Chapter 7

Applications

7.1 Introduction

Theorem 5.4.1 gives a definition of relative rational LS-category depending only on the relative Sullivan model of the considered map. It is nevertheless difficult to compute it generally, owing to the difficulty to determine Ker (h) precisely. Still, there are a few cases where it is possible to simplify the problem, for example if the map is the inclusion of a fibre. We deal with it in section 7.3.

On the other hand it was shown by D. Stanley in [Sta00] that for a spherical fibration with fibre an odd sphere the rational sectional category of the fibration depends only on the order of its Euler class. We show in section 7.4 that the analogous assertion for rational relative category is not true.

First of all however we consider two simple examples that show that the R-category can take up any value, including infinity.

7.2 Two elementary examples

7.2.1 Example 1

The first example that interests us is an odd spherical fibration $S^n \to E \xrightarrow{f} B$, such that the Sullivan model for B is generated by a single element with even degree: x. A Sullivan model for f is therefore given by

$$\Lambda(x) \to \Lambda(x) \otimes \Lambda(a) \quad \text{with} |x| = k, \text{ even; } |a| = n, \text{ odd; } dx = 0, \text{ and } da = x^t$$

(if $da = 0$, there exists a retraction in dimension 0: the morphism which is the identity on $\Lambda(x)$ and which sends a to 0). We construct the standard surjective model $h : \Lambda(x) \otimes \Lambda(a, \tilde{a}) \to \Lambda(x) \otimes \Lambda(a)$ and compute its kernel: it is the ideal generated by $\tilde{a} - x^t$. By computing the homology of $\mathfrak{F}_m$ it is possible to find a Sullivan model for $\Lambda(x) \to \mathfrak{F}_m$:

$$\Lambda(x) \to \Lambda(x) \otimes \Lambda(w_m), \quad \text{with} \quad dw_m = x^{t+m}.$$

Let us suppose that there exists a retraction $r : \Lambda(x) \otimes \Lambda(w_m) \to \Lambda(x)$, then we must have that $r(w_m) = 0$ for degree reasons. But $r(dw_m) = r(x^{t+m}) = x^{t+m}$ because r is a retraction. We therefore obtain a contradiction, which means that there cannot be a homotopy retraction for any dimension m, or that the R-category, and therefore also the relative category, is infinite.

7.2.2 Example 2

As in the first example, we consider an odd spherical fibration $S^n \to E \xrightarrow{f} B$, but this time the Sullivan model for B is given by $\Lambda(x, y)$, with $|x| = k$ even, $dx = 0$, $dy = x^m$. A relative model for f is

$$\Lambda(x, y) \to \Lambda(x, y) \otimes \Lambda(a)$$

with $da = x^t$. As in the first example, we exclude $da = 0$ because then a retraction exists trivially in dimension zero.

Lemma 7.2.1 *For this fibration we have*

- *if* $t \geq m$, $\mathrm{Rcat}_o(f) = 0$*;*
- *if* $t \leq m$, $\mathrm{Rcat}_o(f) = m - t$*.*

Proof. Let us begin with the case $t \geq m$. We can then define $r : \Lambda(x, y) \otimes \Lambda(a) \to \Lambda(x, y)$ as being $r(x) = x, r(y) = y, r(a) = x^{t-m}y$. If $m \geq t$ we proceed by induction on $s \equiv m - t$. The case $s = 0$ has just been considered. In case $s > 0$, there is no homotopy retraction possible for the morphism $\Lambda(x, y) \to \Lambda(x, y) \otimes \Lambda(a)$.

We suppose that for $s \leq q - 1$, $\mathrm{Rcat}_o(f) = s$ and that for $s > q - 1$, $\mathrm{Rcat}_o(f) > q - 1$. We show: if $s = q$, $\mathrm{Rcat}_o(f) = s$ and if $s > q$, $\mathrm{Rcat}_o(f) > q$. Using theorem 5.4.1, we look for a homotopy retraction for $\Lambda(x, y) \to \mathfrak{F}_q$. The standard surjective model for f is $h : \Lambda(x, y) \otimes \Lambda(a, \tilde{a}) \to \Lambda(x, y) \otimes \Lambda(a)$, which has kernel $< x^t - \tilde{a} >$. For $s \geq q$, the elements of the homology of $\mathfrak{F}_q = \frac{\Lambda(x,y)\otimes\Lambda(a,\tilde{a})}{\Lambda^{\geq q}(x,y)\cdot <x^t-\tilde{a}>}$ are the homology classes of

$$[x], [x^2], ..., [x^{t+q-1}];\ [y - x^s a], [x(y - x^s a)], ..., [x^{t+q-2}(y - x^s a)];$$

and of all elements of type $[x^{pt+l}y - x^l y\tilde{a}^p]$ such that $p, l \in \mathbb{Z}$, $l + pt > q + t - 1$, $p \geq 1$ and $0 \leq l \leq q - 2$, all other products being equal to zero. To build a model for $\Lambda(x, y) \to \mathfrak{F}_q$ we use a method given in section 2.4. We construct a graded vector space V degree by degree in order to obtain a quasi-isomorphism $\Phi : \Lambda(x, y) \otimes \Lambda V \xrightarrow{\simeq} \mathfrak{F}_q$ such that $\Phi(x) = [x]$ and $\Phi(y) = [y]$. We define $\Phi_n \equiv \Phi|_{\Lambda(x,y)\otimes\Lambda V^{\leq n}}$.

- We call Φ_0 the morphism of cdgas $\Lambda(x, y) \to \mathfrak{F}_q$, with $\Phi_0(x) = [x]$, $\Phi_0(y) = [y]$. We check that for $0 \leq n \leq (t + q)k - 2$, $H^n(\Phi_0) : H^n(\Lambda(x, y)) \to H^n(\mathfrak{F}_q)$ is an isomorphism. Moreover $\mathrm{Ker}\, H^{(t+q)k-1}(\Phi_0) = 0$.
- We extend Φ_0 to $\Phi_{(t+q)k-1} : \Lambda(x, y)\otimes\Lambda V^{\leq(t+q)k-1} \to \mathfrak{F}_q$ by setting $V^{(t+q)k-1} \equiv < z >$, $dz = x^{t+q}$, $\Phi_{(t+q)k-1}(z) = [x^q a]$, because

$$\mathrm{Ker}\, H^{(t+q)k}(\Phi_0) = \mathbb{Q} \cdot [x^{t+q}],$$

$$H^{(t+q)k-1}(\mathfrak{F}_q) = \mathrm{Im}\, H^{(t+q)k-1}(\Phi_0).$$

- We now proceed by induction, exactly like in section 2.4: supposing that Φ_n is constructed, that $H^l(\Phi_n)$ is an isomorphism for $0 \leq l \leq n$, and that $H^{n+1}(\Phi_n)$ is injective, we extend to Φ_{n+1} by the following procedure. We choose cocycles $\xi_\alpha \in \mathfrak{F}_q^{n+1}$, $\alpha \in A$ and $\psi_\beta \in (\Lambda(x, y) \otimes \Lambda V^{\leq n})^{n+2}$, $\beta \in B$ so that

$$H^{n+1}(\mathfrak{F}_q) = \mathrm{Im}\, H^{n+1}(\Phi_n) \oplus \bigoplus_{\alpha\in A} \mathbb{Q} \cdot [\xi_\alpha],$$

$$\mathrm{Ker}\, H^{n+2}(\Phi_n) = \bigoplus_{\beta \in B} \mathbb{Q} \cdot [\psi_\beta].$$

Note that this means that for every $\beta \in B$ there exists a $\lambda_\beta \in \mathfrak{F}_q$ such that $\Phi_n(\psi_\beta) = d\lambda_\beta$.

We extend Φ_n by choosing the degree $n+1$ of V to be the vector space

$$V^{n+1} \equiv < v_\alpha, \tilde{v}_\beta >_{\alpha \in A, \beta \in B},$$

with

$$\begin{array}{ll} dv_\alpha = 0 & \Phi_{n+1}(v_\alpha) = \xi_\alpha \\ d\tilde{v}_\beta = \psi_\beta & \Phi_{n+1}(\tilde{v}_\beta) = \lambda_\beta \end{array}$$

Since $d^2 = 0$ in $\Lambda(x,y) \otimes \Lambda V^{\leq n}$ and in V^{n+1}, then it is zero for any product in $\Lambda(x,y) \otimes \Lambda V^{\leq n+1}$. Similarly we can show $d\Phi_{n+1} = \Phi_{n+1} d$.

A model for $\Lambda(x,y) \to \mathfrak{F}_q$ is then given by $\Lambda(x,y) \to \Lambda(x,y) \otimes \Lambda(z, v_0, v_1, ...)$ with $dz = x^{t+q}$, $dv_0 = x^{t+q-1}(y - x^{s-q} z)$. The generators are listed in increasing degree order with $|v_i| \geq (q+t+m-1)k - 1$.

If $s > q$, since a homotopy retraction r must have $r(x) \neq 0$, and by degree reasons $r(z) = 0$, we obtain a contradiction $0 = dr(z) = r(dz) = r(x^{t+q}) \neq 0$, therefore $\mathrm{Rcat}(f) > q$.

If $s = q$, we know by hypothesis that $\mathrm{Rcat}(f) \geq q$. Then a homotopy retraction r can be defined if for all i the differential dv_i is either equal to zero or is a sum of products of generators containing always at least an element $y - z$, yz or v_j, $j < i$. Indeed we could then take r to be the identity on $\Lambda(x,y)$, $r(z) = y$, and zero on $\Lambda(v_0, v_1, ...)$.

Let us show the last assumption by induction. It works for v_0, and we suppose it working for $n \leq i - 1$.

Without loss of generality we suppose $dv_i \neq 0$. For reasons of degree, the differential of v_i can be only one of two types:

$$dv_i = \begin{cases} \beta x^b + \xi \\ \beta x^b y + \gamma x^b z + \xi \end{cases}$$

where β, $\gamma \in \mathbb{Q}$, b is an integer ≥ 0, and ξ denotes all terms containing at least a yz or a v_j. Since we have constructed our model following the method in [FHT01], we notice that dv_i does not contain any term of type βx^b. Indeed if $dv_i = \beta x^b + \xi$, then v_i would have been introduced to "kill" $\beta x^b + \xi$ because its differential is zero but it does not correspond to an element of the cohomology of $\mathfrak{F}_q$. But $d(\beta x^b + \xi) = d\xi$ and it would suffice to set $dv_i \equiv \xi$. Indeed we would then have $\beta x^b + \xi = d(\beta x^{b-m} y + v_i)$, where $b \geq m$ because $|v_i| \geq (m+t)k - 1$. In case $dv_i = \beta x^b y + \gamma x^b z + \xi$, since $d^2 v_i = 0$, we deduce $(\beta + \gamma) x^{m+b} + d\xi = 0$. But we have just shown that $d\xi$ does not contain any term of type βx^b, so we must have $\beta = -\gamma$ and $d\xi = 0$. □

7.3 Inclusion of a fibre

Let us suppose in this section that the map we are considering is the inclusion of a fibre $F \xrightarrow{f} E$ for a fibration $E \xrightarrow{p} B$. In this case it is possible to give a sufficient condition for $\mathrm{cat}_o(f) \leq m$ that is fairly simpler than the definition. Although this condition is not necessary, it can be successfully used to compute relative categories in a few examples.

Proposition 7.3.1 *Let $F \xrightarrow{f} E$ be the inclusion of the fibre for a fibration $E \xrightarrow{p} B$, $(\Lambda W, d) \to (\Lambda W \otimes \Lambda V, d)$ be a relative Sullivan model for p and $\tilde{f} : (\Lambda W \otimes \Lambda V, d) \to (\Lambda V, d)$ a representative for f. We define*

$$(\mathfrak{K}_m, d) \equiv \left(\frac{\Lambda W \otimes \Lambda V}{\Lambda^{\geq m}(W \oplus V) \cdot \Lambda^+ W}, d \right)$$

and denote by $\tilde{\pi}' : (\Lambda W \otimes \Lambda V, d) \to (\Lambda X, d)$ a representative of the projection $\pi' : (\Lambda W \otimes \Lambda V, d) \to (\mathfrak{K}_m, d)$. We also denote by $\tilde{k}' : (\Lambda X, d) \to (\Lambda V, d)$ a representative of the morphism $k' : (\mathfrak{K}_m, d) \to (\Lambda V, d)$, induced by $\tilde{f}$. Then $\tilde{k}' \circ \tilde{\pi}'$ is homotopic to $\tilde{f}$.

- *If there exists a homotopy retraction r of $\tilde{\pi}'$ then $\mathrm{Rcat}_o(f) \leq m$;*
- *If moreover $\tilde{f} \circ r \simeq \tilde{k}'$ then $\mathrm{cat}_o(f) \leq m$.*

Proof. The long fibration sequence

$$\ldots \to \Omega B \to F \xrightarrow{f} E \xrightarrow{p} B,$$

becomes the following sequence in rational homotopy:

$$(\Lambda W, d) \to (\Lambda W \otimes \Lambda V, d) \xrightarrow{\tilde{f}} (\Lambda V, d) \to (\Lambda \bar{W}, \bar{d}) \to \ldots,$$

where $(\Lambda W, d) \to (\Lambda W \otimes \Lambda \bar{W}, d)$ is a model of the augmentation $(\Lambda W, d) \to (\mathbb{Q}, 0)$. We begin by constructing a relative Sullivan model $\bar{f}$ of $\tilde{f}$, and then the standard surjective model h of it

$$\begin{tikzcd}
\Lambda W \otimes \Lambda V \arrow[rr, "\tilde{f}"] \arrow[dr, hook, "\bar{f}"] \arrow[ddr, hook, "g"', "\simeq"] & & \Lambda V \\
& \Lambda W \otimes \Lambda V \otimes \Lambda \bar{W} \arrow[ur, "\phi", "\simeq"'] & \\
& \Lambda(W \oplus V) \otimes \Lambda(\bar{W} \oplus \bar{\bar{W}}.) \arrow[u, two heads, "h"'] &
\end{tikzcd}$$

The diagram commutes exactly. Notice that with these notations

$$(\mathfrak{F}_m, d) = \left(\frac{\Lambda(W \oplus V) \otimes \Lambda(\bar{W} \oplus \bar{\bar{W}})}{\Lambda^{\geq m}(W \oplus V) \cdot \mathrm{Ker}\,(h)}, d \right)$$

There exists moreover a morphism j which is a lift in the following commuting diagram:

$$\begin{tikzcd}
\Lambda W \otimes \Lambda V \arrow[rr, equal] \arrow[d, hook, "g", "\simeq"'] & & \Lambda W \otimes \Lambda V \arrow[d, "\tilde{f}"] \\
\Lambda W \otimes \Lambda V \otimes \Lambda(\bar{W} \oplus \bar{\bar{W}}) \arrow[r, two heads, "h"'] \arrow[urr, dotted, "j", "\simeq"'] & \Lambda W \otimes \Lambda V \otimes \Lambda \bar{W} \arrow[r, "\simeq", "\phi"'] & \Lambda V.
\end{tikzcd}$$

Therefore $j(\mathrm{Ker}\,(h)) \subset \mathrm{Ker}\,(\tilde{f}) = \Lambda^+ W \otimes \Lambda V$ and $j(\Lambda^{\geq m}(W \oplus V) \cdot \mathrm{Ker}\,(h)) \subset \Lambda^{\geq m}(W \oplus V) \cdot (\Lambda^+ W \otimes \Lambda V) = \Lambda^{\geq m}(W \oplus V) \cdot \Lambda^+ W$. The morphism j therefore induces a morphism $\Psi : (\mathfrak{F}_m, d) \longrightarrow (\mathfrak{K}_m, d)$. From the previous diagram it is easy to see that

$$\begin{tikzcd}
& \Lambda W \otimes \Lambda V \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\
\mathfrak{F}_m \arrow[rr, "\Psi"'] \arrow[d, "k"] & & \mathfrak{K}_m \arrow[d, "k'"] \\
\Lambda W \otimes \Lambda V \otimes \Lambda \bar{W} \arrow[rr, "\simeq", "\phi"'] & & \Lambda V
\end{tikzcd}$$

commutes, where k and k' are induced respectively by h and $\tilde{f}$. Choosing now Sullivan models for $\mathfrak{F}_m$ and $\mathfrak{K}_m$ and representatives for the various morphisms of the last diagram, we obtain

$$\begin{tikzcd}
& & \Lambda W \otimes \Lambda V \arrow[dll, "\tilde{\pi}"'] \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] \arrow[drr, "\tilde{\pi}'"] & & \\
\Lambda M \arrow[r, "\lambda", "\simeq"'] \arrow[dr, "\tilde{k}"] & \mathfrak{F}_m \arrow[rr, "\Psi"'] \arrow[d, "k"] & & \mathfrak{K}_m \arrow[d, "k'"'] & \Lambda X \arrow[l, "\simeq"', "\lambda'"] \arrow[dl, "\tilde{k}'"] \\
& \Lambda W \otimes \Lambda V \otimes \Lambda \bar{W} \arrow[rr, "\simeq", "\phi"'] & & \Lambda V &
\end{tikzcd}$$

where everything commutes at least up to homotopy. As usual there exists a morphism $\bar{\Psi} : (\Lambda M, d) \to (\Lambda X, d)$ such that $\lambda' \circ \bar{\Psi} \simeq \Psi \circ \lambda$. Since $\lambda' \circ \bar{\Psi} \circ \tilde{\pi} \simeq \Psi \circ \lambda \circ \tilde{\pi} \simeq \lambda' \circ \tilde{\pi}'$, then $\bar{\Psi} \circ \tilde{\pi} \simeq \tilde{\pi}'$. Let us now suppose the existence of a homotopy retraction $r' : (\Lambda X, d) \to (\Lambda W \otimes \Lambda V, d)$ for $\tilde{\pi}'$. We show that $r := r' \circ \bar{\Psi}$ is a homotopy retraction for $\tilde{\pi}$:

$$r \circ \tilde{\pi} = r' \circ \bar{\Psi} \circ \tilde{\pi} \simeq r' \circ \tilde{\pi}' \simeq id_{\Lambda W \otimes \Lambda V}.$$

Moreover if $\tilde{f} \circ r' \simeq \tilde{k}'$, then

$$\phi \circ k \circ \pi \circ r' \circ \bar{\Psi} \simeq k' \circ \Psi \circ \pi \circ r' \circ \bar{\Psi} \simeq k' \circ \pi' \circ r' \circ \bar{\Psi} =$$

$$\tilde{f} \circ r' \circ \bar{\Psi} \simeq \tilde{k}' \circ \bar{\Psi} \simeq k' \circ \lambda' \circ \bar{\Psi} \simeq k' \circ \Psi \circ \lambda = \phi \circ k \circ \lambda \simeq \phi \circ \tilde{k}.$$

Moreover we notice that $\tilde{k}$ is a lifting for the diagram

$$\begin{tikzcd}
& & & \Lambda W \otimes \Lambda V \otimes \Lambda \bar{W} \arrow[d, "\simeq"', "\phi"] \\
\Lambda M \arrow[r, "\simeq", "\lambda"'] \arrow[urrr, dotted, "\tilde{k}"] & \mathfrak{F}_m \arrow[r, "\Psi"] & \mathfrak{K}_m \arrow[r, "k'"] & \Lambda V
\end{tikzcd}$$

and is then unique up to homotopy. Therefore $\tilde{k} \simeq k \circ \pi \circ r' \circ \bar{\Psi} = k \circ \pi \circ r$ as wished and $\mathrm{cat}_o(f) \leq m$. □

Remark. Let us consider a spherical fibre sequence $F \xrightarrow{i} E \xrightarrow{p} B$ with fibre an odd-dimensional sphere S^{2n+1}. Since F is an Eilenberg-MacLane space $K(\mathbb{Q}, 2n+1)$, p is a principal fibration, and therefore it is the inclusion of the fibre of a fibration $B \to K(\mathbb{Q}, 2n+2)$. This means that the previous proposition can be applied to a large class of morphisms.

To compute the R-category or the rational LS-category of an example we need a few properties of classical relative LS-category.

Lemma 7.3.2 *Let $f : E \to X$ be a map, then*

1. $\mathrm{Rcat}(f) \leq \mathrm{cat}(X)$;
2. $\mathrm{cat}(f) \geq \mathrm{cat}(X \cup_f CE)$.

Proof. Without loss of generality we can suppose that f is a cofibration.

1. Let us suppose that $\mathrm{cat}(X) \leq n$. This means that it is possible to cover X with $n+1$ open sets U_i which are contractible in X. Since the base point is contained in E, any one of the U_i is contractible into E as requested by the definition of Rcat.
2. If $\mathrm{cat}(f) \leq n$, there exists a covering of X by $n+1$ open sets U_i, $0 \leq i \leq n$, such that U_0 contains E and can be continuously deformed into E relatively to E. The other open sets are contractible in X. We can now choose an open covering $\{V_i\}_{i=0}^{n}$ for $X \cup_f CE$ as follows: $V_i = U_i$ for $1 \leq i \leq n$ and $V_0 = U_0 \cup_f CE$. To contract V_0 it suffices to first deform U_0 into E. Since the deformation occurs relatively to E it can be extended to all of $U_0 \cup_f CE$ by keeping it constant on CE. At the end of this deformation we are left with CE, which is a cone, and therefore contractible in itself.

□

Example.

- Let us consider the Hopf fibration $S^5 \to \mathbb{C}P(2) \to \mathbb{C}P(\infty)$. We are going to determine the rational relative LS-category of the inclusion of the fibre $\hat{f} : S^5 \to \mathbb{C}P(2)$. It is well-known that $\mathrm{cat}(\mathbb{C}P(2)) = 2$ and therefore $\mathrm{Rcat}_o(f) \leq \mathrm{Rcat}(f) \leq 2$ thanks to lemma 7.3.2. It is now a matter to decide whether the rational R-category is equal to 0,1 or 2. A model for the fibration is
$$(\Lambda(x), d) \longrightarrow (\Lambda(x,y), d) \xrightarrow{\tilde{f}} (\Lambda(y), d),$$
with $|x| = 2$, $dx = 0$, $|y| = 5$ and $dy = x^3$ in $(\Lambda(x,y), d)$. Unfortunately if in this case we try to use the simplified model described in proposition 7.3.1 $(\Lambda(x,y), d) \to (\mathfrak{K}_1, d)$, we find no possible retraction, which gives us no supplementary information. It is therefore necessary to work on a model for $(\Lambda(x,y), d) \to (\mathfrak{F}_1, d)$.
- We consider the Sullivan model for $\tilde{f}$: $(\Lambda(x,y), d) \to (\Lambda(x,y,z), d)$, where $dz = x$, and $\phi : (\Lambda(x,y,z), d) \to (\Lambda(y), d)$ is a weak equivalence such that $\phi(x) = \phi(z) = 0$ and $\phi(y) = y$. We are here in the case considered in lemma 7.2.1 and therefore $\mathrm{Rcat}_o(f) = 2$, which implies that $\mathrm{Rcat}(f) = 2$.

- Let us now determine $\mathrm{cat}_o(f)$. It is obviously equal to or greater than the rational R-category. We show that it is equal to 3. We begin by showing that it is smaller or equal to 3 by using proposition 7.3.1. Let us point out that we cannot use this proposition to show $\mathrm{cat}_o(f) = 2$ since there is no convenient retraction for $(\Lambda(x,y),d) \to (\mathfrak{K}_2,d)$. In the case we are considering, we have

$$(\mathfrak{K}_3,d) = \left(\frac{\Lambda(x,y)}{\Lambda^{\geq 3}(x,y)\cdot\Lambda^+(x)}, d\right).$$

As a vector space, a basis for the homology is given by the homology classes of the elements $[x],[x^2],[xy],[x^2y]$. Since $H^1(\mathfrak{K}_3) = 0$, we can use the same method as described in section 2.4 and in lemma 7.2.1 and we find that a Sullivan model for $\pi : (\Lambda(x,y),d) \to (\mathfrak{K}_3,d)$ is $(\Lambda(x,y),d) \to (\Lambda(x,y)\otimes\Lambda(w,v_1,v_2,...),d)$, with $dw = x^4$. A set of generators for the homology of $(\Lambda(w,v_1,v_2,...),d)$ is given by $\{x,x^2,xy-w,x^2y-xw\}$. The elements v_i are there only to eliminate any superfluous element and have increasing degree, with $|v_1| = 10$. The quasi-isomorphism $\phi : (\Lambda(x,y)\otimes\Lambda(w,v_1,v_2,...),d) \to (\mathfrak{K}_3,d)$ is therefore given by

$$\phi(x) = [x],\ \phi(y) = [y],\ \phi(w) = 0,\ \phi(v_i) = 0 \quad \forall\, i.$$

A representative for $\mathfrak{K}_3 \to \Lambda(y)$ is given by

$$\tilde{k}' : (\Lambda(x,y)\otimes\Lambda(w,v_1,v_2,...),d) \to (\Lambda(y),d),$$

with $\tilde{k}'(y) = y$ and $\tilde{k}' \equiv 0$ for all other elements. Since the derivative of any v_i is a sum of products which always contain either $xy - w$ or a v_j with $j < i$, we can define a retraction $r : (\Lambda(w,v_1,v_2,...),d) \to (\Lambda(x,y),d)$ as being

$$r(x) = x,\ r(y) = y,\ r(w) = xy,\ r(v_i) = 0 \quad \forall\, i.$$

It is easy to check that $\tilde{f} \circ r = \tilde{k}'$ and therefore $\mathrm{cat}_o(f) \leq 3$.

- To show that $\mathrm{cat}_o(f) \neq 2$ we use the standard surjective model for $(\Lambda(x,y),d) \to (\Lambda(x,y)\otimes\Lambda(z),d)$ which is $(\Lambda(x,y),d) \to (\Lambda(x,y,z,\tilde{z}),d)$, with $|z| = 1$ $dz = \tilde{z}$, $d\tilde{z} = 0$. First of all we derive the homology algebra of

$$(\mathfrak{F}_2,d) = \left(\frac{\Lambda(x,y)\otimes\Lambda(z,\tilde{z})}{\Lambda^{\geq 2}(x,y)\cdot < x-\tilde{z} >}, d\right),$$

where $< x - \tilde{z} >$ is the ideal generated by $x - \tilde{z}$. We find that the homology is generated as a vector space by the homology classes of

$$[x],\ [x^2],\ [x^2z - y],\ [x^3z - xy],\ [xy - y\tilde{z}],\ [x^3z - y\tilde{z}].$$

Since $H^1((\mathfrak{F}_2,d)) = 0$, we use again the method of lemma 7.2.1 to derive a Sullivan model for $(\Lambda(x,y),d) \to (\mathfrak{F}_2,d)$. We find the following relative cdga:

$$(\Lambda(x,y),d) \to (\Lambda(x,y)\otimes\Lambda(w,v_1,v_2,...),d))$$

with $dw = 0$, $|w| = 5$, $dv_1 = x^2w$, $dv_2 = xv_1 - yw$. The quasi-isomorphism

$$\Phi : (\Lambda(x,y)\otimes\Lambda(w,v_1,v_2,...),d) \to (\mathfrak{F}_2,d)$$

is given by

$$\Phi(x) = [x],\ \Phi(y) = [y],\ \Phi(w) = [y - x^2z],\ \Phi(v_1) = [-xyz],\ \Phi(v_2) = 0.$$

If there exists a homotopy retract $r : (\Lambda(x, y) \otimes \Lambda(w, v_1, v_2, ...), d) \to (\Lambda(x, y), d)$, for reasons of degree or of compatibility with the differential we must have

$$r(x) = x,\ r(y) = y,\ r(w) = r(v_1) = r(v_2) = 0.$$

Let us now check if $\tilde{f} \circ r$ can be homotopic to a representative $k : (\Lambda(x, y) \otimes \Lambda(w, v_1, v_2, ...), d) \to (\Lambda(x, y) \otimes \Lambda(z), d)$ for the morphism $(\mathfrak{F}_2, d) \to (\Lambda(x, y) \otimes \Lambda(z), d)$. We choose k to be

$$k(x) = x,\ k(y) = y,\ k(w) = y - x^2 z,\ k(v_1) = -xyz,\ k(v_2) = 0.$$

On the other hand we have $(\tilde{f} \circ r)(w) = 0$. If $\tilde{f} \circ r \simeq k$ there exists a morphism of degree -1,

$$h : (\Lambda(x, y) \otimes \Lambda(w, v_1, v_2, ...), d) \to (\Lambda(x, y) \otimes \Lambda(z), d)$$

such that

$$y - x^2 z = dh(w) - h(dw) = dh(w).$$

Since by inspection $y - x^2 z$ is not a boundary, such a morphism h does not exist and therefore $\mathrm{cat}_o(f) > 2$.

7.4 Euler class

Don Stanley showed in [Sta00] that the rational sectional category of an odd spherical fibration depends only on the order of its Euler class. We prove in this section that this is not the case for R-category.

Definition. Let $E \xrightarrow{p} B$ be a fibration with fibre an odd sphere S^{2n+1}, whose Sullivan model is of type $\Lambda(a)$, with $|a| = 2n + 1$. If $(\Lambda X, d) \to (A_{PL}(B), d)$ is a Sullivan model for B, there exists a Sullivan model for p which looks as follows: $(\Lambda X, d) \to (\Lambda X \otimes \Lambda(a), d)$, with $w \equiv da \in \Lambda^+ X$.

We consider the cofibration $f : (\Lambda(\tilde{a}), d) \to (\Lambda(\tilde{a}) \otimes \Lambda X \otimes \Lambda(a), d)$, with $|\tilde{a}| = 2n + 2$, $da = w - \tilde{a} \in \Lambda(\tilde{a}) \otimes \Lambda X \otimes \Lambda(a)$ in order to obtain the following diagram

$$\begin{tikzcd}
\Lambda(\tilde{a}) \arrow[r, "\psi"] \arrow[dr, hook, "f"'] & \Lambda X \arrow[r, "i"] & \Lambda X \otimes \Lambda(a) \\
& \Lambda(\tilde{a}) \otimes \Lambda X \otimes \Lambda(a). \arrow[u, "\phi"', "\simeq"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Here $\phi(a) = 0$, $\phi(\tilde{a}) = w$ and ϕ acts as the identity on ΛX. The morphism g is defined as the identity on $(\Lambda X \otimes \Lambda(a), d)$ and $g(\tilde{a}) =$, while $\psi(\tilde{a}) = w$. If we define a morphism $h : (\Lambda(\tilde{a}) \otimes \Lambda X \otimes \Lambda(a), d) \to (\Lambda X \otimes \Lambda(a), d)$ of degree -1 in the following way:

$$h(\tilde{a}) = a,\ h(a) = 0,\ h(x) = 0 \quad \forall x \in \Lambda X,$$

then we check that

$$i \circ \Phi - g = hd + dh.$$

We verify therefore that the diagram commutes up to homotopy.

Moreover ϕ is a weak equivalence: let $\gamma : \Lambda X \to \Lambda(\tilde{a}) \otimes \Lambda X \otimes \Lambda(a)$ denote the inclusion. Since $\phi \circ \gamma = id_{\Lambda X}$ it is enough to show that γ is a weak equivalence. This is the case because γ is a model for a fibration whose fibre is modeled by $\Lambda(\tilde{a}) \otimes \Lambda(a)$, which is a contractible algebra.

Hence g is a representative for p and f represents a map $B \to K(\mathbb{Q}, 2n+2)$ which corresponds to a cohomology class $\alpha \in H^{2n+2}(B)$. From our construction we deduce that w is a representative for α. We call $\alpha = [w]$ the ***Euler class*** of the fibration p.

To compute the category of a spherical fibration p we can slightly simplify theorem 5.4.1.

Proposition 7.4.1 *Let $(\Lambda X, d) \xrightarrow{f} (\Lambda X \otimes \Lambda(a), d)$ be the relative Sullivan model of a map p, where $da = w \in \Lambda^+ X$. We construct a surjective model $h : \Lambda X \otimes \Lambda(a, \tilde{a}) \to \Lambda X \otimes \Lambda(a)$ of p by setting $da = w - \tilde{a}$, $d\tilde{a} = 0$, in $(\Lambda X \otimes \Lambda(a, \tilde{a}), d)$, and defining h as being the identity on $\Lambda X \otimes \Lambda(a)$ and $h(\tilde{a}) = 0$. We have then* $\mathrm{Ker}\,(h) = <\tilde{a}> = \Lambda X \otimes \Lambda(a) \otimes \Lambda^+(\tilde{a})$.

The following composition

$$\Lambda X \overset{\eta_m}{\to} \Lambda X \oplus \Lambda^{\geq m} X \otimes \Lambda^+(a) \overset{s_m}{\to} \Lambda X \otimes \Lambda(a),$$

where η_m and s_m are inclusions, is equal to f. Choosing a Sullivan model ΛM for $\Lambda X \oplus \Lambda^{\geq m} X \otimes \Lambda^+(a)$ and representatives $\tilde{\eta}_m : (\Lambda X, d) \to (\Lambda M, d)$ and $\tilde{s}_m : (\Lambda M, d) \to (\Lambda X \otimes \Lambda(a), d)$ for η_m and s_m respectively, we have $\tilde{s}_m \circ \tilde{\eta}_m \simeq f$. Then

- $\mathrm{Rcat}(p) \leq m$ *if and only if $\tilde{\eta}_m$ admits a homotopy retraction;*
- $\mathrm{cat}_o(p) \leq m$ *if and only if $\tilde{\eta}_m$ admits a homotopy retraction r such that $f \circ r \simeq \tilde{s}_m$.*

PROOF. We have already shown that $(\Lambda X, d) \to (\Lambda X \otimes \Lambda(a, \tilde{a}), d)$ is a weak equivalence, and therefore that h is a surjective model for p.

The kernel of h takes a particularly simple form: $\mathrm{Ker}\,(h) = <\tilde{a}> = \Lambda X \otimes \Lambda(a) \otimes \Lambda^+(\tilde{a})$. Indeed, in case $|a|$ is odd, if $\xi = \gamma + \alpha a + \sum_{i=1}^{s} \beta_i \tilde{a}^i + \sum_{i=1}^{r} \delta_i a \tilde{a}^i$ is such that $h(\xi) = 0$, we must have $h(\xi) = \gamma + \alpha a = 0$ and therefore $\gamma = 0$, $\alpha = 0$. On the other hand, if $|a|$ is even and $h(\gamma + \sum_{i=1}^{s} \alpha_i a^i + \beta \tilde{a} + \sum_{i=1}^{r} \delta_i a^i \tilde{a}) = \gamma + \sum_{i=1}^{s} \alpha_i a^i = 0$ then $\gamma = \alpha_i = 0$ for all i. In this situation $\mathfrak{F}_m = \frac{\Lambda X \otimes \Lambda(a, \tilde{a})}{\Lambda^{\geq m} X \otimes \Lambda(a) \otimes \Lambda^+(\tilde{a})}$.

Let us now consider the following commutative diagram

$$\begin{tikzcd}
\frac{\Lambda^{\geq m} X \otimes \Lambda(a, \tilde{a})}{\Lambda^{\geq m} X \otimes \Lambda(a) \otimes \Lambda^+(\tilde{a})} \arrow[r, tail] & \frac{\Lambda X \otimes \Lambda(a, \tilde{a})}{\Lambda^{\geq m} X \otimes \Lambda(a) \otimes \Lambda^+(\tilde{a})} \arrow[r, two heads, "\alpha"] & \Lambda X / \Lambda^{\geq m} X \otimes \Lambda(a, \tilde{a}) \\
\Lambda^{\geq m} X \otimes \Lambda(a) \arrow[u, equal] \arrow[r, tail] & \mathfrak{F}_m \times_{\Lambda X / \Lambda^{\geq m} X \otimes \Lambda(a, \tilde{a})} \Lambda X / \Lambda^{\geq m} X \arrow[u, "\beta"] \arrow[r, two heads] & \Lambda X / \Lambda^{\geq m} X \arrow[u, "j", "\simeq"'] \\
\Lambda^{\geq m} X \otimes \Lambda(a) \arrow[u, equal] \arrow[r, tail, "i"] & \Lambda X \oplus \Lambda^{\geq m} X \otimes \Lambda^+(a) \arrow[r, two heads, "q"] & \Lambda X / \Lambda^{\geq m} X, \arrow[u, equal]
\end{tikzcd}$$

where all lines are short exact sequences, α is induced by the morphism $proj \otimes id_{\Lambda(a, \tilde{a})}$: $(\Lambda X \otimes \Lambda(a, \tilde{a}), d) \to (\Lambda X / \Lambda^{\geq m} X \otimes \Lambda(a, \tilde{a}), d)$, i and j are inclusions, q is the restriction of $proj \otimes id_{\Lambda(a)} : (\Lambda X \otimes \Lambda(a) \to \Lambda X/\Lambda^{\geq m}X \otimes \Lambda(a), d)$, and the upper right square is a pull-back. Using the long exact sequences associated to the upper two lines, we deduce that β is a weak equivalence. We now construct a morphism

$$\psi : (\Lambda X \oplus \Lambda^{\geq m}X \otimes \Lambda^+(a), d) \to (\mathfrak{F}_m \times_{\Lambda X/\Lambda^{\geq m}X \otimes \Lambda(a,\tilde{a})} \Lambda X/\Lambda^{\geq m}X, d) \equiv (\mathfrak{P}, d)$$

by using the universal property of pull-backs. It therefore suffices to construct a morphism

$$\zeta : (\Lambda X \oplus \Lambda^{\geq m}X \otimes \Lambda^+(a), d) \to \left(\frac{\Lambda X \otimes \Lambda(a,\tilde{a})}{\Lambda^{\geq m}X \otimes \Lambda(a) \otimes \Lambda^+(\tilde{a})}, d\right)$$

such that $\alpha \circ \zeta = j \circ q$. We take $\zeta(\gamma) = [\gamma]$ for all $\gamma \in (\Lambda X \oplus \Lambda^{\geq m}X \otimes \Lambda^+(a), d)$. We verify that ζ is compatible with the differential: let $\xi \in \Lambda X$, then

- $\zeta(d\xi) = [d\xi] = d[\xi] = d\zeta(\xi)$;
- if $\gamma \in \Lambda^{\geq m}X$, $\zeta(d(\gamma \cdot a^n)) = \zeta(d\gamma \cdot a^n \pm n\gamma \cdot w\, a^{n-1})$ is equal to $[d\gamma \cdot a^n] \pm [n\gamma \cdot w\, a^{n-1}]$ while $d\zeta(\gamma \cdot a^n) = d[\gamma \cdot a^n] = [d\gamma \cdot a^n] \pm n[\gamma \cdot (w - \tilde{a})a^{n-1}] = [d\gamma \cdot a^n] \pm n[\gamma \cdot wa^{n-1}]$.

Notice that ζ is well-defined in both cases $|a|$ odd and $|a|$ even. Moreover for $\xi = \sum_i \gamma_i \otimes \delta_i \in (\Lambda X \oplus \Lambda^{\geq m}X \otimes \Lambda^+(a), d)$, where $\gamma_i \in \Lambda X$, $\delta_i \in \Lambda(a)$ then $\alpha \circ \zeta(\xi) = \alpha([\sum_i \gamma_i \otimes \delta_i]) = \sum_i [\gamma_i] \otimes \delta_i = \sum_{[i|\delta_i \in \mathbb{Q}]} \delta_i[\gamma_i]$, and $j \circ q(\xi) = j(\sum_{[i|\delta_i \in \mathbb{Q}]} \delta_i[\gamma_i]) = \sum_{i|\delta_i \in \mathbb{Q}} \delta_i[\gamma_i]$.

Again we deduce from the long exact sequences in homology that ψ is a weak equivalence. It is also easy to control that there exists a commutative diagram

$$\begin{tikzcd}
 & \Lambda X \arrow[dl, "\eta_m"'] \arrow[dr] & \\
\Lambda X \oplus \Lambda^{\geq m}X \otimes \Lambda^+(a) \arrow[rr, "\simeq", "\beta\circ\psi"'] \arrow[dr, "s_m"'] & & \mathfrak{F}_m \arrow[dl] \\
 & \Lambda X \otimes \Lambda(a). &
\end{tikzcd}$$

The proposition is therefore proved. □

The existence of a homotopy retraction for $(\Lambda X, d) \to (\Lambda X \oplus \Lambda^{\geq m}X \otimes \Lambda^+(a), d)$ has some consequence for the Euler class' behaviour.

Proposition 7.4.2 *With notations from proposition 7.4.1, if there exists a homotopy retraction r for $(\Lambda X, d) \to (\Lambda X \oplus \Lambda^{\geq m}X \otimes \Lambda^+(a), d)$, then for all classes $\{\alpha_i\}_{1\leq i\leq m}$ with $\alpha_i \in H^+(\Lambda X, d)$ we have*

$$\alpha_1 \cdot \ldots \cdot \alpha_m \cdot [w] = 0$$

in $H(\Lambda X, d)$.

Proof. Let λ_i be a representative for the homology class α_i. The element $\lambda_1 \cdot \lambda_2 \cdot \ldots\lambda_m \cdot a \in \Lambda^{\geq m}X \otimes \Lambda^+(a)$ has differential $\pm\lambda_1 \cdot \lambda_2 \cdot \ldots\lambda_m \cdot w$ because λ_i is a cycle. Therefore $\xi := \lambda_1 \cdot \lambda_2 \cdot \ldots\lambda_m \cdot w$ is a border in $(\Lambda^{\geq m}X \otimes \Lambda^+(a), d)$.

On the other hand we can build a weak equivalence $(\Lambda X \otimes \Lambda Y, d) \xrightarrow{\phi} (\Lambda X \oplus \Lambda^{\geq m}X \otimes \Lambda^+(a), d)$, a cofibration $\pi : (\Lambda X, d) \to (\Lambda X \otimes \Lambda Y, d)$ such that $\phi \circ \pi = \eta_m$, and a retraction $r : (\Lambda X \otimes \Lambda Y, d) \to (\Lambda X, d)$. Since $\xi \in (\Lambda X, d)$ we have $\xi = \phi(\bar{\xi})$, where $\bar{\xi} = \lambda_1 \cdot \lambda_2 \cdot \ldots\lambda_m \cdot w \in (\Lambda X \otimes \Lambda Y, d)$. But ξ is a border, so $\bar{\xi}$ must be one too: $\bar{\xi} = d\mu$, with $\mu \in (\Lambda X \otimes \Lambda Y, d)$. On the other hand $dr(\mu) = r(d\mu) = r(\bar{\xi}) = r \circ \pi(\lambda_1 \cdot \lambda_2 \cdot \ldots\lambda_m \cdot w) = \lambda_1 \cdot \lambda_2 \cdot \ldots\lambda_m \cdot w$, so $\lambda_1 \cdot \lambda_2 \cdot \ldots\lambda_m \cdot w$ is a border in $(\Lambda X, d)$. □

D. Stanley has shown in [Sta00] that if α is the Euler class of an odd spherical fibration p, and r is the least integer such that $\alpha^r = 0$, then the rational sectional category of p is equal to r. By analogy one would expect that if r is the minimal integer such that for all $\{\alpha_i\}_{1\leq i\leq r}$ with $\alpha_i \in H^+(\Lambda X, d)$ we have $\alpha_1 \cdot ... \cdot \alpha_r \cdot [w] = 0$, then $\mathrm{Rcat}_o(p) \leq r$. This is nevertheless false, as is shown by the following counter-example:

Example. Let $(\Lambda X, d) = (\Lambda(x_1, x_2, ..., x_n, y), d)$ be a Sullivan algebra such that $dx_i = 0$ for all i, $dy = x_1 \cdot ... \cdot x_n$, n even, $|x_i| = 3$, $|y| = 3n-1$, odd. Let us consider a relative Sullivan algebra $(\Lambda X, d) \to (\Lambda X \otimes \Lambda(a), d)$, with $da = x_1x_2 = w$, $|a| = 5$. It is clear that $w \cdot x_3 \cdot ... \cdot x_{n-1}$ is a cycle but not a border in $(\Lambda X, d)$. Therefore the previous proposition allows us to infer that $\mathrm{Rcat}_o(p) \geq n-2$. On the other hand $w \cdot x_{i_1} \cdot x_{i_2} \cdot ...x_{i_{n-2}}$ is equal to zero or is a border for $i_s \in \{1, 2, ..., n\}$. Indeed if an index is equal to 1 or 2, or if any two indices are equal the product is zero, and $w \cdot x_3 \cdot ... \cdot x_n = dy$. We show that $\mathrm{Rcat}_o(p) \neq n-2$ nevertheless.

By proposition 7.4.1 $\mathrm{Rcat}_o(p) = n-2$ is equivalent to the existence of a homotopy retraction for $(\Lambda(x_1, ..., x_n, y), d) \to (\Lambda(x_1, ..., x_n, y) \oplus \Lambda^{\geq n-2}(x_1, ..., x_n, y) \otimes a, d) \equiv (\mathfrak{M}, d)$. We now examine a Sullivan model $(\Lambda X, d) \to (\Lambda X \otimes \Lambda M, d)$ for this morphism, constructed in the standard way (see [FHT01]). We notice that before dimension $3(n-2)+5 = 3n-1$, $(\Lambda X, d)$ is weakly equivalent to $(\mathfrak{M}, d)$.

In dimension $3n-1$, among other things, $d(a \cdot x_3 \cdot ... \cdot x_n) = x_1 \cdot ... \cdot x_n = dy$ and $d(a \cdot x_2 \cdot x_4 \cdot x_5 \cdot ... \cdot x_n) = 0$ in $(\mathfrak{M}, d)$, therefore $(\Lambda M, d)$ must contain basis elements v, z, such that $dv = x_1 \cdot ... \cdot x_n$ and $dz = 0$. If φ denotes the weak equivalence between $(\Lambda X \otimes \Lambda M, d)$ and $(\mathfrak{M}, d)$, then $\varphi(v) = ax_1...x_n$ and $\varphi(z) = ax_2x_4x_5...x_n$. Moreover in $(\Lambda X \otimes \Lambda M, d)$

$$d(vx_2 + zx_3) = 0$$

and

$$\varphi(vx_2 + zx_3) = (-1)^{n-2}ax_2...x_n + (-1)^{n-1}ax_2...x_n = 0,$$

therefore there exists a $\xi \in (\Lambda X \otimes \Lambda M, d)$ such that $d\xi = vx_2 + zx_3$.

Let us now suppose there exists a retraction $r : (\Lambda X \otimes \Lambda M, d) \to (\Lambda X, d)$. Then the only possibility is $r(v) = y$. By degree reasons the image of z by r can be either zero or y. Since $dz = 0$ it must be zero. We then have $r(vx_2 + zx_3) = yx_2 = r(d\xi) = dr(\xi)$ and thus yx_2 must be a border in $(\Lambda X, d)$, which is not the case. This means that there cannot be a retraction at this point.