Chapter 6

Proof of the main theorem

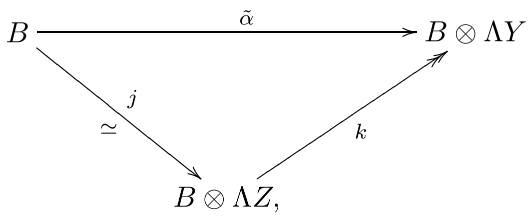
6.1 Introduction

To prove theorem 5.4.1 we introduce a few tools: first of all in section 6.2 we define the *rational cojoin* of two morphisms and show that it models a join of two maps. We can then use it to introduce *Ganea algebras* and *Ganea morphisms* in section 6.3. They actually model relative Ganea spaces and maps as defined in section 3.3. We also construct explicitely the first Ganea algebra. Our proof begins in section 6.4 by showing that for each $m \geq 0$ there exists a morphism which has the m-th Ganea algebras as source and a certain algebra $\mathfrak{F}_m$ as target while letting some diagram commute (up to homotopy). The advantage of this procedure is that while the Ganea algebras are very difficult to actually describe, the cca's $\mathfrak{F}_m$ can be easily built as soon as one chooses a surjective model for the m-th Ganea map, for example the standard surjective model. Of course for the proof of theorem 5.4.1 to be complete we must also exhibit a morphism with $\mathfrak{F}_m$ as source and $\mathfrak{G}_m$ as target, which is what we accomplish in section 6.5, but first we use what is shown in section 6.4 to specialize to the rational LS-category of a space and obtain a new proof of Félix and Halperin's theorem 5.2.3 in subsection 6.4.1.

6.2 Rational cojoin

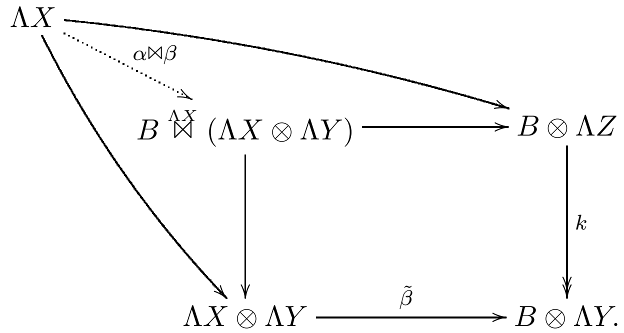
We begin by defining the rational cojoin construction, which we will use repeatedly during our proof.

Definition. Let $(\Lambda X, d) \xrightarrow{\alpha} (B, d)$ be a morphism of cca's, and $(\Lambda X, d) \xrightarrow{\beta} (\Lambda X \otimes \Lambda Y, d)$ be a relative Sullivan cca. We can construct the Sullivan cca $(B \otimes_{\Lambda X} \Lambda X \otimes \Lambda Y, d) = (B \otimes \Lambda Y, d)$ and the inclusion maps $\tilde{\alpha} : (B, d) \to (B \otimes \Lambda Y, d)$, $\tilde{\beta} : (\Lambda X \otimes \Lambda Y, d) \to (B \otimes \Lambda Y, d)$. If neither $\tilde{\alpha}$ nor $\tilde{\beta}$ is surjective, we choose one of them, say $\tilde{\alpha}$, and construct its associated standard surjective morphism as follows:



where j is the inclusion of the base, $Z = Y \oplus \tilde{Y}$, with $d : Y^p \xrightarrow{\cong} \tilde{Y}^{p+1}$, $dy = \tilde{y}$, and $k(b) := b$ for $b \in B$, $k(y) := y$ for $y \in Y$ and $k(\tilde{y}) := dy$ for $\tilde{y} \in \tilde{Y}$.

The map k is surjective, hence a fibration, and we can construct the pull-back of k and $\tilde{\beta}$, which we call the ***rational cojoin of*** $(\Lambda X \otimes \Lambda Y, d)$ and (B, d) ***under*** $(\Lambda X, d)$: $(B, d) \overset{(\Lambda X,d)}{\bowtie} (\Lambda X \otimes \Lambda Y, d)$. The induced morphism $(\Lambda X, d) \to (B, d) \overset{(\Lambda X,d)}{\bowtie} (\Lambda X \otimes \Lambda Y, d)$ is the ***rational cojoin of*** α and β: $\alpha \bowtie \beta$



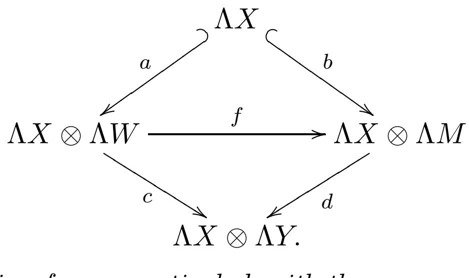
The relation between usual join and rational cojoin becomes clear thanks to the following lemma.

Lemma 6.2.1 *Using notations as in the definition of rational cojoin, if α represents a map $a : E \to X$ and β is a Sullivan model for $b : M \to X$, then any Sullivan representative of $\alpha \bowtie \beta$ also represents $a \bowtie b$. Henceforth $(B, d) \overset{(\Lambda X,d)}{\bowtie} (\Lambda X \otimes \Lambda Y, d)$ has the same rational homotopy type as $(E \bowtie_X M, d)$.*

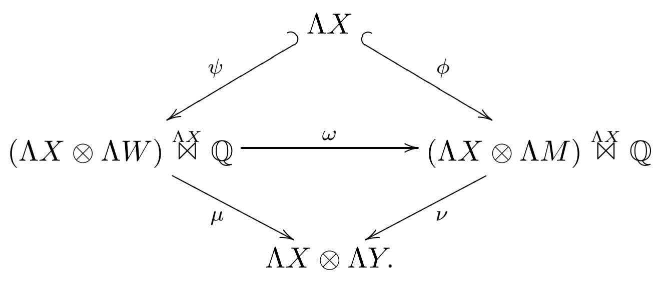
Proof. The proof is straightforward using modelization of adjunction spaces and pull-backs from section 2.6. □

An important property of the rational cojoin is that it is a functorial construction, as the topological join is. We show this in the special case where one of the morphisms involved in the cojoin is an augmentation $\epsilon : (\Lambda X, d) \to (\mathbb{Q}, 0)$.

Lemma 6.2.2 *Let $(\Lambda X, d) \to (\Lambda X \otimes \Lambda Y, d)$, $(\Lambda X, d) \to (\Lambda X \otimes \Lambda M, d)$ and $(\Lambda X, d) \to (\Lambda X \otimes \Lambda W, d)$ be three relative Sullivan models. Let us suppose that there exist such morphisms between these cca's that make the following diagram commute.*

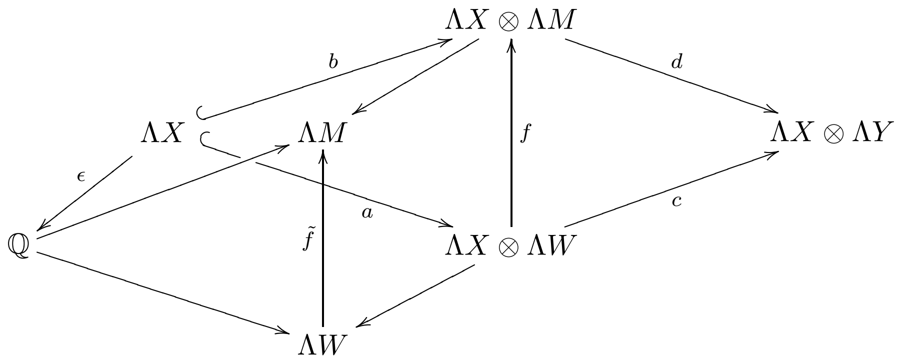


Then we can take the cojoin of a, respectively b with the augmentation $\epsilon : (\Lambda X, d) \to (\mathbb{Q}, 0)$ and we obtain another commutative diagram:

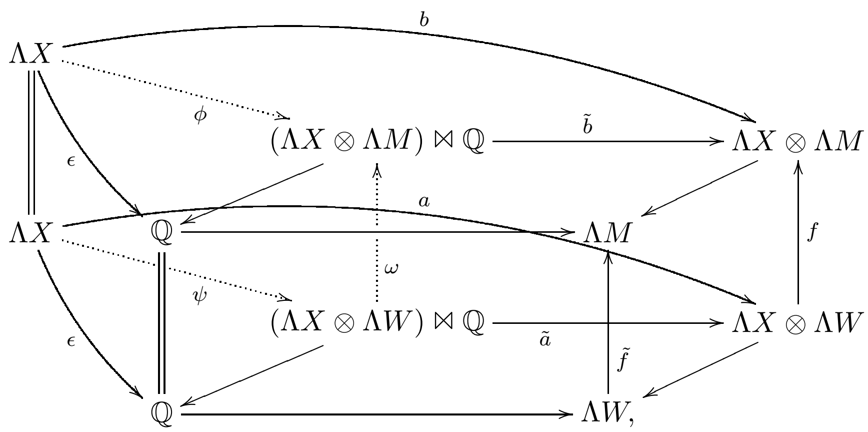


Moreover $\nu \circ \phi = d \circ b$ *and* $\mu \circ \psi = c \circ a$.

PROOF. Following the rational cojoin construction, we first form cca's $((\Lambda X \otimes \Lambda W) \otimes_{(\Lambda X,d)} \mathbb{Q}, d) \cong (\Lambda W, d)$, $((\Lambda X \otimes \Lambda M) \otimes_{(\Lambda X,d)} \mathbb{Q}, d) \cong (\Lambda M, d)$. The morphism f induces a morphism $f \otimes id_{\mathbb{Q}} : ((\Lambda X \otimes \Lambda W) \otimes_{(\Lambda X,d)} \mathbb{Q}, d) \to ((\Lambda X \otimes \Lambda M) \otimes_{(\Lambda X,d)} \mathbb{Q}, d)$, hence a morphism $\tilde{f} : (\Lambda W, d) \to (\Lambda M, d)$ such that the diagram



commutes. Since the morphisms $(\Lambda X \otimes \Lambda W, d) \to (\Lambda W, d)$ and $(\Lambda X \otimes \Lambda M, d) \to (\Lambda M, d)$ are surjective, we can immediately proceed with the second part of the construction, i.e. the pull-back. We obtain the following commuting diagram



where all dotted arrows are induced by the universal property of pull-backs. The lemma is proved by taking $\mu := c \circ \tilde{a}$ and $\nu := d \circ \tilde{b}$. □

REMARKS.

- Notice that if in the previous lemma we have that $d \circ f \simeq c$, then we obtain $\nu \circ \omega \simeq \mu$, while the rest of the diagram still commutes exactly.
- With a few modifications, it can be verified that the proof works also for the cojoin with any morphism $(\Lambda X, d) \to (B, d)$. The main difference is that after the first part of the construction we do not generally obtain surjective maps, and we must therefore replace them by surjective ones as has been done in the definition of relative cojoin.

6.3 Ganea algebras

Using the rational cojoin construction we can in a way analogous to the one used to construct Ganea spaces, build *Ganea algebras*, which are cca's with the same rational homotopy type as the corresponding Ganea space (absolute or relative).

Definition. Let $f : (\Lambda X, d) \to (\Lambda X \otimes \Lambda Y, d)$ be a relative Sullivan model. We define the ***m-th rational Ganea morphism*** of f, $\mathfrak{g}_m(f) : (\Lambda X, d) \to (\mathfrak{G}_m(f), d)$, as being the cojoin of f with the augmentation $\epsilon : (\Lambda X, d) \to (\mathbb{Q}, 0)$ m-times.

Since, according to the definition of rational cojoin we need one of the morphisms involved to be a relative Sullivan model, we replace $\mathfrak{g}_{m-1}(f)$ by a relative Sullivan model before constructing $\mathfrak{g}_m(f)$. The cca $(\mathfrak{G}_m(f), d)$ is called the ***m-th Ganea cca of*** f.

Lemma 6.2.2 shows that there exists a morphism $\mathfrak{q}_m : (\mathfrak{G}_m(f), d) \to (\Lambda X \otimes \Lambda Y, d)$ such that $\mathfrak{q}_m \circ \mathfrak{g}_m = f$. Thus for example, $\mathfrak{g}_0(f) : (\Lambda X, d) \to (\mathfrak{G}_0(f), d) = (\Lambda X \otimes \Lambda Y, d)$ is just f, and $\mathfrak{q}_0$ is the identity $id_{(\Lambda X \otimes \Lambda Y, d)}$. It is clear from lemma 6.2.1 that $(\mathfrak{G}_m(f), d)$ has the same homotopy type as $G_m(f)$ and that any representative for $\mathfrak{g}_m(f)$ is also a representative for $g_m(f)$. Moreover, a representative of $\mathfrak{q}_m$ is a representative of $q_m(f)$.

Remark. If a morphism f represents a map $\{*\} \to X$, we write $(\mathfrak{G}_m(X), d) = (\mathfrak{G}_m(f), d)$, and say that it is the m***-th Ganea cca of*** X.

Lemma 6.3.1 *Using the same notations as in the previous definition, we have that*

$$(\mathfrak{G}_1(f), d) \cong (\mathbb{Q} \oplus \Lambda^+ X \otimes \Lambda Y, d).$$

Moreover, $\mathfrak{g}_1(f) : (\Lambda X, d) \to (\mathfrak{G}_1(f), d)$ *is isomorphic to the inclusion of* $(\Lambda X, d)$ *and* $\mathfrak{q}_1$ *is isomorphic to* $(\mathbb{Q} \oplus \Lambda^+ X \otimes \Lambda Y, d) \to (\Lambda X \otimes \Lambda Y, d)$ *where* $q \oplus \gamma \longmapsto q + \gamma$.

Proof. According to the definition of the rational cojoin, taking $(B, d) = (\mathbb{Q}, 0)$, we first obtain morphisms $(\mathbb{Q}, 0) \to (\mathbb{Q} \otimes \Lambda Y, d) = (\Lambda Y, d)$ and $(\Lambda X \otimes \Lambda Y, d) \to (\Lambda Y, d)$. Notice that the second map is surjective. We can therefore immediately build the pull-back of the two maps: $(\mathfrak{G}_1(f), d) = (\mathbb{Q} \times_{(\Lambda Y, d)} (\Lambda X \otimes \Lambda Y, d)$, and $\mathfrak{g}_1(f)(\xi) = (\epsilon(\xi), \xi)$, while $\mathfrak{q}_1$ is the projection on $(\Lambda X \otimes \Lambda Y, d)$ .We now construct an isomorphism $\phi : (\mathbb{Q} \times_{\Lambda Y} (\Lambda X \otimes \Lambda Y), d) \to (\mathbb{Q} \oplus \Lambda^+ X \otimes \Lambda Y, d)$. Let $(q, \sum_i \alpha_i \otimes \beta_i) \in \mathbb{Q} \times_{\Lambda Y} (\Lambda X \otimes \Lambda Y)$, where $q \in \mathbb{Q}$, $\alpha_i \in \Lambda X$ and $\beta_i \in \Lambda Y$ for all i. Any element of the cojoin can be written under this form. We define

$$\phi : (q, \sum_i \alpha_i \otimes \beta_i) \longmapsto q \oplus (\sum_i \alpha_i \otimes \beta_i - q).$$

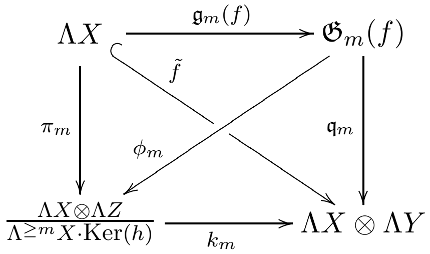
The morphism ϕ is well-defined, because $q = \sum_i \epsilon(\alpha_i) \otimes \beta_i$, and therefore $\sum_i \alpha_i \otimes \beta_i - q \in \Lambda^+ X \otimes \Lambda Y$. One can easily verify that ϕ is a morphism of cca's. Moreover it is an isomorphism, because it admits an inverse $\psi : q \oplus \gamma \longmapsto (q, q + \gamma)$, where $q \in \mathbb{Q}$ and $\gamma \in \Lambda^+ X \otimes \Lambda Y$. Finally we notice that $\phi \circ \mathfrak{g}_1(f)$ is the inclusion of ΛX and $(\mathfrak{q}_1 \circ \psi)(q \oplus \gamma) = \mathfrak{q}_1(q, q + \gamma) = q + \gamma$ as claimed. □

Remark. From now on we will abuse notation and denote by $(\mathfrak{G}_1(f), d)$ the cca $(\mathbb{Q} \oplus \Lambda^+ X \otimes \Lambda Y, d)$ and by $\mathfrak{g}_1(f)$ and $\mathfrak{q}_1$ the isomorphic morphisms.

6.4 First part of the proof

We can now prove one part of the equivalence of the main theorem. It is a consequence of the following proposition:

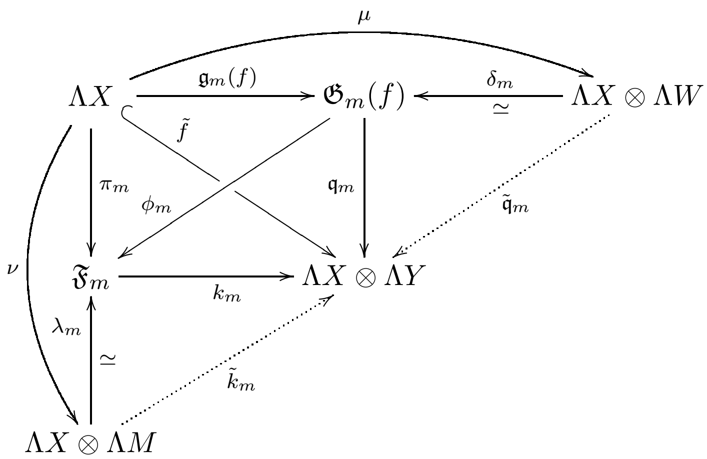
Proposition 6.4.1 *Let us use the same notations as in theorem 5.4.1, then for each m there exists a morphism of cca's $\phi_m : (\mathfrak{G}_m(f), d) \to \left(\frac{\Lambda X \otimes \Lambda Z}{\Lambda^{\geq m} X \cdot \mathrm{Ker}(h)}, d\right)$ which lets the following diagram commute exactly, but for the lower right triangle, which only commutes up to homotopy:*



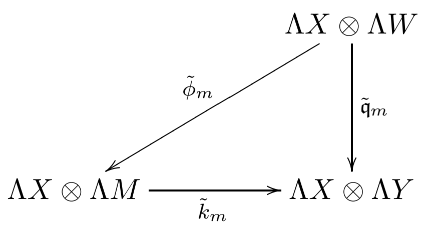
Corollary 6.4.2 *Using notations from theorem 5.4.1,*

- *if π_m admits a homotopy retract r, then $\mathrm{Rcat}_o(f) \leq m$;*
- *if moreover $\tilde{f} \circ r \simeq \tilde{k}_m$ then $\mathrm{cat}_o(f) \leq m$.*

Proof. [of corollary] For simplicity, let us adopt the notation $\mathfrak{F}_m := \frac{\Lambda X \otimes \Lambda Z}{\Lambda^{\geq m} X \cdot \mathrm{Ker}(h)}$. We now choose a relative Sullivan model for $\mathfrak{g}_m(f)$ and π_m in order to obtain a diagram with solid arrows which commutes exactly, but for the lower right triangle, which only commutes up to homotopy:



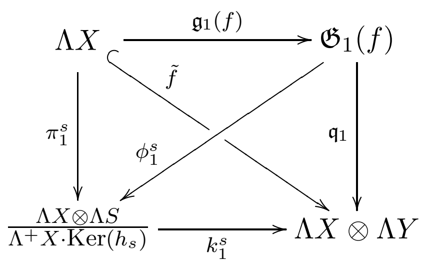
and where moreover $k_m \circ \lambda_m \simeq \tilde{k}_m$ and $\mathfrak{q}_m \circ \delta_m \simeq \tilde{\mathfrak{q}}_m$. By the elementary properties of relative Sullivan models, there exists a lift $\tilde{\phi}_m : (\Lambda X \otimes \Lambda W, d) \to (\Lambda X \otimes \Lambda M, d)$ such that $\tilde{\phi}_m \circ \mu = \nu$ and $\phi_m \circ \delta_m \simeq \lambda_m \circ \tilde{\phi}_m$. Therefore we have also that



commutes up to homotopy. If π_m admits a homotopy retract, there exists a morphism $r : (\Lambda X \otimes \Lambda M, d) \to (\Lambda X, d)$ such that $r \circ \nu \simeq id_{\Lambda X}$ and then defining $r' := r \circ \tilde{\phi}_m$, we verify $r' \circ \mu = r \circ \tilde{\phi}_m \circ \mu = r \circ \nu \simeq id_{\Lambda X}$ i.e. $\mathrm{Rcat}_o(f) \leq m$. If moreover $\tilde{f} \circ r \simeq \tilde{k}_m$, then $\tilde{f} \circ r' = \tilde{f} \circ r \circ \tilde{\phi}_m \simeq \tilde{k}_m \circ \tilde{\phi}_m \simeq \tilde{\mathfrak{q}}_m$. Therefore $\mathrm{cat}_o(f) \leq m$. □

To prove proposition 6.4.1 we are going to need two lemmas: the first one states the same result in a special case: when the surjective model involved is the standard surjective model of $\tilde{f}$ and $m = 1$. In this situation it is indeed possible to construct ϕ_1 explicitely. The second lemma constructs morphisms in both directions between the standard surjective model and any other surjective model, allowing for a generalization of the first lemma to any surjective model. It will then be used to implement an induction process in the general case.

Lemma 6.4.3 *Let $f : E \to X$ be a cofibration and $\tilde{f} : (\Lambda X, d) \to (\Lambda X \otimes \Lambda Y, d)$ a Sullivan model for it. Let $h_s : (\Lambda X \otimes \Lambda S, d) \to (\Lambda X \otimes \Lambda Y, d)$ be the standard surjective model of $\tilde{f}$. Then there exists a morphism $\phi_1^s : (\mathfrak{G}_1(f), d) \to \left(\frac{\Lambda X \otimes \Lambda S}{\Lambda^+ X \cdot \mathrm{Ker}(h_s)}, d\right)$ which lets the following diagram commute exactly:*



Proof. We have shown in lemma 6.3.1, that $\mathfrak{G}_1(f) \cong \mathbb{Q} \oplus \Lambda^+ X \otimes \Lambda Y$. Let us now define $\phi_1^s : \mathbb{Q} \oplus \Lambda^+ X \otimes \Lambda Y \to \frac{\Lambda X \otimes \Lambda S}{\Lambda^+ X \cdot \mathrm{Ker}(h_s)}$ as follows: for $q \oplus \gamma \in \mathbb{Q} \oplus \Lambda^+ X \otimes \Lambda Y$

$$\phi_1^s(q \oplus \gamma) := [q + \gamma].$$

We verify that ϕ_1^s commutes with the differential:

- If $\gamma \in \Lambda^+ X$, then $d\gamma \in \Lambda^+ X$ and we have

$$\phi_1^s(d\gamma) = [d\gamma] = d[\gamma] = d\phi_1^s(\gamma).$$

- If $\gamma = \xi \otimes y$, with $\xi \in \Lambda^+ X$ and $y \in Y$, let us write $z := dy \in \Lambda X \otimes \Lambda Y$. Then

$$\phi_1^s(d(\xi \otimes y)) = \phi_1^s(d\xi \otimes y \pm \xi \otimes z) = [d\xi \otimes y] \pm [\xi \otimes z].$$

  On the other hand we have

$$d\phi_1^s(\xi \otimes y) = d[\xi \otimes y] = [d\xi \otimes y] \pm [\xi \otimes \tilde{y}].$$

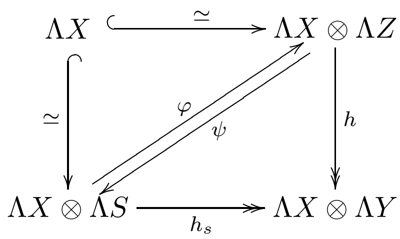
  Since however $h_s(\tilde{y} - z) = dy - z = 0$, we have $\xi \otimes (\tilde{y} - z) \in \mathrm{Ker}\,(h_s)$, and $[\xi \otimes \tilde{y}] = [\xi \otimes z]$. We can proceed similarly for $\gamma = \xi \otimes \alpha$ with $\xi \in \Lambda^+ X$ and $\alpha \in \Lambda^n Y$.

We also verify that $\phi_1^s(\gamma) = [\gamma] = \pi_1^s(\gamma)$ for $\gamma \in \Lambda X$. It remains to show that $k_1^s \circ \phi_1^s = \mathfrak{q}_1$: for $q \oplus \gamma \in \mathbb{Q} \oplus \Lambda^+ X \otimes \Lambda Y$,

$$(k_1^s \circ \phi_1^s)(q \oplus \gamma) = k_1^s([q + \gamma]) = h(q + \gamma) = q + \gamma = \mathfrak{q}_1(q \oplus \gamma).$$

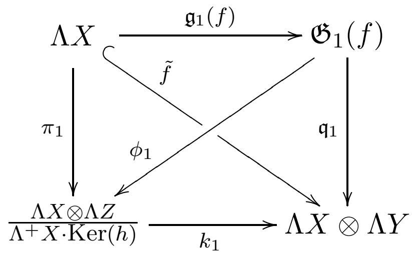
□

Lemma 6.4.4 *If $h_s : (\Lambda X \otimes \Lambda S, d) \to (\Lambda X \otimes \Lambda Y, d)$ is the standard surjective model for the morphism $\tilde{f} : (\Lambda X, d) \to (\Lambda X \otimes \Lambda Y, d)$ from lemma 6.4.3, and $h : (\Lambda X \otimes \Lambda Z, d) \to (\Lambda X \otimes \Lambda Y, d)$ is any surjective model, then there exist morphisms φ, ψ making the following diagram commute exactly:*



PROOF. It is a straightforward consequence of the lifting properties of fibrations and trivial cofibrations, see section 2.7. □

Corollary 6.4.5 *Let $f : E \to X$ be a cofibration and $\tilde{f} : (\Lambda X, d) \to (\Lambda X \otimes \Lambda Y, d)$ a relative Sullivan model for it. Let $h : (\Lambda X \otimes \Lambda Z, d) \to (\Lambda X \otimes \Lambda Y, d)$ be any surjective model of $\tilde{f}$. Then there exists a morphism $\phi_1 : (\mathfrak{G}_1(f), d) \to \left(\frac{\Lambda X \otimes \Lambda Z}{\Lambda^+ X \cdot \mathrm{Ker}(h)}, d\right)$ which lets the following diagram commute exactly:*



PROOF. From lemma 6.4.4 we deduce that $\varphi(\mathrm{Ker}\,(h_s)) \subset \mathrm{Ker}\,(h)$. The morphism φ induces therefore a morphism $\tilde{\varphi} : \left(\frac{\Lambda X \otimes \Lambda S}{\Lambda^+ X \cdot \mathrm{Ker}(h_s)}, d\right) \to \left(\frac{\Lambda X \otimes \Lambda Z}{\Lambda^+ X \cdot \mathrm{Ker}(h)}, d\right)$, such that $k_1 \circ \tilde{\varphi} = k_1^s$. We take $\phi_1 := \tilde{\varphi} \circ \phi_1^s$ and verify that, for any $\gamma \in \Lambda X$,

$$\phi_1(\gamma) = \tilde{\varphi} \circ \phi_1^s(\gamma) = \tilde{\varphi}([\gamma]) = proj \circ \varphi(\gamma) = [\gamma].$$

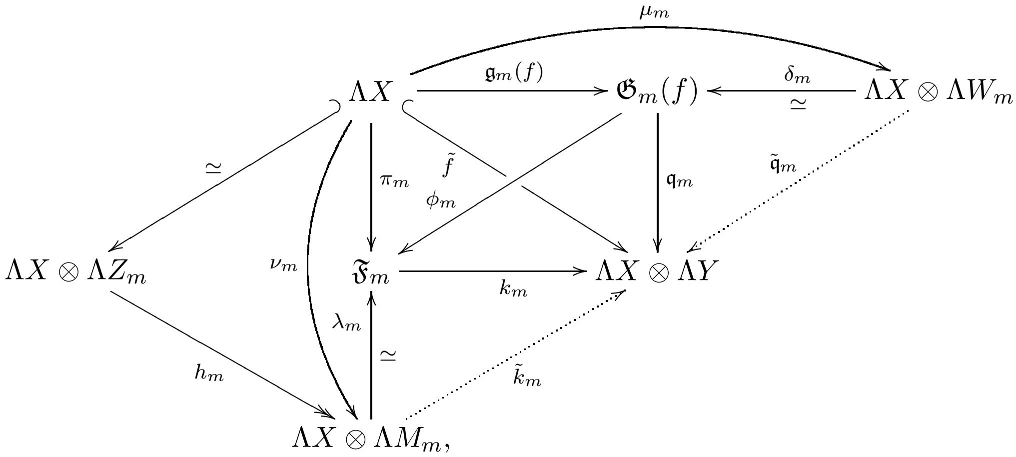
On the other hand we have also that

$$k_1 \circ \phi_1 = k_1 \circ \tilde{\varphi} \circ \phi_1^s = k_1^s \circ \phi_1^s = \mathfrak{q}_1$$

as desired. □

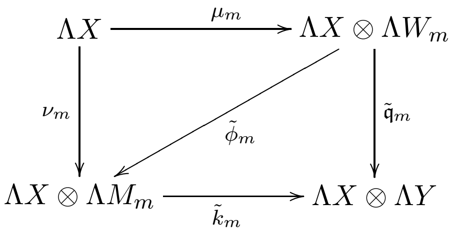
PROOF. [of proposition 6.4.1] Here we can consider directly the general case of h being any surjective model of $\tilde{f}$. We proceed by induction. For $m = 0$ the morphism $h : (\Lambda X \otimes \Lambda Z, d) \to (\Lambda X \otimes \Lambda Y, d)$ induces an isomorphism $\tilde{h} : (\mathfrak{F}_m, d) = \left(\frac{\Lambda X \otimes \Lambda Z}{\mathrm{Ker}(h)}, d\right) \to (\Lambda X \otimes \Lambda Y, d) = (\mathfrak{G}_0(f), d)$. Its inverse gives us ϕ_0.

Let us now proceed with the induction step. We assume that the proposition is true for a certain m. We have therefore the following diagram:

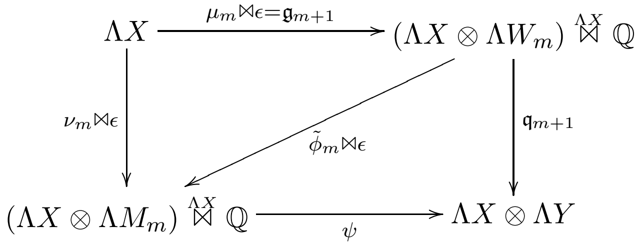


where μ_m and ν_m are relative Sullivan models for $\mathfrak{g}_m(f)$, π_m respectively; h_m is a surjective model of ν_m. The part of the diagram that is made up of solid arrows commutes exactly, but for the lower right triangle, which only commutes up to homotopy. Moreover we have $k_m \circ \lambda_m \simeq \tilde{k}_m$ and $\mathfrak{q}_m \circ \delta_m \simeq \tilde{\mathfrak{q}}_m$. In addition there exists a lift $\tilde{\phi}_m : \Lambda X \otimes \Lambda W_m \to \Lambda X \otimes \Lambda M_m$ such that $\tilde{\phi}_m \circ \mu_m = \nu_m$ and $\lambda_m \circ \tilde{\phi}_m \simeq \phi_m \circ \delta_m$. Therefore we deduce that $\tilde{k}_m \circ \tilde{\phi}_m \simeq \tilde{\mathfrak{q}}_m$.

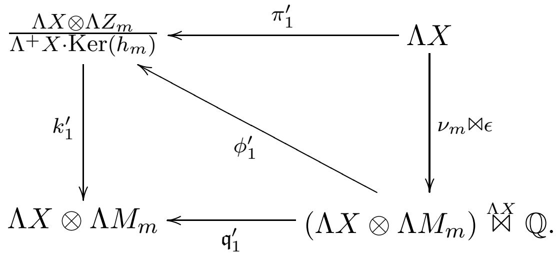
We apply now lemma 6.2.2 to the diagram



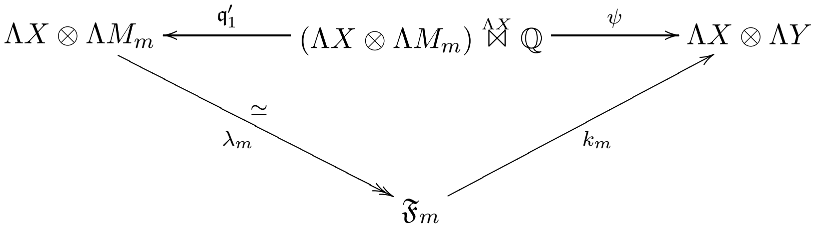
where the upper triangle commutes exactly and the lower one commutes up to homotopy. and use the remark after the lemma to obtain a diagram



after taking the cojoin with the augmentation $\epsilon : \Lambda X \to \mathbb{Q}$. Here again the upper triangle commutes exactly, while the lower triangle commutes up to homotopy. We can now apply lemma 6.4.5 to ν_m to obtain another commutative diagram:



By construction we have in addition that



commutes up to homotopy. To complete the proof it is now sufficient to find a morphism $\theta : \left(\frac{\Lambda X \otimes \Lambda Z_m}{\Lambda^+ X \cdot \mathrm{Ker}(h_m)}, d\right) \to \left(\frac{\Lambda X \otimes \Lambda Z}{\Lambda^{\geq m+1} X \cdot \mathrm{Ker}(h)}, d\right)$ such that $\theta \circ \pi_1' = \pi_{m+1}$ and $k_{m+1} \circ \theta = k_m \circ \lambda_m \circ k_1'$. If such a morphism exists we can indeed define $\phi_{m+1} := \theta \circ \phi_1' \circ (\tilde{\phi}_m \bowtie \epsilon)$, and we verify:

$$\phi_{m+1} \circ \mathfrak{g}_{m+1} = \theta \circ \phi_1' \circ (\phi_m \bowtie \epsilon) \circ \mathfrak{g}_{m+1} = \theta \circ \phi_1' \circ (\nu_m \bowtie \epsilon) = \theta \circ \pi_1' = \pi_{m+1}$$

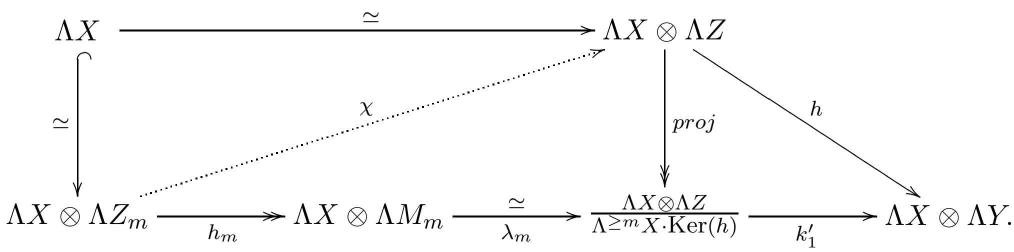
and

$$k_{m+1} \circ \phi_{m+1} = k_{m+1} \circ \theta \circ \phi_1' \circ (\phi_m \bowtie \epsilon) = k_m \circ \lambda_m \circ k_1' \circ \phi_1' \circ (\phi_m \bowtie \epsilon) =$$

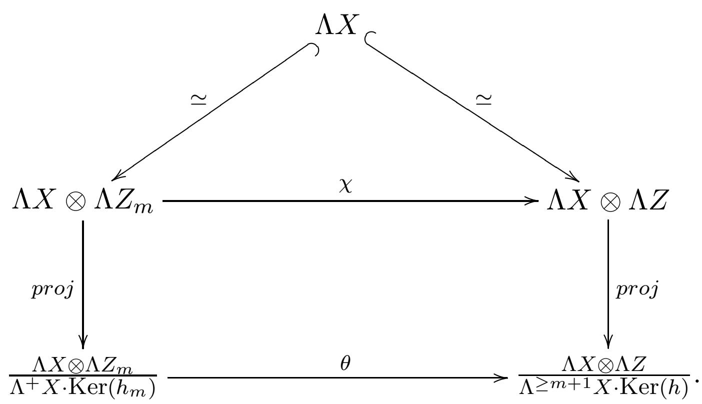
$$k_m \circ \lambda_m \circ \mathfrak{q}_1' \circ (\phi_m \bowtie \epsilon) \simeq \psi \circ (\phi_m \bowtie \epsilon) \simeq \mathfrak{q}_{m+1},$$

as wished.

Notice first of all that we can construct a lift χ in the following commuting diagram:



Since $(proj \circ \chi)(\mathrm{Ker}\,(h_m)) = (\lambda_m \circ h_m)(\mathrm{Ker}\,(h_m)) = 0$, we deduce that $\chi(\mathrm{Ker}\,(h_m)) \subset \mathrm{Ker}\,(proj) = \Lambda^{\geq m} X \cdot \mathrm{Ker}(h)$ and therefore $\chi(\Lambda^+ X \cdot \mathrm{Ker}\,(h_m)) \subset \Lambda^{\geq m+1} X \cdot \mathrm{Ker}(h)$. We can then choose for θ the induced morphism. Consider now the commutative diagram



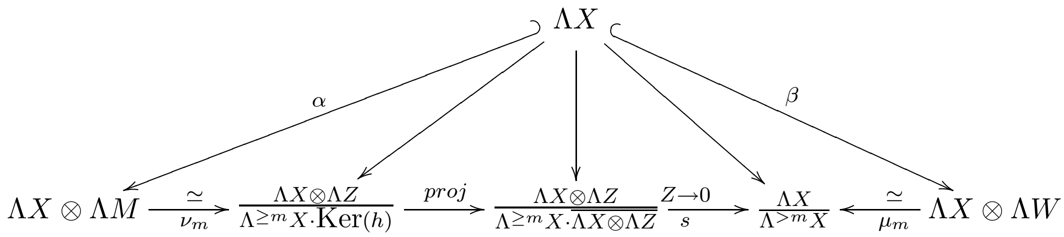
It shows that $\theta \circ \pi_1' = \pi_{m+1}$. On the other hand we see immediately that $k_{m+1} \circ \theta \circ proj = k_m \circ \lambda_m \circ k_1' \circ proj$ and therefore $k_{m+1} \circ \theta = k_m \circ \lambda_m \circ k_1'$. □

6.4.1 Special case: absolute rational category

Before continuing the proof of theorem 5.4.1, we consider here the special case of the map $\{*\} \to X$, or rather its relative Sullivan model $(\Lambda X, d) \to (\Lambda X \otimes \Lambda \bar{X}, d)$ in the rational context. Here the differential d is induced by the differential in $(\Lambda X \otimes \Lambda X \otimes \Lambda \bar{X}, d)$, where $(\Lambda X \otimes \Lambda X, d) \to (\Lambda X \otimes \Lambda X \otimes \Lambda \bar{X}, d)$ is the cofibration associated to the multiplication $(\Lambda X \otimes \Lambda X, d) \to (\Lambda X, d)$.

Notice first of all that in this case the requirement that a homotopy retract r of $(\Lambda X, d) \to (\mathfrak{G}_m(f), d)$ has to fulfill in the definition of the relative rational category is superfluous because, if we denote by δ the quasi-isomorphism $(\Lambda X \otimes \Lambda \bar{X}, d) \to (\mathbb{Q}, 0)$, then for any two morphisms $\alpha, \beta : (\Lambda X, d) \to (\Lambda X \otimes \Lambda \bar{X}, d)$ we verify $\delta \circ \alpha = \delta \circ \beta = \epsilon$, (the augmentation). By the unicity of the homotopy class of liftings, we have therefore that $\alpha \simeq \beta$, that is, every two morphisms from $(\Lambda X, d)$ into $(\Lambda X \otimes \Lambda \bar{X}, d)$ are homotopic.

We have therefore $\mathrm{cat}_o(f) = \mathrm{Rcat}_o(f) = \mathrm{cat}_o(X)$ and we can show theorem 5.4.1 by making use of theorem 5.2.3 and proposition 6.4.1: we suppose X is simply connected as usual. We use the standard surjective model $(\Lambda X \otimes \Lambda \bar{X} \otimes \Lambda \tilde{\bar{X}}, d) \xrightarrow{h} (\Lambda X \otimes \Lambda \bar{X}, d)$. Since $(\Lambda X \otimes \Lambda \bar{X} \otimes \Lambda \tilde{\bar{X}}, d) = (\Lambda X \otimes \Lambda Z, d)$ has no generators in dimension 0, it is an augmented cca, with augmentation ideal $\overline{\Lambda X \otimes \Lambda Z} = \Lambda^+(X \oplus Z)$. This ideal being maximal, it contains $\mathrm{Ker}\,(h)$. We can therefore build the following commuting diagram, where α and β are Sullivan models:



We infer from it the existence of a lift $\tau : (\Lambda X \otimes \Lambda M, d) \to (\Lambda X \otimes \Lambda W, d)$ such that $\tau \circ \alpha = \beta$ and $\mu_m \circ \tau \simeq s \circ proj \circ \nu_m$. Now if there exists a morphism $r : (\Lambda X \otimes \Lambda W, d) \to (\Lambda X, d)$ such that $r \circ \beta \simeq id_{\Lambda X}$ we define $r' := r \circ \tau$ and we verify $r' \circ \alpha = r \circ \tau \circ \alpha = r \circ \beta \simeq id_{\Lambda X}$. This proves

Proposition 6.4.6 *If $\mathrm{cat}_o(X) \leq m$ then there exists a homotopy retract for $(\Lambda X, d) \to (\mathfrak{F}_m, d)$.*

The opposite direction of this inference being shown in section 6.4.1, we have obtained:

Theorem 6.4.7 *The rational LS-category of X is equal to or smaller than m if and only if there exists a homotopy retract for $(\Lambda X, d) \to (\mathfrak{F}_m, d)$.*

6.5 Second part of the proof

Let us now proceed with the second and final part of the proof of theorem 5.4.1. We are going to need the following property of the Ganea cca:

Lemma 6.5.1 *Let $f : (\Lambda X, d) \to (\Lambda X \otimes \Lambda Y, d)$ represent a map $E \to X$. Then*

$$(\mathfrak{G}_m(f), d) \simeq (\mathfrak{G}_{m-1}(X), d) \overset{(\Lambda X, d)}{\bowtie} (\mathfrak{G}_0(f), d).$$

Proof. We know that for "classical" Ganea spaces we have

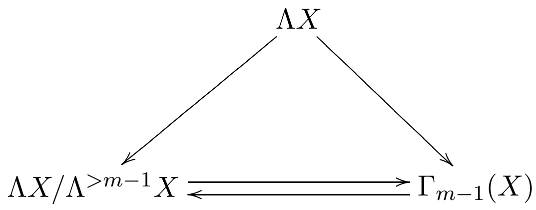
$$G_m f \cong G_{m-1}(X) \bowtie_X G_0(f).$$

We now use lemma 6.2.1 to obtain the wished result. □

On the other hand, by theorem 5.2.3, we can build morphisms

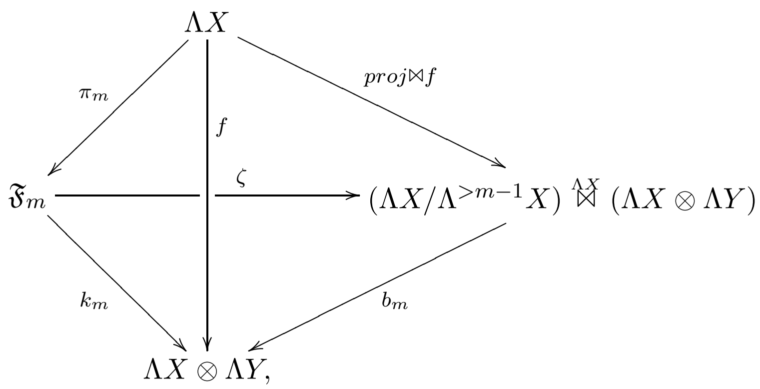
$$(\Lambda X/\Lambda^{>m-1}X, d) \rightleftarrows (\Gamma_{m-1}(X), d)$$

such that the diagram



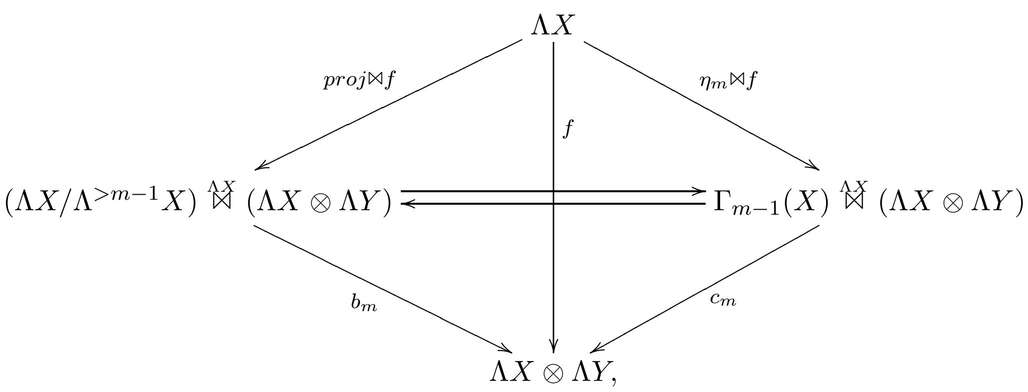
commutes. Bearing in mind that $(\Gamma_{m-1}(X), d)$ is weakly equivalent to $(\mathfrak{G}_{m-1}(X), d)$, we can state the

Proposition 6.5.2 *With the same notations as in section 6.4, if there exists a morphism* $\zeta : (\mathfrak{F}_m, d) \to ((\Lambda X/\Lambda^{>m-1}X), d) \overset{(\Lambda X,d)}{\bowtie} (\Lambda X \otimes \Lambda Y, d)$ *that lets the following diagram commute:*



where b_m is the obvious projection into $\Lambda X \otimes \Lambda Y$ coming from the pull-back, then we have that $\mathrm{Rcat}_o(f) \leq m$ implies the existence of a homotopy retract $r : (\Lambda X \otimes \Lambda M_m, d) \to (\Lambda X, d)$ for π_m. If moreover $\mathrm{cat}_o(f) \leq m$, then $f \circ r \simeq \tilde{k}_m$.

Proof. We use lemma 6.2.2, as well as the second remark following it, and we obtain a commuting diagram



where b_m and c_m are determined in an obvious way by the pull-backs. By composing this diagram with the one from the hypothesis of the proposition, we obtain a morphism from $(\mathfrak{F}_m, d)$ into $(\Gamma_{m-1}(X), d) \overset{(\Lambda X,d)}{\bowtie} (\Lambda X \otimes \Lambda Y, d)$, which according to lemma 6.5.1 is weakly equivalent to $(G_m f, d)$, with all the necessary properties that allow a demonstration of the proposition in the usual way (see for example corollary 6.4.2). $\square$

Therefore it is now sufficient to construct a suitable morphism

$$\zeta : (\mathfrak{F}_m, d) \to (\Lambda X/\Lambda^{>m-1}X, d) \overset{(\Lambda X,d)}{\bowtie} (\Lambda X \otimes \Lambda Y, d)$$

. We begin by giving an explicit construction of the relative cojoin $(\Lambda X/\Lambda^{>m-1}X, d) \overset{(\Lambda X,d)}{\bowtie}$ $(\Lambda X \otimes \Lambda Y, d)$. After the first step of the definition we obtain a commutative diagram

$$\begin{array}{ccc} \Lambda X & \hookrightarrow & \Lambda X \otimes \Lambda Y \\ \downarrow & & \downarrow \mathfrak{p} \\ \Lambda X/\Lambda^{>m-1}X & \xrightarrow{\mathfrak{q}} & \Lambda X/\Lambda^{>m-1}X \otimes \Lambda Y. \end{array}$$

Even though the morphism $\mathfrak{p}$ is surjective, we can use instead of $\mathfrak{q}$ an equivalent surjective morphism:

$$(\Lambda X/\Lambda^{>m-1}X \otimes \Lambda S, d) \underset{\mathfrak{h}}{\twoheadrightarrow} (\Lambda X/\Lambda^{>m-1}X \otimes \Lambda Y, d),$$

where $h_s : (\Lambda X \otimes \Lambda S, d) \to (\Lambda X \otimes \Lambda Y, d)$ is the standard surjective model for f. The morphism $\mathfrak{h}$ is defined as $\mathfrak{h}(\sum_i [x_i] \otimes s_i) = (\mathfrak{p} \circ h_s)(\sum_i x_i \otimes s_i) = \sum_i [x_i] \otimes (\mathfrak{p} \circ h_s(s_i))$, where $x_i \in \Lambda X$, $s_i \in \Lambda S$. It is obviously well-defined. Let us also define a morphism $\mathfrak{p}' : (\Lambda X \otimes \Lambda S, d) \to (\Lambda X/\Lambda^{>m-1}X \otimes \Lambda S, d)$ as being $proj \otimes id_{\Lambda S}$. Therefore $\mathfrak{h} \circ \mathfrak{p}' = \mathfrak{p} \circ h_s$. We can now take the pull-back of $\mathfrak{h}$ with $proj \otimes id_{\Lambda Y} : (\Lambda X \otimes \Lambda Y, d) \to (\Lambda X/\Lambda^{>m-1}X \otimes \Lambda Y, d)$ to obtain $(\Lambda X/\Lambda^{>m-1}X, d) \overset{(\Lambda X,d)}{\bowtie} (\Lambda X \otimes \Lambda Y, d)$. To take advantage of the universal property of pull-backs we must build morphisms α and β that make the following diagram commute:

$$\begin{array}{ccc} \mathfrak{F}_m & \xrightarrow{\alpha} & \Lambda X \otimes \Lambda Y \\ \beta \downarrow & & \downarrow \mathfrak{p} \\ \Lambda X/\Lambda^{>m-1}X \otimes \Lambda S & \underset{\mathfrak{h}}{\twoheadrightarrow} & \Lambda X/\Lambda^{>m-1}X \otimes \Lambda Y \end{array}$$

They would therefore induce the sought after morphism ζ.

- We take $\alpha := k_m$, i.e. the morphism induced by h.
- To define the morphism β we need the map ψ between the surjective model $\Lambda X \otimes \Lambda Z$ and the standard surjective model $\Lambda X \otimes \Lambda S$ which was constructed in lemma 6.4.4. We can now define $\beta([\sum_i x_i \otimes z_i]) := (\mathfrak{p}' \circ \psi)(\sum_i x_i \otimes z_i)$, because $(\mathfrak{p}' \circ \psi)(x) = \mathfrak{p}'(x) = 0$ if $x \in \Lambda^{\geq m}X$.

Let us verify that the preceding diagram commutes: any element of $\Lambda X \otimes \Lambda Z$ is of the form $\sum_i x_i \otimes z_i$ with $x_i \in \Lambda X$ and $z_i \in \Lambda Z$.

- $(\mathfrak{p} \circ \alpha)([\sum_i x_i \otimes z_i]) = \mathfrak{p} \circ h(\sum_i x_i \otimes z_i)$

- $(\mathfrak{h}\circ\beta)([\sum_i x_i\otimes z_i]) = (\mathfrak{h}\circ\mathfrak{p}'\circ\psi)(\sum_i x_i\otimes z_i) = (\mathfrak{p}\circ h_s\circ\psi)(\sum_i x_i\otimes z_i) = (\mathfrak{p}\circ h)(\sum_i x_i\otimes z_i)$.

To prove that the upper triangle of the diagram of proposition 6.5.2 commutes, it suffices to notice that the two maps that induce $proj \bowtie f$ commute with $\alpha\circ\pi_m$ and $\beta\circ\pi_m$ respectively. That the lower triangle commutes is evident from the construction of ζ.