Theorem 5.11 *The number ξ_D (and therefore the number of linearly independent Vassiliev invariants of order D) is asymptotically bounded by*

$$\xi_D < \frac{D!}{1.1^D}.$$

5.6 The segment length inequality

After having estabilished our result it is perhaps worth saying a word about some possibilities left open in the proof of our bound.

The observation made in case 2 of the proof can be generalized somewhat.

Definition 5.12 *Call a* segment *of an LCD a maximal piece of the solid line containing left basepoints only and its length the number of such basepoints.*

Then by the argument above we have the

Theorem 5.13 (Segment length inequality)

$$\sum_{\substack{\text{reduced reg.}\\ \text{LCD's } L \text{ of deg } D}} \prod_{\text{segments of } L} \big(\text{length of segment}\big)! \quad \leq D!$$

Basically our proof was that we bounded the L's with $\leq \delta D$ factors equal to one in the product and used that the rest appears with multiplicity at least $2^{\delta D}$ in the sum. However, many reduced regular LCD's appear with much higher factors and if one were able to control their number (which probably requires much labour) this would improve the base in the denominator we obtain in case 2 (and the total one for all regular LCD's). One might even hope that one can achieve *each* base in case 2 (and therefore as well for the total bound). But, to put an end to our dreams, recall that we will never be able to prove $(D!)^{1-\epsilon}$ (for some $\epsilon > 0$) this way!

6 The dimension of a graded commutative algebra and asymptotics of Vassiliev invariants

Here we discuss the relation between the dimension of a symmetric algebra (with the induced grading) over a graded vector space (latter called henceforth the primitive part of the algebra), and apply it to deduce a lower bound for the number of all Vassiliev invariants.

One of the combinatorial aspects of a commutative graded algebra (CGA) A is the relation between the asymptotical behaviour of its gradeded pieces A_D depending on their primitive parts P_D (d and D will denote the degree). We will make two assumptions on such an algebra:

1) $\deg(a \cdot b) = \deg(a) + \deg(b) \quad \forall a, b \in A$,
2) prime factorization is unique in A.

Consider the commutative graded Hopf algebra $A = \mathcal{A}$ of chord diagrams.

Recently, Chmutov and Duzhin [CD2] obtained the following result for $\mathcal{A}$.

Theorem 6.1 *The dimension of primitive elements in A_D has the asymptotical lower bound $D^{\log_{4+\varepsilon} D}$ for each $\varepsilon > 0$.*

As it was not explained by the authors which base of the logarithm we can choose, we should do this here.

Proof. We have by [CD2, theorem 2.5] the lower bound

$$\sum_{\substack{n=1 \\ n \text{ even}}}^{d} \frac{n^{d-n}}{(d-n)! \cdot 2^{\frac{(d-n+1)(d-n)}{2}} 3^{d-n}}\,.$$

Set $n := d - \log_C d$ for fixed $C > 1$. The summand we consider is (temporarily omitting C for simplicity)

$$\frac{(d-\log d)^{\log d}}{(\log d)! \cdot \sqrt{2}^{\log^2 d} \left(3\sqrt{2}\right)^{\log d}}\,.$$

By Stirling formula this is asymptotically equivalent to

$$\begin{aligned} & \left(\frac{e\,(d-\log d)}{3\sqrt{2}\cdot \log d \cdot d^{\log\sqrt{2}}}\right)^{\log d} \cdot \frac{1}{2\pi\sqrt{\log d}} \\ \geq \quad & \left(d^{1-\log\sqrt{2}-\varepsilon}\right)^{\log d} \qquad \forall \varepsilon > 0 \\ = \quad & d^{\ln d \Big(\underbrace{\frac{1-\frac{\ln\sqrt{2}}{\ln C}}{\ln C}}_{f(\ln C)} - \varepsilon\Big)}\,. \end{aligned} \tag{6.1}$$

Now vary C and try to maximize $f(x)$ over $x := \ln C \in (0, \infty)$. We have

$$f'(x) = -\frac{1}{x^2} + \frac{2\ln\sqrt{2}}{x^3}$$

with the zero $x_0 = \ln 2$. We find

$$f(x_0) = \frac{1}{2\ln 2}\,,$$

so the best expression in (6.1) is

$$d^{\ln d\left(\frac{1}{\ln 4}-\varepsilon\right)} = d^{\log_{4+\varepsilon} d}$$

□

Remark 6.1 As all estimates were sharp and we took the maximum it is very likely that this is the best we can do.

Such a result opens the question which lower bound it implies for the dimension of the space of *all* chord diagrams of degree D, isomorphic [BN2] to the factor space of Vassiliev invariants of degree D modulo such of degree $\leq D-1$.

Generally, the relation between $p_d := \dim P_d$ and $a_d := \dim A_d$ is

$$a_D \;=\; \sum_{\substack{(k_1,\dots,k_D)\,\geq\, 0 \\ \sum_i i\cdot k_i \,=\, D}} \;\prod_{j=1}^{D} \binom{p_j + k_j - 1}{k_j} \tag{6.2}$$

where the D-tuple $k = (k_1, \dots, k_D)$ corresponds to products of k_i factors of degree i. Denote the number of such products by con_k.

Remark 6.2 In terms of the generating functions $a(x)$ of (a_d) and $p(x)$ of (p_d), equation (6.2) can be rewritten as

$$a(x) \;=\; \prod_{d=1}^{\infty} \frac{1}{(1-x^d)^{p_d}} \;= \exp\left(p(x) + \frac{1}{2}p(x^2) + \frac{1}{3}p(x^3) + \cdots\right)\,. \tag{6.3}$$

This relation is well-known in combinatorics. It appears explicitly in Cayley's counting of rooted trees [Ca].

Using this relation, Chmutov and Duzhin gave a lower bound for a_D without discussing details.

The aim of this section is to elucidate a little more the relation between p_d and a_d in general, and to apply it to the special case of chord diagrams, explaining and motivating one possible approach to such sort of problems.

6.1 The dominating partition

In the following we discuss the search of partitions of D which give the maximal contribution in (6.2). This maximal contribution we denote by $dom(D)$. For each D we choose one special such partition (if there are several) and consider the sequence of these partitions.

In general, the asymptotic contribution in (6.2) of a sequence of partitions of D as $D \to \infty$ roughly depends on the asymptotic behaviour of the number of parts d in the partition as function of d and D. So here is a possible strategy: take partitions with different asymptotics of number of parts d and calculate the corresponding contributions.

Ideally, it would be nice to have a result telling us *which* asymptotics of number of parts we have to take to find asymptotically the *dominating* partition in (6.2), that is, the one k giving the summand with the highest contribution con_k, which we will denote by $dom(D)$, but neither I could deduce nor I know of any such (really ingenious) result.

Below we will make an ansatz to come close to the dominating partition. We will use the number of parts of a fixed degree d in the partition to be constant, while $D \to \infty$, except finitely many D where it is 0. We will fix this constant a priori for each length and for given D take so many lengths, until D is exhausted (this of course will not work exactly in general; we will forget about the small rest, or add it to the part in degree one). That is, we consider partitions $((d_{i,D})_{i=1}^{l_D})_{D=1}^{\infty}$ with $d_{i,D} = d_{i,D+1}$ for $1 < i \leq l_D$ and $l_D \leq l_{D+1}$. This turns out to be a good ansatz, i.e. a sequence of partitions asymptotically producing the dominating contribution can always be chosen (by possibly losing a minimum on the quality of the asymptotics) to have this property (note, that some small variations of the partitions in such a sequence will produce asymptotically equal contributions). We will explain this at the end of the subsection.

In the following I shall only briefly discuss three main cases.

1) The case of 1 primitive element per degree. The numbers a_d are the so-called *partition numbers* $p(d)$ [An, Ri], which are known to have the asymptotical behaviour

$$p(d) \asymp \frac{1}{4\sqrt{3}\,d} \cdot \left(e^{\sqrt{2/3}\,\pi}\right)^{\sqrt{d}} .$$

   See [An, page 70]. Here the asymptotics of a_d comes from the abundance of summands on the r.h.s. of (6.2), not from their single contributions.

2) The case of polynomially many primitive elements per degree $p_d = d^i$, $i \in [0, \infty)$ fixed.

   I guess (without having a proof) that the dominating asymptotical contribution comes from partitions of D, where the number of parts equal to d is d^i, $1 \leq d \leq d_0(D)$ for some $d_0(D) \in \mathbf{N}$. This contribution is between $C_{1,2}^{D^{(i+1)/(i+2)}}$ for two constants $C_{1,2} > 1$, which can be chosen arbitrarily close to each other. This is the maximal contribution from all partitions with d^p parts d, where p varies over $[0, \infty)$.

   Note, that already in the case of polynomial behaviour the question of asymptotics of a_d is basically equal to the one of finding the dominating partition in (6.2), since even multiplication by $p(d)$ (and adding $\sqrt{d}$ to the exponent) doesn't give any significant improvement anymore.

3) Asymptotically $p_d > d^i$ for all $i \in \mathbf{N}$. By $2)$ you obtain $a_D \geq e^{D^{1-\varepsilon}}$ for each $\varepsilon > 0$.

So why was that a good ansatz? Here is one justification: For the sequence of dominating partitions either the number of parts equal to d is bounded for all d, or grows beyond any limit for all d as $D \to \infty$ (unless $p_d = 1$ or $p_d = 0$), so that the ratio between the number k_{d_1} of parts equal to d_1 and the number k_{d_2} of parts equal to d_2 converges to the ratio of $(p_{d_1} - 1)/d_1$ and $(p_{d_2} - 1)/d_2$.

In the case $p_d = 1$ we will always have in our sequence of dominating partitions $k_d = 0$ (unless all $p_d = 1$, which is not interesting), and in the case $p_d = 0$ we set $k_d = 0$.

Here is how fast this number can grow for $D \to \infty$. Let us assume that $p_2 > 1$. If not, replace the '2' by some other 'i' with $p_i > 1$.

Theorem 6.2 *In the sequence of dominating partitions the number $k_2(D)$ of parts equal to 2 satisfies for all D the relation*

$$\sum_{l=1}^{k_2(D)-1} \frac{(p_{2l} - 1) \cdot 2l}{\left(\frac{k_2(D)}{k_2(D)-l}\right)^{p_2 - 1} - 1} - 2l \leq D . \tag{6.4}$$

Proof. The last l factors in the expansion

$$\binom{p_d+k_d-1}{k_d} = \frac{p_d}{1}\cdot\frac{p_d+1}{2}\cdot\ldots\cdot\frac{p_d+k_d-1}{k_d}\,. \tag{6.5}$$

of the binomial coefficient $\binom{p_d+k_d-1}{k_d}$ in degree $d=2$ are

$$\left(1+\frac{p_2-1}{k_2+1-l}\right)\cdot\ldots\cdot\left(1+\frac{p_2-1}{k_2}\right)\,. \tag{6.6}$$

The factor coming to the right of (6.5) for $d=2l$ when augmenting k_{2l} by one is

$$1+\frac{p_{2l}-1}{k_{2l}+1}\,. \tag{6.7}$$

Then we must have $(6.6)>(6.7)$. Otherwise by removing the summands equal to 2 and taking one more summand equal to $2l$ we get a larger contribution. We have for all k_2 and $l<k_2$

$$\begin{aligned}(6.6) &\leq \exp\left((p_2-1)\left(\frac{1}{k_2+1-l}+\ldots+\frac{1}{k_2}\right)\right)\\ &\leq \exp\big((p_2-1)(\ln k_2-\ln(k_2-l))\big)\\ &= \left(\frac{k_2}{k_2-l}\right)^{p_2-1}.\end{aligned}$$

Thus

$$k_{2l}\geq\frac{p_{2l}-1}{\left(\frac{k_2}{k_2-l}\right)^{p_2-1}-1}-1$$

for all $l<k_2$. The assertion is immediate, since (k_i) is a partition of D. □

Corollary 6.1 *There exist constants C', C'' so that*

$$\sum_{l=1}^{C'\cdot k_2(D)} 2l\left(\frac{p_{2l}-1}{C''}-1\right)\leq D\,.$$

Very roughly, replacing the sum by an integral, you see that $k_2(D)$ must be bounded above (modulo constants) by something like $F^{-1}(D)$, where $F(l)=p_{2l}\cdot l^2$.

Definition 6.3 An asymptotic is an equivalence class of sequences of naturals modulo the equivalence $(a_i)\asymp(b_i)\iff\lim\limits_{i\to\infty}a_i/b_i=1$. The asymptotic $[(a_i)]$ is higher than the asymptotic $[(b_i)]$ if $\liminf_{i\to\infty}a_i/b_i>1$. This gives a partial ordering among all asymptotics.

Definition 6.4 *Denote by $dom_b(D)$ the highest asymptotic of contributions of a sequence of partitions in (6.2) with bounded number of parts d as $D\to\infty$.*

Note that $dom_b(D)$ is not defined as a sequence itself, only its asymptotical behaviour is determined. Furthermore note, that $dom_b(D)$ is a maximal element in a partial ordering, so it does not need to exist (not even by Zorn's lemma)! The following discussion is under the (naive) assumption that it does really exist.

Corollary 6.2 *Let $dom_b(D)$ grow less fast than any exponential in D. Denote here by $dom_b(D)$ a* special representant *of its asymptotic. Then*

$$dom(D) < \left(dom_b\left(\frac{D}{B(D)}\right)\right)^{B(D)} \tag{6.8}$$

for any sequence $B(D)$ with $B(D)\geq k_2(D)$.

Proof. Take a sequence of partitions of $D/B(D)$ with $k_d(D)/B(D)$ parts d, use the observations directly before theorem 6.2 and note that, expanding the binomial coefficients as in (6.5), the remaining $k_d(D) - k_d(D)/B(D)$ factors in each degree are lower than the first $k_d(D)/B(D)$. □

For example, if you assume the bound in 2) on 43 is the best one, you get by corrollary 6.2

$$dom(D) < C^{D^{\frac{i+1}{i+2}+\frac{1}{(i+1)(i+2)}}},$$

using $B(D) := D^{\frac{1}{i+1}}$, where $B(D)$ can be chosen from corollary 6.1 ignoring C', C''. For large i the additional term is small. If the bound is larger, we get less of an improvement. In 3) on 43 the additional term would be compensated by the choice of ε.

Let us come back to our original ansatz to fix the number of parts equal to d as $D \to \infty$. It is sufficient to consider only the asymptotic behaviour of the number of parts k_d equal to d as $d \to \infty$, not their sequence $(k_d)_{d=1}^{\infty}$ itself. More precizely, we have

Proposition 6.1 Take two sequences of partitions with k_d, k'_d parts equal to d where k_d, k'_d are constant in D. Let $k_d \asymp k'_d$ as $d \to \infty$. Then asymptotically as $D \to \infty$ we have

$$con_k^{1-\varepsilon} \leq con_{k'} \leq con_k^{1+\varepsilon}$$

for all $\varepsilon > 0$.

Proof. Use a similar argument to the one in the proof of corollary 6.2.

6.2 A lower bound for the number of all Vassiliev invariants

We will now follow the strategy in our ansatz and consider the case $p_d = d^{\log_{4+\varepsilon} d}$, which is the relevant bound for Vassiliev invariants. It is my reproduction of the result in [CD2, Appendix] with a small correction.

Theorem 6.5 *For the dimension a_d of the part in degree d of the commutative graded algebra of chord diagrams, we have asymptotically*

$$a_d \geq C^{d/(4+\varepsilon)^{\sqrt{\log_4 d}}}$$

for each constant $C > 1$ and each $\varepsilon > 0$.

Note that the variation of ε makes the choice of C unimportant.

Proof. Look at partitions into equal parts and vary their length: set (in the notation of [CD2, Appendix]) $n \cdot p_n := (C + \varepsilon') \cdot d$. You have as a lower bound the expression

$$f(n) := \left(\frac{n \cdot p_n}{d}\right)^{d/n}.$$

Then use the fact that for each $a > 1, \varepsilon' > 0$

$$\frac{d}{a^{\sqrt{\log_a(Cd)}}} \geq (1-\varepsilon')\frac{d}{a^{\sqrt{\log_a(d)}}} \tag{6.9}$$

and you see that for $a = 4 + \varepsilon$ by a reparametrization of ε you can transform the denominators in (6.9) to the one in the theorem. □

Remark 6.3 Theorem 6.5 was suggested to Chmutov and Duzhin by myself. It can be obtained from our ansatz, but the proof presented here is basically due to Chmutov and Duzhin and is much more elegant.

Note that the proof holds also for the unframed case, as $n > 1$.

Remark 6.4 By some technical calculation you can find that, by being able to vary C by an ε, no improvement by corollary 6.2 would be possible (if this were the dominating asymptotical contribution). E.g., take (using corollary 6.1) $B(D) := (4+\varepsilon)^{\sqrt{\log_{4+\varepsilon} D}}$. So, whatever the dominating contribution of constant number of parts is, any improvement due to considering an unbounded number of parts will be gobbled up by adding any $\varepsilon > 0$ to the base.

In particular, the ansatz of Chmutov and Duzhin can only be (if at all) negligably better than mine. More precisely, one can prove that, if

$$\text{r.h.s. of (6.8)} \ < \ dom_b(D)^{1+\varepsilon} \tag{6.10}$$

and

$$\frac{\log dom_b'(D)}{\log dom_b(D)}$$

is monotonously growing, then (6.10) is also true for dom_b' instead of dom_b.

To see this, take the logarithm on both sides of (6.10) for dom_b and dom_b'.

Remark 6.5 If you go to the bother of taking the derivate of f and find its maximal value, the lower bound for a_d is a little better, but only for fixed ε, in the quality of

$$C^{d\cdot\sqrt{\log_{4+\varepsilon} d}/(4+\varepsilon)^{\sqrt{\log_{4+\varepsilon} d}}} \tag{6.11}$$

for some constant $C > 1$.

My proof of theorem 6.5 works with $k_d := \varepsilon' \cdot d^{\log_{4+\varepsilon} d}$, letting $\varepsilon' \to 0$. The proof suggests that taking a partition with the number of parts d growing somewhat weaker than $\varepsilon' \cdot d^{\log_{4+\varepsilon} d}$ will produce a further improvement. Actually, you can obtain (6.11) by setting $k_d := d^{\log_{4+\varepsilon} d - C'}$ and a lot of highly technical arguments, which we preferred to omit here. In fact, it is a strong challenge to find (and prove!) the asymptotics of the number k_d of parts equal to d in our ansatz, producing the dominating contribution.

On the other hand, however, in the end such attempts wont give much, since taking more awkward asymptotics of the number of parts will make the expressions fairly unwielding (as you can see in remark 6.5) and whatever we try, the improvement will stay small – it turns out that we will never be able to remove that “almost” before the exponential, unless we manage to do the same already with the bound for the primitive invariants.

6.3 The exponential barrier

More precisely, this fact can be formulated as follows.

Theorem 6.6 *If p_d grows less fast than any exponential, then so does a_d.*

Here “grows weaker than any exponential” means that the sequence does not contain a subsequence admitting a lower exponential bound (so we do not need to restrict ourselves to monotone sequences).

Remark 6.6 Looking at the second equality in (6.3), this is just the statement that if the radius of convergence of $p(z)$ is 1, then so it is for $a(z)$. This might have been noticed or implicitly conceived already by Pólya in his celebrated paper [Po]. Such arguments are used in the asymptotical analysis of graphical trees. See, e.g., [HP]. Here we present a proof without use of Pólya theory.

Proof. Note, that we may assume (making if necessary p_d a little bigger) that $p_d = \hat{C}^{d/f(d)}$ with some constant $\hat{C} > 1$ and a monotonous f with $f(1) = 1$, $f(d) < d^{1-\varepsilon}$ for some $\varepsilon > 0$.

First we shall establish that for this p_d the sequence a_d is bounded *above* by an exponential in d. Use

$$\binom{n+k-1}{k} \ < \ n^k$$

on the r.h.s. of (6.2) and observe that the maximal contribution of a partition (=summand) therein is the one from the partition of d into d parts ‘1’, which is an exponential in d. The multiplication with the number of partitions $\sim C^{\sqrt{d}}$ does not change anything essential.

Now assume, that from the growth of p_d primitive elements you would also obtain an exponential *lower* bound $a_d > C'^d$ with a $C' > 1$. Consider now $p_d' := \hat{C}^{d/\sqrt{f(d)}}$ and note that p_d' grows faster than any power of p_d. For p_d' you would have by the above argument an asymptotical exponential upper bound for the corresponding a_d'. But now the contradiction follows from the following lemma.

Lemma 6.7 *If p_d primitive elements produce a_d total elements, then p_d^2 primitive elements produce $\geq \frac{a_d^2}{2}$ total elements. As a consequence, $p_d^{2^k}$ primitive elements produce $\geq a_d^{2^k-\varepsilon''}$ total elements for all $\varepsilon'' > 0$.*

Proof of lemma. Expand the binomial coefficients on the r.h.s. of (6.2) as a product of k_j factors as in (6.5). Now you see that replacing $p_d \mapsto p_d^2$ each factor at least squares itself. It only remains to apply the standard inequality

$$2\sum_{i\vdash d} a_i^2 \geq \left(\sum_{i\vdash d} a_i\right)^2,$$

with a_i being the contributions from the partitions of the degree. □

Of course, for the k we have to choose in the proof of theorem 6.6, when applying lemma 6.7 we have

$$\hat{C}^{d/\sqrt{f(d)}} \geq \left(\hat{C}^{d/f(d)}\right)^{2^k}$$

only for *almost* all d, say for $d \geq d_0$, but omitting factors of the first d_0 degrees divides the contribution by a term bounded above by a polynomial in d, e.g. you can take

$$d^{\sum_{i=1}^{d_0} \hat{C}^{i/f(i)}},$$

which can be compensated by choosing ε'' in the lemma a little bigger (since $d/f(d)$ grows faster than d^ε for some $\varepsilon > 0$), so the argument still works. □

We see that the exponential growth is a very strong “barrier”. If the growth of p_d is less than exponential, then there is a qualitative difference between the growth of the total dimension and the one of the primitive part.

And, once it is broken, the primitive elements become dominating in each degree, so the asymptotics of a_D and p_D is (up to a negligable factor) equal.

In view of all this the decisive question is

Question. Is the exponential asymptotics a lower or an upper bound for Vassiliev invariants?

Answering this question will surely be hard. We saw why for the lower bound it will be so – we are much further away from the exponential bound than theorem 6.5 suggests. On the other hand, for an upper bound the best we can offer at present is something like $D!/1.1^D$ [St6]. Thus also in this case hard work is in store for us . . .

7 The braid index and the growth of Vassiliev invariants

In this section, we use the new approach of braiding sequences to prove exponential upper bounds for the number of Vassiliev invariants on knots with bounded braid index and arborescent knots.

Diagrams refer henceforth to knot diagrams (and not to chord diagrams).

7.1 Braiding sequences

Recall the basic definitions in the context of braiding sequences from §1.7.

Definition 7.1 *For some odd $k \in \mathbf{Z}$, a k-braiding of a crossing p in a diagram D is a replacement of (a neighborhood of) p by the braid σ_1^k (see figure 6). A braiding sequence (associated to a numbered set P of crossings in a diagram D; all crossings by default) is a family of diagrams, parametrized by $|P|$ odd numbers $x_1, \dots, x_{|P|}$, each one indicating that at crossing number i an x_i-braiding is done.*

Any Vassiliev invariant v of degree at most k behaves on a braiding sequence as a polynomial of degree at most k in $x_1, \dots, x_{|P|}$ (see [St4] and [Tr]), and this polynomial is called braiding polynomial of v on this braiding sequence.

Let C be a class of knots and $v : C \longrightarrow \mathbf{Q}$ a map. Extend v to singular knots as described in section 1, equation (1.1). This extension is well-defined on those singular knots, all of whose resolutions result in knots from C.