Chapter 4

Product formulas

4.1 Introduction

As was said in chapter 3 there exist formulas giving upper bounds for the LS-category and the cone-length of a product of spaces. We derive formulas for the cone-length of a product of maps in section 4.2 by constructing new cones out of the spaces involved in the cone decomposition of the two original maps. As a remark we check that the obtained formula is also valid when working with the a-cone-length which was introduced by Scheerer and Tanré [ST99] as long as the class of spaces a satisfies certain conditions. We then use the decomposition for the cone-length of a product of maps to obtain bounds for its F-category, its R-category and its LS-category in section 4.3. To do so we first prove a "quasi-lifting" lemma to allow some of the diagrams involved in the adjunction of a cone to a space to commute up to homotopy.

In this chapter we work in the category of pointed CW-complexes.

4.2 Product formula for relative cone-length

We derive an inequality formula for the cone-length of the product of two maps, depending only on the cone-length of the maps and the cone-length of their source spaces. We use the "bullet construction" which was introduced by Stanley in [Sta98] to construct new cofibration sequences out of the cofibration sequences involved in determining the cone-length of each map and source space.

Definition. Let $B \xrightarrow{i} C$ and $B' \xrightarrow{i'} C'$ be cofibrations, and the following commutative diagram

$$\begin{tikzcd} B \times B' \arrow[r, "id\times i'"] \arrow[d, "i\times id"'] & B \times C' \arrow[d] \\ C \times B' \arrow[r] & X \end{tikzcd}$$

be a pushout. Then we write $(C \times C')^{\bullet} \equiv X$.

Lemma 4.2.1 *Let $E \xrightarrow{f} F$ be a cofibration and*

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "f"'] & G \arrow[d] \\
F \arrow[r] & H
\end{tikzcd}$$

be a pushout. Then for every cofibration $A \xrightarrow{i} X$ there exists a pushout:

$$\begin{tikzcd}
(F \times X)^\bullet \arrow[r] \arrow[d] & (H \times X)^\bullet \arrow[d] \\
F \times X \arrow[r] & H \times X.
\end{tikzcd}$$

Proof. We first notice that the product with a space X and pushout commute. This is an easy consequence of

$$Z^{X \times Y} \cong (Z^Y)^X.$$

Considering the definition of $(F \times X)^\bullet$ and $(H \times X)^\bullet$, we can therefore build the following commutative diagram,

$$\begin{tikzcd}
 & E \times A \arrow[rr] \arrow[dl] \arrow[ddd] & & E \times X \arrow[dl] \arrow[ddr] \arrow[ddd] & \\
F \times A \arrow[rr] \arrow[drrrr, bend right=10] \arrow[ddd] & & (F \times X)^\bullet \arrow[drr, dotted] \arrow[ddd, dotted] & & \\
 & & & & F \times X \arrow[ddd] \\
 & G \times A \arrow[rr] \arrow[dl] & & G \times X \arrow[dl] \arrow[ddr] & \\
H \times A \arrow[rr] \arrow[drrrr, bend right=10] & & (H \times X)^\bullet \arrow[drr, dotted] & & \\
 & & & & H \times X
\end{tikzcd}$$

where the top, the bottom and the left faces of the cube are pushouts, and the broken arrows are induced by pushouts. By lemma 1.2.4 the right face is then also a pushout. Since the right face of the prism

$$\begin{tikzcd}
 & E \times X \arrow[dl] \arrow[dr] \arrow[dd] & \\
(F \times X)^\bullet \arrow[rr] \arrow[dd] & & F \times X \arrow[dd] \\
 & G \times X \arrow[dl] \arrow[dr] & \\
(H \times X)^\bullet \arrow[rr] & & H \times X
\end{tikzcd}$$

is a pushout, we have again by lemma 1.2.4 that the front face of the prism is a pushout. □

Lemma 4.2.2 *Let $A \longrightarrow B \longrightarrow C$ and $A' \longrightarrow B' \longrightarrow C'$ be cofibration sequences. Then there exists a cofibration sequence of the form:*

$$(CA \times CA')^\bullet \longrightarrow (C \times C')^\bullet \longrightarrow C \times C'.$$

Remark. Notice that $(CA \times CA')^\bullet \cong A * A'$, the join of A and A'.

Proof. Apply lemma 4.2.1 to the following pushout

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ CA & \longrightarrow & C \end{array}$$

and to the cofibration $A' \longrightarrow CA'$ to obtain the pushout

$$\begin{array}{ccc} (CA \times CA')^\bullet & \longrightarrow & (C \times CA')^\bullet \\ \downarrow & & \downarrow \\ CA \times CA' & \longrightarrow & C \times CA'. \end{array}$$

Apply again lemma 4.2.1, this time to the pushout

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ CA' & \longrightarrow & C' \end{array}$$

and to the cofibration $B \longrightarrow C$ to obtain the pushout

$$\begin{array}{ccc} (C \times CA')^\bullet & \longrightarrow & (C \times C')^\bullet \\ \downarrow & & \downarrow \\ C \times CA' & \longrightarrow & C \times C'. \end{array}$$

Composing the two pushouts we obtain

$$\begin{array}{ccc} (CA \times CA')^\bullet & \longrightarrow & (C \times C')^\bullet \\ \downarrow & & \downarrow \\ CA \times CA' & \longrightarrow & C \times C' \end{array}$$

which is also a pushout. Since $C(A * A') \cong CA \times CA'$, the result follows. $\square$

We have now enough cofibration sequences to prove our

Theorem 4.2.3 *Let $f : A \to X$, $f' : A' \to X'$ be maps such that $Cl(A)$, $Cl(A')$, $Cl(X, A)$, $Cl(X', A')$ are finite. Then we have for the map $f \times f'$:*

$$Cl(X \times X', A \times A') \leq \max\{Cl(A), Cl(A')\} + Cl(X, A) + Cl(X', A').$$

Proof. Suppose there exist two sequences of cofibration sequences

$$W(i) \longrightarrow A(i-1) \longrightarrow A(i) \qquad U(j) \longrightarrow A'(j-1) \longrightarrow A'(j)\ ,$$

with

- $1 \le i \le s+p$, $A(0) \simeq \{*\}$, $A(s) \simeq A$ $(Cl(A) \le s)$, $A(s+p) \simeq X$ and $A \to A(i) \to X$ is homotopic to f for all $s \le i \le s+p$ $(Cl(X,A) \le p)$;
- $1 \le j \le k+n$, $A'(0) \simeq \{*\}$, $A'(k) \simeq A'$ $(Cl(A') \le k)$, $A'(k+n) \simeq X'$ $(Cl(X',A') \le n)$ and $A' \to A(j) \to X'$ is homotopic to f' for all $k \le j \le k+n$.

The inclusions $A(i) \subset A(s+p)$, $A'(j) \subset A'(k+n)$ are cofibrations. Let us suppose, without loss of generality, that $s \ge k$. We can then define the following subspaces of $A(s+p) \times A'(k+n)$, for $0 \le g \le p+s+n$:

$$M(g) \equiv (A(s) \times A'(k)) \cup \bigcup_{\substack{\theta(g)\le i\le p\\ 0\le j\le k+n\\ i+j=g}} (A(s+i) \times A'(j)) \cup \bigcup_{\substack{0\le i\le s+p\\ 0\le j\le n\\ i+j=g}} (A(i) \times A'(k+j))$$

where $\theta(g) = 0$ for $0 \le g \le k$, and $\theta(g) = g-k$ for $k+1 \le g \le s+p+n$, and the unions are taken over subspaces of $A(s+p) \times A'(k+n)$. We claim that for each $1 \le g \le s+p+n$ there exist a space $V(g)$ and a cofibre sequence $V(g) \longrightarrow M(g-1) \longrightarrow M(g)$, and that moreover we have $M(0) \simeq A \times A'$, $M(s+p+n) \simeq X \times X'$, and $A \times A' \to M(g) \to X \times X'$ is homotopic to $f \times f'$, which proves the proposition under our assumptions.

We define:

$$V(g) \equiv (W(s+g) \times *) \vee \bigvee_{\substack{\theta(g)\le i\le g\\ 1\le j\le g-1\\ i+j=g}} (W(s+i) * U(j)) \vee \bigvee_{\substack{1\le i\le g-1\\ 0\le j\le g\\ i+j=g}} (W(i) * U(k+j)) \vee (* \times U(k+g)).$$

Using lemma 4.2.2 and the cone decompositions of X and X' we obtain the following pushouts:

$$\begin{tikzcd} W(s+g) \times * \arrow[r] \arrow[d] & A(s+g-1) \times * \arrow[d] \\ CW(s+g) \times * \arrow[r] & A(s+g) \times *, \end{tikzcd} \qquad \begin{tikzcd} W(s+i) * U(j) \arrow[r] \arrow[d] & (A(s+i) \times A'(j))^{\bullet} \arrow[d] \\ CW(s+i) \times CU(j) \arrow[r] & A(s+i) \times A'(j), \end{tikzcd}$$

$$\begin{tikzcd} * \times U(k+g) \arrow[r] \arrow[d] & * \times A'(k+g-1) \arrow[d] \\ * \times CU(k+g) \arrow[r] & * \times A'(k+g), \end{tikzcd} \qquad \begin{tikzcd} W(i) * U(k+j) \arrow[r] \arrow[d] & (A(i) \times A'(k+j))^{\bullet} \arrow[d] \\ CW(i) \times CU(k+j) \arrow[r] & A(i) \times A'(k+j), \end{tikzcd}$$

with the obvious associated cofibre sequences. We now only have to check that the spaces we are attaching a cone to are subspaces of $M(g-1)$ for all i and j that we are concerned with.

1. For all g such that $\theta(g) \le g \le p$ we have $\theta(g-1) \le g-1 \le p$, therefore $(A(s+g-1) \times *) \subset M(g-1)$. Analogously we find $(* \times A'(k+g-1)) \subset M(g-1)$.

2. To build $(A(s+i)\times A'(j))^\bullet$ we need spaces $A(s+i)\times A'(j-1)$ and $A(s+i-1)\times A'(j)$. Remember that $\theta(g)\leq i\leq p$, $1\leq j\leq k+n$, $i+j=g$. Then obviously $i\geq\theta(g-1)$, $0\leq j-1\leq k+n$ and $A(s+i)\times A'(j-1)\subset M(g-1)$. For $A(s+i-1)\times A'(j)$ we must consider two cases:

$g\leq k$: We have either $i\geq 1$ and therefore $A(s+i-1)\times A'(j)\subset M(g-1)$, or $i=0$. In the second case the space we consider is

$$A(s-1)\times A'(g)\subset A(s)\times A'(k)\simeq A\times A'\subset M(g-1).$$

$g>k$: Then $\theta(g)=g-k>0$ and $i-1\geq g-1-k=\theta(g-1)$. Hence $A(s+i-1)\times A'(j)\subset M(g-1)$.

3. To build $(A(i)\times A'(k+j))^\bullet$ we need spaces $A(i)\times A'(k+j-1)$ and $A(i-1)\times A'(k+j)$. Here $1\leq i\leq s+p$, $0\leq j\leq n$, $i+j=g$. Obviously $A(i-1)\times A'(k+j)\subset M(g-1)$. If $j\neq 0$, we also see easily that $A(i)\times A'(k+j-1)\subset M(g-1)$. Otherwise we must consider $A(g)\times A'(k-1)$.

$g\leq k$: Then $g\leq s$ and $A(g)\times A'(k-1)\subset A(s)\times A'(k)\subset M(g-1)$.

$g>k$: Then $g=k+r$ with $r>0$ and therefore $s+r\geq g$. This shows that $A(g)\times A'(k-1)\subset A(s+r)\times A'(k-1)$. Now $r=g-k\geq g-1-k=\theta(g-1)$, then $A(s+r)\times A'(k-1)\subset M(g-1)$.

It is now obvious that $M(0)\simeq A\times A'$ and $X\times X'\simeq A(s+p)\times A'(k+n)=M(s+p+n)$, and that $A\times A'\to M(g)\to X\times X'$ is equal to $A\times A'\overset{\cong}{\to}A(s)\times A'(k)\subset M(g)\subset A(s+p)\times A'(k+n)\overset{\cong}{\to}X\times X'$ which is homotopic to $f\times f'$. □

REMARK. In [ST99] Scheerer and Tanré introduced the new absolute invariant $a-Cl$, which is defined like the normal cone-length with the difference that in any cofibration sequence $A\to X(i)\to X(i+1)$ involved in determing it the space A must be an element of a fixed class of spaces a. We can obviously extend this definition to the a-cone-length of a map by making the same modification. For certain types of classes of maps a our product formula remains valid.

Corollary 4.2.4 *Let a be a class of spaces, such that, for all $U,V\in a$ then*

- *if a space W is homeomorphic to U, then $W\in a$,*
- $U*V\in a$,
- $U\vee V\in a$.

Let $f:A\longrightarrow X$, $f':A'\longrightarrow X'$ be two maps with $a-Cl(A)$, $a-Cl(A')$, $a-Cl(f)$, $a-Cl(f')$ finite. Then

$$a-Cl(f\times f')\leq\max\{a-Cl(A),a-Cl(A')\}+a-Cl(f)+a-Cl(f').$$

PROOF. It suffices to notice that in the definition of $V(g)$ in the proof of the previous proposition there only appear wedges of joins of spaces in a. □

We can now relate to the properties of closed classes as demonstrated in [Far95] to show:

Corollary 4.2.5 *If a is a closed class, and f, f', A, A', X, X' are like in the previous corollary, then*

$$a - Cl(f \times f') \leq \max\{a - Cl(A), a - Cl(A')\} + a - Cl(f) + a - Cl(f').$$

4.3 Product formulas for relative categories

If their source spaces have finite cone-length, it is also possible to find upper bounds for the F-category, the R-category and the LS-category of maps: we build maps from the product of two Ganea spaces for different maps f, f' into the Ganea space of the product $f \times f'$ using the cone-decomposition of the latter which was used to prove theorem 4.2.3. We show that such maps let some diagrams (homotopy) commute, which allows us to define a section for the Ganea map of the product $f \times f'$ using sections for the Ganea maps of f and f'.

In the proof of proposition 4.3.2 we will use several times the following lemma, which constructs a map between the homotopy cofibres of two maps while keeping some diagram commuting. It can be seen as a "quasi-lifting lemma".

Lemma 4.3.1 *Let*

$$A \xrightarrow{i} X \xrightarrow{k} X \cup_i CA,$$
$$F \xrightarrow{j} E \xrightarrow{p} B$$

be respectively a cofibration sequence and a fibration, such that there exist maps f and g like in the following diagram, letting it commute exactly

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, tail, "k"'] & E \arrow[d, two heads, "p"] \\ X \cup_i CA \arrow[r, "g"'] & B. \end{tikzcd}$$

If $l : E \to E \cup_F CF$ denotes the obvious inclusion in the homotopy cofibre of j, and $q : E \cup_F CF \to B$ is defined as equal to p on E and to $*$ on CF, then there exists a map $\Phi : X \cup_i CA \to E \cup_F CF$ such that $\Phi \circ k = l \circ f$ and $q \circ \Phi \simeq g \,\mathrm{rel}\, X$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, tail, "k"'] & E \arrow[r, "l"] \arrow[d, two heads, "p"] & E \cup_F CF \arrow[dl, "q"] \\ X \cup_i CA \arrow[r, "g"'] \arrow[urr, dotted, "\Phi"] & B. & \end{tikzcd}$$

Moreover, if we consider the fibration p' associated to q:

$$\begin{tikzcd} E \cup_F CF \arrow[rr, "\lambda", "\simeq"'] \arrow[dr, "q"'] & & E' \arrow[dl, two heads, "p'"] \\ & B, & \end{tikzcd}$$

then there exists a map $\Psi : X \cup_i CA \to E'$ such that

- $\Psi \simeq \lambda \circ \Phi$,
- $\Psi \circ k = \lambda \circ l \circ f$ *and*
- $p' \circ \Psi = g$.

PROOF. We begin by defining an injection $\alpha : A \to A \times I$ as $\alpha(a) = (a, 0)$ and a map $\gamma : A \times I \to X \cup_i CA$, the obvious projection, which make diagram

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "\alpha"'] & X \arrow[r, "f"] \arrow[d, tail, "k"'] & E \arrow[d, two heads, "p"] \\
A \times I \arrow[r, "\gamma"'] & X \cup_i CA \arrow[r, "g"'] & B
\end{tikzcd}$$

commute. By the homotopy lifting property of fibrations, there exists a homotopy $H : A \times I \to E$ such that $H(a, 0) = (f \circ i)(a)$ for all $a \in A$ and $p \circ H = g \circ \gamma$. We therefore notice that for all $a \in A$ we have

$$p \circ H(a, 1) = (g \circ \gamma)(a, 1) = g(*) = *,$$

which means that $H(A \times \{1\}) \subset F$. We are now ready to define

$$\Phi : X \cup_i CA \longrightarrow E \cup_F CF$$

$$\Phi([x]) = [f(x)] \qquad \forall x \in X$$
$$\Phi([a, t]) = \begin{cases} [H(a, 2t)] & 0 \leq t \leq 1/2,\ a \in A \\ [H(a, 1), 2t - 1] & 1/2 \leq t \leq 1,\ a \in A. \end{cases}$$

It is easy to check that Φ is well-defined by using $[H(a, 1)] = [H(a, 1), 0]$ and

$$\Phi([i(a)]) = [f(i(a))] = [H(a, 0)] = \Phi([a, 0]).$$

It therefore remains to show that $g \simeq q \circ \Phi$. We define explicitely the homotopy between these two maps:

$$G : (X \cup_i CA) \times I \longrightarrow B$$

$$G([x], s) = g([x]) = (p \circ f)(x) \qquad x \in X,\ s \in I$$
$$G([a, t], s) = \begin{cases} g([a, (s + 1)t]) & 0 \leq (s + 1)t \leq 1,\ a \in A \\ * & (s + 1)t \geq 1,\ a \in A. \end{cases}$$

It is a well-defined map because, when $(s + 1)t = 1$, we have $g([a, 1]) = g(*) = *$, and $G([a, 0], s) = g([a, 0]) = g([i(a)]) = G([i(a)], s)$. It is easy to check that G is an homotopy between g and $q \circ \Phi$ which is constant on X.

Let us now consider the inclusion $Y \equiv (X \times I) \cup_{i \times id} (CA \times \{0\}) \xrightarrow{\gamma} (X \cup_i CA) \times I \equiv Z$. It is actually a cofibration, because $CA \to CA \times I$ is a cofibration, and it is easy to check that it is a homotopy equivalence. We therefore obtain the following commutative **solid** diagram:

$$\begin{tikzcd}
Y \arrow[r, "(\lambda \circ l \circ f \circ pr_X) \cup (\lambda \circ \Phi)"] \arrow[d, tail, "\simeq"', "\gamma"] & E' \arrow[d, two heads, "p'"] \\
Z \arrow[r, "-G"'] \arrow[ur, dotted, "F"] & B,
\end{tikzcd}$$

where $pr_X : X \times I \to X$ is the projection on the first component. Since Y is closed in Z, (Z, Y) is an NDR-pair and according to section 2.7 there exists then a map $F : Z \to E'$ such that

$$F \circ \gamma = (\lambda \circ l \circ f \circ pr_X) \cup (\lambda \circ \Phi)$$

and

$$p' \circ F = -G.$$

We define now $\Psi : X \cup CA \to E'$ by $\Psi(z) \equiv F(z, 1)$, and we verify

- $\lambda \circ \Phi \simeq \Psi$, because $F([x], t) = (\lambda \circ l \circ f)(x) = \lambda \circ \Phi([x])$ for $x \in X$, and $F([a, s], 0) = \lambda \circ \Phi([a, s])$, where $a \in A$ and $s \in I$. Therefore $F(-, 0) = \lambda \circ \Phi$ and F is a homotopy between $\lambda \circ \Phi$ and Ψ.
- $(\Psi \circ k)(x) = \Psi([x]) = F([x], 1) = (\lambda \circ l \circ f)(x)$, where $x \in X$.
- $(p' \circ \Psi)(z) = (p' \circ F(z, 1)) = -G(z, 1) = G(z, 0) = g(z)$, where $z \in Z$.

□

We can apply this procedure to Ganea spaces and maps:

Proposition 4.3.2 *Let $f : E \to B$ and $f' : E' \to B'$ be two continuous maps and let denote by $F_n \to G_n \xrightarrow{g_n} B$, $F'_n \to G'_n \xrightarrow{g'_n} B'$ and $\tilde{F}_n \to \tilde{G}_n \xrightarrow{\tilde{g}_n} B \times B'$ the n-th Ganea fibration of f, f' and $f \times f'$ respectively. Let also denote by $q_n : E \to G_n$, $q'_n : E' \to G'_n$ and $\tilde{q}_n : E \times E' \to \tilde{G}_n$ the respective injections.*

If $p \equiv \max\{Cl(E), Cl(E')\}$ is finite, then for all $n, m \in \mathbb{N}$ there exists a map $\Phi(n, m) : G_n \times G'_m \to \tilde{G}_{n+m+p}$ such that the diagram

$$\begin{tikzcd} G_n \times G'_m \arrow[rr, "\Phi(n,m)"] \arrow[dr, "g_n \times g'_m"'] & & \tilde{G}_{n+m+p} \arrow[dl, "\tilde{g}_{n+m+p}"] \\ & B \times B' & \end{tikzcd}$$

commutes exactly, and $\Phi(n, m) \circ (q_n \times q'_m) \simeq \tilde{q}_{n+m+p}$.

Proof. We recall that for any Ganea fibration $G_k(g) \xrightarrow{g_k(g)} Y$ of a map $g : X \to Y$, q_k has relative cone length $Cl(q_k) \leq k$, which allows us to construct an (n+m+p)-relative cone decomposition of $g_n \times g'_m$ by using 4.2.3. We therefore consider a sequence of cofibre sequences

$$\begin{gathered} V(1) \longrightarrow M(0) \longrightarrow M(1) \\ \vdots \\ V(n+m+p) \longrightarrow M(n+m+p-1) \longrightarrow M(n+m+p) \end{gathered}$$

with

$$E \times E' \overset{\theta}{\simeq} M(0) \subset M(1) \subset ... \subset M(n+m+p) \overset{\nu}{\simeq} G_n \times G'_m$$

and

$$E \times E' \xrightarrow{i_k} M(k) \xrightarrow{p_k} G_n \times G'_m \simeq q_n \times q'_m.$$

We denote by *incl* any inclusion such as $M(k) \subset M(j)$ where $j \geq k$.

There exists therefore a diagram

$$\begin{tikzcd}
V(1) \arrow[r] & M(0) \arrow[r, "incl"] & M(1) \arrow[r, "incl"] & M(n+m+p) \arrow[r, "\nu", "\simeq"'] & G_n \times G'_m \arrow[d, "g_n \times g'_m"] \\
\tilde{F}_0 \arrow[r] \arrow[dr, "j"'] & E \times E' \arrow[u, "\simeq", "\theta"'] \arrow[rrr, "f \times f'"] \arrow[d, "\simeq"', "\tilde{q}_0"] & & & B \times B' \\
& \tilde{G}_0 \arrow[urrr, "\tilde{g}_0"'] & & &
\end{tikzcd}$$

commuting up to homotopy, because $(g_n \times g'_m) \circ p_{n+m+p} \circ i_{n+m+p} \simeq (g_n \times g'_m) \circ (q_n \times q'_m) \simeq f \times f'$.

We prove now by induction that for each $n + m + p \geq k \geq 0$ there exists a map $\beta(k) : M(k) \to \tilde{G}_k$ such that the following diagram commutes:

$$\begin{tikzcd}
M(k) \arrow[rr, "\beta(k)"] \arrow[dr, "(g_n \times g'_m) \circ p_k"'] & & \tilde{G}_k \arrow[dl, "\tilde{g}_k"] \\
& B \times B', &
\end{tikzcd}$$

while $$\beta(k) \circ incl \circ \theta \simeq \tilde{q}_k.$$

We begin by defining $\bar{\beta}(0) : M(0) \to \tilde{G}_0$ as being $\tilde{q}_0 \circ \eta$, with η the homotopy inverse of θ. We have $(g_n \times g'_m) \circ p_0 \simeq \tilde{g}_0 \circ \bar{\beta}(0)$. Since $\tilde{g}_0$ is a fibration, there exists a map $\beta(0) \simeq \bar{\beta}(0)$ such that $(g_n \times g'_m) \circ p_0 = \tilde{g}_0 \circ \beta(0)$. We have moreover that

$$\bar{\beta}(0) \circ \theta = \tilde{q}_0 \circ \eta \circ \theta \simeq \tilde{q}_0$$

and therefore $$\beta(0) \circ \theta \simeq \tilde{q}_0.$$

We proceed now with the induction step: let us suppose that there exists a map $\beta(k) : M(k) \to \tilde{G}_k$ such that

$$\tilde{g}_k \circ \beta(k) = (g_n \times g'_m) \circ p_k \qquad \text{and} \qquad \beta(k) \circ incl \circ \theta \simeq \tilde{q}_k.$$

Since $p_{k+1} \circ incl = p_k$, the following diagram commutes:

$$\begin{tikzcd}
& V(k+1) \arrow[r] & M(k) \arrow[r, "incl"] \arrow[d, "\beta(k)"'] & M(k+1) \arrow[d, "(g_n \times g'_m) \circ p_{k+1}"] \\
& \tilde{F}_k \arrow[r] & \tilde{G}_k \arrow[r, "\tilde{g}_k"'] & B \times B'.
\end{tikzcd}$$

Using lemma 4.3.1 we build a map

$$\beta(k+1) : M(k+1) \longrightarrow \tilde{G}_{k+1}$$

such that $\beta(k+1) \circ incl = \delta_k \circ \tilde{incl}_k \circ \beta(k)$ and $\tilde{g}_{k+1} \circ \beta(k+1) = (g_n \times g'_m) \circ p_{k+1}$, where $\tilde{incl}_k : \tilde{G}_k \to \tilde{G}_k \cup C\tilde{F}_f$ is the obvious inclusion, and $\delta_k : \tilde{G}_k \cup C\tilde{F}_k \xrightarrow{\simeq} \tilde{G}_{k+1}$ an homotopy equivalence.

On the other hand, we check that

$$\beta(k+1) \circ incl \circ \theta = \delta_k \circ \tilde{incl}_k \circ \beta(k) \circ incl \circ \theta \simeq \delta_k \circ \tilde{incl}_k \circ \tilde{q}_k,$$

and it is easy to see that this last term is actually $\tilde{q}_{k+1}$. The induction step is then over.

We can now define $\bar{\Phi}(n,m) = \beta(n+m+p) \circ \mu$, where μ is the homotopy inverse of $\nu : M(n+m+p) \to G_n \times G'_m$. From the above induction process we infer that

$$\tilde{g}_{n+m+p} \circ \bar{\Phi}(n,m) = \tilde{g}_{n+m+p} \circ \beta(n+m+p) \circ \mu =$$

$$(g_n \times g'_m) \circ p_{n+m+p} \circ \mu = (g_n \times g'_m) \circ \nu \circ \mu \simeq g_n \times g'_m,$$

and we can then find a map $\Phi(n,m) \simeq \bar{\Phi}(n,m)$ such that $\tilde{g}_{n+m+p} \circ \Phi(n,m) = g_n \times g'_m$, as desired.

Moreover, we can see that

$$\Phi(n,m) \circ (q_n \times q'_m) \simeq \bar{\Phi}(n,m) \circ (q_n \times q'_m) \simeq$$

$$\bar{\Phi}(n,m) \circ \nu \circ incl \circ \theta \simeq \beta(n+m+p) \circ incl \circ \theta \simeq \tilde{q}_{n+m+p}.$$

□

Using proposition 4.3.2 we give here upper bounds for the three possible categories, Fcat, Rcat, cat, of a product of maps.

Theorem 4.3.3 *Let f, f' be maps like in proposition 4.3.2. If $p \equiv \max\{Cl(E), Cl(E')\}$ is finite, then*

$$\mathrm{Rcat}(f \times f') \leq \mathrm{Rcat}(f) + \mathrm{Rcat}(f') + \max\{cat(E), cat(E')\}.$$

PROOF. Let $\mathrm{Rcat}(f) = n$, $\mathrm{Rcat}(f') = m$ and s, s' denote sections of g_n and g'_m respectively. We define a map $\tilde{s} : B \times B' \to \tilde{G}_{n+m+p}$ by

$$\tilde{s} = \Phi(n,m) \circ (s \times s'),$$

where $\Phi(n,m)$ is the map of proposition 4.3.2. Then

$$\tilde{g}_{n+m+p} \circ \tilde{s} = \tilde{g}_{n+m+p} \circ \Phi(n,m) \circ (s \times s') =$$

$$(g_n \times g'_m) \circ (s \times s') = id_B \times id_{B'} = id_{B \times B'}.$$

This means that

$$\mathrm{Rcat}(f \times f') \leq \mathrm{Rcat}(f) + \mathrm{Rcat}(f') + \max\{Cl(E), Cl(E')\}.$$

Recall now from [Cor95] that there exist suspensions ΣX, $\Sigma X'$ such that $Cl(E \vee \Sigma X) = cat(E)$ and $Cl(E' \vee \Sigma X') = cat(E')$. Define a map $g : E \vee \Sigma X \to B$ as being f on E and $*$ on ΣX, and a map g' analogously. We now show that

$$\mathrm{Rcat}(g) = \mathrm{Rcat}(f).$$

We define maps $u : E \vee \Sigma X \to E$ as the projection and $v : E \to E \vee \Sigma X$ as the inclusion. We therefore have a commutative diagram:

$$\begin{tikzcd}
\bar{G}_0 \arrow[drr, "\bar{g}_0"'] & E \vee \Sigma X \arrow[l, "\simeq", "\omega"'] \arrow[rr, shift left, "u"] \arrow[dr, "g"] & & E \arrow[ll, shift left, "v"] \arrow[dl, "f"'] \arrow[r, "\simeq", "\lambda"'] & G_0 \arrow[dll, "g_0"] \\
& & B, & &
\end{tikzcd}$$

where g_0 and $\bar{g}_0$ are the 0-th Ganea fibrations of f and g respectively. Defining $\bar{\omega}, \bar{\lambda}$ as being the homotopy inverses of ω, λ respectively, we have $g_0 \circ \lambda \circ v \circ \bar{\omega} \simeq \bar{g}_0$ and $\bar{g}_0 \circ \omega \circ u \circ \bar{\lambda} \simeq g_0$ which according to [ST97] means that $\bar{g}_0$ is a 0-LS-fibration for g_0. Always according to [ST97] we see that the k-th Ganea fibration for g, $\bar{g}_k : \bar{G}_k \to B$, is a k-LS-fibration for g_0, which means that there exist maps u_k, v_k such that the following diagram commutes up to homotopy

$$\begin{tikzcd}
\bar{G}_k \arrow[rr, shift left, "u_k"] \arrow[dr, "\bar{g}_k"'] & & G_k \arrow[ll, shift left, "v_k"] \arrow[dl, "g_k"] \\
& B &
\end{tikzcd}$$

and therefore the existence of a section for g_k is equivalent to the existence of a section for $\bar{g}_k$.

Analogously, we define $g' : E' \vee \Sigma X' \to B'$, and we show that $\mathrm{Rcat}(g \times g') = \mathrm{Rcat}(f \times f')$. It suffices to check that the product $u \times u'$ of the projections and $v \times v'$ of the inclusions make the following diagram commute:

$$\begin{tikzcd}
\bar{\bar{G}}_0 \arrow[drr, "\bar{\bar{g}}_0"'] & (E \vee \Sigma X) \times (E' \vee \Sigma X') \arrow[l, "\simeq", "\omega"'] \arrow[rr, shift left, "u \times u'"] \arrow[dr, "g \times g'"] & & E \times E' \arrow[ll, shift left, "v \times v'"] \arrow[r, "\simeq", "\lambda"'] \arrow[dl, "f \times f'"'] & \tilde{G}_0 \arrow[dll, "\tilde{g}_0"] \\
& & B \times B', & &
\end{tikzcd}$$

and then proceed as in the previous case.

We now have

$$\begin{aligned}
\mathrm{Rcat}(f \times f') = \mathrm{Rcat}(g \times g') &\leq \mathrm{Rcat}(g) + \mathrm{Rcat}(g') + \max\{Cl(E \vee \Sigma X), Cl(E' \vee \Sigma X')\} \\
&= \mathrm{Rcat}(f) + \mathrm{Rcat}(f') + \max\{cat(E), cat(E')\}.
\end{aligned}$$

□

Theorem 4.3.4 *Let f, f' be maps like in proposition 4.3.2. If $p \equiv \max\{Cl(E), Cl(E')\}$ is finite, then*

$$\mathrm{cat}(f \times f') \leq \mathrm{cat}(f) + \mathrm{cat}(f') + \max\{Cl(E), Cl(E')\}.$$

PROOF. We begin like in proof of theorem 4.3.3 by defining a section for $\tilde{g}_{n+m+p}$ as $\tilde{s} = \Phi(n,m) \circ (s \times s')$, with $s \circ f \simeq q_n$ and $s' \circ f' \simeq q'_m$. We can then check that $\tilde{s}$ fulfills Cornea's requirement by using proposition 4.3.2:

$$\tilde{s} \circ (f \times f') = \Phi(n,m) \circ (s \times s') \circ (f \times f') \simeq \Phi(n,m) \circ (q_n \times q'_m) \simeq \tilde{q}_{n+m+p}.$$

□

In the case of the F-category we obtain a well-known result:

Theorem 4.3.5 *Let f, f' be maps like in proposition 4.3.2, then*

$$\mathrm{Fcat}(f \times f') \leq \mathrm{Fcat}(f) + \mathrm{Fcat}(f').$$

Proof. Notice that here we work with the Ganea spaces and maps corresponding to the maps $* \to B$, $* \to B'$ and $* \to B \times B'$, instead of f, f' and $f \times f'$. To avoid confusion we therefore use standard indexation. Let s, s' be maps such that $g_n(B) \circ s \simeq f$ and $g_m(B') \circ s' \simeq f'$. Since $PB \simeq PB' \simeq *$, we have $Cl(PB) = Cl(PB') = 0$. We can therefore define $\tilde{s} : B \times B' \to G_{n+m}(B \times B')$ as being

$$\tilde{s} \equiv \Phi(n, m) \circ (s \times s')$$

where $\Phi(n, m)$ is the map built in proposition 4.3.2. We then see that

$$g_{n+m}(B \times B') \circ \tilde{s} = g_{n+m}(B \times B') \circ \Phi(n, m) \circ (s \times s') = (g_n(B) \times g_m(B')) \circ (s \times s') \simeq f \times f'.$$

$\square$