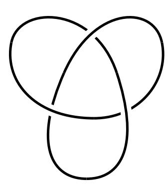
1 Vassiliev Invariants for knots

For many years the nature of knots has fascinated many people, especially mathematicians. Trying to rigorously understand and describe this phenomenon led to the development of knot theory as a branch of mathematics, more precisely of mathematical topology.

Initiated by VASSILIEV's considerations of the homology of knot spaces [Va], in the last years the theory of VASSILIEV invariants has become an interesting and intensively studied field of knot theory. It aims to approach the problem of topological classification of knots by using simpler and easier understandable algebraic and combinatorical methods. In this way this many-sided new field opens interesting relations to classical fields of mathematics, such as the homological algebra, the theory of Lie groups, graph theory and topological quantum field theories (TQFT's).

1.1 The classification problem of knots

Consider a knot, i. e. an oriented embedding of S^1 in $\mathbb{R}^3$. Let us assume, that this embedding is C^1, which particularly means tame, i. e. the knot is isotopic to a piecewise linear embedding. This does not constrain too much the complexity of knots, it only excludes some ugly pathological cases. The following picture shows the two simplest, but non-trivial knots.



trefoil

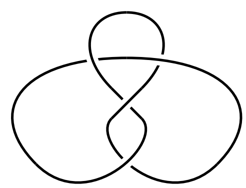


figure 8 knot

Thinking of a knot as of a real object, say a closed piece of rope, one gets an idea of transforming one knot into another by a sequence of pulling and twisting its strands, but not cutting the knot somewhere. One might consider knots as equal, if they are in this sense transformable into each other. This is mathematically described by the notion of (ambient) isotopy. So, in mathematical terms, knots are called topologically equal, if they are ambient isotopic, and to topologically classify a given knot means to determine uniquely its isotopy class.

Trying to classify knots led to the search for knot invariants, which, at least in some particular cases, can distinguish knots. Some approaches have been made to construct such invariants topologically. The most famous invariants of this kind are the polynomial invariants like the ALEXANDER/CONWAY [Al, Co, Ka3] and HOMFLY [H, LM] polynomials, and also CASSON's invariant [AM]. Unfortunately, these topological approaches often meet major difficulties, resulting from the analytical methods they involve.

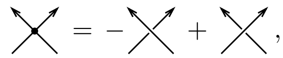
On this background the main advantage of the theory of VASSILIEV invariants appears to be, that it offers a new view of the topological structure of knots in a combinatorial and therefore more discrete and directly accessible way and makes it possible to apply simpler algebraic methods for exploring them.

1.2 The filtration of the knot space

Consider the linear space $\mathcal{V}$, (freely) generated by all the (isotopy classes of) knot embeddings. There is a self-suggesting way how to denote them. Every embedding class of a knot is determined by its projection on a 2-plane in $\mathbb{R}^3$, where all the crossings are transversal and equipped with the additional information, whether they are over- $(\times)$ or under- $(\times)$ crossings. Let $\mathcal{V}^p$ be the space of singular knots with exactly p double points $\times$ (up to isotopy).

Except for the last section, it is convenient to assume it to be a vector space over a field of characteristic zero, else it can be a module over any commutative ring with unit.

$\mathcal{V}^p$ can be identified with a linear subspace of $\mathcal{V}$ by resolving the singularities into the difference of an over- and an undercrossing via the rule

 (1.1)

where all the rest of the knot projections are assumed to be equal (one can show that the result does not depend on the order in which all the double points are resolved). This yields a filtration of $\mathcal{V}$

$$\mathcal{V} = \mathcal{V}^0 \supset \mathcal{V}^1 \supset \mathcal{V}^2 \supset \mathcal{V}^3 \supset \ldots \tag{1.2}$$

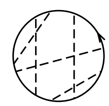
There exists a combinatorial description of the graded vector space,

$$\bigoplus_{i=0}^{\infty} \left(\mathcal{V}^i \Big/ \mathcal{V}^{i+1} \right) \tag{1.3}$$

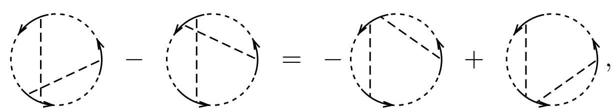
associated to this filtration, namely

$$\mathcal{V}^i \Big/ \mathcal{V}^{i+1} \;\simeq\; \mathcal{L}in\,\{ \text{ chord diagrams of degree } i \,\} \Big/ \begin{smallmatrix} 4T \text{ relation} \\ FI \text{ relation} \end{smallmatrix}\,, \tag{1.4}$$

where the chord diagrams (CD's) are objects like this (an oriented circle with finitely many dashed chords in it)

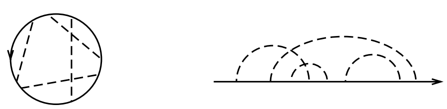


(up to isotopy) and are graded by the number of chords. The $4T$ (4 term) relations have the form

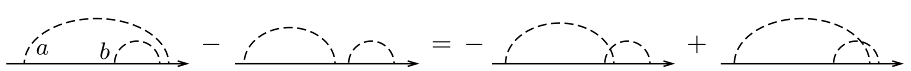


and the FI (framing independence) relation requires, that each chord diagram with an *isolated* chord (i. e., a chord not crossed by any of the others) is zero.

Chord diagrams appear in two forms – circular and linearized. In the latter case they are also called *linearized chord diagrams* or LCD's.



In the linearized form, which we will use henceforth, the 4T relation looks this way:



term 1 term 4 term 3 term 2 $(4T(a,b))$

(Note that the fixed end of chord a can also be to the right or within the chord b.) The meaning of the underlines '$-$' in the previous picture is the following: if in a picture an interval of the baseline is marked by a '$-$', we will not allow basepoints of other chords to end within it, whereas in the other case this can be.

The map which yields this isomorphism is given by the following simple way of how to assign a chord diagram D_K to a singular knot K. Connect in the parameter space of K (which is an oriented S^1) pairs of points with the same image by a chord. Actually, the idea to describe singular knots in this way led to the representation (1.4), and, more generally, many of the further algebraic statements are based on this idea. Once the chord diagrams have been introduced, the $4T$ and FI relations become self-suggesting. Look e. g. at the FI relation. If one takes a singular knot corresponding to a diagram with an isolated chord and resolves the singularity corresponding to this chord, one gets exactly an ambient isotopy relation of (singular) knots.

Note that (1.4) implies the finite dimensionality of the filtration (1.2), which considerably simplifies further algebraic considerations.

Definition 1.1 *A* VASSILIEV *invariant of degree m is an element $V \in \mathcal{V}^*$ (where $\mathcal{V}^*$ denotes henceforth the dual space of $\mathcal{V}$), s. t.*

$$V\Big|_{\mathcal{V}^{p+1}} \equiv 0 \quad \textit{and} \quad V\Big|_{\mathcal{V}^{p}} \not\equiv 0\,.$$

It is possible in some sense to consider (1.1) as a way to “differentiate” a knot, and in this language VASSILIEV invariant corresponds to a polynomial invariant, i. e. a function with a vanishing derivative.

This notion is interesting in connection with many other known knot invariants, such as the CONWAY and the HOMFLY polynomials, which have originally come about by topological considerations, but which fulfill certain relations closely connected to (1.1). E. g., the CONWAY polynomial $C_K(z)$ of a knot K satisfies the (skein) relation

$$C\left(\dot\times\right)(z) := C\left(\times_+\right)(z) - C\left(\times_-\right)(z) = z\,C\left(\,\right)\left(\,\right)(z)\,, \tag{1.5}$$

from which one concludes, that

Lemma 1.2 *The coefficient $K \mapsto [C_K(z)]_{z^k}$ of z^k in the Conway polynomial is a* VASSILIEV *invariant of degree at most k.*

By similar arguments the same is true for the HOMFLY polynomial in the reparametrization introduced by Jones [J] $F_K(N, e^z)$ (and therefore, setting $N = 2$, for the Jones polynomial [J2] as well) and the Kauffman polynomial [Ka2] in his version called by Kauffman the Dubrovnik polynomial [Ka]. See [BL].

In this way Vassiliev invariants generalize many known (topological) invariants, and that is another reason why Vassiliev invariants are so interesting.

Note, that the grading of the chord diagrams is preserved by $4T$, so the linear space $\mathcal{A}^r$ obtained by taking the direct sum over all i of the right hand side of (1.4) is a finite dimensionally graded space. Let G_m denote the degree-m-piece of $\mathcal{A}^r$.

By definition all VASSILIEV invariants of degree $\le m$ are sensitive with respect to knot classification only *maximally* up to $\mathcal{V}^{m+1}$. However, one does not know yet, whether *all* Vassiliev invariants are capable of a *complete* topological classification of knots.

Conjecture 1.3 VASSILIEV *invariants separate knots.*

By now conjecture 1.3 has been proved at least for pure braids [BN3, BN4] (see section 1.5), but for knots we know little about it. As it stands it sounds very appealing, but unfortunately, we cannot even yet affirm the following weaker

Question 1.1 *(see [BN2, sect. 7.2]) Do Vassiliev invariants distinguish knot orientation?*

This is one of the hardest problems in knot theory. Yet, there are no easily definable invariants, as quantum and skein invariants, which distinguish knot orientation. As pointed out by BIRMAN [Bi], the fact itself that non-invertible knots exist has been proved only in the 60’s by TROTTER [Tr]. In [St4] I tried to enlighten the problems with detecting orientation with Vassiliev invariants in an indepentent way from Bar-Natan’s computational arguments (see section 1.6).

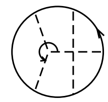
1.3 The Algebra $\mathcal{A}$

Let us for a moment forget about the FI relation and consider $\mathcal{A}$ as the space, obtained from $\mathcal{A}^r$ by not factoring out the FI relation.

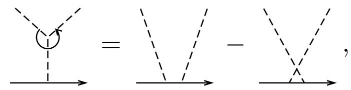
For this space $\mathcal{A}$ at least 3 other descriptions are known (for the proof, that they are all isomorphic, see [BN2]).

1) $$\mathcal{A} = \mathcal{L}in\left\{\begin{array}{c}\text{CCD's (Chinese character or}\\ \text{FEYNMAN diagrams)}\end{array}\right\}\Big/ STU \text{ relation},$$

where a Chinese character diagram is something like

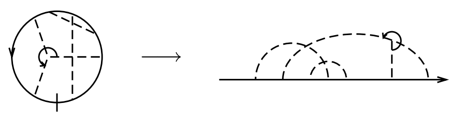


(like a chord diagram, but with oriented internal trivalent vertices allowed, which represent singular triple points) and the STU relation is



2) $$\mathcal{A} = \mathcal{L}in \, \{ \text{ linearized CCD's } \} \Big/ STU \text{ relation},$$

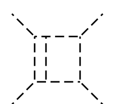
where a linearized Chinese character diagram is a Chinese character diagram, with the solid line cut somewhere



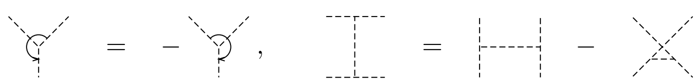
and both spaces are graded by half the number of trivalent vertices (internal or on the solid line) and

3) $$\mathcal{A} = \mathcal{L}in \, \{ \text{ CC's (Chinese characters) } \} \Big/ AS \text{ and } IHX \text{ relation},$$

where the Chinese characters are objects like this



(Chinese character diagrams with the solid line removed) and graded by half the number of vertices (univalent or trivalent), and the AS (antisymmetry) and the IHX relations are



Using the different representations of $\mathcal{A}$ one can define multi- and comultiplication, which make $\mathcal{A}$ into a graded commutative and cocommutative HOPF algebra [MM], including also a sort of ADAMS operations ψ^q, $q \in \mathbb{Z}$.

1.4 Weight systems

We have defined the Vassiliev invariants in terms of spaces of knot embeddings, but we found an easier description of these spaces by diagrams. Then is it possible to describe Vassiliev invariants by combinatorical objects defined entirely with the help of our diagram spaces?

Definition 1.4 *A weight system of degree m is an element $W \in (G_m\mathcal{A})^*$. Let $\mathcal{W}^m$ be the linear space of all weight systems of degree m.*

It is easy to assign to a Vassiliev invariant V of deg $\leq m$ a weight system W of degree m. Consider the (graded) map W_* given by

$$\mathcal{V}_m := (\mathcal{V}^m)^* \xrightarrow{W_m} (G_m\mathcal{A})^* =: \mathcal{W}^m \tag{1.6}$$

$$W_m(V)\,(\text{diagram of deg } m) \quad := \quad V\,\left(\begin{array}{c}\text{one singular knot with } m \text{ singula-}\\ \text{rities, that represents this diagram}\end{array}\right) \tag{1.7}$$

The kernel of this map is by definition $\mathcal{V}_{m-1} = (\mathcal{V}^{m-1})^*$, i. e. the space of Vassiliev invariants of deg $\leq m-1$, so we have the exact sequence

$$0 \longrightarrow \mathcal{V}_{m-1} \hookrightarrow \mathcal{V}_m \xrightarrow{W_m} \mathcal{W}_m \, .$$

Theorem 1.5 (Fundamental Theorem of Vassiliev invariants) *The sequence*

$$0 \longrightarrow \mathcal{V}_{m-1} \hookrightarrow \mathcal{V}_m \xrightarrow{W_m} \mathcal{W}_m \longrightarrow 0$$

is exact, and hence splits (over a field), i. e. there is a (graded) map V

$$V_m : \mathcal{W}_m \longrightarrow \mathcal{V}_m \, ,$$

such that

$$Im(\, Id_{\mathcal{V}_m} - V \circ W \,) \subset \mathcal{V}_{m-1} \quad \textit{and} \quad W_m \circ V_m = Id_{\mathcal{W}_m} \, .$$

Kontsevich's result is exactly the construction of the split homomorphism V for characteristic zero. In some sense the map V can be considered as "integrating" a weight system. If we want to do this, we need to consider another crucial object, closely related to this theorem. Namely, it turns out, that proving that theorem is equivalent to constructing a *universal* VASSILIEV *invariant*, i. e. a map

$$\mathcal{V}^* \xrightarrow{V'} \mathcal{A}^{**}_* \, ,$$

where $\mathcal{A}^{**}$ is the graded completion of $\mathcal{A}$, such that for every knot $K \in \mathcal{V}^m$ and any Vassiliev invariant $V \in \mathcal{V}_m$ there holds

$$\big(W(V)\big)\,(\,V'(K)\,) \quad = \quad V(K)\, .$$

Therefore, universal Vassiliev invariants play a basic role in the theory of Vassiliev invariants. Universal Vassiliev invariants have been constructed by many people an in different ways, e. g. [BN, Pi]. Though it is known, that the choice of a universal Vassiliev invariant is not unique, surprisingly many of the ansatzes seem to generate the same special one.

Summarizing, Vassiliev invariants form a commutative and co-commutative Hopf algebra $\mathcal{A}^*$ which is the dual algebra to the Hopf algebra $\mathcal{A}$ of *chord diagrams* (CD's) modulo the 4T (4 term) relation.

The primitive (as Hopf algebra elements) Vassiliev invariants $\mathcal{P}(\mathcal{A}^*)$ are the ones behaving additively under connected knot sum. Since co-commutative co-associative Hopf algebras over a field of characteristic zero are primitively generated [MM, corr. 4.18], the projection from $\mathcal{A}^*$ onto $\big(\mathcal{P}(\mathcal{A})\big)^*$ gives an isomorphism between $\big(\mathcal{P}(\mathcal{A})\big)^*$ and $\mathcal{P}(\mathcal{A}^*)$. $\mathcal{P} = \mathcal{P}(\mathcal{A})$ is the linear subspace generated by *connected* chord diagrams. $\mathcal{A}$ and $\mathcal{A}^*$ are finite-dimensionally graded by the number of chords (which we will denote by D).

The degree-1-piece $I := G_1\mathcal{A}$ of $\mathcal{A}$ is one-dimensional and the algebra $\mathcal{A}^r := \mathcal{A}/I\mathcal{A}$ has as dual the *framing-independent* Vassiliev invariants (i. e., exactly the ideal generated by FI relations).

Numerically little is known about $\mathcal{A}$ and $\mathcal{A}^r$. The dimension of their graded pieces $G_D\mathcal{A}$ and $G_D\mathcal{A}^r$ is known up to degree $D \leq 9$ [BN2][1]. In general one knows some asymptotical bounds for $\dim G_D\mathcal{A}^r$. It was recently found [CD2] that $\dim G_D\mathcal{P}$ grows faster than any polynomial in D. The best upper bound known up to now is the one obtained by NG [Ng, corollary 4.3][2] : $(D-2)!/2$ for $D > 5$ (see also Ng and STANFORD [NS]), who improved the bound $(D-1)!$ by CHMUTOV and DUZHIN [CD].

In section 5 we're going to prove the upper bound $D!$/any given polynomial in D for both $G_D\mathcal{A}$ and $G_D\mathcal{A}^r$ and later improve it to $D!/1.1^D$.

1.5 VASSILIEV invariants for braids and string links

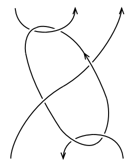
In the same way as for knots, the idea of constructing VASSILIEV invariants can be generalized to other "knot-like objects", i. e. certain classes of 1-dimensional embeddings into $\mathbb{R}^3$, factorized by an appropriate notion of isotopy, such as tangles, (pure) braids and string links. The resulting diagram spaces $\mathbf{AT}$ (for tangles), $\mathbf{AB}$ (for braids),

[1] Some more computational information on these algebras and their generalizations to string links [BN4] and braids [BN3] can be found in [BN7].

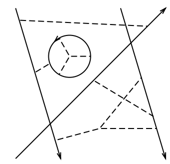
[2] Ng proved the bound only for the non-framed case, however, with a minor modification the proof for a somewhat worse but asymptotically equal bound also works for the framed case.

$\mathbf{AP} = \mathcal{A}^{sl}$ (for string links) and $\mathcal{A}^{pb} = \mathbf{AP}^{hor}$ (for pure braids) are constructed in the same way as for knots and have a similar algebraic structure (here the multiplication is given by stacking up) and properties as the knot diagram algebra $\mathcal{A}$. Here are some typical examples for the several embedding classes and for diagrams, corresponding to these classes.

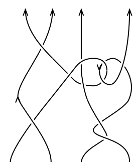
A tangle is an oriented embedding $T : [0,1] \times \mathbb{Z}_n \cup S^1 \times \mathbb{Z}_m \hookrightarrow [0,1] \times \mathbb{R}^2$, such that $T(\{0,1\} \times \mathbb{Z}_n) \subset \{0,1\} \times \mathbb{R}^2$.



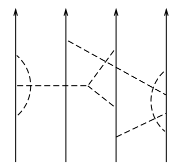
Here is a singular tangle diagram.



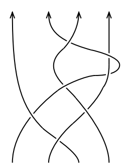
A string link is an oriented embedding $S : [0,1] \times \mathbb{Z}_n \hookrightarrow [0,1] \times \mathbb{R}^2$, such that $S(\{i\} \times \mathbb{Z}_n) \subset \{i\} \times \mathbb{R}^2$ for $i = 0, 1$.



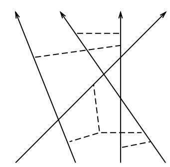
The diagram of a string link contains no cyclic full lines, and all strands are vertical and point to the top.



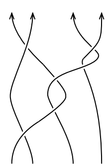
Here is a braid. It is a string link with monotonous 1st component.



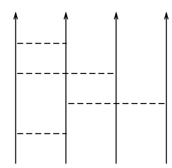
The diagram of a braid contains no chords ending on one strand only.



A pure braid is a braid, which preserves the order of the strands.



The diagram of a pure braid can be simplified to a diagram only with horizontal chords.



$\mathcal{A}^{sl}$ is known to have also a formulation as a space of coloured Chinese characters, i. e. Chinese characters with coloured univalent vertices [BN4].

1.6 Constructing a universal VASSILIEV invariant

There are at least 4 different approaches to this task: a naive topological approach (see [BS]), which fails but comes close, KONTSEVICH's integral formula [Ko], a physical and an algebraic approach. Originally due to DRINFEL'D [Dr] and elucidated further by BAR-NATAN [BN], latter is the probably most elegant solution. (See also [K]).

1.7 Braiding sequences

The combinatorial structure of chord diagrams and Chinese character diagrams considerably simplified our understanding of Vassiliev invariants and was the main tool in the proof of a series of results [BG, Vo]. Despite being therefore much celebrated, this approach has some serious defects. Although many ways exist to prove the Fundamental theorem [BS], they are all rather complicated and at some point unnatural, and their connections are not yet completely understood. So the integration of the (series of) weight system(s) to a Vassiliev invariant is far from being routine work.

But even for itself, although simpler and much friendlier to work with, the combinatorial structure of chord diagrams is far from being easily understandable [BN7].

In an attempt to create an alternative to the (defects of the) classical approach and generalizing some ideas of Dean [De], Trapp [Tr] and Stanford [Sa], in [St4] I introduced the notion of a braiding sequence. It offered a simple direct understanding of the behaviour of Vassiliev invariants on special knot classes, something, which was never worked out using the classical approach. Beside some other facts, it gave relatively simple proofs that the dimension of the space of Vassiliev invariants of degree $\leq n$ on certain knot classes is finite (arborescent knots), and in some cases even exponential upper bounds in n for this dimension (e. g., rational knots, closed 3 braids), something, which was not yet achieved by chord diagrams.

Moreover, while the Kontsevich-Drinfel'd approach (used in [CD]) works only over zero (field) characteristic, our arguments with braiding sequences hold for any zero divisor free ring, in particular the fields $\mathbb{Z}_p$, p prime.

Definition 1.6 *For some odd $k \in \mathbb{Z}$, a k-braiding of a crossing p in a diagram D is a replacement of (a neighborhood of) p by the braid σ_1^k (see figure 1). A braiding sequence (associated to a numbered set P of crossings in a diagram D; all crossings by default) is a family of diagrams, parametrized by $|P|$ odd numbers $x_1, \dots, x_{|P|}$, each one indicating that at crossing number i an x_i-braiding is done.*

Any Vassiliev invariant v of degree at most k behaves on a braiding sequence as a polynomial of degree at most k in $x_1, \dots, x_{|P|}$ (see [St4] and [Tr]), and this polynomial is called the braiding polynomial of v on this braiding sequence.

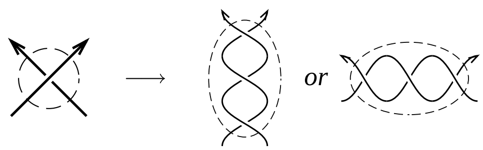


Figure 1. Two ways to do a -3-braiding at a crossing.

2 The results of this thesis

The subject of the present thesis are combinatorics of chord diagrams and asymptotics of Vassiliev invariants.

In sections 3 and 4 we will derive some (purely) enumerative results on special kinds of chord diagrams. Although not directly related to Vassiliev invariants, these results provide a glimpse of the combinatorial complexity of chord diagrams – already for chord diagrams with properties, which are easy to define, the enumeration is rather hard and requires additional ideas.

We show consecutively how to count in a non-brute force way all chord diagrams of given degree, all chord diagrams up to mirroring, all chord diagrams with an isolated chord (the ones sent to zero by the FI relation), all chord diagrams with an (isolated) chord of length one, chord diagrams, whose intersection graph is connected and those for which it is a tree.

In section 5 we will use combinatorial techniques to relate the enumeration of special chord diagrams to the enumeration of Vassiliev invariants and will prove the asymptotical upper bound $D!/1.1^D$ for the number of Vassiliev invariants in degree D.

The basic idea for this improvement is to work with linearized chord diagrams (LCD's) and the order of chord basepoints from left to right.

In section 6 we will use the techniques of section 5 and the result of Chmutov and Duzhin [CD2] to deduce a lower bound for the number of all Vassiliev invariants and discuss the relation between the asymptotics of primitive and all Vassiliev invariants. At the same time, we give a summary on what we know about the asymptotics of Vassiliev invariants.

Finally, in section 7 we use the rather different approach of braiding sequences to prove exponential upper bounds for the number of Vassiliev invariants on knots with bounded braid index and arborescent knots.

Parts of this work can be found in several papers of mine [St2, St6, St8, St9, St10].

Acknowledgement. I would wish to thank to T. Fiedler, S. Duzhin, S. Chmutov, E. Vogt and Y. Rong for their helpful remarks.