Chapter 4

Delay in joint projects

4.1 Introduction

Delay is an important aspect of many situations in which a public good is to be provided through private contributions. This was first pointed out as an empirical observation by Olson (1982, p. 39-40) with respect to delayed formation of organized interest groups. Other salient examples include delayed adoption of standards by societies or certain industries (see Farrell and Saloner, 1988), and the pivotal role played by delay in the political economy of policy reform as considered by Drazen (2000).

Typically, authors have explained delay in private provision games by introducing private information. In Bliss and Nalebuff (1984) a public good is to be provided completely by one individual. Individuals who have private information about their costs, play a waiting game to see if someone will come forward to provide the good. Gradstein (1992) retains the assumption of private information about contribution costs but considers a production technology where the public good is produced by the number of contributing individuals with decreasing returns. Both find inefficient delay if the number of individuals is finite. Similarly, in applied contributions such as Alesina and Drazen (1991), delay is driven by some waiting game which originates in private information.

However, there seem to be many situations with inefficient delay in which full information seems to be a valid approximation. Therefore it is worthwhile studying alternative causes of delay. In this chapter I concentrate on the role of convex contribution costs and their interaction with other potential aspects of the situation. Convexity of contribution costs is a reasonable assumption in many settings. Depending on the nature of the contributing unit and the form of the contributions, their causes may be, for example, increasing disutility of labor or decreasing marginal utility of remaining income.

I study the case of a discrete project that has to be completed by means of private contributions and may be epitomized by the proverbial example of the building of a bridge by a group of people. Benefits only start flowing once the bridge is completed. No agent can be excluded from the benefits of the project and side payments are not allowed. All players have perfect and complete information. The problem is analyzed in continuous time, which makes derivations of completion times in equilibrium easier and avoids equilibria which may only be artifacts of a discrete setting. The model is solved for the set of symmetric open-loop (OLE) and Markov perfect (MPE) equilibria.

The convexity of costs usually implies the optimality of project completion in some positive time. However, in such a setting, individual contributions may be postponed for two reasons. First, convexity of costs implies an incentive to spread out contributions in order to decrease marginal costs. If individual costs and benefits are not perfectly aligned with their social counterparts, this creates a social inefficiency. Second, the dynamic nature of the situation adds a time dimension to the players' incentives to free ride on others' contributions. The paper analyzes how individual heterogeneity, order of moves and the absence of commitment devices will affect delay in such a setting.

The findings of the model are as follows. In a perfectly symmetric setting with commitment, a continuum (with respect to completion time) of symmetric completing equilibria exists which contains the social optimum as the lower bound completion time. This Pareto-dominates all other symmetric equilibria, so that it is the natural outcome. If commitment is retained but players are asymmetric or players do not commit to their contribution paths simultaneously, inefficient delay occurs for distributional reasons. Finally, in a symmetric setting without commitment, inefficient delay occurs in equilibrium. No player can contribute efficiently fast, since this will leave him open to exploitation by the other players later on. Furthermore, unlike the private provision of a continuously divisible public good, individual contributions are strategic complements, which make partial harmonization of strategies seem beneficial and this is important in many policy-related applications. The results are in line with the observation that, in many real life examples, delay seems to be closely related to asymmetric players and missing compensation mechanisms as well as to the lack of commitment devices.

The chapter relates to two further strands of literature. First, there is the connection with the the private provision games of a continuously divisible public good in dynamic settings analyzed by Fershtman and Nitzan (1991), Wirl (1996) and Itaya and Shimomura (2001). This literature finds that without commitment the provision level may be increased or decreased, depending on the set of admissible strategies.

Second, and more importantly, Admati and Perry (1991) and Marx and

Matthews (2000) have considered voluntary contributions to a joint project, with the key feature of a pay-off function that is discontinuous at completion of the project. Marx and Matthews analyze the case where contributors have linear costs and know both their own and the total sum of contributions, but cannot observe the individual contributions of other players. With linear costs, if the project is carried out in equilibrium, inefficiency from delay is caused only by the time assumed to elapse between the players' interactions. Admati and Perry consider the case where two players with convex costs contribute alternately to a joint project. The MPE derived in section four can be regarded as a differential game n-player counterpart of their analysis. While they are concerned with the question of whether socially desirable projects will be carried out or not, the present analysis focusses on delay and makes it possible to consider completion times and their comparative statics explicitly. Furthermore, it allows insights to be made into important structural properties of the equilibrium, such as strategic complementarity of individual contributions.

The chapter is organized as follows. Section two sets up the model and solves for the social planner's optimum. In section three the symmetric equilibria of the situation with symmetric players and commitment is studied. Section four discusses how the commitment case changes, when asymmetry and non-simultaneous moves are considered. Section five derives the Markov perfect equilibria of the situation without commitment. Section six concludes.

4.2 The model

A group of n players contributes to a joint project. Individual contributions x_i, $i = 1, ..., n$, are immediately sunk. Once total contributions reach the project size K the project is completed and from this point T onwards, it starts yielding an infinite continuous stream of benefits D_i for individual i. The individual convex cost functions are, for tractability, assumed to be quadratic, $C_i(x_i) = \frac{c_i}{2}x_i^2$. For now, I consider the case of identical individuals so that for all i, $c_i = c$ and $D_i = D$. Furthermore, r denotes the discount rate and parameter values are assumed such that it is not profitable to carry out the project individually.

The social planner maximizes the representative individual's intertemporal problem (SWP) given by

$$\max_{x(t)} J = -\int_0^T \frac{c}{2}x^2 e^{-rt}dt + \frac{D}{r}e^{-rT} \tag{4.1}$$

s.t.

$$\dot{k} = nx(t) \tag{4.2}$$

$$x(t) \geq 0 \tag{4.3}$$

$$k(0) = 0 \tag{4.4}$$

$$k(T) = K. \tag{4.5}$$

Alternatively, a project whose size is proportional to the number of users can be considered. In this case (4.5) is replaced by $k(T) = nK$, and parameters have to be assumed that, even for $n = 1$, make it beneficial to carry out the project. The solution of the SWP is summarized in the following proposition.

Proposition 6 *The socially optimal contribution paths to the project are given as $x^*(t) = \frac{\lambda^*}{c}e^{rt}$, where λ^*, which is constant over time, denotes the* *costate variable of the social planner's problem. The optimal completion time is given by* $T^* = r^{-1} \ln \left(\frac{D + \frac{rK}{n}\sqrt{\frac{c}{2}D}}{D - \frac{c}{2}\frac{[Kr]^2}{n^2}} \right)$.

Proof of proposition 6: See appendix.□

Naturally, with positive discounting, the optimal policy is to increase contributions exponentially at rate r until completion, in order to keep the present value of the marginal cost of an additional unit of contribution unchanged. Only projects for which $D > \frac{c}{2}\frac{[Kr]^2}{n^2}$ have a positive present value and should be carried out. If the project size is proportional to the number of participants, K is replaced by nK in T^* and the optimal time is independent of the number of participants. If the size of the project is fixed, $\lim_{n->\infty} T^* = 0$, i.e., the project should be carried out immediately. As the number of contributors becomes very large, the costs can be shared among all of them, and thus the convexity of individual costs no longer has any bite.

4.3 The non-cooperative solution with commitment

I consider now the open-loop equilibria of the differential game, where n players contemplate their contributions to the joint project individually. In the open-loop case, the players choose their complete contribution paths $x_i(t)$ at the beginning of the game. Thus, the open-loop assumption implies that all players are able to commit themselves to their chosen paths over the entire contribution period. For an extensive discussion of strategy space and information sets in differential games see Dockner et al. (2000). A Nash Equilibrium is given by a vector of n optimal time paths, such that $J^i(x_1^*(t), ..., x_i^*(t), ..., x_n^*(t)) \geq J^i(x_1^*(t), ..., x_i(t), ...x_n^*(t))$, $\forall i$. Each individual

i solves the problem (IMP)

$$\max_{x_i(t)} J^i = -\int_0^T \frac{c}{2} x_i^2 e^{-rt} dt + \frac{D}{r} e^{-rT} \tag{4.6}$$

s.t.

$$\dot{k} = \sum_{j=1}^{n} x_j, \tag{4.7}$$

$x_i \geq 0$, (4.4) and (4.5), where the other contributions are taken as given in (4.7). The symmetric equilibria are given as follows:

Proposition 7 *There exists a continuum of symmetric open loop equilibria in which all players commit to contribution paths growing at rate r. The completion time of these equilibria ranges from T^* to $\bar{T}$, where $\bar{T} = r^{-1} \ln \left(\frac{D + \frac{rK}{n} \sqrt{\left(n - \frac{1}{2}\right) cD}}{D - (n - \frac{1}{2}) c \left[\frac{rK}{n}\right]^2} \right)$. Additionally, there is a non-completing equilibrium, in which no player ever contributes.*

Proof of proposition 7: Consider the 2-player case. If player 1 chooses to build half the project in the efficient time and to contribute zero afterwards, player 2 is left with the problem of completing a half-size project with half the social benefits (his own only). Thus, the individual problem is just a scaled down version of the social problem. The individual costs and benefits are perfectly aligned with the social ones. Player 2's best response is to complete the project in the efficient time. If player one builds her half in any $\tilde{T} \in [T^*, \bar{T}]$, it is always optimal for player two to contribute her half up to that time as well, since speeding up not only has higher marginal costs for the own given share of one half, but also necessitates taking over some of player 1's share. However, if player one chooses to stretch her share's contribution beyond $\bar{T}$, then it pays for player two to take over some of player one's share. Consequently, no $T > \bar{T}$ can be an equilibrium. In the appendix it is shown that the cut-off point is indeed given by $\bar{T}$. The logic extends straightforwardly to the n-player case. The non-completing equilibrium follows directly from the assumption that carrying out the entire project individually is not profitable. If all players expect the others not to contribute, their best responses are also not to contribute and the project will not be undertaken.$\square$

The efficient equilibrium Pareto-dominates the non-completing and the other symmetric completing equilibria, so that, with pre-play communication, reaching the first best seems a plausible outcome. Thus, with perfectly symmetric players and commitment no inefficiencies from delay are likely to occur. Apparently, there are two decisive features that trigger the result. First, since players can commit not to contribute after T^*, the other players have no hope of free-riding on the others' contributions afterwards. Second, due to symmetry, private and social costs and benefits are perfectly aligned for one player if all other players choose to contribute according to the first best path.

Note that, if the project has a strictly positive net value, further continua of asymmetric equilibria also exist. For a given non-equal distribution of project size shares, the completion times of these equilibria are on an interval $[T', T'']$, where $T^* < T'$ and $T'' < \bar{T}$. The less equal the distribution of project shares, the smaller $[T', T'']$, since higher share individuals have an incentive to delay their contributions, while for lower share individuals taking over other agents' burdens is more attractive.

4.4 Sequential moves and asymmetry

Let me now consider the robustness of the optimality results under commitment. I will discuss two modifications to the above setting, where, for simplicity, I focus on the two-player setting, though the considerations extend directly to the n-player case.

First, suppose that the game is changed to a Stackelberg setting in the sense that player 1 gets to choose her contribution path first before player 2 chooses hers. As in the static private provision game of a discrete public good, the Stackelberg leader is in a position to extract some rent from the follower. In a static setting all rent is extracted from the follower.[1] Here, however, the leader will leave some rent for the follower. The reason is that the follower has another instrument to react to the leader's attempt to shift a bigger share of the project's size to the follower. This instrument is to delay contributions in order to smooth marginal costs. This will hurt the leader, since completion is delayed. Thus the leader faces a trade-off between shifting a larger share of the project's size to the follower and the induced completion delay. While, for the given model, the situation can no longer be solved out in closed form, it can be shown that the leader's profits are increasing in the follower's share if her own share is one half. Thus, the leader will always shift some burden, and this will delay completion. Obviously, this argument only holds, if the project creates positive profits that can be shifted.

Similarly, if players are asymmetric, either with respect to their cost parameter c_i or to their benefits D_i, even with commitment the project will not be completed in the socially optimal time unless side payments are possible. The reason is that, in such a situation, the individual problem is no longer a scaled down version of the social problem. People whose marginal cost is lower will not contribute enough. Similarly, players with lower benefits will

[1]Note, that, if benefits or costs and thus the willingness to contribute, depend on income levels, such exploitation may be mitigated through a strategic transfer from the follower to the leader, see Buchholz et al. (1997).

not contribute fast enough.

Again, it is interesting to consider the static counterpart. If players are sufficiently asymmetric in the productivity of their contributions, the discrete public good may not be provided, even though it would be socially desirable. This may also happen in the dynamic situation, but, typically, there is a number of projects for which the inefficiencies arise from delayed completion.

Thus, both the possibility of committing first to a contribution path and asymmetry among the players will typically cause the optimality result under commitment to break down. The players' interests are no longer perfectly aligned or, in the Stackelberg setting, are even partly opposed. Thus, distributional reasons prevent the parties from achieving the first best solution. In the Stackelberg case, the outright aim is to exploit the other party, in the asymmetry case, it is the inability of the less productive, or the party who profits more to compensate the other through redistribution. In fact, such asymmetries are present in many real life examples, and in many situations they constitute an important reason for excessive delay.

4.5 The Markov perfect equilibrium

Let me now turn to the situation where no commitment at the beginning of the game is possible. For this, I consider the Markov perfect equilibria (MPE) of the game as set up in section 3. The strategies of the players, $\phi(k, t)$, are now allowed to depend on time and the evolution of the state variable k, the progress of the project. Thus, they are rules conditioned on these variables. This implies that, at any instance, players reoptimize their contributions based on the sum of all contributions made up to that time, so that these strategies are time consistent. An MPE is given by a vector of n optimal rules, such that $J^i(\phi_1^*(k,t), ..., \phi_i^*(k,t), ..., \phi_n^*(k,t)) \geq J^i(\phi_1^*(k,t), ..., \phi_i(k,t), ...\phi_n^*(k,t))$, $\forall i$. In fact, since the problem is independent of time, the strategies will only depend on the project's progress, $\phi(k)$. Again, there is a non-completing equilibrium in which no player ever contributes and a completing equilibrium, which is given in

Proposition 8 *(i) The following strategies constitute a symmetric MPE:* $\phi(k) = \frac{1}{c}\ [\beta + \gamma k]$*, where* $\beta = \frac{2\sqrt{cD(n-1/2)} - crK}{2n-1}$ *and* $\gamma = \frac{cr}{(2n-1)}$ *.*

(ii) The completion time of this MPE is given by $T_{MP} = \frac{c}{\gamma n} \ln\left(1 + \frac{\gamma K}{\beta}\right)$ *and* $T_{MP} > T^*$*.*

Proof of proposition 8: See appendix.□

The result shows that, unless it is possible to commit, a strategic incentive to delay contributions exists. The intuition is straightforward. Although any one of the players would like to contribute faster, doing so involves a time-consistency problem. Contributing more heavily early on and then reducing or even stopping contributions later is not credible. Later on it is in the player's own interest to contribute to completing the project. Since the other players know this, they would be in a position to exploit the player who contributed heavily early on by reducing their later contributions. Thus, in order to prevent the others from free riding each player chooses to delay her contributions inefficiently. If delay is sufficiently severe, projects cease to be individually profitable in equilibrium and the completing equilibrium disappears. Thus, some socially desirable projects will not be carried out. This parallels the findings of Admati and Perry (1991), who use a discrete set-up, in which players alternate their contributions to the project sequentially.

Considering the comparative static properties of the completing MPE with respect to the number of players reveals that, for the fixed size project, the cost-sharing effect dominates the free-riding effect. However, considering the proportional size project, which implies controlling for the cost-sharing effect, demonstrates that the time-consistency problem is aggravated and delay increased. Eventually, for all n bigger than some critical value, the completing equilibrium breaks down.

There are two further interesting aspects of the completing MPE. First, comparing T_{MP} with the upper bound of the continuum of symmetric open-loop equilibria, $\bar{T}$, shows that $T_{MP} < \bar{T}$. Thus, there is a range of equilibria with commitment that have longer completion times and consequently leave everybody worse off. The reason for this can be found in the coordination problem and the impossibility of re-optimizing under commitment.

Second, since $\gamma > 0$, individual contributions are strategic complements. This contrasts with the findings of Fershtman and Nitzan (1991) for the continuously divisible public good, where they are substitutes. The strategic complementarity has important consequences for alliance formations in such settings. Consider a subgroup of agents forming a coalition by harmonizing their strategies. While agents outside the coalition will always profit from such harmonization, with strategic complementarity, the subgroup forming the coalition will typically also increase its welfare (Gaudet and Salant, 1991). Here, the conjecture is that the harmonization of the own strategy with a partner weakens the time-consistency problem. Free-riding within the coalition is prevented through harmonization, free-riding outside the coalition is reduced due to strategic complementarity. Both effects lead to faster completion. Thus, partial harmonization leaves everybody better off.

4.6 Conclusion

I have studied a situation in which a public project is provided through private contributions with convex costs. A continuum of symmetric completing equilibria exists with symmetric players and commitment. Since the first best solution is among them, this is the natural outcome with potential preplay communication. If asymmetric players are allowed for, or if players do not simultaneously commit to their contribution paths, delay will occur in equilibrium. Alternatively, the project may not even be carried out at all. These inefficient equilibria with delay have their origins in distributional concerns, i.e. the first mover's incentive to exploit the followers and the missing compensation mechanisms in the case of asymmetric costs and benefits. The results parallel the static provision game of a discrete public good to some extent. However the dichotomy provision versus non-provision is complemented by the delay dimension's inefficiency. The possibility of stretching out contributions to reduce costs eases some of the strong distributional implications of some of the static equilibria.

A time consistency problem arises if agents cannot commit to a contribution path. This causes all agents to delay their contributions so that inefficiently late completion of the project results. If an agent contributes efficiently fast early on, she is open to exploitation through the others later on. These others can reduce their late contributions, knowing that the completion of the project is also in the interest of the early heavy contributor, who will end up contributing relatively heavily later on as well. Delay relative to efficient completion increases in the number of agents if the size of the project is proportional, since the dynamic free riding incentive becomes stronger. Surprisingly, there is a range of symmetric commitment equilibria whose completion times are longer than without commitment. Finally, individual contributions are strategic complements in the completing MPE. This justifies the conjecture that a subgroup of agents who form a coalition by harmonizing their contribution decisions reduces inefficient delay and will leave everybody better off.

The results are valid in many microeconomic settings, such as the joint development of some non-patentable innovation by a group of firms, the writing of a joint paper by a group of scientists, or the attempt of a group of lobbyists to change a legislator's opinion. They also have straightforward policy implications for macroeconomic issues. Consider the examples of policy reform or stabilizations. The successful completion of such projects often necessitates contributions, such as wage restraints or reduced fiscal spending from various parties over a prolonged period of time. For these measures to be successful, it is often important that combined contributions of the actors involved reach a threshold value, say to qualify for support from the IMF or to regain the confidence of financial markets. The groups involved often face convex contribution costs originating from financial market imperfections or tax smoothing arguments. Then the present model directly allows observed delay or even failure to be traced to cost and benefit asymmetries with missing compensation mechanisms, non-simultaneous moves and the absence of commitment devices. Furthermore, the results indicate that, if some parties involved, such as a group of sectorial unions in the case of a wage restraint, or regional governments in the case of reducing fiscal spending, can harmonize their strategies, earlier and successful completion is more likely to result.

4.7 Appendix

Proof of proposition 6: The Hamiltonian of the SWP is given as

$$H = -e^{-rt}\frac{c}{2}\left[x(t)\right]^2 + \lambda^*(t)x(t). \tag{4.8}$$

Consequently, the necessary conditions are then given by

$$\frac{\partial H}{\partial x} = -e^{-rt}cx(t) + \lambda^*(t) = 0 \tag{4.9}$$

$$\dot{\lambda}^* = -\frac{\partial H}{\partial k} = 0 => \lambda^*(t) = const \tag{4.10}$$

$$[H]_{t=T} + \frac{\partial\left[\frac{D}{r}e^{-rT}\right]}{\partial T} = 0 \tag{4.11}$$

The optimal individual contribution path follows directly from (4.9). Solving the system of (4.9), (4.10) and (4.11) by making use of the initial and terminal conditions (4.4) and (4.5) delivers the resulting completion time given in proposition 1.□

Proof that $\bar{T}$ is the cut-off point in proposition 7 : For the proof it is helpful first to establish a lemma which establishes the fact that under commitment individual actions can be perfectly summarized by the individual's planned project share and the planned termination time.

Lemma 1 *For a given total project share a player plans to contribute, her strategy is perfectly characterized by the terminal time chosen. The contributions will start in t=0 and grow exponentially at rate r until the chosen termination time is reached, i.e.* $x_t = \frac{\lambda}{c}e^{rt}\ \forall\ t \in [0, T]$ *and* $x_t = 0\ \forall\ t > T$. *The value of* λ *depends on project share* δ *and the planned* T *in the following way:* $\delta K = \int_0^T \frac{\lambda}{c}e^{rt}dt$.

Proof: The exponential contribution path follows directly from cost minimization. The present value of an additional marginal contribution should be kept constant. The relationship between δK, T and λ satisfies that the project share is actually reached in the planned time.□

These general properties of the players' contribution paths can now be used to prove proposition 2. For simplicity, I normalize the project size to one and demonstrate, for the two player case, that $\bar{T}$ is indeed the cut-off value above which it pays to take over some of the other players' share, but below which it does not. Suppose that player two chooses a project share of one half and some $T_2 > \bar{T}$. In this case it is never a best response for player 1 to choose the same completion time but some completion time smaller than T_2. To see this, consider the change in the net present value of costs and benefits that result from reducing completion time. The NPV of player 1's costs is

$$NC_1(T_1) = \int_0^{T_1} \frac{c}{2} \left(\frac{\lambda'}{c} e^{rt} \right)^2 e^{-rt} dt, \tag{4.12}$$

where the value of λ' has to be derived from the increased overall burden by taking over part of player two's contribution. If player 1 expects player 2 to choose $T_2 > \bar{T}$, then, by lemma 1, he also expects an exponential contribution path for player 2 fulfilling $\frac{1}{2} = \int_0^{T_2} \frac{\lambda_2}{c} e^{rt} dt$. This gives $\lambda_2 = \frac{rc}{2\left[e^{rT_2}-1\right]}$, so that player two's cumulated contribution evolves according to $\dot{k}_2(t) = \frac{r}{2\left[e^{rT_2}-1\right]} e^{rt}$, which can be solved as $k_2(t) = \frac{\left[e^{rt}-1\right]}{2\left[e^{rT_2}-1\right]}$. Therefore player 1 concludes for himself that, if he chooses to complete at some $T_1 < T_2$, the completion constraint will read

$$\frac{1}{2} + \left(\frac{1}{2} - \frac{\left[e^{rT_1} - 1\right]}{2\left[e^{rT_2} - 1\right]} \right) = \int_0^{T_1} \frac{\lambda'}{c} e^{rt} dt \tag{4.13}$$

$$\frac{1}{2} + \left(\frac{1}{2} - \frac{\left[e^{rT_1} - 1\right]}{2\left[e^{rT_2} - 1\right]} \right) = \frac{\lambda'}{rc} \left[e^{rT_1} - 1\right]$$

$$\frac{rc\left(1 - \frac{\left[e^{rT_1}-1\right]}{2\left[e^{rT_2}-1\right]}\right)}{\left[e^{rT_1} - 1\right]} = \lambda'$$

$$rc\left(\frac{1}{[e^{rT_1}-1]}-\frac{1}{2\,[e^{rT_2}-1]}\right)=\lambda'$$

Substituting into the cost function and solving the integral gives

$$NC_1(T_1)=\frac{cr}{2}\left[\frac{1}{(e^{rT_1}-1)}-\frac{1}{[e^{rT_2}-1]}+\frac{[e^{rT_1}-1]}{4\,[e^{rT_2}-1]^2}\right]. \tag{4.14}$$

Then, the change in costs is given as

$$\frac{\partial NC_1(T_1)}{\partial T_1}=\frac{cr}{2}re^{rT_1}\left(\frac{1}{4\,[e^{rT_2}-1]^2}-\frac{1}{(e^{rT_1}-1)^2}\right) \tag{4.15}$$

The change in the NPV of the returns is given by $\frac{\partial}{\partial T_1}[\frac{D}{r}e^{-rT_1}]=-De^{-rT_1}$. Evaluating these expressions at $T_1=T_2$ a condition is derived for when it is better to choose some smaller T_1 :

$$De^{-rT_2}>\frac{cr}{2}re^{rT_2}\left(-\frac{1}{4\,[e^{rT_2}-1]^2}+\frac{1}{(e^{rT_2}-1)^2}\right) \tag{4.16}$$

This holds as an equality precisely at $T_2=\bar{T}$, so that it pays to take over some share, if $T_2>\bar{T}$, and to choose the same time, $T_1=T_2$, if $T^*\leq T_2<\bar{T}$. The same reasoning can be applied in the general n-player case to derive $\bar{T}$ as given in proposition 2.□

Proof of proposition 8: (i) First note that IMP can be rewritten as

$$\max_{x_i}\ J=\frac{D}{r}-\int_0^T e^{-rt}\left[\frac{c}{2}x_i^2+D\right]dt \tag{4.17}$$

s.t. the given constraints. To solve for the MPE, neglecting the constant term, consider the Hamilton-Jacobi-Bellman equation of player i

$$-V_t(t,k)+rV(t,k)=\max_{\phi_i}\left[-\frac{c}{2}\phi_i^2-D+V_k\left(\phi_i+\sum_{j\neq i}\phi_j\right)\right]. \tag{4.18}$$

Observe that the problem at hand is completely time independent, in the sense that only the number of players and the missing contributions to completion matter for the current value of the project to the individual. Thus,

$V_t(t,k) = 0$. Furthermore, the first order condition of the maximization problem on the right hand side is given as $\phi_i = \frac{V_k}{c}$. Re-substituting gives

$$rV(t,k) = -\frac{c}{2}\left(\frac{V_k}{c}\right)^2 - D + V_k\left(\frac{V_k}{c} + \sum_{j\neq i}\phi_j\right), \tag{4.19}$$

and assuming symmetry

$$rV(t,k) = (n-\frac{1}{2})\frac{V_k^2}{c} - D. \tag{4.20}$$

This non-linear differential equation can be solved as follows. The quadratic nature of the problem leads to the conjecture that the value function itself may be quadratic, $V(k) = \alpha + \beta k + \frac{1}{2}\gamma k^2$. Substituting this into the above gives

$$r\alpha + r\beta k + \frac{1}{2}r\gamma k^2 = \frac{n-\frac{1}{2}}{c}\left(\beta^2 + 2\beta\gamma k + \gamma^2 k^2\right) - D \tag{4.21}$$

This can only hold, if the following three equations hold

$$\frac{1}{2}r\gamma - \frac{n-\frac{1}{2}}{c}\gamma^2 = 0 \tag{4.22}$$

$$r\beta - \frac{n-\frac{1}{2}}{c}2\beta\gamma = 0 \tag{4.23}$$

$$r\alpha - \frac{n-\frac{1}{2}}{c}\beta^2 + D = 0 \tag{4.24}$$

From (4.22) it must be that either $\gamma = 0$ or $\gamma = \frac{cr}{2(n-(1/2))}$. If $\gamma = 0$, $\beta = 0$ and consequently $\alpha = -D/r$. This corresponds to the non-completing perfect Markov equilibrium, in which no one ever contributes. If $\gamma = \frac{cr}{2(n-(1/2))}$, (4.23) always holds for any β. Now we can use the boundary condition $\alpha + \beta K + \frac{1}{2}\gamma K^2 = 0$, such that $\alpha = -\beta K - \frac{1}{2}\frac{cr}{2(n-(1/2))}K^2$. Substituting into (4.24) and solving the resulting quadratic equation gives $\beta = \frac{2\sqrt{cD(n-(1/2))} - crK}{2(n-(1/2))}$ as the positive solution. Consequently, the project's parameter must fulfill $2\sqrt{cD\left(n-(1/2)\right)} \geq crK$ for the completing equilibrium to exist.

Now consider the corresponding evolution of k :

$$\dot{k} = \frac{n}{c}\left[\gamma k(t) + \beta\right]. \tag{4.25}$$

This can be solved as

$$k(t) = -\frac{\beta}{\gamma} + Ze^{\frac{\gamma n}{c}t} \tag{4.26}$$

where Z is a constant that can be determined by the initial condition $k(0) = 0$, such that $Z = \frac{\beta}{\gamma}$. Resubstituting delivers the progress of the project as a function of time $k(t) = -\frac{\beta}{\gamma} + \frac{\beta}{\gamma}e^{\frac{\gamma n}{c}t}$. Setting $k(t) = K$ this can be solved for completion time T:

$$T_{MP} = \frac{c}{\gamma n}\ln\left(1 + \frac{K\gamma}{\beta}\right).$$

(ii) Without loss of generality normalize the benefits, $D = 1$. Then the claim $T_{MP} > T^*$.becomes

$$\frac{2n-1}{rn}\ln\left[\frac{2\sqrt{c(n-(1/2))}}{2\sqrt{c(n-(1/2))}-crK}\right] > \frac{1}{r}\ln\left[\frac{1+\frac{rK}{n}\sqrt{c/2}}{1-\frac{c}{2}\frac{r^2K^2}{n^2}}\right] \tag{4.27}$$

$$<=> \quad \left[\frac{2\sqrt{c(n-(1/2))}}{2\sqrt{c(n-(1/2))}-crK}\right]^{\frac{2n-1}{n}} > \left[\frac{1+\frac{rK}{n}\sqrt{c/2}}{1-\frac{c}{2}\frac{r^2K^2}{n^2}}\right]$$

$$<=> \quad \left[\frac{1}{1-\frac{crK}{2\sqrt{c(n-(1/2))}}}\right]^{\frac{2n-1}{n}} > \left[\frac{1}{1-\frac{rK}{n}\sqrt{\frac{c}{2}}}\right].$$

This will always be true, if

$$\frac{crK}{2\sqrt{c(n-(1/2))}} > \frac{rK}{n}\sqrt{\frac{c}{2}}$$

$$<=> n > \sqrt{2(n-(1/2))},$$

which holds for all $n \geq 2.\square$