Chapter 3

Classical LS-category

3.1 Introduction

In this chapter by "space" we mean a path-connected pointed CW-complex, whose base-point is denoted $*$. By "map" we mean a continuous pointed map.

The *Lusternik-Schnirelmann category* and the *cone-length* of a space are two homotopy invariants which have been thoroughly studied since 1934, when the LS-category was introduced by Lusternik and Schnirelmann [LS34]. The original definition was given as the number of open sets needed to cover a space X while being contractible in X. Later a second definition was given by G. Whitehead [Whi56] in terms of a possible deformation of the diagonal map, then a third one by Ganea [Gan67] asking for a homotopy section to some maps he defined. A great amount of work has been dedicated to LS-category, as is apparent in the reviews of James [Jam78], [Jam95]. This invariant was even studied in the rational homotopy context by Félix and Halperin [FH83].

As for the cone-length it was originally called strong category [Fox41] and was defined as the minimum number of open subsets, contractible in *themselves*, which are needed to cover X. For it to be homotopy invariant one then had to minimize this value for all spaces of the same homotopy type as X. The main results about cone-length were given by Ganea [Gan67] and Cornea [Cor95], [Cor94].

In section 3.2 we state the three equivalent definitions for LS-category and the most commonly used definition for the cone-length of a space. We finish by giving upper bounds for the LS-category and the cone-length of a product of spaces. In section 3.3 we extend the LS-category first to a cofibration, then to any map. We give three possible generalizations: the F-category, which was first given by Fox [Fox41], the LS-category, introduced by Fadell and Husseini [Fad85], [FH96] and the R-category, a slight modification of LS-category. For each one of them we give three equivalent definitions: in terms of coverings, of homotopy sections to Ganea maps and of deformations of the diagonal. Finally we give the definition of the cone-length of a map as it was introduced by Marcum [Mar98].

3.2 Absolute LS-category and cone-length

The Lusternik-Schnirelmann category of a topological space was first introduced by Lusternik and Schnirelmann [LS34] to give a lower bound on the number of critical points of a smooth function $f : X \to \mathbb{R}$, where X is a manifold. The definition that is most commonly accepted nowadays is

Definition. Let X be a space, then the ***LS-category*** of X, noted $\text{cat}(X)$, is m, where $(m+1)$ is the minimum number of open subsets of X that are contractible in X which are needed to cover X.

An equivalent definition was given later by G. Whitehead [Whi56] in terms of a possible deformation of the diagonal map, then a third one by Ganea [Gan67], asking for the existence of a homotopy section to a certain map. To state them we first need to introduce *fat wedges* and *Ganea spaces and maps.*

Definitions.

- The ***n-th fat wedge*** of a space X, noted $T^n(X)$ is defined inductively as follows:

  $$T^1(X) := \{*\}; \qquad T^n(X) := (T^{n-1}(X) \times X) \cup (X^{n-1} \times \{*\}).$$

  That is, $T^n(X)$ is the subset of X^n whose elements have at least one coordinate equal to the basepoint.

- The ***n-th Ganea map*** associated to X, $g_n(X)$ is defined inductively as follows:

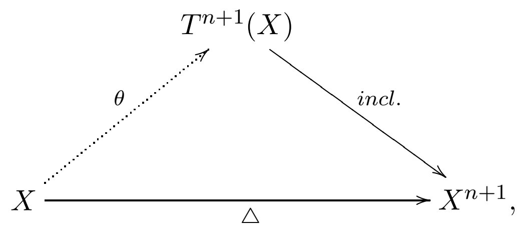
  $$g_0(X) : G_0(X) \equiv \{*\} \to X; \qquad g_n(X) \equiv g_{n-1} \bowtie g_0(X) : G_n(X) \to X,$$

  whereby the source space $G_n(X) \equiv G_{n-1}(X) \bowtie_X G_0(X)$ of the n-th Ganea map is called the ***n-th Ganea space*** of X.

We can now state a theorem about the equivalence of the three definitions of LS-category we have hinted at.

Theorem 3.2.1 *The following assertions are equivalent:*

- *$cat(X) \leq n$,*
- *there exists a map θ such that the following diagram commutes up to homotopy:*



  where $\triangle$ is the diagonal map: $\triangle(x) = (x, x, ..., x)$ for all $x \in X$.

- *the n-th Ganea map admits a homotopy section.*

We prove this well-known result in a more general form in the next section, when dealing with the category of maps.

We now give the most commonly used definition for cone-length, which was proved to be equivalent to the original one for the class of spaces we are considering.

Definition. Let X be a space. Suppose there exist cofibration sequences of spaces

$$Z(i) \longrightarrow X(i) \longrightarrow X(i+1) \quad \text{where} \quad 0 \leq i \leq n$$

such that $X(0) \simeq *$, $X(n) \simeq X$. Then we say that X ***has cone length smaller or equal to n***, and we write $Cl(X) \leq n$.

If there exist no such cofibration sequences for any n, we say that the cone length of X is infinite: $Cl(X) = \infty$.

$Cl(X)$ is strongly related to $cat(X)$ since it is always equal either to $cat(X)$ or to $cat(X) + 1$.

There exist a "product inequality" for LS-cat as well as for cone-length

Theorem 3.2.2 *Let X and Y be spaces, then*

- $cat(X \times Y) \leq cat(X) + cat(Y)$;
- $Cl(X \times Y) \leq Cl(X) + Cl(Y)$.

Deciding when equality or strict inequality holds has been largely studied, for example while trying to prove the famous *Ganea conjecture*: $cat(X \times S^n) = cat(X) + 1$, where X is any space and S^n is a sphere of dimension $n \geq 1$. While shown to hold for simply connected rational topological spaces by Jessup [Jes87] and Hess [Hes91], a counter-example was given by Iwase in [Iwa98], showing how intuition can sometimes fail us in this field.

3.3 Relative LS-category and cone-length

There have been several generalizations of LS-category and cone-length from spaces to maps. We will speak about *absolute* category or cone-length when meaning the category or cone-length of a space, and about a *relative* invariant when considering maps.

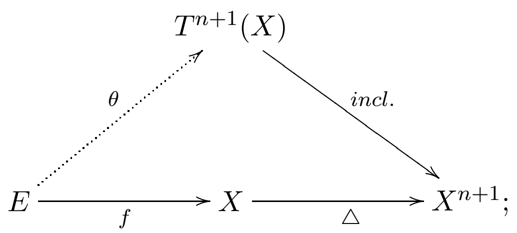
The first extension of LS-category was given by Fox [Fox41] and studied by Bernstein and Ganea [BG12]. However we are more interested in a second relativization of LS-category and choose therefore to call this first invariant F-category, keeping the name LS-category for the second one.

Definition. Let $f : E \to X$ be a map. We say that its F-category is smaller or equal to m ($\mathrm{Fcat}(f) \leq m$) if the space E can be covered by $m+1$ open subspaces: $E = \bigcup_{i=0}^{m} U_i$ such that for each i, $f|_{U_i} \simeq *$.

As it is the case for the absolute category, there are equivalent definitions.

Theorem 3.3.1 *For a map $f : E \to X$ we have equivalence between:*

1. $\mathrm{Fcat}(f) \leq m$;
2. *there exists a map θ such that the following diagram commutes up to homotopy:*



3. *there exists a map $s : E \to G_m(X)$ such that $g_m(X) \circ s \simeq f$.*

This is a well-known result, whose proof is analogous to the proof of theorem 3.3.2.

Instead of keeping the "absolute" fat wedge and Ganea spaces, another way of proceeding to extend LS-category is to generalize them to "relative" fat wedge and Ganea spaces. Moreover, depending on how many conditions we impose in the definition, we obtain two different relative categories: the relative LS-category *cat*, which was introduced by Fadell and Husseini [FH96], [Fad85], and Cornea [Cor98], and the R-category *Rcat*, which is always smaller than or equal to the previous one. We begin by considering the special case of a cofibration and then relate to this case to give a general definition.

Let $f : E \to X$ be a cofibration.

Definitions.

- The ***R-category*** of f, $\mathrm{Rcat}(f)$, is m, where $(m+1)$ is the minimum number of open subsets of X that are necessary to cover X, under the condition that one of these subsets can be continuously deformed into E while the other open sets are contractible in X. What we mean by "an open subset $U \subset X$ can be continuously deformed into E" is that there exists a homotopy $H : U \times I \to X$ such that $H_0 = id_U$ and $\mathrm{Im}\,(H_1) \subset E$. The subset U does not necessarily contain E.

- If in the definition of R-category we add the condition that the open set which is deformed into E must contain E and that the homotopy must be relative to E, then we obtain the ***LS-category*** of f, $\mathrm{cat}(f)$. It is a somewhat weaker statement than saying that E is a strong deformation retract of U, because the deformation takes place in X.

- The ***n-th fat wedge*** of the map f, or ***n-th relative fat wedge*** of f, is denoted $T^n(f)$ and is defined as being E for $n = 1$, and

$$T^n(f) := (T^{n-1}(f) \times X) \cup (X^{n-1} \times \{*\}),$$

  i.e. it is the subspace of X^n such that its elements have either the first coefficient in E, or one of the other coefficients equal to the base point.

- The n-th Ganea map of f is defined as being f itself for $n = 0$, and inductively:

$$g_n(f) := g_{n-1}(f) \bowtie g_0(X),$$

  with $g_n(f) : G_n(f) \to X$, where $G_n(f)$ is the ***n-th Ganea space*** of f. The map $q_n : E \to G_n(f)$, which is part of the homotopy push-out forming the join, is such that $g_n(f) \circ q_n \simeq f$.

Remarks.

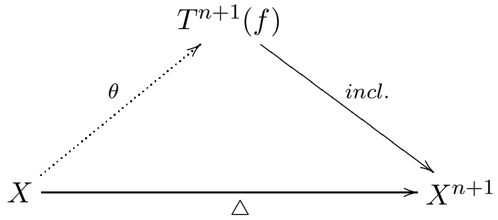
- The R-category and the LS-category of the map $* \to X$ and the F-category of the identity id_X are all equal to $\mathrm{cat}(X)$, which explains why we speak about a "generalization".

- When it is clear whether we are considering absolute or relative categories, and for which space, respectively for which map, we will drop the indices X or f.

- Up to now we have defined category only for cofibrations. The next two theorems will allow us to give a more general definition.

Theorem 3.3.2 *Let $f : E \to X$ be a cofibration. The following propositions are equivalent:*

1. $\mathrm{Rcat}(f) \leq n$;
2. *there exists a map $\theta : X \to T^{n+1}(f)$ such that the following diagram commutes up to homotopy:*



3. *$g_n(f)$ admits a homotopy section.*

Proof.

1 ⇔ 2 Let $X = \bigcup_0^n U_i$ with $h_0 : U_0 \times I \to X$, such that $h_0(x, 0) = x$ and $h_0(x, 1) \in E$ for all $x \in U_0$, $h_i : U_i \times I \to X$, such that $h_i(x, 0) = x$ and $h_i(x, 1) = *$ for all $x \in U_i$, $1 \leq i \leq n$, which is possible because X is path connected. Since CW-complexes are normal, for each U_i there exist closed subspaces F_i and G_i and an open subspace O_i, such that $F_i \subset O_i \subset G_i \subset U_i$ and $X = \bigcup_0^n F_i$. Moreover there are maps $f_i : X \to I$ such that $f_i(x) = 1$ for all $x \in F_i$ and $f_i(x) = 0$ for all $x \in X - G_i$. We can define a map $\theta : X \to T^{n+1}(X)$ as follows:

$$\theta(x) = (\theta_0(x), \theta_1(x), ..., \theta_n(x)),$$

$$\theta_i(x) = \begin{cases} x & x \in X - G_i \\ h_i(x, f_i(x)) & x \in U_i \end{cases} .$$

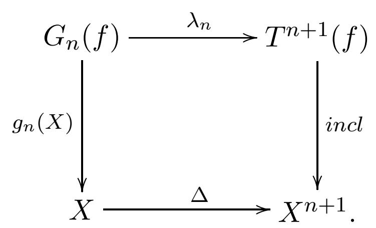
It is well-defined because there exists an i such that $x \in F_i$, and therefore either $h_0(x, 1) \in E$, if $i = 0$, or $h_i(x, 1) = *$. It remains to verify that there exists a homotopy $H : X \times I \to X^{n+1}$ such that $incl \circ \theta \simeq \Delta$. We define it as

$$H(x, t) := (H_0(x, t), H_1(x, t), ..., H_n(x, t)),$$

$$H_i(x, t) = \begin{cases} x & x \in X - G_i \\ h_i(x, f_i(x) \cdot t) & x \in U_i \end{cases} .$$

Inversely let us suppose that $H = (H_0, ..., H_n)$ is an homotopy between Δ and $incl \circ \theta$, with $\theta : X \to T^{n+1}(X)$. There exists an open set U containing the base point which is contractible in X and an open set V containing E which can be continuously deformed into E. We take $U_0 := H_0^{-1}(V, 1)$, $U_i := H_i^{-1}(U, 1)$ and $h_0 := H_0|(U_0 \times I)$ $h_i := H_i|(U_i \times I)$ for all i. Therefore if $x \in X$ we have $\theta(x) \in T^{n+1}$ or more precisely either $H_0(x, 1) \in E \subset V$, i.e. $x \in U_0$, or there exists an i such that $H_i(x, 1) = * \in U$, i.e. $x \in U_i$.

2 ⇔ 3 We show that there exists a homotopy pull-back:



– First of all we notice that the join of the inclusions $X^k \times T^l(X) \to X^{k+l}$ and $T^k(f) \times X^l \to X^{k+l}$ is homotopy equivalent to the inclusion $T^{k+l}(f) \to X^{k+l}$. We follow [Cuv98] and construct the standard homotopy pull-back of the two inclusions:

$$M := \{(w_1, w_2, ..., w_{k+l}) \in (X^I)^{k+l} | (w_1(0), w_2(0), ..., w_k(0)) \in T^k(f),$$

$$(w_{k+1}(1), ..., w_{k+l}(1)) \in T^l(X)\}.$$

It is then possible to define a homotopy equivalence $\varphi : M \to T^k(f) \times T^l(X)$ as being

$$\varphi(w_1, w_2, ..., w_{k+l}) := (w_1(0), ..., w_k(0), w_{k+1}(1), ..., w_{k+l}(1)).$$

It is indeed possible to contract each path w_i to its beginning- or end-point. We can therefore use the space $T^k(f) \times T^l(X)$ to complete the construction of the join, i.e. construct a homotopy push-out of the inclusion of $T^k(f) \times T^l(X)$ in $T^k(f) \times X^l$ and in $X^k \times T^l(X)$. Since these last two maps are cofibrations we obtain the inclusion

$$(T^k(f) \times X^l) \bigcup_{T^k(f) \times T^l(X)} (X^k \times T^l(X)) \equiv T^{k+l}(f) \longrightarrow X^{k+l}.$$

Of course if we choose $E = *$, we obtain that the join of the inclusions $X^k \times T^l(X) \to X^{k+l}$ and $T^k(X) \times X^l \to X^{k+l}$ is homotopy equivalent to $T^{k+l}(X) \to X^{k+l}$.

– We now define $\tilde{G}_n(f)$ as the standard homotopy pull-back of $incl : T^{n+1}(f) \to X^{n+1}$ and $\Delta : X \to X^{k+1}$. We show that $\tilde{G}_n(f) \simeq G_n(f)$.

It is easy to see that $\tilde{G}_n(f)$ is homotopy equivalent to the homotopy pull-back of $incl : T^{n+1}(f) \times X^1 \to X^{n+2}$ and $\Delta : X \to X^{n+2}$. Similarly, $\tilde{G}_0(X)$ is homotopy equivalent to the homotopy pull-back of $X^{n+1} \times T^1(X) \to X^{n+2}$ and $\Delta : X \to X^{n+2}$. Applying the join theorem I [Doe98] to these two homotopy pull-backs, we obtain a new homotopy pull-back

$$\begin{array}{ccc} \tilde{G}_n(f) \bowtie_X \tilde{G}_0(X) & \longrightarrow & (T^{n+1}(f) \times X^1) \bowtie_{X^{n+2}} (X^{n+1} \times T^1(X)) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta} & X^{n+2} \end{array}$$

where the vertical maps result from the two joins. Therefore we have shown that

$$\tilde{G}_{n+1}(f) \simeq \tilde{G}_n(f) \bowtie_X \tilde{G}_0(X)$$

for all $n \geq 0$. On the other hand we already know that

$$G_{n+1}(f) \simeq G_n(f) \bowtie_X G_0(X).$$

Actually $\tilde{G}_0(f) = \{(x, w, y) \in X \times X^I \times E | w(0) = x, w(1) = y\}$. Since $G_0(f) = E$ is homotopic to $\tilde{G}_0(f)$ through the map $e \mapsto (e, const_e, e)$, which also means that $G_0(X) \simeq \tilde{G}_0(X)$, a simple induction gives $G_n(f) \simeq \tilde{G}_n(f)$ for all n.

From now on we can therefore consider $G_n(f)$ as the standard homotopy pull-back of the first diagram of the proof. It remains to show that the definition in terms of fat wedge and the one in terms of Ganea spaces are equivalent. If $\theta : X \to T^{n+1}$ is such that $incl \circ \theta \simeq \Delta$ the following diagram homotopy commutes:

$$\begin{array}{ccc} X & \xrightarrow{\theta} & T^{n+1}(f) \\ {\scriptstyle id_X}\downarrow & & \downarrow{\scriptstyle incl} \\ X & \xrightarrow{\Delta} & X^{n+1}. \end{array}$$

By the properties of homotopy pull-backs there exists therefore a map $s : X \to G_n(f)$ such that $g_n(f) \circ s \simeq id_X$ as desired.

If inversely we have a homotopy section s for $g_n(f)$, we define $\theta := \lambda_n \circ s$ and we verify that $incl \circ \theta = incl \circ \lambda_n \circ s \simeq \Delta \circ g_n(f) \circ s \simeq \Delta$.

□

There exist a Whitehead-type and a Ganea-type definition for the LS-category of a cofibration too:

Theorem 3.3.3 *Let $f : E \to X$ be a cofibration. The following propositions are equivalent:*

1. $\mathrm{cat}(f) \leq n$;

2. *there exists a map $\theta : X \to T^{n+1}(f)$ such that the following diagram commutes up to a homotopy $H = (H_0, H_1, ..., H_n) : X \times I \to X^{n+1}$:*

$$\begin{array}{ccccc} & & T^{n+1}(f) & & \\ & {\scriptstyle \theta}\nearrow & & \searrow{\scriptstyle incl.} & \\ X & & \xrightarrow{\quad \Delta \quad} & & X^{n+1}, \end{array}$$

   with H such that $H_0(-,t)|_E = f$ for all $t \in I$;

3. *$g_n(f)$ admits a homotopy section s, such that $s \circ f \simeq q_n$.*

PROOF.

1 ⇔ 2 The proof is similar to the one for the corresponding case of Theorem 3.3.2. The only differences are first of all that, given an appropriate covering for X, the homotopy h_0 has the additional property $h_0(e,t) = e$ for all $e \in E \subset U_0$ and $t \in I$. Moreover, we must have $E \subset F_i$, which is possible because of the normality of CW-complexes. We define θ and H like in the previous proof and we verify that if $e \in E$, $H_0(e,t) = h_0(e, f_0(e)t) = h_0(e,t) = e = f(e)$.

In the other direction if H is a homotopy between $incl \circ \theta$ and Δ with $pr_1 \circ H(-,t)|_E = f$, then $E \subset U_0 = H_0^{-1}(E, 1)$ and for all $e \in E$, $h_0(e,t) = H_0(e,t) = f(e) = e$.

2 ⇔ 3 Here also the proof is similar to the one of Theorem 3.3.2. It was shown that $G_n(f)$ can be considered as the standard homotopy pull-back of $\Delta : X \to X^{n+1}$ and $incl : T^{n+1}(f) \to X^{n+1}$, i.e.

$$G_n(f) = \{(x, w, (y_0, ..., y_n)) \in X \times {X^{n+1}}^I \times T^{n+1}(f) \,|w(0) = \Delta(x), w(1) = (y_0, ..., y_n)\}.$$

In this case the natural inclusion $q_n : E \to G_n(f)$ can be taken as being $q_n(e) = (e, const_{(e,...,e)}, e, ..., e)$, where $const_x$ is the constant path at the point x. Indeed if we go back to the proof of Theorem 3.3.2, we deduce that $q_0(e) = (e, const_e, e)$. On the other hand it is possible to follow step by step a proof of Cuvilliez in [Cuv98], 1.5, to show that the inclusion $G_k(f) \to G_n(f)$ is homotopic to $j_{k,n} : (x, w, y_0, ..., y_k) \mapsto (x, (w, const_x, const_x, ..., const_x), y_0, ..., y_k, x, ..., x)$. Since we are only interested on a condition stating that the composition of two maps should be equal to q_n *up to homotopy*, we can take $q_n := j_{0,n} \circ q_0$.

Let us now suppose that there exists a map θ as in condition 2. It induces a map $s : X \to G_n(f)$, $s(x) = (x, H(x, -), incl \circ \theta(x))$, where H is the homotopy between Δ and $incl \circ \theta$. There is a homotopy between q_n and $s \circ f$ given by $G : E \times I \to G_n(f)$, $G(e, t) := (e, \gamma_H^t, H(e, t))$, where γ_H^t is the portion of the path $H(e, -)$ between 0 and t. The homotopy is well-defined since $H_0(-, t)|_E = f$ ensures that $H(e, t) \in T^{n+1}(f)$ for all t.

On the other hand suppose we are given a homotopy section $s' : X \to G_n(f)$ for $g_n(f)$ such that $s' \circ f \simeq q_n$. We show that $g_n(f)$ can be considered as a fibration, which means that there exists a map $s \simeq s'$ which is an exact section for $g_n(f)$. Let us replace the map $incl : T^{n+1}(f) \to X^{n+1}$ by its associated fibration $m_n : \bar{T}^{n+1} \to X^{n+1}$, with $\bar{T}^{n+1} = \{(y_o, ..., y_n, \gamma) \in T^{n+1}(f) \times (X^{n+1})^I | \gamma(0) = (y_0, ..., y_n)\}$ and $m_n(y_0, .., y_n) = \gamma(1)$. Then the pull-back of Δ and m_n is

$$\begin{array}{ccc} B_n & \xrightarrow{\delta_n} & \bar{T}^{n+1} \\ \downarrow{\scriptstyle p_n} & & \downarrow{\scriptstyle m_n} \\ X & \xrightarrow[\Delta]{} & X^{n+1} \end{array}$$

with $B_n := \{(x, y_0, ..., y_n, \gamma) \in X \times \bar{T}^{n+1} | \gamma(1) = \Delta(x)\}$ and $p_n : (x, y_0, ..., y_n, \gamma) \longmapsto x$ is a fibration. Is is evident that B_n is homeomorphic to $G_n(f)$ over X by the homotopy equivalence $(x, y_0, ..., y_n, \gamma) \overset{\phi}{\longmapsto} (x, \gamma^{-1}, (y_0, ..., y_n))$ with obvious inverse ψ, while $g_n(f) \circ \phi = p_n$ and $p_n \circ \psi = g_n(f)$, therefore $g_n(f)$ is equivalent to a fibration.

We now use the following classical result: if $p : E \to B$ is a fibration and we are given a map $\alpha : A \to B$ and maps $\beta_1 \simeq \beta_2 : A \to E$ such that $p \circ \beta_1 = p \circ \beta_2 = \alpha$ then there exists a homotopy H between β_1 and β_2 such that $p \circ H(-, t) = \alpha$. In our case, take $p = g_n(f)$, $\alpha = f$, $\beta_1 = s \circ f$ and $\beta_2 = q_n$. Let us now construct a new map $\bar{s}$ which is homotopy equivalent to s, and such that $\bar{s} \circ f = q_n$: consider the solid arrows diagram

$$\begin{array}{ccc} (E \times I) \cup (X \times \{0\}) & \xrightarrow{H \cup s} & G_n(f) \\ \downarrow{\scriptstyle (f \times id) \cup (id \times incl)} & \nearrow{\scriptstyle F} & \\ X \times I & & \end{array}$$

Since f is a cofibration, there exists a homotopy F that makes it commute exactly. We take $\bar{s} := F(-,1)$. Therefore F is a homotopy between s and $\bar{s}$. Moreover if we restrict F to $E \times I$ it is equal to H, i.e. it is of the form $F(e,t) = H(e,t) = (e, H_1(e,t), H_2(e,t))$ for all $e \in E$. We can now choose for $\theta := \lambda_n \circ \bar{s}$, where λ_n is the projection $G_n(f) \to T^{n+1}(f)$ from the pull-back, and we verify

$$incl \circ \theta = incl \circ \lambda_n \circ \bar{s} \overset{K\circ\bar{s}}{\simeq} \Delta \circ g_n(f) \circ \bar{s} \overset{\Delta\circ g_n(f)\circ F^{-1}}{\simeq} \Delta \circ g_n(f) \circ s = \Delta.$$

The homotopy G between $incl \circ \theta$ and Δ is given by $K \circ \bar{s} + \Delta \circ g_n(f) \circ F^{-1}$, where K is the homotopy between $incl \circ \lambda_n$ and $\Delta \circ g_n(f)$. We check that $pr_1 \circ G|_{E\times I} = f$:

- $K(\bar{s}(e),t) = K((e, id_{\Delta e}, \Delta e), t)$ for all $e \in E$ because $\bar{s} \circ f = q_n$. On the other hand $K((x, \gamma, y_o, ..., y_n), t) = \gamma(t)$ for $((x, \gamma, y_o, ..., y_n), t) \in G_n(f) \times I$, therefore $K((e, id_{\Delta e}, \Delta e), t) = id_{\Delta e}(t) = \Delta e$.
- $\Delta \circ g_n(f)(F^{-1}(e,t)) = \Delta \circ g_n(f)(F(e, 1-t)) = \Delta \circ g_n(f)(e, H_1(e, 1-t), H_2(e, 1-t)) = \Delta(e)$.

□

To show that Rcat and cat are homotopy invariants we need to define a *homotopy retract* for a map f and to show that it has R-category less than or equal to Rcat(f), and LS-category less than or equal to cat(f)..

Definitions.

- Let $f : E \to X$ be a map between spaces X and Y. We say that a map $g : A \to Y$ is a ***homotopy retract*** for f if there exists a homotopy commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{i} & E & \xrightarrow{j} & A \\ \downarrow g & & \downarrow f & & \downarrow g \\ Y & \xrightarrow[k]{} & X & \xrightarrow[l]{} & Y \end{array}$$

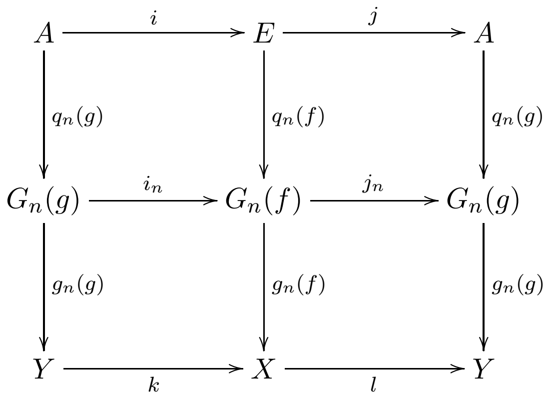
  such that the horizontal lines are homotopic to id_A and id_Y respectively.

- If moreover in the previous diagram we have $i \circ j \simeq id_A$ and $k \circ l \simeq id_Y$ we say that f is ***homotopically equivalent*** to g.

Lemma 3.3.4 *Suppose the cofibration $g : A \to Y$ is a homotopy retract for the cofibration $f : E \to X$, then*

1. Rcat(g) $\leq$ Rcat(f);
2. cat(g) $\leq$ cat(f).

Proof. We use notations as in definition 3.3. For all $n \geq 0$ we build a homotopy commutative diagram



such that $l \circ k \simeq id_Y$ using functoriality of standard homotopy pull-backs and push-outs applied to the diagram from definition 3.3 (see lemma 1.2.1).

1. If there is a map $s : X \to G_n(f)$ with $g_n(f) \circ s \simeq id_X$ we can define $s' \equiv j_n \circ s \circ k$ and verify

$$g_n(g) \circ s' = g_n(g) \circ j_n \circ s \circ k \simeq l \circ g_n(f) \circ s \circ k \simeq l \circ k \simeq id_Y.$$

2. Moreover if $s \circ f \simeq q_n(f)$, then

$$s' \circ g = j_n \circ s \circ k \circ g \simeq j_n \circ s \circ f \circ i \simeq j_n \circ q_n(f) \circ i \simeq q_n(g) \circ j \circ i \simeq q_n(g).$$

□

An easy consequence of this lemma is the invariance under homotopy of the R-category of a map.

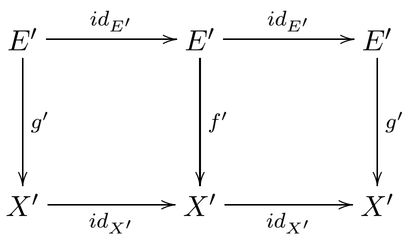
Theorem 3.3.5 *Let $f, g : E \to X$ be homotopically equivalent cofibrations, and $f', g' : E' \to X'$ be homotopic cofibrations, then*

$$\mathrm{Rcat}(f) = \mathrm{Rcat}(g), \qquad \mathrm{Rcat}(f') = \mathrm{Rcat}(g')$$

and

$$\mathrm{cat}(f) = \mathrm{cat}(g), \qquad \mathrm{cat}(f') = \mathrm{cat}(g').$$

Proof. The map f is a retract of g and g is a retract of f, so according to lemma 3.3.4, $\mathrm{Rcat}(g) \leq \mathrm{Rcat}(f)$ and $\mathrm{Rcat}(f) \leq \mathrm{Rcat}(g)$, and analogously for $\mathrm{cat}(f)$ and $\mathrm{cat}(g)$. Moreover we can build a homotopy commutative diagram



therefore f' and g' are homotopically equivalent. □

REMARK. Notice that if one takes as definition for LS-category, respectively for R-category the definition in terms of Ganea spaces, it can be applied to any map, not necessarily a cofibration, and the previous theorem is still valid.

We can now give a general definition of the category of a map for maps that are not necessarily cofibrations.

Definition. Let $f : E \to B$ be any map. It admits a standard associated cofibration $\tilde{f} : E \to X$. We say that the ***R-category***, respectively the ***LS-category***, of f is equal to n if the ***R-category***, respectively the ***LS-category*** of $\tilde{f}$ is equal to n.

Equivalently, one can construct Ganea spaces and maps starting from f because this construction does not require f to be a cofibration. We then apply the Ganea definition for relative category to f.

The equivalence between the two definitions follows from theorem 3.3.5 because f and $\tilde{f}$ are homotopically equivalent. Of course one can choose any cofibration associated to f as the map $\tilde{f}$.

The cone-length of a space was generalized to maps by Marcum [Mar98].

Definition. Let $f : A \to X$ be a map between spaces A, X. Suppose there exist cofibration sequences

$$Z(i) \longrightarrow X(i) \longrightarrow X(i+1)$$

where $0 \leq i \leq n$, $X(0) \simeq A$, $X(n) \simeq X$. If the composition

$$A \longrightarrow X(i) \longrightarrow X \text{ is homotopic to } f$$

with $A \to X(i)$ the map resulting from composition of all cofibrations $X(k) \to X(k+1)$ for $0 \leq k \leq i$, with the homotopy equivalence between A and $X(0)$, then we say that f ***has cone length smaller or equal to n*** and we write $Cl(f) = Cl(X, A) \leq n$.

If there exist no such cofibration sequences for any n, we say that the cone length of X relative to A is infinite: $Cl(X, A) = \infty$.

Again we see here immediately that $Cl(* \to X) = Cl(X)$ and therefore this relative cone-length is a generalization of the absolute one.