

Quantum Speed Limit for Non-Markovian Dynamics

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We derive a Margolus-Levitin-type bound on the minimal evolution time of an arbitrarily driven open quantum system. We express this quantum speed limit time in terms of the operator norm of the nonunitary generator of the dynamics. We apply these results to the damped Jaynes-Cummings model and demonstrate that the corresponding bound is tight. We further show that non-Markovian effects can speed up quantum evolution and therefore lead to a smaller quantum speed limit time.

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What is the maximal speed of evolution of a quantum system? This question is of fundamental importance in virtually all areas of quantum physics, where the determination of the minimal duration of a process is of interest. The latter include quantum communication [1], computation [2], and metrology [3], as well as nonequilibrium thermodynamics [4] and optimal control theory [5]. The maximal rate of evolution may be characterized by the quantum speed limit time defined as the minimal time a system needs to evolve between two states separated by a given angle [6–11]. For closed systems, and orthogonal states, the latter can be obtained by combining the results of Mandelstam-Tamm (MT) [12] and Margolus-Levitin (ML) [13], and reads $\tau_{\text{QSL}} = \max\{\pi\hbar/(2\Delta E), \pi\hbar/(2E)\}$. The MT bound depends on the variance ΔE of the energy of the initial state while the ML bound on its mean energy E with respect to the ground state; the quantum speed limit time is thus given by the ML expression whenever $E \leq \Delta E$. The fact that τ_{QSL} is determined by the initial energy of the system reflects the fact that the Hamiltonian is the generator of unitary Schrödinger dynamics. It is worth emphasizing that the existence of a speed limit time is a purely quantum effect which vanishes when \hbar goes to zero. Generalizations of the MT and ML findings to nonorthogonal states and to driven systems have been provided in Refs. [14–19]. Since all systems are unavoidably connected to their surroundings, a determination of the speed limit time for open quantum systems appears necessary; the latter indeed plays a key role in the analysis of environmental effects on, e.g., the maximal rate of information transfer [1], of gate operations [2], and of entropy production [4], as well as on the efficiency of optimal control algorithms [5]. Recently, the quantum speed limit time has been derived for open systems described by positive nonunitary maps and applications to dephasing in noisy channels and quantum parameter estimation have been discussed [20,21]. In both approaches, the speed limit time was obtained in terms of the variance of the generator

of the evolution, which reduces to the MT expression in the case of closed system dynamics. To our knowledge, no ML type bound has been proposed for open quantum systems to date.

In this Letter, we use a geometric approach to derive a quantum speed limit time valid for open system dynamics with possibly time-dependent nonunitary generators. In contrast to previous studies, we obtain a bound that depends on the mean of the generator; it therefore reduces to the ML formula for unitary processes. MT type bounds are usually derived with the help of the Cauchy-Schwarz inequality. The latter invariably leads to expressions containing the variance of the generator, and is hence not suitable for getting ML type bounds. We here solve this technical challenge by making use of the von Neumann trace equality [22–24]. In the following, we obtain a quantum speed limit time that depends on the operator norm of the nonunitary generator and show that the so obtained ML bound is not only sharper than the MT bound, it is also tight. By employing both inequalities, we are able to obtain a unified quantum speed limit time for generic positive open system dynamics. We further apply these results to investigate the influence of non-Markovianity on the rate of quantum evolution. Non-Markovian (or memory) effects become important when the relaxation time of the system is comparable to the relaxation time of the environment [25,26]. They have been shown to play a central role in the creation of steady state entanglement [27], in the description of quantum coherence in photosynthetic systems [28], in quantum metrology [29], and optimal control theory [30]. Recent experiments with photons in a controllable non-Markovian environment have been reported in Refs. [31,32]. Interestingly, we will show that non-Markovian dynamics can lead to smaller quantum speed limit times.

Geometric approach.—We consider a possibly driven open quantum system and ask for the minimal time that is necessary for it to evolve from an initial state ρ_0 to a final

state ρ_τ . Without loss of generality, we assume that the initial state is pure, $\rho_0 = |\psi_0\rangle\langle\psi_0|$. The case of an initially mixed state can be treated by purification in a sufficiently enlarged Hilbert space [33]. Note that under non-unitary dynamics, the final state ρ_τ will be generally mixed. The basis of our geometric approach is provided by the Bures angle $\mathcal{L}(\rho_0, \rho_\tau)$ between initial and final states of the quantum system [33,34],

$$\mathcal{L}(\rho_0, \rho_\tau) = \arccos\left(\sqrt{\langle\psi_0|\rho_\tau|\psi_0\rangle}\right). \quad (1)$$

The Bures angle is a generalization to mixed states of the angle in Hilbert space between two state vectors [35].

To evaluate the quantum speed limit time, we consider the dynamical velocity with which the density operator of the system evolves [8]. The latter is given by the time derivative of the geometric Bures angle (1),

$$\begin{aligned} \frac{d}{dt} \mathcal{L}(\rho_0, \rho_t) &\leq \left| \frac{d}{dt} \mathcal{L}(\rho_0, \rho_t) \right| \\ &= \frac{1}{\sqrt{1 - \langle\psi_0|\rho_\tau|\psi_0\rangle}} \frac{|\langle\psi_0|\dot{\rho}_t|\psi_0\rangle|}{2\sqrt{\langle\psi_0|\rho_\tau|\psi_0\rangle}}. \end{aligned} \quad (2)$$

Using the definition (1), Eq. (2) can be written as

$$2 \cos(\mathcal{L}) \sin(\mathcal{L}) \dot{\mathcal{L}} \leq |\langle\psi_0|\dot{\rho}_t|\psi_0\rangle|. \quad (3)$$

Expression (3) will serve as the starting point for our unified derivation of ML and MT type bounds on the rate of quantum evolution, using, respectively, the von Neumann trace inequality and the Cauchy-Schwarz inequality.

Margolus-Levitin bound.—To illustrate the use of the von Neumann trace inequality, we begin by providing a derivation of the ML bound in the case of driven unitary dynamics. The generator is here given by the time-dependent Hamiltonian H_t of the system, and its density operator obeys the von Neumann equation,

$$\dot{\rho}_t = \frac{1}{i\hbar} [H_t, \rho_t]. \quad (4)$$

Substituting Eq. (4) into Eq. (3), we have

$$2 \cos(\mathcal{L}) \sin(\mathcal{L}) \dot{\mathcal{L}} \leq \frac{1}{\hbar} |\langle\psi_0|[H_t, \rho_t]|\psi_0\rangle|. \quad (5)$$

Equation (5) can be estimated from above with the help of the triangle inequality and we obtain

$$2 \cos(\mathcal{L}) \sin(\mathcal{L}) \dot{\mathcal{L}} \leq \frac{1}{\hbar} (|\text{tr}\{H_t \rho_t \rho_0\}| + |\text{tr}\{\rho_t H_t \rho_0\}|). \quad (6)$$

To proceed, we introduce the von Neumann trace inequality for operators which reads [22–24],

$$|\text{tr}\{A_1 A_2\}| \leq \sum_{i=1}^n \sigma_{1,i} \sigma_{2,i}. \quad (7)$$

Inequality (7) holds for any complex $n \times n$ matrices A_1 and A_2 with descending singular values, $\sigma_{1,1} \geq \dots \geq \sigma_{1,n}$

and $\sigma_{2,1} \geq \dots \geq \sigma_{2,n}$. The singular values of an operator A are defined as the eigenvalues of $\sqrt{A^\dagger A}$ [36]. For a Hermitian operator, they are given by the absolute value of the eigenvalues of A , and are positive real numbers. If A_1 and A_2 are simple (positive) functions of density operators acting on the same Hilbert space, Eq. (7) remains true for arbitrary dimensions [24]. The singular values of the operators A and A^\dagger are moreover identical. By taking $A = (H_t \rho_t)^\dagger = \rho_t^\dagger H_t^\dagger = \rho_t H_t$, and combining Eqs. (6) and (7) we thus find

$$2 \cos(\mathcal{L}) \sin(\mathcal{L}) \dot{\mathcal{L}} \leq \frac{2}{\hbar} \sum_i \sigma_i p_i = \frac{2}{\hbar} \sigma_1, \quad (8)$$

where σ_i are the singular values of $H_t \rho_t$ and $p_i = \delta_{i,1}$ those of the initial pure state ρ_0 . For a Hermitian operator, the operator norm is given by the largest singular value $\|A\|_{\text{op}} = \sigma_1$, while the trace norm is equal to their sum $\|A\|_{\text{tr}} = \sum_i \sigma_i$ [36]. As a consequence,

$$2 \cos(\mathcal{L}) \sin(\mathcal{L}) \dot{\mathcal{L}} \leq \frac{2}{\hbar} \|(H_t \rho_t)\|_{\text{op}} \leq \frac{2}{\hbar} \|(H_t \rho_t)\|_{\text{tr}}. \quad (9)$$

We note, in addition, that the trace norm is given by

$$\|(H_t \rho_t)\|_{\text{tr}} = \text{tr}\{|H_t \rho_t|\} = \langle H_t \rangle, \quad (10)$$

when the instantaneous eigenvalues of H_t are all positive (the latter can always be realized by properly choosing the zero of energy [13]). Integrating Eq. (9) over time from $t = 0$ to $t = \tau$, we arrive at the inequality

$$\tau \geq \frac{\hbar}{2E_\tau} \sin^2[\mathcal{L}(\rho, \rho_\tau)], \quad (11)$$

with the time-averaged energy $E_\tau = (1/\tau) \int_0^\tau dt \langle H_t \rangle$. Equation (11) is the ML bound for driven closed systems.

The above derivation can be easily extended to arbitrary time-dependent nonunitary equations of the form

$$\dot{\rho}_t = L_t(\rho_t), \quad (12)$$

with positive generator L_t [37]. The latter are trace class (super-)operators in a complex Banach space and need generally not be Hermitian [26]. However, for symmetric norms, such as the Schatten p norm $\|L_t\|_p = [\sum_i \lambda_i^p]^{1/p}$, that we here consider [38], $\|L_t^\dagger\| = \|L_t\|$. As a result, all previously used definitions and inequalities remain valid [39]. Substituting Eq. (12) into Eq. (3) we then have

$$2 \cos(\mathcal{L}) \sin(\mathcal{L}) \dot{\mathcal{L}} \leq |\langle\psi_0|L_t(\rho_t)|\psi_0\rangle|, \quad (13)$$

which is the nonunitary generalization of Eq. (5). Noting that $\langle\psi_0|L_t(\rho_t)|\psi_0\rangle = \text{tr}\{L_t(\rho_t)\rho_0\}$ and employing the von Neumann trace inequality (7), we obtain

$$2 \cos(\mathcal{L}) \sin(\mathcal{L}) \dot{\mathcal{L}} \leq \sum_i \lambda_i p_i = \lambda_1, \quad (14)$$

where λ_i are the singular values of the operator $L_t(\rho_t)$. Equation (14) can again be estimated from above by the operator norm and the trace norm to yield

$$2 \cos(\mathcal{L}) \sin(\mathcal{L}) \dot{\mathcal{L}} \leq \|(L_t(\rho_t))\|_{\text{op}} \leq \|[L_t(\rho_t)]\|_{\text{tr}}. \quad (15)$$

Integrating Eq. (15) over time, we eventually find

$$\tau \geq \max\left\{\frac{1}{\Lambda_{\tau}^{\text{op}}}, \frac{1}{\Lambda_{\tau}^{\text{tr}}}\right\} \sin^2[\mathcal{L}(\rho, \rho_{\tau})], \quad (16)$$

where we have defined $\Lambda_{\tau}^{\text{op, tr}} = (1/\tau) \int_0^{\tau} dt \|(L_t(\rho_t))\|_{\text{op, tr}}$. Equation (16) provides a ML type bound on the rate of quantum evolution valid for arbitrary positive driven open system dynamics.

Mandelstam-Tamm bound.—We next derive a unified bound for the quantum speed limit time for open systems by generalizing the method presented in Ref. [21] based on the relative purity. To this end, we rewrite Eq. (3) as

$$2 \cos(\mathcal{L}) \sin(\mathcal{L}) \dot{\mathcal{L}} \leq |\text{tr}\{L_t(\rho_t)\rho_0\}|. \quad (17)$$

The latter can be estimated from above with the help of the Cauchy-Schwarz inequality for operators:

$$2 \cos(\mathcal{L}) \sin(\mathcal{L}) \dot{\mathcal{L}} \leq \sqrt{\text{tr}\{L_t(\rho_t)L_t(\rho_t)^{\dagger}\}\text{tr}\{\rho_0^2\}}. \quad (18)$$

Since ρ_0 is a pure state, $\text{tr}\{\rho_0^2\} = 1$, and we obtain

$$2 \cos(\mathcal{L}) \sin(\mathcal{L}) \dot{\mathcal{L}} \leq \sqrt{\text{tr}\{L_t(\rho_t)L_t(\rho_t)^{\dagger}\}} = \|[L_t(\rho_t)]\|_{\text{hs}}, \quad (19)$$

where $\|A\|_{\text{hs}} = \sqrt{\text{tr}\{A^{\dagger}A\}} = \sqrt{\sum_i \sigma_i^2}$ is the Hilbert-Schmidt norm [36]. Integrating Eq. (11) over time leads to the following MT type bound for nonunitary dynamics,

$$\tau \geq \frac{1}{\Lambda_{\tau}^{\text{hs}}} \sin^2[\mathcal{L}(\rho, \rho_{\tau})], \quad (20)$$

where $\Lambda_{\tau}^{\text{hs}} = (1/\tau) \int_0^{\tau} dt \|[L_t(\rho_t)]\|_{\text{hs}}$. For unitary processes, $\Lambda_{\tau}^{\text{hs}}$ is equal to the time-averaged variance of the energy. For initially pure states, relative purity and fidelity are identical, and the bound (20) thus reduces to the one derived in Ref. [21], since $\sin^2 x \geq |\cos x - 1|$, $0 \leq x \leq \pi/2$; they are, however, different for initially mixed states. Combining Eqs. (16) and (20), we obtain

$$\tau_{\text{QSL}} = \max\left\{\frac{1}{\Lambda_{\tau}^{\text{op}}}, \frac{1}{\Lambda_{\tau}^{\text{tr}}}, \frac{1}{\Lambda_{\tau}^{\text{hs}}}\right\} \sin^2[\mathcal{L}(\rho, \rho_{\tau})]. \quad (21)$$

Equation (21) provides a unified expression for the quantum speed limit time for generic (positive) open system dynamics. It represents a general extension of the MT and ML result, $\tau_{\text{QSL}} = \max\{\pi\hbar/(2\Delta E), \pi\hbar/(2E)\}$.

We may go a step further by noting that for trace class operators the following inequality holds (see Ref. [39], Theorem 1.16),

$$\|A\|_{\text{op}} \leq \|A\|_{\text{hs}} \leq \|A\|_{\text{tr}}. \quad (22)$$

As a result, $1/\Lambda_{\tau}^{\text{op}} \geq 1/\Lambda_{\tau}^{\text{hs}} \geq 1/\Lambda_{\tau}^{\text{tr}}$ and we can therefore conclude that the ML-type bound based on the operator norm of the nonunitary generator provides the sharpest

bound on the quantum speed limit time. We will show below that the bound can be attained and is hence tight.

Non-Markovian effects.—We may use the above results to investigate the influence of non-Markovian dynamics on the quantum speed limit time. To this end, we consider the exactly solvable damped Jaynes-Cummings model for a two-level system resonantly coupled to a leaky single mode cavity [40,41]; the environment is supposed to be initially in a vacuum state. The non-Markovian properties of the damped Jaynes-Cummings model have recently been studied experimentally in a solid-state cavity QED system [42]. The nonunitary generator of the reduced dynamics of the system is

$$L_t(\rho_t) = \gamma_t \left(\sigma_- \rho_t \sigma_+ - \frac{1}{2} \sigma_+ \sigma_- \rho_t - \frac{1}{2} \rho_t \sigma_+ \sigma_- \right), \quad (23)$$

where $\sigma_{\pm} = \sigma_x \pm i\sigma_y$ are the Pauli operators and γ_t the time-dependent decay rate. By assuming that there is only one excitation in the combined atom-cavity system, the environment can be described by an effective Lorentzian spectral density of the form

$$J(\omega) = \frac{1}{2\pi} \frac{\gamma_0 \lambda}{(\omega_0 - \omega)^2 + \lambda^2}, \quad (24)$$

where ω_0 denotes the frequency of the two-level system, λ the spectral width, and γ_0 the coupling strength. The time-dependent decay rate is then explicitly given by

$$\gamma_t = \frac{2\gamma_0 \lambda \sinh(dt/2)}{d \cosh(dt/2) + \lambda \sinh(dt/2)}, \quad (25)$$

where $d = \sqrt{\lambda^2 - 2\gamma_0 \lambda}$. In the interaction picture, the reduced density operator of the system at time t reads

$$\rho_t = \begin{pmatrix} \rho_{11}(0) |e^{-\Gamma_t}| & \rho_{10}(0) e^{-\Gamma_t/2} \\ \rho_{10}^*(0) e^{-\Gamma_t^*/2} & 1 - \rho_{11}(0) |e^{-\Gamma_t}| \end{pmatrix}, \quad (26)$$

where $\Gamma_t = \int_0^t dt' \gamma_{t'}$. We shall examine the case where the system starts in the excited state, $\rho_{11}(0) = 1$ and $\rho_{10}(0) = 0$. For vanishing coupling, the system is isolated and in a stationary state. For finite coupling, the two-level system is driven by the bath. The correlation time of the bath is $\tau_B = \lambda^{-1}$, while the decay time of the system is equal to $\tau_S = \gamma_0^{-1}$. The non-Markovian properties of the model have been investigated in Refs. [43–47]: The dynamics is Markovian in the weak-coupling regime, $\gamma_0 < \lambda/2$. For large time scale separation $\tau_B \ll \tau_S$, or, equivalently, $\gamma_0 \ll \lambda$, the decay rate is constant, $\gamma_t = \gamma_0$. The dynamics is non-Markovian for strong coupling, $\gamma_0 > \lambda/2$, which corresponds to an imaginary parameter d . In this regime, the decay rate γ_t is an oscillatory function of time.

Figure 1 shows the quantum speed limit time (21) for the two-level system as a function of the coupling strength γ_0 , obtained for the three different norms, in the case $\tau = 1$. We can distinguish two different phases. The speed limit time exhibits a plateau independent of γ_0 for moderate

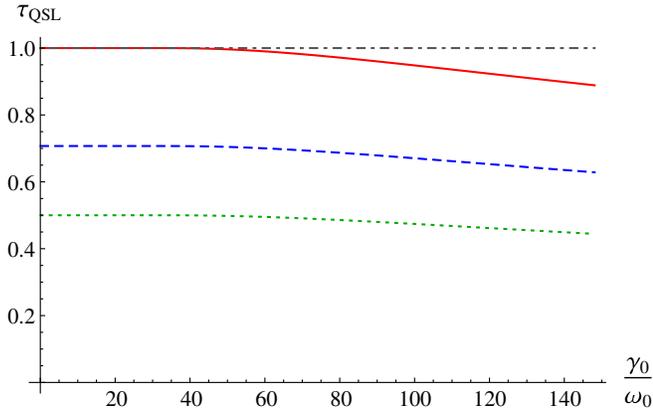


FIG. 1 (color online). Quantum speed limit time τ_{QSL} , Eq. (21), for the damped Jaynes-Cummings model as a function of the coupling strength γ_0 . The three curves are obtained for the trace norm (green dotted line), Hilbert-Schmidt norm (blue dashed line), and the operator norm (red solid) of the nonunitary generator $L(\rho_t)$, Eq. (23). The black (dash-dotted) line represents the actual driving time $\tau = 1$. Parameters are $\lambda = 50$ and $\omega_0 = 1$.

coupling and then decreases for large coupling amplitudes. Our second observation is that the ML bound based on the operator norm is sharper than the MT bound based on the Hilbert-Schmidt norm, in agreement with Eq. (22). It is also sharper than the ML bound based on the trace norm. Remarkably, the operator-norm bound is *tight* as it reaches the actual driving time τ over a large range of coupling strengths.

The above behavior can be explained by evaluating the singular values of the operator $\hat{\rho}_\tau$ in the strong Markovian limit $\gamma_\tau = \gamma_0$. The two values are equal and given by $|\hat{\rho}_{11}| = |\hat{\rho}_{00}| = |\gamma_\tau \exp(-\int_0^\tau dt \gamma_t)| = \gamma_0 \exp(-\gamma_0 \tau)$. For small coupling $\gamma_0 \tau \ll 1$, the singular values are thus proportional to the coupling strength $|\hat{\rho}_{11}| \simeq \gamma_0$. For larger coupling such that $\gamma_0 \tau \gg 1$, the singular values are independent of γ_0 , $|\hat{\rho}_{11}| \simeq 0$. The plateau seen in Fig. 1 is hence a signature of Markovian dynamics and follows from the time independence of the decay rate. The height of the plateau can be determined by computing the time averaged norm of $\hat{\rho}_t$ plotted in Fig. 2;

$$\frac{1}{\tau} \int_0^\tau dt \|\hat{\rho}_t\| = \frac{n}{\tau} [1 - \exp(-\gamma_0 \tau)], \quad (27)$$

with $n = 1, \sqrt{2}$, and 2 for the operator, Hilbert-Schmidt, and trace norms, respectively. The constant n is equal to $2^{1/p}$ for the general Schatten p norm. Equation (27) hence shows that the operator norm ($p = \infty$) is the only p norm for which the plateau reaches the actual driving time τ . Furthermore, the increase of the norm of the rate $\hat{\rho}_t$ in the strong coupling regime, $\gamma_0 > \lambda/2$ appears as a consequence of the (oscillatory) time dependence of the decay rate γ_t , and is thus a purely non-Markovian effect. We therefore reach the interesting conclusion that non-Markovian dynamics can increase the rate of evolution

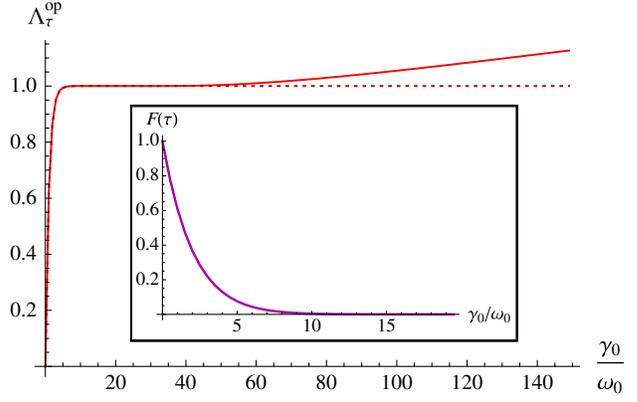


FIG. 2 (color online). Time-averaged operator norm of the nonunitary generator $L(\rho_t)$ of the damped Jaynes-Cummings model (23) (continuous line). The dotted line corresponds to the Markovian result, Eq. (27). The inset shows the fidelity $F(\rho_0, \rho_\tau) = \cos(\mathcal{L}(\rho_0, \rho_\tau))$. Same parameters as in Fig. 1.

of a quantum system, and thus reduce the quantum speed limit time below its Markovian value.

Discussion.—Non-Markovian dynamics has been characterized by an increase of the distinguishability of quantum states quantified by their trace distance [44]. The time derivative, $\sigma_t = \partial_t \|\rho_t^1 - \rho_t^2\|_{\text{tr}}/2$, of the trace distance between states $\rho_t^{1,2}$ has been shown to be negative (decrease of distinguishability) for Markovian processes and positive (increase of distinguishability) for non-Markovian processes. The latter has been interpreted as an information backflow from the environment to the system [44–47]. For the damped Jaynes-Cummings model (23) with $\rho_0^1 = |1\rangle\langle 1|$ and $\rho_0^2 = |0\rangle\langle 0|$, $\sigma_t = -\gamma_t \exp(-\Gamma_t)$ [44,45,48] and the singular values of the nonunitary generator are thus given by $|\sigma_t|$. In the non-Markovian regime, $\gamma_t < 0$ ($\sigma_t > 0$), and the rate Γ_t is hence a decreasing function of time. As a result, σ_t grows with time through its exponential dependence on Γ_t . The decrease of the quantum speed limit (which is inversely proportional to the singular values) is therefore a consequence of the backflow of information to the system. The latter is observable provided that the prefactor $-\gamma_t$ does not decrease too rapidly. More precisely, the calculation of the time derivative of σ_t , $\dot{\sigma}_t = (\gamma_t^2 - \dot{\gamma}_t) \exp(-\Gamma_t)$, reveals that non-Markovian effects speed up quantum evolution $\dot{\sigma}_t > 0$, provided that $\dot{\gamma}_t < \gamma_t^2$, that is, if the rate of change of γ_t is smaller than its square. When this condition is not met, the dynamics of the system counters the effect of the information backflow and prevents the speed-up [49]. We emphasize that the above discussion is not limited to the damped Jaynes-Cummings model, but applies to a general class of two-level systems obeying a time-local master equation of the form, $\dot{\rho}_t = -(\gamma_t/2) \times (L^\dagger L \rho_t + \rho_t L^\dagger L - 2L \rho_t L^\dagger)$, with arbitrary Lindblad-type operator L [50]. The latter includes the important dephasing channel model ($L = \sigma_z$) which has been recently investigated experimentally [32].

Conclusions.—We have derived a quantum speed limit time that generalizes the familiar MT and ML results to generic time-dependent (positive) dynamics of open quantum systems. In particular, using the von Neumann trace inequality, we have obtained an expression of the speed limit time in terms of the operator norm of the nonunitary generator of the evolution. We have demonstrated that the latter bound is sharper than any bound based on a Schatten p -norm, such as the trace and Hilbert-Schmidt norms, and that it is moreover tight. Applying these results to the damped Jaynes-Cummings model has additionally shown that non-Markovian effects, and the associated information backflow from the environment, can lead to faster quantum evolution, and hence to smaller quantum speed limit times.

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