Fractional Quantum Field Theory, Path Integral, and Stochastic Differential Equation for Strongly Interacting Many-Particle Systems

Hagen Kleinert*
Institut für Theoretische Physik, Freie Universität Berlin, 14195 Berlin, Germany and ICRANeT Piazzale della Repubblica, 10 -65122, Pescara, Italy

While free and *weakly* interacting particles are well described by a second-quantized nonlinear Schrödinger field, or relativistic versions of it, with various approximations, the fields of *strongly* interacting particles are governed by *effective actions*, whose quadratic terms are extremized by fractional wave equations. Their particle orbits perform universal Lévy walks rather than Gaussian random walks with perturbations.

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Quantum-mechanical physics is explained with high accuracy by Schrödinger theory. The wave equation for many particles can convenienty be reformulated as a second-quantized *field theory*, with an action that is the sum of quadratic and an interacting term

$$\mathcal{A}=\mathcal{A}_2+\mathcal{A}_{\rm int}, \tag{1}$$

where the term $\mathcal{A}_2$ has typically the form

$$\mathcal{A}_2=\int d^Dxdt\psi^*(\mathbf{x},t)[i\partial_t+\hbar^2\nabla^2/2-V(\mathbf{x})]\psi(\mathbf{x},t), \tag{2}$$

with D being the space timension, m the mass, and $V(\mathbf{x})$ some external potential. The interaction term $\mathcal{A}_{\rm int}$ may be approximated in molecular systems by a fourth-order term in the field

$$\frac{1}{2}\int d^Dxd^Dx'dt\psi^*(\mathbf{x}',t)\psi^*(\mathbf{x},t)V_{12}(\mathbf{x},\mathbf{x}')\psi(\mathbf{x},t)\psi(\mathbf{x}',t), \tag{3}$$

where $V_{12}(\mathbf{x},\mathbf{x}')$ is some two-body potential.

If relativistic velocities are present, the field is generalized to a scalar Klein-Gordon field, or a quantized Dirac field. In molecular physics, the fourth-order term is due to the exchange of a minimally coupled quantized photon field and is proportional to e^2, where e is the electric charge. The field equations may be studied with any standard method of quantum field theory, and corrections can be derived using perturbation theory in powers of $\alpha\equiv e^2/\hbar\approx 1/137$. Since α is very small, this appeoch is quite successful.

If time is continued analytically to imaginary values $t=i\tau$, one is faced with the so-called Euclidean version of quantum field theory. Then perturbation theory may be understood as developing a theory of particle physics from an expansion around Gaussian random walks. Indeed, the relativistic scalar free-particle propagator of mass m in $D+1$-dimensional euclidean energy-momentum space $p^\mu=(\mathbf{p},p_4)$, has the form

$$G(p)=\frac{1}{\mathbf{p}^2+p_4^2+m^2}=\int_0^\infty ds\, e^{-sm^2}e^{-s(\mathbf{p}^2+p_4^2)}, \tag{4}$$

where the energy has been continued analytically to $p_4=-iE$. The Fourier transform of $e^{-s(\mathbf{p}^2+p_4^2)}$ is the distribution of Gaussian random walks of length s in $D+1$ euclidean dimensions

$$P(\mathbf{x},x_4)=(4\pi s)^{-(D+1)/2}e^{-(|\mathbf{x}|^2+x_4^2)/4s}, \tag{5}$$

which makes the propagator (4) a superposition of such walks with lenghts distributed like e^{-sm^2} [1–3]. This propagator is the relativistic version of the free-field propagator of the action (2). The second-quantized field theory described by (1) accounts for grand-canical ensembles of orbits with their two-body interactions [4].

Gaussian random walks are a natural and rather universal starting point for many stochastic processes. For instance, they form the basis of the most important tool in the theory of financial markets, the Black-Scholes option price theory [5] (Nobel Prize 1997), by which a portfolio of assets is hoped to remain steadily growing through hedging. In fact, the famous *central-limit theorem* permits us to prove that many independent random movements of finite variance always pile up to display a Gaussian distribution [6].

However, since the last stock market crash and the still ongoing financial crisis it has become clear that realistic distributions belong to a more general universality class, the so-called Lévy stable distribution. They are the univarsal results of a pile up of random movements of infinite variance [7]. They account for the fact that rare events, which initiate crashes, are much more frequent than in Gaussian distributions. These are events in the so-called Lévy tails $\propto 1/|x|^{1+\lambda}$ of the distributions, whose description requires a Hamiltonian [3]

$$H={\rm const}\,(p^2)^{\lambda/2}. \tag{6}$$

Such tail-events are present in the self-similar distribution of matter in the universe [8–10], in velocity distributions of many body sytems with long-range forces [11], and in the distributions of windgusts [12], oceanic moster waves [13], and earthquakes [14], with often catastrophic consequences. They are a consequence of rather general maximal entropy assumptions [15]. In the limit $\lambda \to 2$, the Lévy distributions reduce to Gaussian distributions.

The purpose of this note is to point out, that such distributions occur quite naturally also in many-particle systems, provided the interactions are very strong [16]. They have been observed in numerous experiments at second-order phase transitions. The most accurate measurement of this type was done in a satellite (the so called Infrared Atronomical Satellite IRAS) by studying the singularity of the specific heat of superfluid ^{4}He near the critical temperature [17]. The observation agreed extremely well with the theoretical strong-coupling prediction [18].

The field of a strongly interacting N-body system is usually a multivalued function. Singularities perforate the space via vortex lines (for instance in type II superconductors or in superfluid ^{4}He), or via line-like defects in the displacement field of a world-crystal formulation of Einstein(-Cartan) gravity [19]. If the positions of two particles are exchanged, one obtains a factor $+1$ for bosons or -1 for electrons. In two dimensions, one may even obtain a general phase $e^{i\phi}$ (anyons) [22].

A strongly interacting field system has a conformally invariant Green function [20–22]

$$G(\mathbf{p}, p_4) = [p_4^{1-\gamma} \phi(\mathbf{p}^2/p_4^z)]^{-1}. \tag{7}$$

If the dimension D differs only by a very small amount ϵ from the critical dimension D_c, where the theory is scale-invariant, i.e., $D = D_c + \epsilon$, then γ is of order ϵ and z differs from unity by a similar amount. Such a power behavior is assured near D_c if the Gell-Mann-Low function [23] has an infrared-stable fixed point in the renormalization flow of the coupling constant. Very close to the critical dimension, a lowest approximation to $G(\mathbf{p}, p_4)$ is

$$G(\mathbf{p}, p_4) = \{p_4^{1-\gamma}[1 + D_\lambda(\mathbf{p}^2/p_4^z)^{\lambda/2}]\}^{-1}, \tag{8}$$

where λ is close to 2, and D_λ is a generalization of the diffusion constant in the Fokker-Planck equation.

Time-independent propagators involve the limit $p_4 \to 0$, where the correlation function behaves like

$$G(\mathbf{p}, 0) \propto |\mathbf{p}|^{-2+\eta}. \tag{9}$$

The index η is the *anomalous dimension* of the field, which is also of order ϵ. The existence of this limit in (8) fixes the scaling relation

$$\lambda = 2(1-\gamma)/z = 2 - \eta. \tag{10}$$

See Appendix for the calculation of the exponents to order ϵ. The Green function (8) determines the probability distribution of particle after a time t via the *double fractional Fokker-Planck equation*

$$[\hat{p}_4^{1-\gamma} + D_\lambda(\hat{\mathbf{p}}^2)^{\lambda/2}]P(\mathbf{x}, t) = \delta(t)\delta^{(D)}(\mathbf{x}), \tag{11}$$

where $\hat{p}_4 \equiv \partial_t$, $\hat{\mathbf{p}} \equiv i\partial_\mathbf{x} \equiv i\boldsymbol{\nabla}$. A convenient definition of the fractional derivatives uses the same formula as in the dimensional continuation of Feynman diagrams $(-\boldsymbol{\nabla}^2)^{\lambda/2} = \Gamma[\lambda/2]^{-1} \int d\sigma \sigma^{-\lambda/2-1} e^{\lambda \boldsymbol{\nabla}^2/2}$ [24, 25]. The solution of (11) is given in the literature [26] and reads

$$\frac{t^{-\gamma}}{\pi^{D/2}|\mathbf{x}|^{D/2}} H_{2,3}^{2,1}\left(\frac{|\mathbf{x}|^\alpha}{2^\alpha D_\lambda t^{1-\gamma}}\middle|_{(1,1),(D/2,\alpha/2);(1,\alpha/2)}^{(1,1);(1-\gamma,1-\gamma)}\right), \tag{12}$$

where $H_{2,3}^{2,1}$ is a *Fox H-function* [27]. In the limits $\gamma = 0$ and $\alpha = 2$, this reduces to the standard quantum mechanical Gaussian expression $(4\pi D_\lambda t)^{-D/2} e^{-|\mathbf{x}|^2/4D_\lambda t}$. For $\gamma = 0, \alpha = 1$, the result is

$$P(\mathbf{x}, t) = \frac{D_\lambda t}{\pi^{(D+1)/2}|\mathbf{x}|^{D+1}} H_{1,1}^{1,1}\left(\frac{D_\lambda^2 t^2}{|\mathbf{x}|^2}\middle|_{(0,1)}^{(1/2-D/2,1)}\right), \tag{13}$$

which is simply the Cauchy-Lorentz distribution function

$$[\Gamma(D/2+1/2)/\pi^{(D+1)/2}] D_\lambda t / [(D_\lambda t)^2 + |\mathbf{x}|^2]^{D/2+1/2}.$$

The probability (11) may be calculated from the *doubly fractional canonical path integral* over fluctuating orbits $t(s), \mathbf{x}(s)$ $p_4(s), \mathbf{p}(s)$ viewed as functions of some pseudotime s [28]:

$$\{\mathbf{x}_b t_b s_b | \mathbf{x}_a t_s s_a\} = \int \mathcal{D}\mathbf{x} \mathcal{D}t \mathcal{D}\mathbf{p} \mathcal{D}p_4 e^{-\mathcal{A}}, \tag{14}$$

where $\mathcal{A}$ is the euclidean action of the paths $t(s), \mathbf{x}(s)$:

$$\mathcal{A} = \int ds[i(\mathbf{p}\mathbf{x}' - ip_4 t') - \mathcal{H}(\mathbf{p}, p_4)]. \tag{15}$$

Here $t'(s) \equiv dt(s)/ds$, $\mathbf{x}'(s) \equiv d\mathbf{x}(s)/ds$, and $\mathcal{H}(\mathbf{p}, p_4) = p_4^{1-\gamma} + D_\lambda(\hat{\mathbf{p}}^2)^{\lambda/2}$. At each s, the integrals over the components of $\mathbf{p}(s)$ and $p_4(s)$ run from $-\infty$ to ∞, whereas those over $p_4(s)$ run from $-i\infty$ to $i\infty$. At the end we obtain $P(\mathbf{x}, t)$ from the integral $\int_0^\infty ds \{\mathbf{x}\, t\, s | \mathbf{0}\, 0\, 0\}$.

If $\gamma = 0$, the path integral over $p_4(s)$ yields the functional $\delta[t'(s) - 1]$, which brings (14) to the canonical path integral

$$(\mathbf{x}_b t_b | \mathbf{x}_a t_a) = \int \mathcal{D}\mathbf{x} \mathcal{D}\mathbf{p} e^{-\mathcal{A}'}, \tag{16}$$

with

$$\mathcal{A}' = \int dt[i\mathbf{p}\dot{\mathbf{x}} - D_\lambda(\hat{\mathbf{p}}^2)^{\lambda/2}]. \tag{17}$$

Now $P(\mathbf{x}, t) = (\mathbf{x}t|\mathbf{0}\, 0)$ satisfies the ordinary fractional Fokker-Planck equation

$$[\hat{p}_4 + D_\lambda(\hat{\mathbf{p}}^2)^{\lambda/2}]P(t, \mathbf{x}) = \delta(t)\delta^{(D)}(\mathbf{x}). \tag{18}$$

This has been discussed at length in recent literature [29].

At this place it is worth mentioning that the probability can be written as a superposition $\int_0^\infty (d\sigma/\sigma) f_\lambda(\sigma t^{-2/\lambda}) P_{\rm G}(\sigma, \mathbf{x})$ of Gaussian distributions $P_{\rm G}(\sigma, \mathbf{x}) = (4\pi\sigma)^{-D/2} e^{-\mathbf{x}^2/4\sigma}$ with weight

$$f_\lambda(\sigma) = S_D \sum_{n=1}^\infty \frac{(-1)^n \sigma^{-n\lambda/2}}{(n+1)!\Gamma(D-1-n\lambda/2)} D_\lambda^{n/\lambda}, \tag{19}$$

where $S_D = 2\pi^{D/2}/\Gamma(D/2)$ is the surface of a sphere in D dimensions.

If $\gamma \neq 0$, the above functional δ-function is softened, and the relation between the pseudotime s and the physical time becomes stochastic. It is governed by the probability distribution that solves the path integral the

$$\{t_b s_b | t_a s_a\} = \int \mathcal{D}t\mathcal{D}p_4 \exp\left\{\int ds\,[p_4 t' - p_4^{1-\gamma})]\right\}. \quad (20)$$

For imaginary $p_4 = -iE$, we define a *noise Hamiltonian* $\tilde{H}(\eta)$ which has the property that [28, 30]

$$e^{-p_4^{1-\gamma}} = \int_{-\infty}^{\infty} d\eta e^{-p_4\eta - \tilde{H}(\eta)}. \quad (21)$$

The inverse of the Fourier integral yields the *noise probability* $P(\eta) = \int_{-i\infty}^{i\infty} dp_4 e^{p_4\eta - p_4^{1-\gamma}}$, and a probability functional [31]:

$$P[\eta] \equiv e^{-\int ds \tilde{H}(\eta)} = \int \mathcal{D}p_4 \exp\left[\int ds\,(p_4\eta - p_4^{1-\gamma})\right]. \quad (22)$$

Using this we may solve the stochastic differential equation of the Langevin type

$$t'(s) = \eta(s), \quad (23)$$

in which the noise $\eta(s)$ has a zero expectation value for each s, and the correlation functions for $n = 2, 4, 6, \ldots$:

$$\langle \eta(s_1) \ldots \eta(s_{2n}) \rangle \equiv \int \mathcal{D}\eta\, \eta(s_1) \ldots \eta(s_{2n}) P[\eta]. \quad (24)$$

If $\gamma = 0$, the solution of (22) is $P[\eta] = \delta[\eta(s) - 1]$, implying that $\eta(s)$ ceases to fluctuate, and (23) becomes $t'(s) \equiv 1$, so that $t \equiv s$.

In the past, many nontrivial Schrödinger equations (for instance that of the $1/r$-potential) have been solved with path integral methods by re-formulating them on the pseudotime axis s, that is related to the time t via a *space-dependent differential equation* $t'(s) = f(x(t))$. This method, invented by Duru and Kleinert [32] to solve the path integral of the hydrogen atom, has recently been applied successfully to various Fokker-Planck equations [33, 34]. The stochastic differential equation (23) may be seen as a stochastic version of the Duru-Kleinert transformation that promises to be a useful tool to study non-Markovian systems.

Certainly, the solutions of Eq. (18) can also be obtained from a stochastic differential equation

$$\dot{\mathbf{x}} = \boldsymbol{\eta}, \quad (25)$$

whose noise is distributed with a fractional probability

$$P[\boldsymbol{\eta}] = \int \mathcal{D}^D x e^{\int dt (i\mathbf{p}\cdot\boldsymbol{\eta} - D_\lambda (\mathbf{p}^2)^{\lambda/2})}. \quad (26)$$

Experimentally, a system with in the strong-coupling limit can be produced by forming a Bose-Einstein condensate (BEC) in a magnetic field whose strength is tuned to a Feshbach resonance [35] of the two-particle interaction. In a BEC, the four-field term in the interaction (3) is local and parametrized by $V_{12}(\mathbf{x}, \mathbf{x}) \propto g\delta(\mathbf{x} - \mathbf{x}')$. At the Feshbach resonance, the bare coupling strength g goes to infinity [36], and the renormalized coupling g_R, multiplied by $6\mu^{-\epsilon}/(4\pi)^2$, converges to a fixed point $g^* \approx 0.503$ [see Fig. 17.1 in Ref. [21]), where μ is some mass scale.

The theoretical tool to describe the physics in this regime is the effective action $\Gamma[\Psi, \Psi^*]$. This a functional of the classical expectation values of the quantum fields $\Psi(t, \mathbf{x}) \equiv \langle \psi(t, \mathbf{x}) \rangle$, and contains all information of the full quantum theory [21, 47]. It is the Legendre transform of the generating functional $Z[\eta, \eta^*] = \int \mathcal{D}\psi\mathcal{D}\psi^* e^{-\mathcal{A} - \eta^*\psi - \eta\psi^*}$ of the full quantum theory, and is extremal on the physical field expections. All its vertex functions can be found from the functional derivatives of $\Gamma[\Psi, \Psi^*]$. In the strong-coupling limit, the effective interaction changes the interaction (3) to an anomalous power law $\Gamma^{\rm int}[\Psi, \Psi^*] = (g_c/2) \int dt d^D x\, |\Psi(t, \mathbf{x})|^{\delta+1}$. The power δ is a critical exponent that is measured experimentally by the relation $B = |\Psi|^\delta$. Its value is determined by η via the so-called hyperscaling relation [37] $\delta = (D + 2 - \eta)/(D - 2 + \eta)$. The value of g_c is related to the critical value $g^* \approx 0.503$ by $g_c \mu^{-\eta D/(D-2+\eta)} = (2g^*)^{(\delta-1)/2}(4\pi)^2/24 \approx 6.7$. As a possible application we may study the behavior of a triangular lattice of vortices which form in a rotating Bose-Einstein condensate [38], and letting the magnetic field approach a Feshbach resonance.

The results may then be compared with a calculation based on a new field equation that generalizes the famous Gross-Pitaevskii equation [39]

$$\left[\hat{E} - \frac{1}{2m}\hat{\mathbf{p}}^2 - g|\Psi(t, \mathbf{x})|^2\right] \Psi(t, \mathbf{x}) = 0. \quad (27)$$

The new equation is obtained by extremizing the effective action $\Gamma[\Psi, \Psi^*] = \Gamma_0[\Psi, \Psi^*] + \Gamma^{\rm int}[\Psi, \Psi^*]$, where

$$\Gamma_0 \equiv \int dt d^D x\, \Psi^\dagger(t, \mathbf{x})[\hat{E}^{1-\gamma} - D_\lambda(\hat{\mathbf{p}}^2)^{\lambda/2}]\Psi(t, \mathbf{x}). \quad (28)$$

By forming $\delta\mathcal{A}^{\rm eff}/\delta\Psi^\dagger(t, \mathbf{x})$, we obtain what may be called the *fractional Gross-Pitaevskii equation*:

$$\left[\hat{E}^{1-\gamma} - D_\lambda(\hat{\mathbf{p}}^2)^{1-\eta/2} - \frac{\delta+1}{4\mu^\eta} g_c |\Psi(t, \mathbf{x})|^{\delta-1}\right]\Psi(t, \mathbf{x}) = 0. \quad (29)$$

The fractional Schrödinger equation has many problems, such as the nonvalidity of the quantum superposition law, the violation of unitarity of the time evolution, and the violation of probability conservation which can produce nonsensical probabilities > 1 [29]. However, these problems exist only if we restrict ourselves to the free effective action (28), but this is meaningless, since the entire theory is only defined by the effective action in the strong-coupling limit — and this contains necessarily additional nonquadratic terms. Hence it does not possess free quasiparticles as in the time-honored Landau theory of Fermi liquids [40]. There is always an interaction that invalidates the standard discussion of Schrödinger equations. In fact, the theory of high-T_c superconductivity must probably be built as a true strong-coupling theory of this type with electrons being non-Fermi liquids [40].

The relativistic version of the entire discussion is simpler since it is based on the euclidean Green function (9) in which $\mathbf{p}$ denotes the $D-1$-dimensional vectors $(\mathbf{p}, p_4)$. The Fourier transform is the distribution fulfilling the Fokker-Planck equation

$$[\partial_s + (\hat{\mathbf{p}}^2)^{1-\eta/2}] P(s, \hat{\mathbf{x}}) = \delta(s)\delta^{(D+1)}(\mathbf{x}). \tag{30}$$

and possessing the path integral representation

$$P(s, \hat{\mathbf{x}}) = \int \mathcal{D}\mathbf{x}\mathcal{D}\mathbf{p} e^{\int ds [i\mathbf{p}\dot{\mathbf{x}} - (\hat{\mathbf{p}}^2)^{1-\eta/2}]}. \tag{31}$$

The ϵ-expansion is now around $D_c = 4$ in powers of $\epsilon = -(D - D_c)$. The critical exponent η is small of order ϵ^2: $\eta = \epsilon^2/50 + \cdots \approx 0.04$. It can be ignored for $\epsilon = 1$. The power δ in the interaction is $3 + \epsilon + 23\epsilon^2/50 + \cdots \approx 4.76$ [41].

The time-independent *fractional Gross-Pitaevskii equation* reads now

$$\left[(\hat{\mathbf{p}}^2)^{1-\eta/2} + \frac{\delta+1}{4\mu^\eta} g_c |\Psi(\mathbf{x})|^{\delta-1}\right]\Psi(\mathbf{x}) = 0, \tag{32}$$

with $g_c \approx 27$. For a $d = D - 1$ -dimensional vortex in $D = 3$ dimensions, it is solved by $\tilde{\Psi}(\mathbf{x}) = a|\mathbf{x}_\perp|^{-A}$ with $A = (2-\eta)/(\delta-1) = D/2 - 1 + \eta/2 \approx 1/2$ and for $\mu = 1$: $[(\delta+1)a^{\delta-1}/4]g_c = -{}^d c_{\lambda+A-d}{}^d c^{-1}_{A-d} \approx 0.2$, $\lambda = 2 - \eta$ [25].

To compare our theory with experimental data, we must study the BEC in the scale-invariant strong-coupling limit. This is reached either by going to the temperature T_c of the second-order phase transition, or by raising the magnetic field B towards the field strength B_c of a Feshbach resonance. Then the coherence length ξ grows like $\xi \propto |t|^{-\nu}$ where $\nu \approx 2/3$ [21, 42], and $t \equiv 1 - T/T_c$ or $t \equiv 1 - B/B_c$ ([35]). If the BEC is enclosed in a weak harmonic trap, this adds in the brackets of (27) a term $\propto |\mathbf{x}|^2 = R^2$. This is normally observed by the condensate density going to zero linearly like $1 - r^2 \equiv 1 - R^2/R_{\rm b}^2$ near the border $R_{\rm b}$ (in the Thomas-Fermi approximation) [43]. For B near B_c (or T near T_c), however, the anomalous power δ will lead to the steeper approach to zero $(1 - r^2)^{2\beta}$ where $2\beta \equiv \nu(D - 2 + \eta) = 1 - 3\epsilon/10 + \cdots \approx 0.7$, plotted in Fig. 1, as will be shown immediately. In addition, the central region is depleted.

Let us study the appearance of a reduced mass $\hat{m}^2 \propto (1 - r^2)$ in the trap. The effective action will introduce it

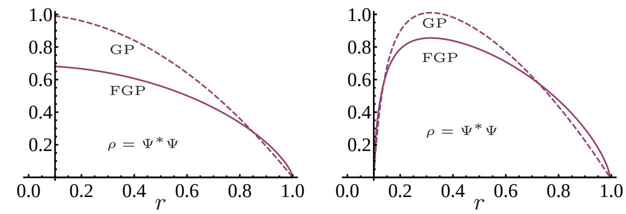

Figure 1: Condensate density from Gross-Pitaevskii equation (27) (GP,dashed) and its fractional version (29 (FGP), both in Thomas-Fermi approximation where the gradients are ignored. The FGP-curve shows a marked depletion of the condensate. On the right hand, a vortex is included. The zeros at $r \approx 1$will be smoothened by the gradient terms in (32).

into (32) in the form $\mu^{2-\eta}(\hat{m}^2)^{\nu(2-\eta)} f(|\Psi|^2/(\hat{m}^2)^{2\beta})$ with a Taylor series of $f(x)$ (note that $\nu(2-\eta) = 1 + \frac{\epsilon}{5} + \cdots \approx 1.3$). For small $\hat{m}^2$, this may be resummed to a Widom type expression $[(\delta+1)/4\mu^\eta]g_c|\Psi|^{\delta-1}w(\hat{m}^2/|\Psi|^{1/\beta})$ [21]. This explains the earlier-stated steeper falloff $|\Psi|^2 \propto (\hat{m}^2)^{2\beta}$ of the density profiles in Fig. 1. The Widom function $w(\hat{m}^2/|\Psi|^{1/\beta})$ can be expanded as 1 plus a power series in $(\hat{m}^2)^{\omega/2\nu} \propto \xi^{-\omega}$ which contains the Wegner critical exponent $\omega \approx 0.8$ that governs the *approach to scaling* [45]. Thereby the interaction term $|\Psi|^{\delta-1}$ is modified to $|\Psi|^{\delta-1}(1 + \text{const} \times \xi^{-\omega}|\Psi|^{-\omega\nu/\beta})$. Similarly, the kinetic terms $(\hat{\mathbf{p}}^2)^{1-\eta/2}$ in (32) and (29) approach the scaling limit like $(\hat{\mathbf{p}}^2)^{1-\eta/2}[1 + \text{const} \times \xi^{-\omega}(\hat{\mathbf{p}}^2)^{-\omega/2} + \dots]$ [46].

A further observable phenomenon is that the resonance frequency of a forced collective oscillation will depend on the field strength B near the Feshbach resonance [44].

Summarizing we have seen that a many-body system with strong couplings between the constituents satisfies a more general form of the Schrödinger equation, in which the momentum and the energy appear with a power different from $\alpha = 2$ and $\gamma = 0$, respectively. The associated Green function can be written as a path integral over fluctuating time and space orbits that are functions of some pseudotime s. This is a Markovian object, but non-Markovian in the physical time t that is related to s by a stochastic differential equation of the Langevin type. The particle distributions can also be obtained by solving a Langevin type of equation in which the noise has correlation functions whose probability distribution is specified.

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Appendix: The lowest-order critical exponents can be extracted directly from the one-loop-corrected inverse Green function $G^{-1}(E, \mathbf{p})$ in $D = 2 + \epsilon$ dimensions after a minimal subtraction of the $1/\epsilon$ -pole at [48]:

$$E-\mathbf{p}^2+a\left(\tfrac{1}{3}\mathbf{p}^2-E\right)^{D-1}. \tag{33}$$

For $\mathbf{p}=0$, this has a power $-(-E)^{1-a\epsilon}$, so that $\gamma=a\epsilon$. For $E=0$, on the other hand, we obtain $(-\mathbf{p}^2)^{1-a\epsilon/3}$, so that $(1-\gamma)/z-1\approx\gamma/3$.

* Electronic address: h.k@fu-berlin.de

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