Chapter 8

Sectional category

8.1 Introduction

All of the topological spaces which we consider in this chapter are simply connected with homology of finite type over $\mathbb{Q}$.

Another LS-category-type invariant is the sectional category of a fibration. Since it can also be defined in terms of the existence of a homotopy section for a kind of Ganea space, its rationalization is likely to have a simple characterization in terms of the Sullivan model of the fibration, just as the relative LS-category had. We show that this is true in this chapter, and even more: that it is possible to define an invariant generalizing the sectional category as well as the R-category and still find a satisfactory characterization of its rational counterpart. Our main result in this section is therefore a generalization of part of theorem 5.4.1.

We begin section 8.2 by giving a definition in terms of coverings of the sectional category of a fibration and of the new invariant: the sectional category of a sequence of maps. Then we introduce the fat wedges and the Ganea maps associated to a sequence of maps and show that the classical definition of the new sectional category is equivalent to a definition in terms of the existence of homotopy sections for the Ganea maps and to a definition using the generalized fat wedges. We define the *rational sectional category* of a sequence of morphisms in section 8.3 and we state a theorem giving an equivalent definition using the Sullivan models of each of the morphisms involved. We prove the theorem in section 8.4: first of all we model the generalized fat wedge and the diagonal map in order to build a model Γ_m for the generalized m-th Ganea space. It is then possible to construct a morphism $\Gamma_m \to \mathfrak{F}_m$, where the target is described in terms of Sullivan models. To build a morphism in the opposite direction we must construct inductively another model $\mathfrak{H}_m$ for the Ganea space G_m.

8.2 Classical sectional category

Originally sectional category is defined for a fibration $p : E \to B$ in [Sch66]. We introduce the sectional category of $\mathfrak{E}$, where $\mathfrak{E}$ is a sequence of maps $\mathfrak{E} = (f_0, f_1, ...)$, with $f_i : E_i \to B$, that reduces to $\text{secat}(p)$ when $\mathfrak{E} = (p, p, p, ...)$ and to $\text{Rcat}(f : E \to B)$ when $\mathfrak{E} = (f, * \to B, * \to B, ...)$.

Definitions.

- Let $p : E \to B$ be a fibration, then the ***sectional category*** of p, noted $\text{secat}(p)$, is

m, where $(m+1)$ is the minimum number of open subsets of B over which p has a section, which are needed to cover B.

- Let $\mathfrak{E}$ be a sequence of maps $(f_0, f_1, ...)$ with $f_i : E_i \to B$ for $0 \leq i$, then the ***sectional category*** of $\mathfrak{E}$ is the integer m, where $m+1$ is the minimum number of open subsets $U_i \subset B$, $0 \leq i \leq m$, such that $B = \cup_{i=0}^{m} U_i$ and for each $0 \leq i \leq m$, f_i has a homotopy section over U_i.

- Let $\mathfrak{E}$ be a sequence of maps $\mathfrak{E} = (f_0, f_1, ...)$. Then we denote by $\mathfrak{E}_{i,j}$, with $i \leq j$ the sequence $(f_i, f_{i+1}, ..., f_j, 0, 0, ...)$.

- Let $\mathfrak{E}$ be a sequence of cofibrations $\mathfrak{E} = (f_0, f_1, ...)$, $f_i : E_i \to X$. The ***m-th fat wedge*** of $\mathfrak{E}_{i,j}$, with $i, j, m \in \mathbb{N}$, $i \leq j$ is $T_m(\mathfrak{E}_{i,j}) \equiv \{(x_i, x_{i+1}, ..., x_j) \in X^{j-i+1}$ such that $|\{s | x_s \in E_s\}| \geq m\}$.

- In the following standard homotopy pull-back

$$\begin{tikzcd}
G_{i,j}(\mathfrak{E}) \arrow[r] \arrow[d, "g_{i,j}(\mathfrak{E})"'] & T_1(\mathfrak{E}_{i,j} \arrow[d, hook] \\
X \arrow[r, "\Delta"] & X^{j-i+1}
\end{tikzcd}$$

the space $G_{i,j}(\mathfrak{E})$ is the (i,j)***-th Ganea space associated to*** $\mathfrak{E}$ and the map $g_{i,j}(\mathfrak{E})$ is the (i,j)***-th Ganea map associated to*** $\mathfrak{E}$.

- The ***n-th Ganea map*** associated to $\mathfrak{E}$: $g_n(\mathfrak{E})$, is the $n+1$ fold join

$$f_0 \bowtie f_1 \bowtie ... \bowtie f_n : G_n(\mathfrak{E}) \equiv E_0 \bowtie_B E_1 \bowtie_B ... \bowtie_B E_n \to B.$$

$G_n(\mathfrak{E})$ is called the ***n-th Ganea space associated to*** $\mathfrak{E}$.

- If $\mathfrak{E} = (p, p, ...)$, we often write $g_n(p) : G_n(p) \to B$ instead of $g_n(\mathfrak{E}) : G_n(\mathfrak{E}) \to B$.

We use Ganea spaces and maps to give a second definition of the sectional category.

Proposition 8.2.1 *1. Let $p : E \to B$ be a fibration. Then* $\mathrm{secat}(p) \leq m$ *if and only if the mth Ganea map $g_m(p)$ admits a homotopy section.*

2. Let $\mathfrak{E} = (f_0, f_1, ...)$ be a sequence of maps $f_i : E_i \to B$, $i \geq 0$, then $\mathrm{secat}(\mathfrak{E}) \leq m$ *if and only if the mth Ganea map $g_m(\mathfrak{E})$ admits a homotopy section.*

PROOF. A proof of the particular case 1 is given in [Jam78]. The second part of the proposition can be proved in an analogous way. □

REMARK. It is now clear why $\mathrm{Rcat}(f : E \to B) = \mathrm{secat}(f, * \to B, * \to B, ...)$.

Notice however that our n-th Ganea spaces are defined only up to homotopy. We would like to choose them without ambiguity, and we show therefore that $G_n(\mathfrak{E})$ is weakly equivalent to $G_{0,n}(\mathfrak{E})$. We then take this last space as standard Ganea space. This allows us to give a third definition of secat in terms of fat wedges. We can moreover take advantage of the construction of Ganea spaces in terms of fat wedges to rationalize the concept of sectional category.

Lemma 8.2.2 *Let $\mathfrak{E} = \{f_0, f_1, ...\}$, $f_i : E_i \to X$ be a sequence of cofibrations. Then the join of the inclusions $X^{k+1} \times T_1(\mathfrak{E}_{k+1,k+l}) \to X^{k+l+1}$ and $T_1(\mathfrak{E}_{0,k}) \times X^l \to X^{k+l+1}$ is weakly equivalent to $T_1(\mathfrak{E}_{0,k+l})$.*

Proof. We follow [Cuv98] by first constructing the standard homotopy pull-back of the inclusions:

$$\mathfrak{P} = \{(w_0, w_1, ..., w_{k+l}) \in (X^I)^{k+l+1} | (w_0(0), ..., w_k(0)) \in T_1(\mathfrak{E}_{0,k}),$$
$$(w_{k+1}(1), ..., w_{k+l}(1)) \in T_1(\mathfrak{E}_{k+1,k+l})\}.$$

We then define a map $\varphi : \mathfrak{P} \to T_1(\mathfrak{E}_{0,k}) \times T_1(\mathfrak{E}_{k+1,k+l})$ as

$$\varphi(w_0, w_1, ..., w_{k+l}) = (w_0(0), ..., w_k(0), w_{k+1}(1), ..., w_{k+l}(1)).$$

It is actually an homotopy equivalence, since it is possible to contract each path w_i to its beginning or its endpoint. We therefore use the space $T_1(\mathfrak{E}_{0,k}) \times T_1(\mathfrak{E}_{k+1,k+l})$ to complete the construction of the join. We must construct a homotopy push-out of the inclusion of $T_1(\mathfrak{E}_{0,k}) \times T_1(\mathfrak{E}_{k+1,k+l})$ in $T_1(\mathfrak{E}_{0,k}) \times X^l$ and in $X^{k+1} \times T_1(\mathfrak{E}_{k+1,k+l})$. Since these inclusions are cofibrations we obtain

$$(T_1(\mathfrak{E}_{0,k}) \times X^l) \cup_{T_1(\mathfrak{E}_{0,k}) \times T_1(\mathfrak{E}_{k+1,k+l})} (X^{k+1} \times T_1(\mathfrak{E}_{k+1,k+l})) = T_1(\mathfrak{E}_{0,k+l}).$$

□

The previous lemma allows us to prove a property of Ganea spaces.

Proposition 8.2.3 *Let $\mathfrak{E} = \{f_0, f_1, ...\}$, $f_i : E_i \to X$ be a sequence of cofibrations. Then*

$$G_{0,k}(\mathfrak{E}) \bowtie_X G_{k+1,k+l}(\mathfrak{E}) \simeq G_{0,k+l}(\mathfrak{E}).$$

Proof. We again follow [Cuv98]. We apply the join theorem I [Doe98] to the following two homotopy pull-backs:

$$\begin{tikzcd}
G_{0,k}(\mathfrak{E}) \arrow[r] \arrow[d, "g_{0,k}(\mathfrak{E})"'] & T_1(E_{0,k}) \times X^l \arrow[d, hook] \\
X \arrow[r, "\Delta"'] & X^{k+l+1}
\end{tikzcd}$$

$$\begin{tikzcd}
G_{k+1,k+l}(\mathfrak{E}) \arrow[r] \arrow[d, "g_{k+1,k+l}(\mathfrak{E})"'] & X^{k+1} \times T_1(E_{k+1,k+l}) \arrow[d, hook] \\
X \arrow[r, "\Delta"'] & X^{k+l+1}
\end{tikzcd}$$

We obtain another homotopy pull-back:

$$\begin{tikzcd}
G_{0,k}(\mathfrak{E}) \bowtie_X G_{k+1,k+l}(\mathfrak{E}) \arrow[r] \arrow[d, "g_{k+1,k+l}(\mathfrak{E})"'] & (T_1(E_{0,k}) \times X^l) \bowtie_{X^{k+l+1}} (X^{k+1} \times T_1(E_{k+1,k+l})) \arrow[d, hook] \\
X \arrow[r, "\Delta"'] & X^{k+l+1}
\end{tikzcd}$$

Using lemma 8.2.2 we conclude. □

Corollary 8.2.4 *Let $\mathfrak{E} = \{f_0, f_1, ...\}$, $f_i : E_i \to X$ be a sequence of cofibrations. Then*

$$G_k(\mathfrak{E}) \simeq G_{0,k}(\mathfrak{E}).$$

Proof. It is clear from the definition of $G_k(\mathfrak{E})$ that $G_k(\mathfrak{E}) \simeq G_{k-1} \bowtie_X E_k$ and $g_k(\mathfrak{E}) \simeq g_{k-1}(\mathfrak{E}) \bowtie f_k$. Using proposition 8.2.3 we have also $G_{0,k}(\mathfrak{E}) \simeq G_{0,k-1} \bowtie_X G_{k,k}(\mathfrak{E})$ and $g_{0,k}(\mathfrak{E}) \simeq g_{0,k-1} \bowtie g_{k,k}(\mathfrak{E})$. Since on the other hand we have $G_0(\mathfrak{E}) = E_0 = G_{0,0}(\mathfrak{E})$, $g_0(\mathfrak{E}) = f_0 = g_{0,0}(\mathfrak{E})$ and $G_{k,k}(\mathfrak{E}) = E_k$, $g_{k,k}(\mathfrak{E}) = f_k$, we can use induction and conclude. □

Since it can easily be verified that the n-th Ganea space of a sequence of maps $\mathfrak{E}$ is homotopy equivalent to the n-th Ganea space of the same sequence where any number of maps have been replaced by their associated cofibration or fibration, we have:

Proposition 8.2.5 *Let $\bar{\mathfrak{E}}$ be a sequence of maps. We construct a new sequence $\mathfrak{E}$, where each map is replaced by its associated cofibration, then the **sectional category of** $\mathfrak{E}$ is smaller or equal to m if the $(0, m)$-Ganea map $g_{0,m}(\mathfrak{E})$ associated to $\mathfrak{E}$ admits a homotopy section.*

From now on we will restrict ourselves to sequences of cofibrations without loss of generality. There is an equivalent definition in terms of the fat wedge:

Proposition 8.2.6 *Let $\mathfrak{E}$ be a sequence of cofibrations. Then* $\operatorname{secat}(\mathfrak{E}) \leq m$ *if and only if there exists a map $\theta : X \to T_1(\mathfrak{E}_{0,m})$ such that the following diagram commutes up to homotopy*

$$\begin{tikzcd} & T_1(\mathfrak{E}_{0,m}) \arrow[dr, hook] & \\ X \arrow[ur, dotted, "\theta"] \arrow[rr, "\Delta"] & & X^{m+1}. \end{tikzcd}$$

Proof. Simply follow the end of the proof of theorem 3.3.2. □

8.3 Rational sectional category

We define the rational sectional category of a sequence of maps $\mathfrak{E}$ by rationalizing the classical definition in terms of the fat wedge.

Definition. Let $\mathfrak{E} = (f_0, f_1, ...)$ be a sequence of cofibrations $f_i : E_i \to X$. Let moreover $(\Lambda X, d) \to (A_{PL}(X), d)$ be a Sullivan model for X, $(\Lambda Y, d) \to (A_{PL}(T_1(\mathfrak{E}_{0,m})), d)$ a Sullivan model for $T_1(\mathfrak{E}_{0,m})$ and $\alpha : ((\Lambda X)^{\otimes m+1}, d) \to (\Lambda X, d)$, $\beta : ((\Lambda X)^{\otimes m+1}, d) \to (\Lambda Y, d)$ be representatives for the diagonal and the inclusion of the fat wedge respectively. We say that the ***rational sectional category of*** $\mathfrak{E}$ is smaller or equal to m if there exists a morphism $\gamma : (\Lambda Y, d) \to (\Lambda X, d)$ such that the following diagram commutes up to homotopy

$$\begin{tikzcd} & \Lambda Y \arrow[dl, dotted, "\gamma"'] & \\ \Lambda X & & (\Lambda X)^{\otimes m+1} \arrow[ul, "\beta"'] \arrow[ll, "\alpha"'] \end{tikzcd}$$

We write $\mathrm{secat}_o(\mathfrak{E}) \le m$.

Let us recall that it is always possible to find a surjective model associated to a relative Sullivan model $(\Lambda X, d) \to (\Lambda X \otimes \Lambda Y, d)$. It is called standard surjective model, and is a morphism $h : (\Lambda X \otimes \Lambda Z, d) \to (\Lambda X \otimes \Lambda Y, d)$, where $Z = Y \oplus \tilde{Y}$, $d : Y \simeq \tilde{Y}$, $dy = \tilde{y}$, $h|_{\Lambda X \otimes \Lambda Y} = id$, $h(\tilde{y}) = dy$ for all $\tilde{y} \in \tilde{Y}$. It makes the diagram

$$\begin{tikzcd} \Lambda X \arrow[rr] \arrow[dr, "g"'] & & \Lambda X \otimes \Lambda Y \\ & \Lambda X \otimes \Lambda Z \arrow[ur, "h"'] & \end{tikzcd}$$

commute, where g is the inclusion of the base. We can now state our main theorem

Theorem 8.3.1 *Let $\mathfrak{E} = (f_0, f_1, ...)$ be a sequence of cofibrations $f_i : E_i \to X$. If $\bar{f}_i : (\Lambda X, d) \to (\Lambda X \otimes \Lambda Y_i, d)$ is a Sullivan model for f_i and $h_i : (\Lambda X \otimes \Lambda Z_i, d) \to (\Lambda X \otimes \Lambda Y, d)$ is a standard surjective model for f_i, then* $\mathrm{secat}_o(\mathfrak{E}) \le m$ *if and only if the morphism*

$$(\Lambda X, d) \longrightarrow \left(\frac{\Lambda X \otimes \bigotimes_{i=0}^{m} \Lambda Z_i}{\prod_{i=0}^{m} \mathrm{Ker}\,(h_i)}, d \right) \equiv (\mathfrak{F}_m(\mathfrak{E}), d),$$

defined by the inclusion followed by the projection, admits a homotopy retract.

8.4 Proof of the Theorem

The first step to prove theorem 8.3.1 is to find a model for $T_1(\mathfrak{E}_{0,m}) \to X^{m+1}$ and use it to construct a cca $(\Gamma_m(\mathfrak{E}), d)$ with the same rational homotopy type as $G_{0,m}(\mathfrak{E})$. To do so we follow [FH83], where an algebra of the same rational homotopy type as $G_m(X)$ is constructed. It is then possible to build a morphism $(\Gamma_m(\mathfrak{E}), d) \to (\mathfrak{F}_m(\mathfrak{E}), d)$ which, given a homotopy retract for $(\Lambda X, d) \to (\mathfrak{F}_m(\mathfrak{E}), d)$, allows the construction of a homotopy retract for $(\Lambda X, d) \to (\Gamma_m(\mathfrak{E}), d)$.

NOTATION. We denote by $S_*(X)$ the singular chain complex over the space X, and by A_{PL} the Sullivan functor. Let $T_1^i(\mathfrak{E}_{0,m}) \equiv \{(x_0, x_1, ...x_m) \in X^{m+1}$ such that $x_i \in E_i\}$, then by $\tilde{S}_*(T_1(\mathfrak{E}_{0,m}))$ we mean the singular simplexes whose image is included in $T_1^i(\mathfrak{E}_{0,m})$ for some i, and $\tilde{A}_{PL}(T_1(\mathfrak{E}_{0,m})) \equiv A_{PL}(\tilde{S}_*(T_1(\mathfrak{E}_{0,m})))$.

Model of the fat wedge

Proposition 8.4.1 *Let $\mathfrak{E} = (f_0, f_1, ...)$ be a sequence of cofibrations $f_i : E_i \to X$. Then both morphisms*

$$\left(\frac{A_{PL}(X)^{\otimes m+1}}{\mathrm{Ker}\,(A_{PL}(f_0)) \otimes \mathrm{Ker}\,(A_{PL}(f_1)) \otimes ... \otimes \mathrm{Ker}\,(A_{PL}(f_m))}, d \right) \xrightarrow{\phi_{m+1}} (\tilde{A}_{PL}(T_1(\mathfrak{E}_{0,m})), d)$$

$$(A_{PL}(T_1(\mathfrak{E}_{0,m})),d) \xrightarrow{res} (\tilde{A}_{PL}(T_1(\mathfrak{E}_{0,m})),d)$$

are quasi-isomorphisms, where res is the restriction to simplices in $\tilde{S}_*(T_1(\mathfrak{E}_{0,m}))$*, and* ϕ_{m+1} *is induced by the morphism* $\lambda^{m+1} = \lambda_1^{m+1} \cdot \ldots \cdot \lambda_{m+1}^{m+1}$*. Here* λ_i^{m+1} *is the composition*

$$(A_{PL}(X),d) \xrightarrow{A_{PL}(pr_i)} (A_{PL}(X^{m+1}),d) \xrightarrow{A_{PL}(incl)} (A_{PL}(T_1(\mathfrak{E}_{0,m})),d) \xrightarrow{res} (\tilde{A}_{PL}(T_1(\mathfrak{E}_{0,m})),d)$$

with pr_i *the projection on the i-th component, incl the inclusion and res the restriction.*

PROOF. We follow the corresponding proof in [FH83].

- We begin by showing that res is a quasi-isomorphism. It is enough to prove that the inclusion $\tilde{S}_*(\mathfrak{E}_{0,m}) \to S_*(\mathfrak{E}_{0,m})$ is a quasi-isomorphism. For simplicity we restrict ourselves to the case $m = 1$. It is easy to generalize the proof for any m. Since we are dealing with CW-complexes and cofibrations, there exist open subsets $U_i \subset X$ such that $E_i \overset{j_i}{\subset} U_i$ and there exists a map $r_i : U_i \to E_i$ such that $r_i \circ j_i = id_{E_i}$ and $j_i \circ r_i \simeq id_{U_i}$ rel E_i, for $i = 0, 1$. Hence $E_0 \times X \overset{\simeq}{\subset} U_0 \times X$, $X \times E_1 \overset{\simeq}{\subset} X \times U_1$ and $(E_0 \times X) \cup (X \times E_1) \overset{\simeq}{\subset} (U_0 \times X) \cup (X \times U_1)$. therefore we can work with U_i instead of E_i as far as homology is concerned.

  Let $\tilde{S}_*((U_0 \times X) \cup (X \times U_1))$ denote the singular simplices whose image is contained either in $U_0 \times X$ or in $X \times U_1$. Then $\tilde{S}_*((U_0 \times X) \cup (X \times U_1)) = S_*(U_0 \times X) + S_*(X \times U_1) \subset S_*((U_0 \times X) \cup (X \times U_1))$. Finally this inclusion is a quasi-isomorphism ([Rot88], proof of thm 6.17, p.117).

- Let us turn to ϕ_{m+1}. We show that it is well-defined: Let us choose any simplex $\sigma \in \tilde{S}_*(T_1(\mathfrak{E}_{0,m}))$, i.e. such that there exists an i with $\sigma(\Delta) \subset T_1^i(\mathfrak{E}_{0,m})$, which means that $pr_i \circ incl \circ \sigma(\Delta) \subset E_i$. If $\psi_i \in \mathrm{Ker}\,(A_{PL}(f_i))$ for a certain i, then for all elements $\sigma \in S_*(E_i)$ we have $\psi_i(f_i \circ \sigma) = 0$. Now $\lambda_i^{m+1}(\psi_i)(\sigma) = \psi_i(f_i \circ pr_i \circ incl \circ \sigma)$. Since $pr_i \circ incl \circ \sigma \in S_*(E_i)$, then $\lambda_i^{m+1}(\psi_i) = 0$, and for $\psi_j \in \mathrm{Ker}\,(A_{PL}(f_j))$, $0 \leq j \leq m$ it is obvious that $\lambda^m(\psi_0 \otimes \psi_1 \otimes \ldots \otimes \psi_m) = 0$.

  It remains to show that ϕ_{m+1} is a quasi-isomorphism. We proceed by induction. For $m = 0$,

$$(\frac{A_{PL}(X)}{\mathrm{Ker}\,(f_0)},d) \xrightarrow{\simeq} (A_{PL}(E_0),d) = (A_{PL}(T_1(\mathfrak{E}_{0,0})),d) = (\tilde{A}_{PL}(T_1(\mathfrak{E}_{0,0})),d),$$

  where the isomorphism is induced by $A_{PL}(f_0)$. We suppose the assumption is true for all $n \leq (m-1)$ and show it for $n = m$. We write $[\tilde{\mathfrak{F}}_m, d) \equiv \Big(\frac{A_{PL}(X)^{\otimes m+1}}{\mathrm{Ker}\,(f_0)\otimes\ldots\otimes\mathrm{Ker}\,(f_m)}, d\Big)$. Consider the following commutative diagram:

$$\begin{tikzcd}
\tilde{A}_{PL}(T_1(\mathfrak{E}_{0,m})) \arrow[r, hook, "F_1,F_2"] & \tilde{A}_{PL}(T_1(\mathfrak{E}_{0,m-1}) \times X) \oplus A_{PL}(X^m \times E_m) \arrow[r, two heads, "G_1-G_2"] & \tilde{A}_{PL}(T_1(\mathfrak{E}_{0,m-1}) \times E_m) \\
\tilde{\mathfrak{F}}_m \arrow[u, "\phi_{m+1}"] \arrow[r, hook, "(K_1,K_2)"'] & \tilde{\mathfrak{F}}_{m-1} \otimes A_{PL}(X) \oplus A_{PL}(X)^{\otimes m} \otimes A_{PL}(E_m) \arrow[u, "\phi_m \cdot id \oplus \gamma^m \cdot id_{E_m}", "\simeq"'] \arrow[r, two heads, "H_1-H_2"'] & \tilde{\mathfrak{F}}_{m-1} \otimes A_{PL}(E_m). \arrow[u, "\phi_m \cdot id", "\simeq"']
\end{tikzcd}$$

  - The morphisms F_1, F_2, G_1, G_2 are restrictions;

- K_2 is induced by $id_{A_{PL}(X)^{\otimes m}} \otimes A_{PL}(f_m)$; $H_1 = id_{\tilde{\mathfrak{F}}_{m-1}} \otimes A_{PL}(f_m)$;
- K_1 is the projection
$$\tilde{\mathfrak{F}}_m \to \frac{A_{PL}(X)^{\otimes m+1}}{\mathrm{Ker}\,(f_0) \otimes ...\mathrm{Ker}\,(f_{m-1}) \otimes A_{PL}(X)} = \frac{A_{PL}(X)^{\otimes m}}{\mathrm{Ker}\,(f_0) \otimes ...\mathrm{Ker}\,(f_{m-1})} \otimes A_{PL}(X);$$
- $H_2 = proj \otimes id_{A_{PL}(E_m)}$;
-
$$\phi_m \cdot id(\varphi \otimes \psi) = (\tilde{A}_{PL}(pr_1) \circ \phi_m(\varphi)) \cdot (res \circ A_{PL}(pr_2)(\psi));$$
- $\gamma^m(\varphi_1 \otimes ... \otimes \varphi_m) = \prod_{i=1}^m A_{PL}(pr_i)(\varphi_i)$;
- $\gamma^m \cdot id_{E_m}(\varphi \otimes \psi) = A_{PL}(pr_1) \circ \gamma^m(\varphi) \cdot A_{PL}(pr_2)(\psi)$;
- $\phi_m \cdot id_{E_m}(\varphi \otimes \psi) = A_{PL}(pr_1) \circ \phi_m(\varphi) \cdot A_{PL}(pr_2)(\psi)$.

It is easy but somewhat long to check that the diagram commutes, that its lines are exact and that the two rightmost vertical arrows are quasi-isomorphisms. Therefore the leftmost arrow is also a quasi-isomorphism.

□

Our next step consists in replacing the algebras and ideals involved in the left hand side of the formula appearing in proposition 8.4.1 by others that are weakly equivalent and can be constructed starting directly with the f_i's Sullivan models.

Lemma 8.4.2 *Let $f : X \to E$ be a continuous cofibration, and $A_{PL}(f) : (A_{PL}(E), d) \to (A_{PL}(X), d)$ its associated algebra morphism under the Sullivan functor. There exists a Sullivan model for $A_{PL}(f)$ and its associated standard surjective model as in the following commutative diagram:*

$$\begin{tikzcd}
A_{PL}(X) \arrow[rr, two heads, "A_{PL}(f)"] & & A_{PL}(E) \\
\Lambda X \arrow[u, "\simeq"'] \arrow[rr] \arrow[dr, hook, "\simeq"'] & & \Lambda X \otimes \Lambda Y \arrow[u, "\simeq"'] \\
& \Lambda X \otimes \Lambda Z. \arrow[ur, "h"'] &
\end{tikzcd}$$

Moreover there exists a quasi-isomorphism $j : (\Lambda X \otimes \Lambda Z, d) \xrightarrow{\simeq} (A_{PL}(X), d)$ which restricts to a quasi-isomorphism $\mathrm{Ker}\,(h) \to \mathrm{Ker}\,(A_{PL}(f))$.

Proof. Using the properties of closed model categories, we deduce the existence of a lift $\Lambda X \otimes \Lambda Z \xrightarrow{j} A_{PL}(X)$ which is a quasi-isomorphism. Since f is a cofibration, $A_{PL}(f)$ is surjective and we have the following commutative diagram with exact lines:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}\,(A_{PL}(f)) \arrow[r] & A_{PL}(X) \arrow[r, "A_{PL}(f)"] & A_{PL}(E) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ker}\,(h) \arrow[r] \arrow[u, "j|_{\ker(h)}"'] & \Lambda X \otimes \Lambda Z \arrow[r, "h"'] \arrow[u, "\simeq", "j"'] & \Lambda X \otimes \Lambda Y \arrow[r] \arrow[u, "\simeq"] & 0.
\end{tikzcd}$$

Considering the long exact sequence in homology we infer that the leftmost morphism is a quasi-isomorphism. □

We can apply this lemma to all the cofibrations f_i and we obtain a quasi-isomorphism

$$\frac{\bigotimes_{i=0}^{m}(\Lambda X \otimes \Lambda Z_i)}{\bigotimes_{i=0}^{m}\operatorname{Ker}(h_i)} \xrightarrow{\simeq} \frac{A_{PL}(X)^{\otimes m+1}}{\bigotimes_{i=0}^{m}\operatorname{Ker}(A_{PL}(f_i))}.$$

Notation. To simplify the notation, in the next proposition we let

$$(\Lambda, d) \equiv \left(\bigotimes_{i=0}^{m}(\Lambda X \otimes \Lambda Z_i), d\right)$$

$$\text{and} \qquad I \equiv \bigotimes_{i=0}^{m}\operatorname{Ker}(h_i).$$

The next proposition shows that from now on we can use the projection $(\Lambda, d) \to (\Lambda/I, d)$ instead of the morphism $(A_{PL}(X^{m+1}), d) \overset{A_{PL}(incl)}{\to} (A_{PL}(T_1(\mathfrak{E}_{0,m})), d)$ because any representative of the first morphism induces a representative of the second one.

Proposition 8.4.3 *If $(\Lambda W, d) \to (\Lambda/I, d)$ is a Sullivan model and $\lambda : (\Lambda, d) \to (\Lambda W, d)$ is a representative for the projection $(\Lambda, d) \to (\Lambda/I, d)$ then there exists a quasi-isomorphism $(\Lambda W, d) \overset{\eta}{\to} (A_{PL}(T_1(\mathfrak{E}_{0,m})), d)$ such that the following diagram commutes up to homotopy:*

$$\begin{tikzcd}
A_{PL}(X^{m+1}) \arrow[r, "A_{PL}(incl)"] & A_{PL}(T_1(\mathfrak{E}_{0,m})) \\
\Lambda \arrow[u, "\simeq"] \arrow[r, "\lambda"] & \Lambda W \arrow[u, "\eta"', "\simeq"]
\end{tikzcd}$$

Proof. Let us consider the diagram

$$\begin{tikzcd}
 & & \bigotimes_{i=0}^{m} A_{PL}(X) \arrow[drr, "\simeq"] & & \\
 & \Lambda \arrow[ur, "\simeq"] \arrow[rrr, "\simeq"] \arrow[dl, "\lambda"'] \arrow[d] & & & A_{PL}(X^{m+1}) \arrow[d] \\
\Lambda W \arrow[r, "\simeq"] & \Lambda/I \arrow[r, "\bigotimes_i j_i", "\simeq"'] & \frac{A_{PL}(X)^{\otimes m+1}}{\operatorname{Ker}(A_{PL}(f_0))\otimes\ldots\otimes\operatorname{Ker}(A_{PL}(f_m))} \arrow[r, "\phi_{m+1}", "\simeq"'] & \tilde{A}_{PL}(T_1(\mathfrak{E}_{0,m})) & A_{PL}(T_1(\mathfrak{E}_{0,m})) \arrow[l, "res"', "\simeq"]
\end{tikzcd}$$

The j_i's are copies of the quasi-isomorphism j from lemma 8.4.2. It can easily be checked that everything commutes at least up to homotopy, therefore, by the lifting property of Sullivan models, there exists a morphism $\eta : \Lambda W \to A_{PL}(T_1(\mathfrak{E}_{0,m}))$ which, when added to the diagram, lets it commute up to homotopy. □

Model of the diagonal

The diagonal map $\Delta : X \to X^{m+1}$ can be represented by the multiplication

$$\mu : (\Lambda X^{\otimes m+1}, d) \to (\Lambda X, d).$$

The associated relative Sullivan model is $(\Lambda X^{\otimes m+1}, d) \to (\Lambda X^{\otimes m+1} \otimes \Lambda \bar{X}^{\otimes m}, d)$, where $\bar{X}^p \cong X^{p+1}$, $\bar{x} \mapsto x$. Since the algebras ΛZ_i in the standard surjective models for $A_{PL}(f_i)$ are acyclic, it is actually possible to replace $\Lambda X^{\otimes m+1}$ by Λ and we obtain another model for the diagonal:

$$\begin{tikzcd}
\Lambda \arrow[rr, "\varphi"] \arrow[dr, hook] & & \Lambda X \\
& \Lambda \otimes \Lambda \bar{X}^{\otimes m} \arrow[ur, "\Xi", "\simeq"'] &
\end{tikzcd}$$

where φ is the multiplication on $\Lambda X^{\otimes m+1}$ and sends Z_i to 0 for all i.

Generalized Ganea algebra

We have now all necessary ingredients to construct a commutative algebra whose rational homotopy type is the same as $G_{0,m}(\mathfrak{E})$'s.

Proposition 8.4.4 *With notations as in the previous paragraph, we let*

$$(\Gamma_m(\mathfrak{E}), d) \equiv (\Lambda/I \otimes \Lambda \bar{X}^{\otimes m}, d) \equiv (\Lambda/I \otimes_\Lambda (\Lambda \otimes \Lambda \bar{X}^{\otimes m}), d).$$

This algebra has the same rational homotopy type as $G_{0,m}(\mathfrak{E})$.

Proof. It suffices to use the definition of $G_{0,m}(\mathfrak{E})$ as the standard homotopy pull-back of the fat-wedge along the diagonal. In rational homotopy this translates into taking the following push-out:

$$\begin{tikzcd}
\Lambda \arrow[r, hook] \arrow[d] & \Lambda \otimes \Lambda \bar{X}^{\otimes m} \arrow[d] \\
\Lambda/I \arrow[r] & \Gamma_m(\mathfrak{E}).
\end{tikzcd}$$

□

Corollary 8.4.5 *With the same notations as in the previous paragraph, we have that if*

$$(\Lambda X, d) \longrightarrow \left(\frac{\Lambda X \otimes \bigotimes_{i=0}^{m} \Lambda Z_i}{\prod_{i=0}^{m} \mathrm{Ker}\,(h_i)}, d \right) \equiv (\mathfrak{F}_m(\mathfrak{E}), d)$$

admits a homotopy retract, then $\mathrm{secat}_o(\mathfrak{E}) \leq m$.

Proof. We use the push-out from the preceding lemma to construct a morphism $\alpha : (\Gamma_m(\mathfrak{E}), d) \to (\mathfrak{F}_m(\mathfrak{E}), d)$. It is induced by the following two morphisms:

$$(\Lambda/I, d) \longrightarrow (\mathfrak{F}_m(\mathfrak{E}), d),$$

induced by the multiplication on the ΛX's, and

$$(\Lambda \otimes \Lambda \bar{X}^{\otimes m}, d) \longrightarrow (\mathfrak{F}_m(\mathfrak{E}), d),$$

equal to the morphism Ξ from the diagram in the previous subsection, followed by the inclusion of $(\Lambda X, d)$ in $(\Lambda X \otimes \bigotimes_{i=0}^{m} \Lambda Z_i, d)$ and the projection on $(\mathfrak{F}_m(\mathfrak{E}), d)$.

We can define a morphism $(\Lambda X, d) \to (\Gamma_m(\mathfrak{E}), d)$ as being the inclusion $(\Lambda X, d) \to (\Lambda X^{\otimes m+1}, d)$ putting an element $\xi \in \Lambda X$ in any factor ΛX of $\Lambda X^{\otimes m+1}$, say factor number i. It is then straightforward to verify that the following diagram is commutative

$$\begin{tikzcd} & \Lambda X \arrow[dl] \arrow[dr] & \\ \Gamma_m(\mathfrak{E}) \arrow[rr, "\alpha"] & & \mathfrak{F}_m(\mathfrak{E}). \end{tikzcd}$$

Therefore any homotopy retract for the rightmost morphism induces an homotopy retract for the leftmost one. $\square$

Opposite direction

To prove the second half of theorem 8.3.1 we use proposition 8.2.3 which states that

$$G_{0,m}(\mathfrak{E}) \simeq G_{0,m-1}(\mathfrak{E}) \bowtie_X G_{m,m}(\mathfrak{E}).$$

We then proceed by induction, with the induction hypothesis that for each n there exists a morphism φ_n and a commutative diagram

$$\begin{tikzcd} & \Lambda X \arrow[dl] \arrow[dr] & \\ \mathfrak{F}_n(\mathfrak{E}) \arrow[rr, "\varphi_n"] & & \mathfrak{H}_n, \end{tikzcd}$$

where $\mathfrak{H}_n$ is an algebra of the same rational homotopy type as $G_{0,n}(\mathfrak{E})$. This proves the theorem.

In case $n = 0$, we have $(\mathfrak{F}_0(\mathfrak{E}), d) = \left(\frac{\Lambda X \otimes \Lambda Z_0}{\operatorname{Ker}(h_0)}, d\right)$ and we can take

$$\varphi_0 = h_0 : (\mathfrak{F}_0(\mathfrak{E}), d) \to (\Lambda X \otimes \Lambda Y_0, d) \equiv (\mathfrak{H}_0, d).$$

For the induction step we suppose that the hypothesis is valid for $n = m-1$ and show that this is still true for $n = m$. If we take $Z \equiv \bigoplus_{i=0}^{m-1} Z_i$, it is clear that by taking the rational cojoin with the relative Sullivan model $i : (\Lambda X \otimes \Lambda Z, d) \to (\Lambda X \otimes \Lambda Z \otimes \Lambda Y_m, d)$

we obtain a diagram

$$\begin{tikzcd}[column sep=small]
& \Lambda X \arrow[d, hook, "\simeq"'] & \\
& \Lambda X \otimes \Lambda Z \arrow[dl] \arrow[dr] & \\
\mathfrak{F}_{m-1}(\mathfrak{E}) \overset{\Lambda X\otimes \Lambda Z}{\bowtie} \Lambda X \otimes \Lambda Z \otimes \Lambda Y_m \arrow[rr, "\varphi_{m-1}\bowtie i"] & & \mathfrak{H}_{m-1} \overset{\Lambda X\otimes \Lambda Z}{\bowtie} \Lambda X \otimes \Lambda Z \otimes \Lambda Y_m,
\end{tikzcd}$$

where $(\Lambda X, d) \to (\Lambda X \otimes \Lambda Z, d)$ and $(\Lambda X \otimes \Lambda Y_m, d) \to (\Lambda X \otimes \Lambda Z \otimes Y_m, d)$ are quasi-isomorphisms and, thanks to proposition 8.2.3, we notice that $(\mathfrak{H}_{m-1} \overset{\Lambda X\otimes \Lambda Z}{\bowtie} \Lambda X \otimes \Lambda Z \otimes \Lambda Y_m, d)$ has the same rational homotopy type as $G_{0,m}(\mathfrak{E})$ and can therefore be chosen as $(\mathfrak{H}_m, d)$. It therefore now suffices to find a morphism $\xi : (\mathfrak{F}_m(\mathfrak{E}), d) \to (\mathfrak{F}_{m-1}(\mathfrak{E}) \overset{\Lambda X\otimes \Lambda Z}{\bowtie} \Lambda X \otimes \Lambda Z \otimes \Lambda Y_m, d)$ that preserves the maps from $(\Lambda X, d)$. To do so we need an explicit construction of $(\mathfrak{F}_{m-1}(\mathfrak{E}) \overset{\Lambda X\otimes \Lambda Z}{\bowtie} \Lambda X \otimes \Lambda Z \otimes \Lambda Y_m, d)$. Let us therefore consider the following commutative diagram:

$$\begin{tikzcd}
\Lambda X \otimes \Lambda Z \arrow[rr, hook] \arrow[dr, dotted, "a"] \arrow[dd, "proj"'] & & \Lambda X \otimes \Lambda Z \otimes \Lambda Y_m \arrow[dd, "j"] \\
& \mathfrak{F}_{m-1}(\mathfrak{E}) \overset{\Lambda X\otimes \Lambda Z}{\bowtie} \Lambda X \otimes \Lambda Z \otimes \Lambda Y_m \arrow[ur] \arrow[dd, "i"] & \\
\mathfrak{F}_{m-1}(\mathfrak{E}) \arrow[rr] \arrow[dr, "\simeq"'] & & \mathfrak{F}_{m-1}(\mathfrak{E}) \otimes \Lambda Y_m \\
& \mathfrak{F}_{m-1}(\mathfrak{E}) \otimes \Lambda Z_m \arrow[ur, two heads, "f_m"'] &
\end{tikzcd}$$

where f_m is a surjective morphism constructed in the standard way, j is the projection on $\Lambda X \otimes \Lambda Z$ and the identity on ΛY_m, $proj$ is the projection and i is the inclusion. Notice that $(\mathfrak{F}_{m-1}(\mathfrak{E}) \otimes \Lambda Y_m, d) \simeq (\mathfrak{F}_{m-1}(\mathfrak{E}) \otimes_{\Lambda X \otimes \Lambda Z} \Lambda X \otimes \Lambda Z \otimes \Lambda Y_m, d)$ is the algebra resulting from the first part of the process to build a rational cojoin, i.e. from the push-out. The morphism i is replaced by a surjective morphism f_m and the rational cojoin we are looking for is of the same rational homotopy type as the pull-back of f_m and j. Moreover there exists an obvious induced morphism a as in the diagram. Now to construct ξ it is necessary to find maps α and β making the next diagram commute:

$$\begin{tikzcd}
\mathfrak{F}_m(\mathfrak{E}) \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \Lambda X \otimes \Lambda Z \otimes \Lambda Y_m \arrow[d, "j"] \\
\mathfrak{F}_{m-1}(\mathfrak{E}) \otimes \Lambda Z_m \arrow[r, two heads, "f_m"'] & \mathfrak{F}_{m-1}(\mathfrak{E}) \otimes \Lambda Y_m .
\end{tikzcd}$$

They are defined in the following way:

- The morphism α is induced by $h_m \otimes id : (\Lambda X \otimes \Lambda Z_m \otimes \Lambda Z, d) \to (\Lambda X \otimes \Lambda Y_m \otimes \Lambda Z, d)$ followed by the isomorphism switching ΛY_m and ΛZ. It is well-defined because $(h_m \otimes id)(\mathrm{Ker}\,(h_m) \otimes \bigotimes_{i=0}^{m-1} \mathrm{Ker}\,(h_i)) = 0$.
- We can consider β as a morphism into $\left(\frac{\Lambda X \otimes \Lambda Z \otimes \Lambda Z_m}{\bigotimes_{i=0}^{m-1} \mathrm{Ker}\,(h_i) \otimes \Lambda Z_m}, d\right) = (\mathfrak{F}_{m-1}(\mathfrak{E}) \otimes \Lambda Z_m, d)$. Then it is induced by the morphism equal to the identity on $\Lambda X \otimes \Lambda Z \otimes \Lambda Z_m$ because $\bigotimes_{i=0}^{m-1} \mathrm{Ker}\,(h_i) \otimes \mathrm{Ker}\,(h_m) \subset \bigotimes_{i=0}^{m-1} \mathrm{Ker}\,(h_i) \otimes \Lambda Z_m$.

The diagram commutes:

- For $\gamma \in \Lambda X \otimes \Lambda Z$, $j \circ \alpha([\gamma]) = j(\gamma) = [\gamma]$; for $\delta \in \Lambda Z_m$, $j \circ \alpha([\delta]) = (j \circ h_m)(\delta)$. If $h_m(\delta) = \sum_k x_k \otimes y_k$ with $x_k \in \Lambda X$ and $y_k \in \Lambda Y_m$, then $j \circ \alpha([\delta]) = \sum_k [x_k] \otimes y_k$.
- On the other hand, for $\gamma \in \Lambda X \otimes \Lambda Z$, $f_m \circ \beta([\gamma]) = f_m([\gamma]) = [\gamma]$; for $\delta \in \Lambda Z_m$, with $h_m(\delta) = \sum_k x_k \otimes y_k$ with $x_k \in \Lambda X$ and$y_k \in \Lambda Y_m$, we have $f_m \circ \beta([\delta]) = f_m(\delta) = \sum_k [x_k] \otimes y_k$.

There is therefore an induced morphism ξ as wished. To verify that it preserves the maps coming from $(\Lambda X, d)$, we notice first of all that there exists a morphism $(\Lambda X, d) \to (\mathfrak{F}_{m-1}(\mathfrak{E}) \overset{\Lambda X \otimes \Lambda Z}{\bowtie} \Lambda X \otimes \Lambda Z \otimes \Lambda Y_m, d)$, which is the inclusion of $(\Lambda X, d)$ in $(\Lambda X \otimes \Lambda Z, d)$ followed by the morphism a. It is then easy to verify that the morphisms

$$(\Lambda X, d) \longrightarrow (\Lambda X \otimes \Lambda Z, d) \xrightarrow{a} (\mathfrak{F}_{m-1}(\mathfrak{E}) \overset{\Lambda X \otimes \Lambda Z}{\bowtie} \Lambda X \otimes \Lambda Z \otimes \Lambda Y_m, d)$$

and

$$(\Lambda X, d) \longrightarrow (\mathfrak{F}_m(\mathfrak{E}), d) \xrightarrow{\xi} (\mathfrak{F}_{m-1}(\mathfrak{E}) \overset{\Lambda X \otimes \Lambda Z}{\bowtie} \Lambda X \otimes \Lambda Z \otimes \Lambda Y_m, d)$$

are induced by the same maps, and therefore they must be equal.