Chapter 5

Rational LS-category

5.1 Introduction

To enable us to use the powerful methods of rational homotopy, which were summarized in chapter 2, we consider only spaces which are simply connected with homology of finite type over $\mathbb{Q}$. As we did in chapter 2, we use the term "cca" to denote a commutative cochain algebra. To save space in diagrams we usually write down only the algebra without the differential "d", but unless otherwise stated all diagrams are diagrams of cca's

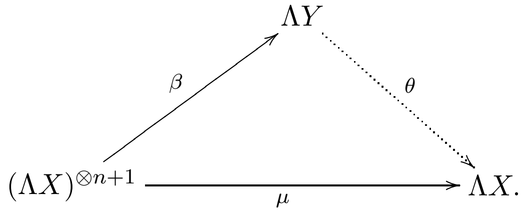
We expose in section 5.2 how the absolute LS-category has been defined in the rational context by Félix and Halperin [FH83]. We then state their fundamental theorems 5.2.2, 5.2.3, which allow computing the rational category of a space directly in the context of commutative cochain algebras. In section 5.3 we give Félix and Halperin's rationalization of the F-category of maps, as well as a theorem 5.3.1 allowing its determination directly in the category of cca's, then we propose a way to rationalize the R-category and the relative LS-category. We finish by stating our main theorem 5.4 in section 5.4, which is actually an equivalent definition of the rational R-category and relative LS-category in the cca setting.

5.2 Absolute case

All definitions and results in this section are taken from [FH83].

Definitions.

- Let $(\Lambda X, d) \xrightarrow{\simeq} (A_{PL}(X), d)$ be a minimal model for X and $\mu : ((\Lambda X)^{\otimes n+1}, d) \to (\Lambda X, d)$ a representative for the diagonal map $X \to X^{n+1}$. Take a minimal model $(\Lambda Y, d) \xrightarrow{\simeq} (A_{PL}(T^{n+1}), d)$ for the fat wedge T^{n+1} and a representative for the inclusion of the fat wedge into X^{n+1}, $\beta : ((\Lambda X)^{\otimes n+1}, d) \to (\Lambda Y, d)$. The ***rational LS-category*** of X, $\mathrm{cat}_o(X)$ is the smallest integer n, such that there exists a morphism of cca's $\theta : (\Lambda Y, d) \to (\Lambda X, d)$, letting the following diagram commute up to homotopy:



- Let ΛX be a Sullivan model and $\phi : (\Lambda X, d) \to (B, d)$ a morphism of cca's. We say that ϕ ***admits a homotopy retraction*** r if there exists a representative for ϕ: $\psi : (\Lambda X, d) \to (\Lambda Y, d)$ and morphism $r : (\Lambda Y, d) \to (\Lambda X, d)$ such that $r \circ \psi \simeq id_{\Lambda X}$. Equivalently, for any representative ψ there exists a morphism r such that $r \circ \psi \simeq id_{\Lambda X}$.

Remarks.

- Félix and Halperin chose to rationalize the definition of LS-category in terms of the fat-wedge. The next results however show that one could have rationalized the definition in terms of Ganea spaces and would have obtained an equivalent definition.

- Since we are considering only simple connected spaces,

$$\operatorname{cat}_o(X) = \operatorname{cat}(X_{\mathbb{Q}}),$$

where $X_{\mathbb{Q}}$ is the *rationalization* of X, which explains why we are talking about the "rationalization" of the LS-category.

Lemma 5.2.1 *Let $(\Lambda X, d) \xrightarrow{\simeq} (A_{PL}(X), d)$ be a minimal model for X. We define the cca $(\Gamma_m(X), d)$ as follows:*

$$\Gamma_m(X) := \frac{(\Lambda X)^{\otimes m+1}}{(\Lambda^+ X)^{\otimes m+1}} \otimes (\Lambda \bar{X})^{\otimes m} = \frac{(\Lambda X)^{\otimes m+1}}{(\Lambda^+ X)^{\otimes m+1}} \otimes_{(\Lambda X)^{\otimes m+1}} (\Lambda X)^{\otimes m+1} \otimes (\Lambda \bar{X})^{\otimes m},$$

i.e. as the tensor product of the projection $((\Lambda X)^{m+1}, d) \to ((\Lambda X)^{m+1}/(\Lambda^+ X)^{m+1}, d)$ and of $((\Lambda X)^{m+1} \to (\Lambda X)^{m+1} \otimes (\Lambda \bar{X})^m, d)$, the cofibration associated to the multiplication $((\Lambda X)^{m+1}, d) \to (\Lambda X, d)$. Then $(\Gamma_m(X), d)$ represents the same rational homotopy type as the Ganea space $G_m(X)$, and any morphism representing the inclusion $((\Lambda X)^{\otimes m+1} \otimes (\Lambda \bar{X})^{\otimes m}, d) \to (\Gamma_m(X), d)$ also represents $g_m(X)$.

We state now two theorems from [FH83] that provide a purely algebraic way to compute the rational category of a space from its minimal Sullivan model.

Theorem 5.2.2 *There exists a homotopy equivalence*

$$(\Gamma_m(X), d) \cong ([\Lambda X/\Lambda^{>m} X] \oplus V, d)$$

where $V = \sum_{p \geq 1} V^p$, the differential in V is zero, and

$$V \cdot ([\Lambda X/\Lambda^{>m} X] \oplus V)^+ = 0.$$

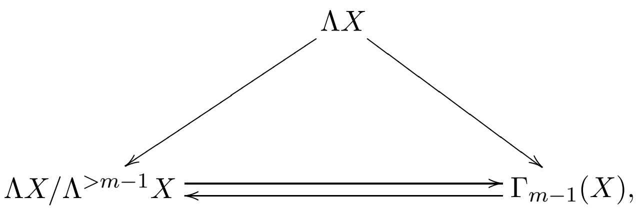
Theorem 5.2.3 *Using notations of Lemma 5.2.1, then the following assertions are equivalent:*

1. $\operatorname{cat}_o(X) \leq m$*;*

2. *there exists a morphism $\eta_m : (\Lambda X, d) \to (\Gamma_m(X), d)$ admitting a homotopy retraction;*

3. *the projection $(\Lambda X, d) \to (\Lambda X/\Lambda^{>m} X, d)$ admits a homotopy retraction.*

Actually there exist morphisms in both directions between $(\Gamma_m(X), d)$ *and* $(\Lambda X/\Lambda^{>m}X, d)$ *making the following diagram commute*



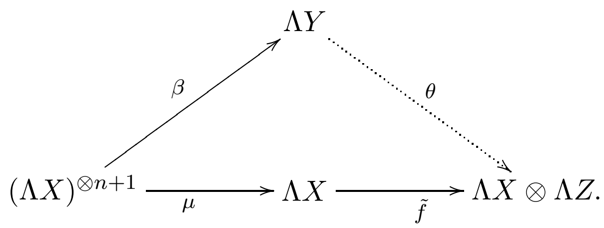
which obviously induces the equivalence between points 2 and 3.

Remark. The equivalence between point (1) and point (2) represents the equivalence between the fat-wedge definition and the Ganea definition. Point (3) is a major result that allows an easier calculation of rational category. Our purpose is to give a corresponding algebraic formula determining the LS-category of a map.

5.3 Relative case: definitions

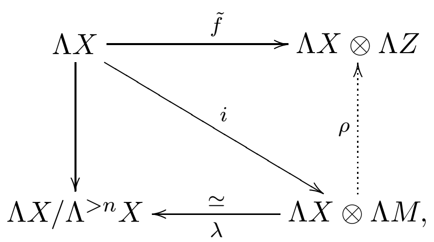
In [Fél89] a relative rational category was already defined, it is a rationalization of what we call the classical F-category (see section 3.3):

Definition. Let $\tilde{f} : (\Lambda X, d) \to (\Lambda X \otimes \Lambda Z, d)$ be a relative Sullivan model for a cofibration $f : E \to X$ and $\mu : ((\Lambda X)^{\otimes n+1}, d) \to (\Lambda X, d)$ a representative for the diagonal map $X \to X^{n+1}$. Choose a minimal model $(\Lambda Y, d) \to (A_{PL}(T^{n+1}(X)), d)$ for the fat wedge $T^{n+1}(X)$ and a representative $\beta : ((\Lambda X)^{\otimes n+1}, d) \to (\Lambda Y, d)$ for the inclusion of the fat wedge into X^{n+1}. The ***rational G-category*** of X, $\mathrm{Fcat}_o(X)$ is the smallest integer m, such that there exists a morphism of cca's $\theta : (\Lambda Y, d) \to (\Lambda X \otimes \Lambda Z, d)$, that makes the following diagram commute up to homotopy:



An algebraic condition involving only the Sullivan model of f, and allowing to compute $\mathrm{Fcat}_o(f)$ was given in [Fél89]:

Theorem 5.3.1 *With notations as above,* $\mathrm{Fcat}_o(f) \leq n$ *if and only if there exists a morphism* ρ *such that the following diagram commutes up to homotopy:*



where λ *is a Sullivan model, and* i *is a representative for the projection.*

Remark. As it is the case for the absolute LS-category, $\mathrm{Fcat}_o(f) = \mathrm{Fcat}(f_{\mathbb{Q}})$, where $f_{\mathbb{Q}}$ is the rationalization of the map f.

On the other hand it is also possible to rationalize the R-category or the LS-category of a map. We choose to rationalize the definition involving Ganea spaces and do not show that it is equivalent to a rational version of the fat-wedge definition, although this is certainly true: using the *spatial realization functor* from cca's to CW-complexes, it certainly follows that

$$\mathrm{Rcat}_o(f) = \mathrm{Rcat}(f_{\mathbb{Q}}) \quad \text{and} \quad \mathrm{cat}_o[f] = \mathrm{cat}(f_{\mathbb{Q}}).$$

On the other hand it was shown in section 3.3 that the definitions in terms of Ganea spaces and fat wedges are equivalent in the category of topological spaces. Use of the A_{PL} functor would then prove the assertion.

Definitions.

- Let $f : E \to X$ be a cofibration, $(\Lambda X, d)$ and $(\Lambda W, d)$ be Sullivan models of spaces X, respectively $G_m(f)$, and $\tilde{f} : (\Lambda X, d) \to (\Lambda X \otimes \Lambda Y, d)$ a relative Sullivan model of f. Moreover let $\gamma : (\Lambda X, d) \to (\Lambda W, d)$ represent the Ganea map $g_m(f)$, and $\delta : (\Lambda W, d) \to (\Lambda X \otimes \Lambda Y, d)$ represent q_n. Then $\delta \circ \gamma$ represents f and therefore it is homotopic to $\tilde{f}$.

$$\begin{array}{ccc} \Lambda X & \xrightarrow{\gamma} & \Lambda W \\ & \searrow^{\tilde{f}} & \downarrow{\scriptstyle \delta} \\ & & \Lambda X \otimes \Lambda Y \end{array}$$

  We say that the ***relative R-category*** of f ($\mathrm{Rcat}_o(f)$) is equal to or smaller than m if γ admits a homotopy retraction $r : (\Lambda W, d) \to (\Lambda X, d)$.

- If in the definition of relative R-category we require moreover that $\tilde{f} \circ r \simeq \delta$, we say that the ***relative LS-category of*** f is equal to or smaller than m ($\mathrm{cat}_o(f) \leq m$).

- Let $f : (\Lambda X, d) \to (\Lambda X \otimes \Lambda Y, d)$ be a relative Sullivan model. We can then construct the commuting diagram

$$\begin{array}{ccccc} \Lambda X & & \xrightarrow{\quad f \quad} & & \Lambda X \otimes \Lambda Y \\ & {\scriptstyle g,\ \simeq}\searrow & & \nearrow{\scriptstyle h} & \\ & & \Lambda X \otimes \Lambda Z, & & \end{array}$$

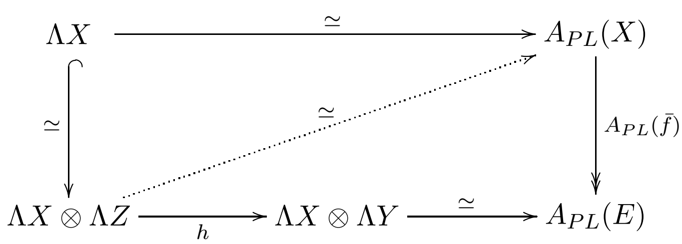
  where g is the inclusion of the base, $Z = Y \oplus \tilde{Y}$, with $d : Y^p \xrightarrow{\cong} \tilde{Y}^{p+1}$, $dy = \tilde{y}$, and h is defined as follows:

$$\begin{array}{rlcl} h: & \Lambda X \otimes \Lambda Z & \longrightarrow & \Lambda X \otimes \Lambda Y \\ & x \in X & \longmapsto & x \\ & y \in Y & \longmapsto & y \\ & \tilde{y} \in \tilde{Y} & \longmapsto & dy. \end{array}$$

  It is easy to verify that h is surjective, and it is therefore called the ***standard surjective model of*** f.

- Any morphism $h : (\Lambda X \otimes \Lambda Z, d) \to (\Lambda X \otimes \Lambda Y, d)$ that is surjective and such that the diagram in the previous definition commutes is called a ***surjective model for*** f.

REMARK. If f is a Sullivan model for a cofibration $\bar{f} : E \to X$, then the solid arrows diagram



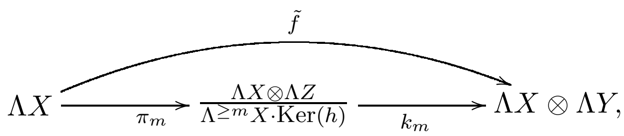
commutes, $A_{PL}(\bar{f})$ is surjective, and there exists a lift, drawn as a pointed arrow, which makes the whole diagram commute. Here A_{PL} denotes the Sullivan functor. This shows that h also represents $\bar{f}$. Notice however that $(\Lambda X \otimes \Lambda Z, d) \to A_{PL}(X)$ is generally not a minimal model even if $(\Lambda X, d) \to A_{PL}(X)$ is.

5.4 Main theorem

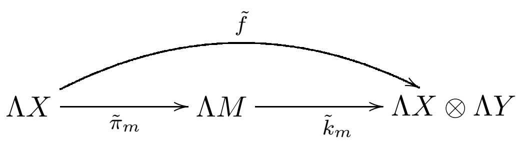
Theorem 5.4.1 *Let $f : E \to X$ be a cofibration, $(\Lambda X, d) \to A_{PL}(X)$ a Sullivan model for X, and $\tilde{f} : (\Lambda X, d) \to (\Lambda X \otimes \Lambda Y, d)$ a Sullivan model for it. Let $h : (\Lambda X \otimes \Lambda Z, d) \to (\Lambda X \otimes \Lambda Y, d)$ be any surjective model for $\tilde{f}$. Define*

$$\pi_m : (\Lambda X, d) \to \left(\frac{\Lambda X \otimes \Lambda Z}{\Lambda^{\geq m} X \cdot \operatorname{Ker} h}, d \right)$$

as the map induced by the inclusion of the base. Then



where k_m is induced by h, commutes. Let us choose a Sullivan model $(\Lambda M, d) \xrightarrow{\simeq} \left(\frac{\Lambda X \otimes \Lambda Z}{\Lambda^{\geq m} X \cdot \operatorname{Ker}(h)}, d \right)$ and representatives $\tilde{\pi}_m$ and $\tilde{k}_m$ for π_m and k_m respectively. They let the diagram



commute up to homotopy. Then

- $\operatorname{Rcat}_o(f) \leq m$ *if and only if $\tilde{\pi}_m$ admits a homotopy retraction;*
- *$cat_o(f) \leq m$ if and only if $\tilde{\pi}_m$ admits a homotopy retraction r, such that $\tilde{f} \circ r \simeq \tilde{k}_m$.*

Chapter 6 is devoted to the proof of this theorem.