  $1 \pmod{k + 4}$  and the first cut will be good by Lemma 4 (iii).
- e)  $n_1 \geq k + 4$  and  $n_2 \geq k + 4$  and  $n_1 \not\equiv 0$  or  $1 \pmod{k + 4}$ ; then the first cut will be good by Lemma 4 (ii).
- f)  $n_1 \geq k + 4$  and  $n_2 \geq k + 4$  and  $n_2 \not\equiv 0$  or  $1 \pmod{k + 4}$ ; then analogously the second cut will be good.
- g)  $n_1 \geq k + 4$  and  $n_2 \geq k + 4$  and  $n_1 \equiv n_2 \equiv 1 \pmod{k + 4}$  and  $n_3 < k + 3$ ; then the first cut will be good by Lemma 4 (iii).

**Subcase B.3:** Suppose none of the above holds, this means we have  $n_1 \equiv n_2 \equiv 1 \pmod{k + 4}, n_3 = k + 3$

We will find in each possible configuration either a cut with one resulting subpolygon of size  $k + 7$  or a pair of consecutive cuts.

We call two reflex vertices opposite to each other if they rectangularly see each other and the edges incident to them (considered as rays emanating from these vertices) represent all 4 main compass directions.

Observe that in the case of two opposite reflex vertices, as well as in the case of two neighboring reflex vertices which both rectangularly see a third reflex vertex, one finds consecutive cuts.

**Subcase B.3.1:**  $C$  is right of  $B$

This is either the left or the right configuration shown in Figure 8. We consider the highest reflex vertex  $D$  below the horizontal line thru  $C$  such that  $D$  is visible both from  $A$  and  $B$ . If there are two such vertices take, say, the left one. Given there is no such vertex the vertical line extensions thru  $A$  and  $B$  define consecutive cuts. But if we have a vertex  $D$  we also have consecutive cuts by the above observation. Note that in all these cuts the subpolygons containing  $P_3$  have size  $\geq k + 5$  and the remaining parts have size  $\geq k + 5$  as well, since each contains  $P_1$  completely. Thus, based on the corollary at least one of the cuts is good.

**Subcase B.3.2:**  $C$  is left of  $A$ .

If  $C$  rectangularly sees the upper neighbor of  $A$ , then we connect  $C$  with this neighbor (even if it is convex) by an L-shaped cut and obtain a subpolygon containing  $P_3$  of size  $k + 7$ . Otherwise there must be a reflex vertex in  $P_1$  which is opposite to  $A$  and we are done.

**Subcase B.3.3:**  $C$  is right of  $A$  and left of  $B$ .

In this case we can apply the same argument as in subcase B.3.2 to  $P_2$  with the roles of  $A$  and  $C$  exchanged.

**Case C:** Suppose that  $\deg(R) = 4$  and assume w.l.o.g. that  $P_1$  and  $P_2$  (resp.  $P_3$  and  $P_4$ ) are left (resp. right) neighbors of  $R$ . Since  $\sum_{i=1}^4 n_i = n + 4$  at least one of the subsums  $n_1 + n_2$  or  $n_3 + n_4$  is less than or equal to  $\frac{n+4}{2}$ . By symmetry, we can assume that this holds for the subsum  $n_3 + n_4$ . Then there is an L-shaped cut

such that the polygon  $P'_3$  on the right side of this cut has  $n_3 + n_4 - 2$  vertices and consists of  $P_3, P_4$  and a portion of  $R$ . Now the analysis of Case B can be applied, with  $P'_3$  taking the place of  $P_3$  in that analysis.  $\square$

## 4 Upper bounds on $r(n, h, 1)$ and $r(n, h, 2)$

In this section we will prove the following result.

**Theorem 5**  $\lfloor \frac{3n+4h+4}{16} \rfloor$   $T_1$ -guards are always sufficient to cover any rectilinear  $(n, h)$ -polygon.

In fact we prove that these guards can be chosen to be polygon edges or edge extensions. Moreover, in the whole section we will deal with the stronger definition of orthogonal visibility: a point  $x$  in a polygon  $P$  is orthogonally visible from a  $T_k$ -guard ( $k \geq 1$ )  $G$  if there is a line of  $G$  such that the perpendicular from  $x$  to this line is contained in  $P$ .

**Lemma 6** Let  $R_1$  and  $R_2$  be adjacent rectangles in  $\mathbf{R}$  separated by the extension of some horizontal polygon edge  $e$ . Then the following holds:

- (i). If  $R_2$  is an upper (resp. lower) neighbor of  $R_1$  and the arc connecting them is directed from  $R_2$  to  $R_1$  then  $R_2$  is the only upper (resp. lower) neighbor of  $R_1$ . Consequently, if  $\text{indeg}(R_1) = 2$  then  $\text{outdeg}(R_1) = 0$ .
- (ii). If  $G$  is a  $T_1$ -guard on the edge  $e$  and its extension then  $G$  can watch any rectangle  $R$  which can be reached by a directed path in  $\mathbf{R}$  starting from  $R_1$  or  $R_2$ .

**Proof:** (i). This follows from the assumption about the general position. (ii). We observe that according to (i) any directed path in  $\mathbf{R}$  is also strictly directed in the geometrical sense (either upwards or downwards). Furthermore on a directed path the rectangles get more and more narrow.  $\square$

**Lemma 7** If  $R_0R_1\dots R_m$  is a directed path in  $\mathbf{R}$  and  $R_{m+1}$  is another rectangle with an arc directed to  $R_m$  then there is a vertical  $T_1$ -guard covering all rectangles  $R_i$  ( $0 \leq i \leq m + 1$ ).

**Proof:** Note that  $R_m$  and  $R_{m+1}$  have a vertical polygon edge  $e$  in common. Since the path from  $R_0$  to  $R_m$  is strictly directed in the geometrical sense with the rectangles becoming more and more narrow,  $e$  can be extended to  $R_0$ .  $\square$

We define the *frame* of  $\mathbf{R}$  to be the largest subgraph  $\mathbf{F}$  such that for every vertex  $R$  in  $\mathbf{F}$ ,  $\deg_{\mathbf{F}}(R) \geq 2$ . If there isn't any nonempty subgraph  $\mathbf{F}$  fulfilling the above condition (i.e. if  $\mathbf{R}$  is a tree) then we define some arbitrary fixed leaf of  $\mathbf{R}$  to be the frame. Thus,  $\mathbf{R}$  consists of its frame and some attached trees. Denote by  $\mathbf{T}$  the set  $\mathbf{R} \setminus \mathbf{F}$  of non-frame nodes. For any  $R \in \mathbf{T}$  there is a unique path  $p(R)$  in  $\mathbf{R}$  connecting it to the frame. A node  $R \in \mathbf{T}$  with degree  $\geq 3$  is called a *primary branch* if for any  $R' \in \mathbf{T}$  such that  $R \in p(R')$ ,  $R$  is the first node of degree  $\geq 3$  on  $p(R')$ .

Let  $R_0 \in \mathbf{T}$  be a leaf and  $p(R_0) = R_0 R_1 \dots R_m$  with  $R_m \in \mathbf{F}$ . We define the *branching distance* of  $R_0$  to be the minimal number  $l$  ( $1 \leq l \leq m$ ) such that  $\deg(R_l) \geq 3$ , or  $m$  if there is no such number.

Let  $G_1, \dots, G_l$  be a family of  $T_1$ -guards in an  $(n, h)$ -polygon  $P$  and  $D$  a rectilinear region covered by them (called a district of the guards). Usually,  $D$  will be smaller than the maximal possible region covered by  $G_1, \dots, G_l$ . Deleting  $D$  from  $P$  we obtain a number (say  $c'$ ) of connected regions which are  $(n_1, h_1), \dots, (n_{c'}, h_{c'})$ -polygons denoted by  $P_1, \dots, P_{c'}$ .

The deletion of  $D$  will be called a *reduction* if  $l + \sum_{i=1}^{c'} \lfloor \frac{3n_i + 4h_i + 4}{16} \rfloor \leq \lfloor \frac{3n + 4h + 4}{16} \rfloor$ , i.e. if the deletion allows to apply induction. Note, that this definition also makes sense if  $D$  is the whole polygon: then we have  $c' = 0$ , the sum over an empty set is also 0 and we get  $l \leq \lfloor \frac{3n + 4h + 4}{16} \rfloor$ . In the proof we will show that in most situations one can find a reduction by a district of a single guard (i.e.  $l = 1$ ). There will be only one special geometrical configuration where a reduction by a district of two guards is necessary.

The following measures *gain* and *gain*<sup>+</sup> will help to formulate sufficient conditions for a district to cause a reduction. Using the notations above we define

$$\text{gain}(D) := 3(n - n') + 4(h - h') + 4(1 - c')$$

where  $n' = \sum_{i=1}^{c'} n_i$ ,  $h' = \sum_{i=1}^{c'} h_i$ . Furthermore let  $\alpha_i$  be the residue  $3n_i + 4h_i + 4 \pmod{16}$  for any  $1 \leq i \leq c'$ . Then we define

$$\text{gain}^+(D) := 3(n - n') + 4(h - h') + 4(1 - c') + \sum_{i=1}^{c'} \alpha_i$$

**Lemma 8** *Let  $D$  be a district of a family of  $T_1$ -guards  $G_1, \dots, G_l$  in a polygon  $P$ . If  $\text{gain}_P^+(D) \geq l \cdot 16$  then the deletion of  $D$  is a reduction.*

**Proof:** We will make use of the fact that  $\lfloor \frac{3n_i + 4h_i + 4}{16} \rfloor = \lfloor \frac{3n_i + 4h_i + 4 - \alpha_i}{16} \rfloor$ .

$$l + \sum_{i=1}^{c'} \left\lfloor \frac{3n_i + 4h_i + 4}{16} \right\rfloor \leq \left\lfloor \frac{16l}{16} \right\rfloor + \sum_{i=1}^{c'} \left\lfloor \frac{3n_i + 4h_i + 4 - \alpha_i}{16} \right\rfloor$$

$$\begin{aligned}
&\leq \left\lfloor \frac{16l + 3n' + 4h' + 4c' - \sum_{i=1}^{c'} \alpha_i}{16} \right\rfloor \\
&\leq \left\lfloor \frac{\text{gain}_P^+(D) + 3n' + 4h' + 4c' - \sum_{i=1}^{c'} \alpha_i}{16} \right\rfloor \\
&\leq \left\lfloor \frac{3n + 4h + 4}{16} \right\rfloor
\end{aligned}$$

□

It will be very helpful to represent  $\text{gain}(D)$  using the number  $r = \frac{n}{2} + h - 1$  of nodes in  $\mathbf{R}(P)$ . Thus  $n = 2(r - h + 1)$  and  $n' = 2(r' - h' + c')$  where  $r'$  is the total number of nodes in the graphs  $\mathbf{R}(P_i)$ ,  $1 \leq i \leq c'$  and we get

$$\text{gain}(D) = 6(r - r') - 2(h - h') + 10(1 - c').$$

The triple  $(\delta_r, \delta_h, \delta_c)$ , where  $\delta_r = r - r'$ ,  $\delta_h = h - h'$ ,  $\delta_c = 1 - c'$ , will be called the *type* of  $D$ .

**Lemma 9 (Expansion Lemma)** *Let  $G$  be a horizontal  $T_1$ -guard in a polygon  $P$  and  $D$  a district of  $G$ . Let  $P_1$  be a polygon representing a connected component of  $P \setminus D$ , and  $e$  be a horizontal edge that bounds  $P_1$  from above and is shared between  $P_1$  and  $D$ . Let  $R$  be the rectangle of  $P_1$  that contains  $e$ . Let  $\overline{D}$  be the expansion of  $D$  by  $R$  and all rectangles reachable from  $R$  on directed paths in  $\mathbf{R}(P_1)$ . If the edge  $e$  is (orthogonally) visible from  $G$  (see Figure 9, where  $G$  runs across the top of the figure), then  $\overline{D}$  is also a district of  $G$  and the following holds:*

- (i).  $\text{gain}(\overline{D}) \geq \text{gain}(D) + 6$
- (ii). if  $\text{indeg}_{P_1}(R) = 0$  then  $\text{gain}(\overline{D}) \geq \text{gain}(D) + 8$ .

**Proof:** Since  $G$  covers the whole horizontal width of  $R$ , it follows from Lemma 6 (ii) that any rectangle reachable on a directed path in  $\mathbf{R}(P_1)$  from  $R$  will be covered by  $G$ . Let  $\mathbf{S}$  be the subtree of  $\mathbf{R}(P_1)$  formed by  $R$  and all nodes reachable from there on a directed path. Let  $B$  denote the set of rectangles in  $\mathbf{S}$  that have two lower neighbors and  $b = |B|$ . The tree  $\mathbf{S}$  has at least  $2b + 1$  nodes. If we add by breadth first search the rectangles of  $\mathbf{S}$  to  $D$  starting with  $R$ , then for each rectangle from  $B$  either the number of connected components of the remaining polygon increases by 1 (say,  $b_1$  times) or the number of holes decreases by 1 ( $b_2 = b - b_1$  times). In contrast, adding a rectangle which has no two lower neighbors neither changes  $\delta_h$  nor increases the number of connected components. So we have

$$\text{gain}(\overline{D}) \geq \text{gain}(D) + 6(2b + 1) - 10b_1 - 2b_2 \geq \text{gain}(D) + (12 - 10)b + 6 \geq \text{gain}(D) + 6$$

Figure 9: Illustrating Lemma 9

Now, suppose that  $\text{indeg}_{P_1}(R) = 0$ . We consider the three possibilities  $\text{outdeg}_{P_1}(R) = 0, 1$  or  $2$ .

If  $\text{outdeg}_{P_1}(R) = 0$  then  $P_1$  consists of  $R$  only and adding  $R$  to  $D$  we reduce the number of connected components of  $P \setminus D$  by one, giving  $\text{gain}(\overline{D}) = \text{gain}(D) + 10$ . If  $\text{outdeg}_{P_1}(R) = 1$  then let  $R'$  be this unique neighbor of  $R$  in  $P_1$ . Adding  $R$  to  $D$  we get a district  $D'$  with  $\text{gain}(D') = \text{gain}(D) + 6$  and, moreover, we can apply this lemma once more to  $D'$  and the rectangle  $R'$  in  $P \setminus D'$ . Thus we get  $\text{gain}(\overline{D}) \geq \text{gain}(D') + 6 = \text{gain}(D) + 12$

Finally, if  $\text{outdeg}_{P_1}(R) = 2$  then  $R \in B$  and thus  $b \geq 1$ . Our claim follows immediately from the inequality in the first part of the proof.  $\square$

The proof of the theorem now follows from the next three lemmata which show that each non-trivial polygon is reducible.

**Lemma 10** *If  $R_0 \in \mathbf{T}$  is a leaf with branching distance  $\geq 3$  then there is some reduction with  $R$  in the reduction district.*

**Proof:** Let  $R_0, R_1, R_2$  be the first three rectangles on the path  $p(R_0)$ . Since  $\text{deg}(R_1) = \text{deg}(R_2) = 2$ , the deletion of the region  $D = R_0 \cup R_1 \cup R_2$  neither disconnects the remaining polygon nor changes the number of holes and we get  $\text{gain}(D) = 6 \cdot 3 = 18$ . Hence, it is sufficient to show that there is a guard  $G$  covering  $D$ . Let us consider the directed versions of the edges  $\{R_0, R_1\}$  and  $\{R_1, R_2\}$ .

- If both arcs are directed from  $R_1$  to  $R_0$  and  $R_2$  then a guard placed on a horizontal boundary of  $R_1$  covers  $D$  by Lemma 6 (ii).

- If the two arcs form a directed path then a guard on a horizontal boundary of the first rectangle of the directed path will cover  $D$  by Lemma 6 (ii).
- If both edges are directed towards  $R_1$  then there is a vertical guard covering  $D$  by Lemma 7.

□

**Lemma 11** *If all leaves in  $\mathbf{R}$  have branching distance  $< 3$  and  $R$  is a primary branching then there is a reduction such that  $R$  or a part of  $R$  is in the reduction district.*

**Proof:** The proof of this lemma is much more complicated than the proof of the preceding lemma. It requires a rather long case inspection and several tricky arguments. However this is not surprising because both lemmata together yield a new proof for simply connected polygons (cf. [1]).

Let  $R$  be a primary branching with neighbors  $R_1, R_2, R_3$  (and possibly  $R_4$ , if  $\deg(R) = 4$ ) in  $\tilde{\mathbf{R}}$ . W.l.o.g. we can assume that  $R_1$  is the (unique) neighbor of  $R$  on the path  $p(R)$  and, moreover, that  $R_1$  is a left lower neighbor of  $R$ . By the assumption there are leaves  $L_2, L_3$  (and possibly  $L_4$ ) such that for any  $i \geq 2$  we have either  $L_i = R_i$  or  $L_i$  is a neighbor of  $R_i$  and  $\deg(R_i) = 2$ . Let  $\mathbf{N}$  be the set of rectangles consisting of  $R_2, R_3, (R_4 \text{ if } \deg(R) = 4)$  and the leaves  $L_2, L_3, (L_4)$  provided they do not coincide with some  $R_i$ . We have to distinguish the following cases:

**Case A:** Suppose that for all rectangles in  $\mathbf{N}$  there is a directed path from  $R$  to them.

Then we choose a horizontal boundary of  $R$  for placing the guard and by Lemma 6 (ii) this guard covers a district  $D$  consisting of  $R$  and all rectangles from  $\mathbf{N}$ . Clearly, the type of this district is  $(\delta_r, 0, 0)$  and  $\delta_r \geq 3$ . This implies  $\text{gain}(D) \geq 18$  and we are done.

**Case B:** Suppose that for some  $i_0 \geq 2$  there is an arc  $R_{i_0} \rightarrow R$  in  $\mathbf{R}$ , i.e.  $R_{i_0}$  is wider than  $R$ .

W.l.o.g. we may assume that  $i_0 = 2$ . Furthermore we can assume that  $R_2$  is an upper neighbor of  $R$ , because otherwise by Lemma 6 (i)  $R_2$  would be the only lower neighbor of  $R$  contradicting that  $R_1$  is also a lower neighbor.

**Subcase B.1:** Suppose that  $L_2 = R_2$ .

Since  $\deg(R) \geq 3$  and since there is only one upper neighbor,  $R_3$  has to be a right lower neighbor. Dependently on whether  $L_3 \neq R_3$  or  $L_3 = R_3$ , we place a guard on the extended common vertical edge of  $L_3$  and  $R_3$  or on the extended common vertical edge of  $R$  and  $R_3$  and define a district  $D$  consisting of  $R, R_2, R_3$  and  $L_3$ . Thus, the type of  $D$  is  $(4, 0, 0)$  or  $(3, 0, 0)$  and we are done.

Figure 10: Illustration of subcase B.2

**Subcase B.2:** Suppose that  $L_2 \neq R_2$ .

Placing a guard  $G$  on the extended horizontal edge which separates  $L_2$  from  $R_2$  we define a district  $D$  consisting of these two rectangles, see Figure 10. Since  $gain(D) = 12$  does not suffice, we apply the expansion lemma. Indeed, the whole upper boundary of  $R$  is orthogonally visible from  $G$ . Hence adding to  $D$  the rectangle  $R$  and all rectangles reachable from  $R$  via a directed path in  $\mathbf{R}$  we get a new district  $\overline{D}$  with  $gain(\overline{D}) \geq 12 + 6 > 16$  and we are done.

**Case C:** Suppose that neither case A nor case B are valid, i.e. for any  $i \geq 2$  there is an arc from  $R$  to  $R_i$  in  $\mathbf{R}$  and there is some  $i_0 \geq 2$  such that  $L_{i_0} \neq R_{i_0}$  and the arc between them is directed from  $L_{i_0}$  to  $R_{i_0}$ . Again, w.l.o.g. we assume  $i_0 = 2$ . Let  $e$  be the common vertical polygon edge of  $R$  and  $R_2$  and  $A$  the lower (resp. upper) polygon vertex of this edge if  $R_2$  is an upper (resp. lower) neighbor of  $R$ . We place a vertical guard  $G$  on the full extension  $\bar{e}$  of  $e$  and define a district  $D$  dependently on whether  $A$  is a reflex vertex or not.

**Subcase C.1:** Suppose that  $A$  is not a reflex vertex.

Then in a first step we define a district  $D$  of type  $(2, 0, 0)$  consisting of  $L_2, R_2$  and the remaining segment (i.e. below  $R_2$ ) of the edge  $e$ , see Figure 11 – the left picture. Denoting this segment by  $e'$ , it is an edge of the polygon  $P' = P \setminus (L_2 \cup R_2)$ . Let  $\varphi$  be the rotation of the plane by  $90^\circ$  such that  $e'' = \varphi(e')$  is a top edge in the rotated polygon  $P'' = \varphi(P')$ , see Figure 11 – the right picture.

Now, we consider the horizontal rectangular decomposition of  $P''$  (i.e. the rotation of the vertical rectangular decomposition of  $P'$ ) and denote by  $S$  the rectangle containing  $e''$ . Restricting the guard  $G$  to  $P'$ , resp. via rotation to  $P''$ , it is placed on the top edge  $e''$  of  $S$ . So we can apply the expansion lemma in this situation and we get a district  $\overline{D}$  with  $gain(\overline{D}) \geq gain(D) + 6 = 18$ .

The trick of first cutting out a district of small gain, then rotating the polygon and applying the expansion lemma will be used several more times. Since in con-

Figure 11: Illustration of subcase C.1

trast to the original expansion lemma, we expand here the district in a horizontal direction, we will refer to this trick as the horizontal expansion lemma.

**Subcase C.2:** Suppose that  $A$  is a reflex vertex.

We consider the horizontal polygon edge  $f$  which determines the upper boundary of the rectangle  $R$  and denote the right polygon vertex on this edge by  $B$ , see Figure 12. Let  $S$  be the rectilinear rectangle spanned by  $A$  and  $B$  (in general,  $S$  is not a rectangle of the rectangular decomposition).

**Subcase C.2.1:** Suppose that  $S \subseteq P$ , i.e. there are no vertices or edges of  $P$  in the interior of  $S$ .

We define a district  $D$  consisting of  $L_2$ ,  $R_2$  and  $S$ . Clearly, this district is covered by  $G$ . Since general position was assumed, one can be sure that the deletion of  $G$  neither disconnects the remaining region  $P' = P \setminus D$  nor changes the number of holes and, furthermore, there is a cut separating the  $(8,0)$ -polygon  $D$  from the  $(n',h')$ -polygon  $P'$ . This implies  $n' + 8 = n + 2$  or equivalently  $\delta_n = 6$  and consequently  $gain(D) = 3\delta_n + 4\delta_h + 4\delta_c = 18$ .

**Subcase C.2.2:** Suppose that  $S \not\subseteq P$ .

**Subcase C.2.2.1:** Suppose that  $R_2$  is a right neighbor of  $R$ .

We will show that summing up all current assumptions we will obtain the following unique situation:

$R$  has two right neighbors  $R_2$  and  $R_3$  both of degree two. Furthermore, we have the following arcs in  $\mathbf{R}$ :  $L_2 \rightarrow R_2 \leftarrow R \rightarrow R_3 \leftarrow L_3$ . In fact, if  $R_2$  were the only right

Figure 12: Illustration of subcase C.2.1:  $S \subset P$ 

neighbor of  $R$  then either subcase C.1 ( $A$  is not a reflex vertex) or subcase C.2.1 ( $S \subseteq P$ ) would apply. Hence, there is a second right neighbor  $R_3$  and since case B is not valid we have an arc  $R \rightarrow R_3$ . Furthermore if  $R_3$  were a leaf or if  $R_3 \neq L_3$  and  $R_3 \rightarrow L_3$  the vertex  $A$  would not be reflex and subcase C.1. would be valid. So we obtain the configuration  $L_2 \rightarrow R_2 \leftarrow R \rightarrow R_3 \leftarrow L_3$  and a guard placed on  $e$  and its full extension vertically crosses all these rectangles. Thus, defining a district consisting of  $L_2, R_2, R_3$  and  $L_3$  we obtain a reduction of type  $(4, 0, 0)$ .

**Subcase C.2.2.2:** Suppose that  $R_2$  is a left neighbor of  $R$ .

Since  $R_1$  is a left lower neighbor of  $R$ ,  $R_2$  must be a left upper neighbor. This subcase is the hardest one. We will analyse it separately as Case E. It will be very useful to exclude several configurations on the right side of  $R$  before (Case D). To do this, let  $\mathbf{N}'$  be the set of all right neighbors of  $R$  (i.e.  $R_3$  and possibly  $R_4$ , if  $\deg(R) = 4$ ) and of the leaves  $L_3$  ( $L_4$ ) if they do not coincide with  $R_3$  ( $R_4$ ).

**Case D:** Suppose we have all assumptions made in subcase C.2.2.2 and moreover  $|\mathbf{N}'| \geq 2$ .

We again examine the cases A, B, and C taking into account the right neighbors only.

**Subcase D–A:** Suppose that all rectangles in  $\mathbf{N}'$  are reachable from  $R$  on directed paths. Consider the L-cut starting vertically from the more narrow left neighbor of  $R$  to the opposite side of  $R$  and then turning to the right side, see Figure 13 where  $R_2$  is more narrow than  $R_1$ . This L-cut removes an  $m$ -gon  $D$  with  $m = 2 \cdot |\mathbf{N}'| + 4 \geq 8$  that can be covered by a horizontal guard in  $R$ . So we get  $\delta_n \geq 6$ ,  $\delta_h = \delta_c = 0$  and consequently  $\text{gain}(D) \geq 18$ .

**Subcase D–B:** If there is a right neighbor  $R_{i_0}$  with an arc  $R_{i_0} \rightarrow R$  in  $\mathbf{R}$  then this is a proper subcase of Case B and so we are done.

Figure 13: Illustration of subcase D–A

**Subcase D–C:** If there is a right neighbor  $R_{i_0}$  with arcs  $R \rightarrow R_{i_0} \leftarrow L_{i_0}$  we are in the situation of Subcase C.2.2.1.

**Case E:** Suppose, we have all assumptions made in subcase C.2.2.2 and moreover  $|\mathbf{N}'| < 2$  (the negation of D).

We recall that these assumptions together imply the following configuration:  $R$  has a left lower neighbor  $R_1$  (which lies on the unique path connecting  $R$  with the frame), a left upper neighbor  $R_2$  with an attached leaf  $L_2$  such that  $R \rightarrow R_2 \leftarrow L_2$  and exactly one right neighbor  $R_3$  which is a leaf and we have  $R \rightarrow R_3$ . Furthermore we know that the lower vertex  $A$  of the common vertical edge  $e$  of  $R, R_1$  and  $R_2$  is reflex and that the interior of the rectangle  $S$  spanned by  $A$  and  $B$  (the right vertex of the horizontal polygon edge bounding  $R$  from above) contains some vertex.

We place a guard onto the full extension  $\bar{e}$  of  $e$  and define a first district  $D_1$  to consist of the guard position itself plus the rectangles  $R_2$  and  $L_2$ . The vertical cut from  $A$  (which is part of  $D_1$ ) causes us to have either  $\delta_h = 1$  and  $\delta_c = 0$ , or  $\delta_h = 0$  and  $\delta_c = -1$ .

**Subcase E.1:** Suppose that by deleting  $D_1$  we get  $\delta_h = 1$  and  $\delta_c = 0$ .

We have  $gain(D_1) = 2 \cdot 6 - 2 = 10$  and in  $P \setminus D_1$  and applying the rotated version of Lemma 9 on both sides of the guard position we obtain a district  $\overline{D_1}$  of gain  $\geq 10 + 2 \cdot 6 > 16$ .

**Subcase E.2:** Suppose that by deleting  $D_1$  we get  $\delta_h = 0$  and  $\delta_c = -1$ .

We have  $gain(D_1) = 2 \cdot 6 - 10 = 2$  and get two polygons  $P_l$  and  $P_r$  to the left and to the right side of the vertical cut from  $A$ . Let  $R_l$  (resp.  $R_r$ ) be the rectangles of the vertical decomposition of  $P_l$  (resp.  $P_r$ ) which contain the vertical cut from  $A$ . Note that for both rectangles one can apply the rotated version of Lemma 9, see Figure

Figure 14: Illustration of subcase E.2

14.

**Subcase E.2.1:** Suppose that in the vertical rectangular decomposition graph of  $P_r$  we have  $\text{indeg}(R_r) \neq 1$ .

An application of Lemma 9 (ii) to  $P_r$  increases the gain by  $\geq 8$  and hence we obtain a district  $\overline{D_1}$  of gain  $\geq 2 + 8 + 6 = 16$ .

**Subcase E.2.2:** Suppose that in the vertical rectangular decomposition graph  $\mathbf{R}'$  of  $P_r$  we have  $\text{indeg}(R_r) = 1$ .

Applying twice the rotated version of Lemma 9 we get a district  $\overline{D_1}$  consisting of  $R_2, L_2, R_r$  and  $R_l$ . Note that the gain of this district is  $2 + 2 \cdot 6 = 14$ . The assumption  $\text{indeg}(R_r) = 1$  implies that if we take a chord in  $P_r$  parallel to the guard, and shift it rightwards starting at the guard's location, then the first vertex of  $P_r$  that this chord will encounter is a reflex vertex on the upper or lower side of  $R_r$ . It is impossible that this vertex is  $B$  because of our assumption that the rectangle  $S$  contains a polygon vertex. Let  $C$  be the highest of all polygon vertices in the interior of  $S$  (the left one if there are two highest ones) and let  $f'$  be the horizontal edge turning from  $C$  to the right, see Figure 15. If  $R'$  denotes the rectangle in the vertical rectangular decomposition of  $P_r$  that is placed between  $f$  and  $f'$  then  $\text{indeg}(R') = 2$ , i.e. the right side of  $R'$  is either the vertical cut of  $B$  and  $B$  is a reflex vertex or the vertical cut from the right vertex  $C'$  of  $f'$  and  $C'$  is a reflex vertex, see Figure 16 for all possible configurations.

Note that otherwise we would get a contradiction either to the fact that  $D$  is a highest vertex in the interior of  $S$  or to the fact that  $R$  has exactly one right neighbor  $R_3$  with  $R \rightarrow R_3$ . Extending  $R'$  horizontally to the left (up to  $R_r$ ) and adding the extended rectangle to  $\overline{D_1}$  we get a district  $D_2$  increasing  $\delta_r$  by 1. Moreover either  $\delta_c$  decreases by 1 or  $\delta_h$  increases by 1. In the second case we are done because we

Figure 15: Illustration of subcase E.2.2

Figure 16: The four possible configurations on the right side of  $R'$

get  $gain(D_2) = gain(\overline{D_1}) + 6 - 2 = 14 + 6 - 2 > 16$ . In the first case we have only  $gain(D_2) = gain(\overline{D_1}) + 6 - 10 = 14 + 6 - 10 = 10$ . Let  $P_1, P_2, P_3$  be the three  $(n_1, h_1)-, (n_2, h_2)-, (n_3, h_3)-$  polygons representing  $P \setminus D_2$  where  $P_1$  is the polygon on the right side of  $R'$  and  $P_2$  the polygon below the horizontal cut from  $C$ . Note that either  $P_1$  is a simple rectangle (Figure 16 (a) and (d)) or  $R_3$  is a leaf in the horizontal rectangular decomposition of  $P_1$  (Figure 16 (c)) or it can be extended (downward) to a leaf  $R'_3$  of  $\mathbf{R}(P_1)$  (Figure 16 (b)).

For  $i \in \{1, 2, 3\}$  let  $\alpha_i$  be the residue  $3n_i + 4h_i + 4 \pmod{16}$ .

**Subcase E.2.2.1:** Suppose that  $\alpha_1 \geq 6$ .

Then we get  $gain^+(D_2) = gain(D_2) + \sum_{i=1}^3 \alpha_i \geq gain(D_2) + \alpha_1 \geq 16$  and we are done.

**Subcase E.2.2.2:** Suppose that  $\alpha_1 < 6$ .

Now we place a second guard horizontally on the edge  $f$  and its extension. Note that we have to find a common district of  $gain^+$  at least 32. If  $P_1$  is a rectangle we add it to  $D_2$ . For the resulting district  $D_3$  we have one rectangle more and one connected component ( $P_1$ ) less and hence  $gain(D_3) = gain(D_2) + 6 + 10 = 26$ .

If  $P_1$  is not a rectangle we add to  $D_2$  the leaf  $R_3$  respectively  $R'_3$ . The new district  $D_3$  has one rectangle more and the polygon  $P'_1 = P_1 \setminus R_3$  (respectively  $P_1 \setminus R'_3$ ) has one rectangle or equivalently two vertices less. Hence, the residue  $\alpha'_1$  of  $P'_1$  is  $\alpha_1 - 6 \pmod{16} \geq 10$ , and consequently  $gain^+(D_3) \geq gain(D_2) + 6 + \alpha'_1 \geq 26$ .

Finally, we consider the rectangle  $R''$  in the horizontal rectangular decomposition of  $P_2$  placed between the vertical cut from  $A'$  and the vertical edge from  $C$ , see Figure 17. Obviously,  $R''$  is covered by the horizontal guard and Lemma 9 can be applied. Note that this application does not change  $\alpha'_1$  and thus for the resulting district  $D$  we get  $gain^+(D) \geq gain(D_3) + 6 + \alpha'_1 \geq 32$ . This completes our case inspection.  $\square$

We note that applying Lemma 10 and Lemma 11 we can reduce the problem to polygons  $P$  such that  $\mathbf{R}(P)$  consists only of its frame and leaves or paths of length 2 attached to the frame. In the following we show how to find a place for a reduction in such a polygon.

We need the following definition: An *extremal hole edge* is a polygon edge  $e$  on the boundary of a hole such that

1.  $e$  connects two reflex vertices and
2. in the partition of  $P$  induced by extending  $e$  in both directions until it hits the boundary, the region containing  $e$  is simply-connected.

We remark that if a polygon has more than one hole, then among all, say, northernmost hole edges there is not necessarily an extremal edge, see Figure 18.

**Lemma 12** *If a rectilinear polygon has holes, then it has an extremal hole edge.*

Figure 17: Illustration of subcase E.2.2.2

Figure 18: No northernmost extremal hole edges

**Proof:** Let us call an edge a reflex edge if it connects two reflex vertices. Clearly, any hole of an  $(n, h)$ -polygon  $P$  has at least 4 reflex edges. Let  $E_h$  denote the set of all horizontal reflex edges of holes in  $P$ . We show that  $E_h$  contains an extremal edge. First observe that  $E_h$  contains a non-empty subset  $E'_h$  of reflex cut edges. A horizontal reflex edge is a cut edge if both extensions to the east and the west hit the outer boundary of  $P$ . To see that there are such edges one defines the following hole merging procedure. One can merge two holes if an edge extension of a reflex edge of one hits the other hole. In this case we merge the holes by adding this one-sided edge extension as a wall to them. If the extension hits the hole itself one adds to the hole the connected component enclosed by the hole and the one-sided edge extension. We search through the set  $E_h$  and apply the procedure whenever it is possible. Remark that this procedure does not create new reflex edges and we are eventually left with a polygon  $P'$  which has at least one hole. The set of horizontal reflex edges in  $P'$  corresponds exactly to those reflex edges in  $E'_h$ . Now to find the extremal edge in  $P$  it is clearly sufficient to show the following fact:

Given a polygon  $Q$  with a distinguished horizontal edge  $e$  on the outer boundary and the property that all horizontal reflex edges are cut edges, there is always an extremal horizontal edge  $e'$  such that in the partition of  $Q$  induced by  $e'$  the simply connected part  $Q_{e'}$  containing  $e'$  does not contain  $e$ .

This can be proved by induction on the number  $h$  of holes. It is true for  $h = 1$  since the hole has at least 2 extremal edges. If we have more than one hole take any horizontal reflex edge  $e''$  and consider  $Q_{e''}$ . There are two cases to distinguish. Firstly, suppose  $Q_{e''}$  is simply connected. Then if  $Q_{e''}$  does not contain  $e$  we are done, otherwise either there is another horizontal reflex cut edge of the same hole which is extremal or choose any one of these edges, say  $d$ , and apply the induction hypothesis to  $Q_d$  with the extension of  $d$  being the new distinguished boundary edge. Given that  $Q_{e''}$  is not simply connected we can apply the induction hypothesis to it with the extension of  $e'$  being the new distinguished boundary edge if  $e \notin Q_{e''}$ .  $\square$

**Lemma 13** *Let  $P$  be a polygon to which Lemma 10 and Lemma 11 cannot be applied. W.l.o.g. let  $e$  be a horizontal extremal hole edge bounding the hole from above and let  $R \in \mathbf{R}$  be the rectangle having  $e$  on its boundary. Then there is a reduction such that  $R$  or a rectangular part of  $R$  is in the district of the reduction.*

**Proof:** We note that  $R$  has two lower neighbors  $R_l$  and  $R_r$ . If there are also upper neighbors  $S_1$  and  $S_2$  of  $R$  then because  $e$  is extremal, each of them is either leaf or of degree two and adjacent to some leaf  $L_1$  or  $L_2$ . Analogously to the proof of Lemma 13 let  $\mathbf{N}$  be the set consisting of all upper neighbors of  $R$  and all leaves adjacent to these neighbors. Again we distinguish three cases:

**Case A:** Suppose that any rectangle of  $\mathbf{N}$  is reachable from  $R$  on a directed path in  $\mathbf{R}$  (note that this condition holds also if  $\mathbf{N}$  is empty). We place a horizontal guard onto the full extension of  $e$ . Clearly, it covers a district

$D$  consisting of  $R$  and all rectangles of  $\mathbf{N}$ . Thus, the type of  $D$  is  $(1 + |\mathbf{N}|, 1, 0)$  and its gain is  $6 + 6 \cdot |\mathbf{N}| - 2 \geq 4$ . Moreover for both  $R_l$  and  $R_r$  the expansion lemma can be applied, so the expanded district  $\overline{D}$  has a gain  $\geq 4 + 2 \cdot 6 = 16$ .

**Case B:** Suppose that there is (exactly) one upper neighbor  $S_1$  and an arc  $R \leftarrow S_1$ .

Placing a horizontal guard onto the upper boundary of  $S_1$  and extending it as far as possible we can cover  $R$  and all rectangles of  $\mathbf{N}$  and hence we can proceed further as in Case A.

**Case C:** Suppose that there is (at least) one upper neighbour  $S_1$  adjacent to a leaf  $L_1$  and arcs  $R \rightarrow S_1 \leftarrow L_1$ .

W.l.o.g. let  $S_1$  be a left neighbor of  $R$ . Placing a vertical guard onto the common vertical polygon edge  $f$  of  $R$  and  $S_1$  and its extension one can cover a district  $D$  consisting of  $L_1, S_1$  and that part of  $R$  which is bounded by  $f$  on the left side and by the extension of the left boundary of  $R_r$  on the right side. So after deleting  $D$  the remaining part of  $R$  forms together with  $R_r$  one rectangle in the rectangular decomposition and thus  $D$  is of type  $(3, 1, 0)$  and one has  $gain(D) = 16$ .  $\square$

We close this section proving the  $\lfloor \frac{n}{6} \rfloor$  upper bound for  $T_2$ -guards. For technical convenience in the inductive proof we introduce a slight reformulation of the bound. For any  $(n, h)$ -polygon  $P$  we define a characteristic number  $\chi(P)$  as follows:

$$\chi(P) = \begin{cases} 1 & \text{if } n = 4 \text{ and } h = 0 \\ 0 & \text{else} \end{cases}$$

**Theorem 14** For any  $(n, h)$ -polygon  $P$  we have  $r(P, 2) \leq \lfloor \frac{n+2\chi(P)}{6} \rfloor$ .

To prove this theorem one goes along similar lines as in the proof of Theorem 5 where in contrast to the above proof the lemmata for reducing simply connected parts becomes rather trivial. For reducing holes the existence of extremal edges is also essential. Roughly speaking one can use the second arm of a  $T_2$ -guard to cover one rectangle more.

Since we want to prove another bound than in Theorem 5 we have to change the definitions of reductions, types and of gain. To avoid confusions with Theorem 5 we will use the notations  $gain_2$ , and  $2$ -reductions. (Note that the definitions depend on the bound one wants to prove rather than on the guard type, so a more precise notation would be  $gain_{\lfloor \frac{n}{6} \rfloor}$  and  $\lfloor \frac{n}{6} \rfloor$ -reduction.)

Let  $G$  be a  $T_2$ -guard in an  $(n, h)$ -polygon  $P$  covering a district  $D$  and let  $P_1, \dots, P_{c'}$  be the  $(n_1, h_1), \dots, (n_{c'}, h_{c'})$ -polygons that are the connected components of  $P \setminus D$ . The deletion of  $D$  will be called a  $2$ -reduction if  $1 + \sum_{i=1}^{c'} \lfloor \frac{n_i+2\chi(P_i)}{6} \rfloor \leq \lfloor \frac{n+2\chi(P)}{6} \rfloor$ , i.e. if the deletion of  $D$  allows us to apply induction.

Define  $\delta_n, \delta_r, \delta_h, \delta_c$  as before and  $\delta_\chi = \chi(P) - \sum_{i=1}^{c'} \chi(P_i)$ , i.e. analogously as  $-\delta_c$  describes the increase of the number of connected components after deleting  $D$ ,  $-\delta_\chi$  describes the increase of the number of connected components that are  $(4, 0)$ -polygons. For shortness, such components will be called *rectangle components*. The tuple  $(\delta_r, \delta_h, \delta_c, \delta_\chi)$  will be called the *2-type* of  $D$ . Now, we can introduce the *gain<sub>2</sub>* of a district as follows:

$$\text{gain}_2(D) = \delta_n + 2\delta_\chi = 2(\delta_r - \delta_h + \delta_c + \delta_\chi)$$

**Lemma 15** *Let  $D$  be a district of a  $T_2$ -guard  $G$  in a polygon  $P$ . If  $\text{gain}_2(D) \geq 6$  then the deletion of  $D$  is a 2-reduction.*

The proof of this lemma is analogous to the proof of Lemma 8. In contrast, the following analog to Lemma 9 contains some essential differences.

**Lemma 16 (Expansion Lemma)** *Let  $G$  be a horizontal  $T_1$ -guard in a polygon  $P$  and  $D$  a district of  $G$ . Let  $P_1$  be one of the connected components of  $P \setminus D$  and  $R$ , and  $e$  be a horizontal edge that bounds  $P_1$  from above and is shared between  $P_1$  and  $D$ . Let  $R$  be the rectangle of  $P_1$  that contains  $e$ . Let  $\overline{D}$  be the expansion of  $D$  by  $R$  and all rectangles reachable from  $R$  on directed paths in  $\mathbf{R}(P_1)$ . If the edge  $e$  is (orthogonally) visible from  $G$  (see Figure 9, where  $G$  runs across the top of the figure), then  $\overline{D}$  is also a district of  $G$  and the following holds: Either  $\text{gain}_2(\overline{D}) \geq \text{gain}_2(D) + 2$  or  $\text{gain}_2(\overline{D}) = \text{gain}_2(D)$  and  $P_1 \setminus \overline{D}$  consists of  $(|\mathbf{S}| + 1)/2$  rectangle components.*

**Proof:** Let  $B$  denote the set of rectangles in  $\mathbf{S}$  that have two lower neighbors and  $b = |B|$ . Then  $\mathbf{S}$  has at least  $2b + 1$  nodes. If we add by breadth first search the rectangles of  $\mathbf{S}$  to  $D$  starting with  $R$  then for each rectangle from  $B$  either the number of connected components of the remaining polygon increases by 1 (say,  $b_1$  times) or the number of holes decreases by 1 ( $b_2 = b - b_1$  times). In contrast, adding a rectangle that does not have two lower neighbors neither changes  $\delta_h$  nor increases the number of connected components. Thus, after deleting all rectangles of  $\mathbf{S}$  from  $P_1$  the number of remaining connected components (and especially the number  $b_3$  of rectangle components) is bounded by  $b_1 + 1$ . So we have

$$\text{gain}(\overline{D}) \geq \text{gain}(D) + 2 \cdot (|\mathbf{S}| - b_1 - b_2 - b_3)$$

Note that

$$|\mathbf{S}| - b_1 - b_2 - b_3 \geq 2b + 1 - b_1 - b_2 - (b_1 + 1) = 2b_1 + 2b_2 + 1 - 2b_1 - b_2 - 1 \geq 0$$

and the left side is equal to zero iff  $|\mathbf{S}| = 2b + 1$ ,  $b_2 = 0$  and  $b_3 = b_1 + 1$ . This implies  $b_3 = (2b_1 + 2)/2 = (2b + 2)/2 = (|\mathbf{S}| + 1)/2$ , which completes the proof.  $\square$

Now we will show that for any polygon one can find a 2-reduction. Obviously, the deletion of any district with 2-type  $(3, 0, 0, 0)$  is a 2-reduction. The following observations will be very helpful to extend the results for  $T_1$ -guards to  $T_2$ -guards.

Figure 19: Illustration of the amplification lemma

**Lemma 17 (Amplification Lemma)** *Let  $D$  be a district of a  $T_1$ -guard  $G$  and suppose that in  $P \setminus D$  there is a rectangle component  $R$  (see Figure 19). Then  $G$  can be amplified to a  $T_2$ -guard  $G'$  covering the district  $D' = D \cup R$  with  $gain_2(D') = gain_2(D) + 6$ .*

**Proof:** Since  $R$  was obtained by the deletion of  $D$  from  $P$  there must be a common point  $A$  on the boundaries of  $P, R$  and  $D$ . Let  $l$  be the perpendicular from  $A$  to  $G$ . Because orthogonal covering is always assumed,  $l$  is included in  $P$  and moreover it is possible to extend  $l$  in such a way that it crosses the entire height of  $R$ . Clearly,  $G$  together with this extended segment forms a  $T_2$ -guard orthogonally covering  $D \cup R$ . By extending  $D$  in this way, one more rectangle is covered, there is one less connected component remaining, and one less rectangle component remaining. Collectively, these changes increase the  $gain_2$  by 6.  $\square$

Let  $D$  be a district of a  $T_1$ -guard of type  $(3, 0, 0)$ . If the remaining polygon  $P' = P \setminus D$  is not a 4-gon then the 2-type of  $D$  is  $(3, 0, 0, 0)$  and hence the deletion of  $D$  is also a 2-reduction. Otherwise, if  $P'$  is a 4-gon then  $P$  must be an 8-gon which clearly can be covered by a  $T_2$ -guard and thus  $P$  is also 2-reducible in this case.

Let  $D$  be a district of a  $T_1$ -guard (w.l.o.g. horizontal) of type  $(2, 0, 0)$ ,  $P_1$  a connected component of  $P \setminus D$  and  $R \in \mathbf{R}(P_1)$  such that the new expansion lemma can be applied. Then we either get  $gain_2(\overline{D}) \geq gain_2(D) + 2 \geq 6$  (which implies a 2-reduction) or  $gain_2(\overline{D}) = gain_2(D) = 4$  and all  $(|\mathbf{S}| + 1)/2$  connected components of  $P_1 \setminus \overline{D}$  are 4-gons. In the latter case one can apply the amplification lemma to get a district  $D^*$  with  $gain_2(D^*) = gain_2(\overline{D}) + 6 = 10$ .

**Lemma 18** *If  $R_0 \in \mathbf{T}$  is a leaf with branching distance  $\geq 3$  then there is some 2-reduction with  $R$  in the reduction district.*

**Proof:** In this situation one can always find a  $T_1$ -guard with a district of type  $(3, 0, 0)$  (see proof of Lemma 10), so we are done.  $\square$

**Lemma 19** *If all leaves in  $\mathbf{R}$  have branching distance  $< 3$  and  $R$  is a primary branching then there is a reduction such that  $R$  or a part of  $R$  is in the reduction district.*

**Proof:** Let us return to the case inspection in the proof of Lemma 11. In case A, B and C.1 there are  $T_1$  guards with districts either of type  $(3, 0, 0)$  or of type  $(2, 0, 0)$  and such that the new expansion lemma (or its rotated version) can be applied. Taking into account the observations above, we are done with these cases and only case C.2 remains (see Figure 12). As in the proof of Lemma 11, we start with a vertical  $T_1$ -guard on the extension  $\bar{e}$  of the edge  $e$  and a district  $D$  consisting of  $R_2$  and  $L_2$  and the guard position. Depending on whether  $\bar{e}$  disconnects the polygon or reduces one hole,  $D$  has the type  $(2, 0, -1)$  or  $(2, 1, 0)$ . Thus the 2-type of  $D$  is either  $(2, 0, -1, \delta_\chi)$  where  $\delta_\chi \in \{0, -1, -2\}$  or  $(2, 1, 0, 0)$ . Furthermore, one can apply the new expansion lemma on the right and on the left side of  $\bar{e}$ . If (before expanding) on one side (resp. on both sides) there is only a rectangle component, i.e.  $\delta_\chi = -1$  (resp.  $\delta_\chi = -2$ ) then the expansion on this side (resp. to both sides) removes one (resp. two) rectangle(s) which is also a connected component and especially a connected component being a 4-gon. Thus the extended district has the 2-type  $(3, 0, 0, 0)$  (resp.  $(4, 0, 1, 0)$ ) which implies a sufficient  $gain_2$  of 6 (resp. 10).

Now we can assume that  $D$  is of 2-type  $(2, 0, -1, 0)$  or  $(2, 1, 0, 0)$  and hence  $gain_2(D) = 2$ . Applying the new expansion lemma on both sides of  $\bar{e}$  we either increase the  $gain_2$  twice by 2 (and we are done) or we know that after this step at least on one side there remains a rectangle component. In this case one can apply the amplification lemma increasing the  $gain_2$  by 6 and we are done.  $\square$

The proof of Theorem 14 will be completed by a lemma that shows how to reduce the number of holes.

**Lemma 20** *Let  $P$  be a polygon to which Lemma 18 and Lemma 19 cannot be applied. W.l.o.g. let  $e$  be a horizontal extremal hole edge bounding a hole from above and let  $R \in \mathbf{R}$  be the rectangle having  $e$  on its boundary. Then there is a reduction such that  $R$  or a rectangular part of  $R$  is in the district of the reduction.*

**Proof:** We switch back to the proof of Lemma 11 and note that  $R$  has two lower neighbors  $R_l$  and  $R_r$ . If there are also upper neighbors  $R_1$  and  $R_2$  of  $R$  then because  $e$  is extremal, each of them is either a leaf or of degree 2 and adjacent to some leaf  $L_1$  or  $L_2$ . Let us start assuming that the set  $\mathbf{N}$  consisting of all upper neighbors of  $R$  and all leaves adjacent to these neighbors is not empty and run through the case inspection under this additional assumption.

Figure 20: Constuction of the districts  $\overline{D'}$  and  $\overline{E}$ 

In **Case A** and **Case B** we have a horizontal guard which first covers a district  $D$  consisting of  $R$  and all rectangles in  $\mathbf{N}$ . Hence  $D$  is of type  $(1 + |\mathbf{N}|, 1, 0)$  and of 2-type  $(1 + |\mathbf{N}|, 1, 0, 0)$ . So we get  $gain_2(D) \geq 2$  and moreover the new expansion lemma can be applied twice. If both applications increase the  $gain_2$  by 2 we are done. Otherwise at least one application causes a rectangle component which can be covered by the amplification lemma, giving a sufficiently large gain.

In **Case C** a vertical guard will be placed onto the full extension  $\bar{e}$  of the edge  $e$  which covers first a district  $D$  consisting of  $R_1, L_1$  and  $\bar{e}$ . Depending on whether the lower vertex of  $e$  is reflex or not, we have  $gain_2(D) = 2$  and  $D$  can be expanded twice or  $gain_2(D) = 4$  with one possible expansion. Again either one gets a sufficient  $gain_2$  by the expansion or there remains a rectangle component which will be covered by amplification of the guard.

Finally, we show how to proceed if the set  $\mathbf{N}$  is empty. First, we place a horizontal guard onto the upper boundary of  $R$  and define a district  $D = R$  of 2-type  $(1, 1, 0, 0)$  and with  $gain_2(D) = 0$ . Obviously, two expansions with respect to  $R_l$  and  $R_r$  are possible. Let  $\overline{D}$  be the new district after the expansions, then we have  $gain_2(\overline{D}) \geq gain_2(D) = 0$ . If there is a rectangle component in  $P \setminus \overline{D}$  we can get a sufficiently large  $gain_2$  by the amplification lemma. Otherwise both expansions increase the  $gain_2$  at least by 2 and we get  $gain_2(\overline{D}) \geq gain_2(D) + 2 \cdot 2 = 4$ . Note that we are done if one of the expansions adds more than 2 to the  $gain_2$ , so we can assume that the application of Lemma 16 to  $R_l$  (as well as to  $R_r$ ) increases the gain exactly by 2. As was shown in the proof of Lemma 16 this increase is  $\geq 2 \cdot (|\mathbf{S}| - b_1 - b_2 - b_3)$  where  $\mathbf{S}$  is the set of all rectangles in  $\mathbf{R}$  reachable from  $R_l$  on a directed path,  $b = b_1 + b_2$  is the number of rectangles in  $\mathbf{S}$  with two lower neighbors and  $b_3$  denotes the number of rectangle components in the remaining polygon which is 0 in this case. Since  $|\mathbf{S}| \geq b + 1$ , the only possibility to get exactly 2 for the increase of the  $gain_2$  is  $b = 0$  and  $|\mathbf{S}| = 1$ , i.e.  $R_l$  has exactly one lower neighbor  $R'_l$  with an arc  $R'_l \rightarrow R_l$ , and  $R_r$  has exactly one lower neighbor  $R'_r$  with an arc  $R'_r \rightarrow R_r$ .

Figure 21:  $R_0$  is a rectangle component in  $P \setminus \overline{D'}$  but not in  $P \setminus \overline{E}$

Now, we choose the extension  $\overline{e}$  of the common vertical edge  $e$  of  $R_l$  and  $R'_l$  for amplifying  $G$ . Let  $D'$  be the district of the new  $T_2$ -guard consisting of  $\overline{D}$  and  $\overline{e}$  (see Figure 20, the darkly shaded region in the left picture). Again we have to distinguish the two cases whether the lower vertex  $A$  of  $e$  is reflex or not.

**Case 1:**  $A$  is a reflex vertex.

Then the 2-type of  $D'$  is  $(3, 2, 0, 0)$  or  $(3, 1, -1, 0)$  and thus  $gain_2(D') = 2$ . Furthermore one can expand  $D'$  twice. Let  $\overline{D'}$  denote the district obtained in this way.

**Subcase 1.1:** One of the two expansions increases the  $gain_2$  by more than 2 or each expansion increases the  $gain_2$  by 2, then obviously  $gain_2(\overline{D'}) = 6$ .

**Subcase 1.2:** The application on the left side of  $\overline{e}$  does not increase the  $gain_2$ . Then there is a rectangle component  $R_0$  in  $P \setminus \overline{D'}$  on the left side of  $\overline{e}$ . Consider a vertical  $T_1$  guard  $H$  on  $\overline{e}$  covering a district  $E$  consisting of  $\overline{e}$ ,  $R$  and  $R_l$ . The district  $E$  has  $gain_2(E) = 0$ , and expanding  $E$  on both sides of  $\overline{e}$  one gets an extended district  $\overline{E}$  with  $gain_2(\overline{E}) \geq 0$ . Note that the rectangle  $R_0$  is a rectangle component of  $P \setminus \overline{E}$  (see Figure 20, the right picture), and we may thus apply amplification to obtain a  $T_2$ -guard covering the district  $E' = \overline{E} \cup R$  with  $gain_2(E') = gain_2(\overline{E}) + 6 \geq 6$ .

**Subcase 1.3:** The application on the right side of  $\overline{e}$  does not increase the  $gain_2$ . Then there is a rectangle component  $R_0$  in  $P \setminus \overline{D'}$  on the left side of  $\overline{e}$ . We will proceed as in Subcase 1.2 and we will be successful if  $R_0$  will be also a rectangle component in  $P \setminus \overline{E}$ . There is one (and only one) exceptional situation, namely if  $R_0$  is a neighbor of  $R_r$  (see Figure 21). Then in  $P \setminus \overline{E}$   $R_0$  and  $R_r$  together form a  $(6, 0)$ -polygon. However, by adding the horizontal arm to  $H$  which covers  $R_0$ , we also cover  $R_r$  and hence the  $gain_2$  increases by 6 (we have eliminated two rectangles

and one connected component). We note that the exceptional situation described above is the only one because we have  $\overline{D'} \setminus \overline{E} = R_r$ .

**Case 2:**  $A$  is not a reflex vertex.

Then the 2-type of  $D'$  is  $(3, 1, 0, 0)$  and thus  $gain_2(D') = 4$ . Furthermore, it is possible to expand on the right side of  $\bar{e}$ . One can handle this situation analogously to Case 1, repeating the inspection of the subcases under the pretense that the application of Lemma 16 on the left side of  $\bar{e}$  increases the  $gain_2$  by exactly 2.

This finishes the proof of this lemma and also the proof of Theorem 14.  $\square$

## 5 Conclusion

We have studied generalized guarding in rectilinear polygons with holes, obtaining general lower bounds and some specific upper bounds. We have found that in the rectilinear world there is a strong difference between odd and even  $k$ . Surprisingly, for  $k \geq 3$ , we have not found lower bounds where increasing  $h$  makes polygons require more guards, and we in fact believe that increasing  $h$  makes polygons require *less* guards. However, we are unable to establish this, and leave this question unsettled.

We note here that our lower bound constructions give the same bounds even if the usual visibility (rather than rectangle visibility) is used, and the  $T_k$ -guards are not rectilinearly embedded; the upper bound arguments (obviously) also hold in this more general situation. The fourth author has previously shown that the even- $k$  upper bound of  $r(n, 0, k) \leq \lfloor \frac{n}{k+4} \rfloor$  holds in this situation [13]; his result is implied by Theorem 1.

There are many questions related to this paper which are yet to be answered. Aside from the usual questions about tight bounds for the generalized guarding problem both for rectilinear and general polygons, we want to mention the following:

- What is the lower bound on  $r(n, h, k)$  when  $\frac{n}{h}$  is small (lots of rectangular holes)?
- Are there lower bound examples that have a different structure but illustrate the same bounds as our constructions? We conjecture that there are no such examples.
- What are the exact bounds for rectilinear polygons with holes expressed as a function only of  $n$  and  $k$ ? (Wessel showed a lower bound of  $\lfloor \frac{3n+4}{14} \rfloor$  for  $k = 1$  [14].)
- To prove Lemma 3, we need only guards that are trees with at most  $k$  edges, while the lower bounds hold even for nonrectilinear trees of diameter  $k$ . How can one exploit the full power of diameter- $k$  trees to get a better upper bound? What is the situation for guards that are *paths* of diameter (length)  $k$ ?

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