Chapter 2

Rational homotopy

2.1 Introduction

The main result in this thesis takes place in the rational homotopy setting. We give therefore some definitions, notations and useful results from this field that will make easier understanding chapters 5 to 8.

Recall that one can consider algebraic topology as striving to classify spaces up to homotopy, while making use of algebraic homotopy invariants such as homotopy or homology groups. Owing to the difficulty of this task, one can decide to put spaces in larger classes than homotopy type, more precisely, one can classify them according to *rational* homotopy type. In this case there is actually an equivalence of categories between classes of spaces and a category of algebraic objects and morphisms: weak equivalence classes of commutative cochain algebras, in short cca's. It becomes therefore interesting to try and translate any problem about rational homotopy type of topological spaces into a problem involving cca's in case it would be easier to solve in this category.

In section 2.2 we define the basic objects needed to develop rational homotopy: *commutative cochain algebras*. We define also *simplicial objects* in a category, which are used in section 2.3 to construct the functor A_{PL}, which realizes an equivalence of categories between the category of rational homotopy types of simply connected spaces X with $H^*(X;\mathbb{Q})$ of finite type, and the category of weak equivalence classes of commutative cochain algebras with rational homology of finite type. Sullivan algebras and relative Sullivan algebras are classes of cca's which possess particularly nice properties. We introduce them in sections 2.4 and 2.5 respectively, giving also theorems about existence of Sullivan models for cca's and cca morphisms, meaning that any cca and cca morphism can be replaced by a weakly equivalent (relative) Sullivan algebra. In section 2.6 we construct cca models for adjunction spaces, for the homotopy fiber of a given map and for the pull-back of two maps using as bricks the Sullivan models of the spaces and of the maps involved. These constructions are used in chapters 6, 7 and 8 to define *Ganea algebras*, which model Ganea spaces. We use them to prove the main theorem of this thesis. For more details or for proofs of the results given in sections 2.2 to 2.6 see [FHT01]. Finally we introduce *closed model categories* in section 2.7 to justify our handling of surjective cca morphisms as fibrations and of relative Sullivan algebras as cofibrations.

2.2 Definitions

2.2.1 Rational homotopy type

Definitions.

- Let $f : X \to Y$ be a continuous map between simply connected spaces. If any of the following equivalent conditions is fulfilled we say that f is a ***rational homotopy equivalence***:
  - $\pi_*(f) \otimes \mathbb{Q}$ is an isomorphism;
  - $H_*(f; \mathbb{Q})$ is an isomorphism;
  - $H^*(f; \mathbb{Q})$ is an isomorphism.
- If there exists a chain of rational homotopy equivalences

$$X \longleftarrow Z(0) \longrightarrow Z(1) \longleftarrow ... \longleftarrow Z(k) \longrightarrow Y$$

  between the simply connected spaces X and Y, we say that they have the same ***rational homotopy type***.

2.2.2 Commutative cochain algebras

In this section the ground ring is $\mathbb{Q}$.

Definitions.

- A ***commutative cochain algebra*** (cca) (A, d) is a graded vector space $\{A^p\}_{p\geq 0}$ such that
  - there exists a ***multiplication*** map $A \times A \to A$, $a \times b \mapsto ab$, which is associative, linear and of degree 0, and has an identity $1 \in A^0$;
  - the multiplication is ***commutative***, i.e. $ab = (-1)^{|a||b|} \cdot ba$, where $|a|$ is the degree of $a \in A$;
  - there exists a ***differential*** $d : A \to A$, i.e. a linear map of degree $+1$, such that $d^2 = 0$;
  - the differential d is a ***derivation***, i.e. $d(ab) = (da)b + (-1)^{|a|} \cdot a(db)$ for all $a, b \in A$.
- A morphism $\phi : (A, d) \to (B, d)$ of cca's is a morphism of graded vector spaces $A \to B$ of degree 0, such that
  - $\phi(aa') = \phi(a)\phi(a')$ for all $a, a' \in A$;
  - $\phi(da) = d\phi(a)$ for all $a \in A$.
- An ***augmentation*** is a morphism of cca's $(A, d) \to (\mathbb{Q}, 0)$, where $(\mathbb{Q}, 0)$ is the cca equal to $\mathbb{Q}$ in degree 0, and 0 otherwise, with differential $d \equiv 0$.
- Let V be a graded vector space. We define $TV \equiv \bigoplus_{i=0}^{\infty} V^{\otimes i}$, where $V^{\otimes 0} \equiv \mathbb{Q}$, and $V^{\otimes i} \equiv V \otimes V \otimes ... \otimes V$ i-times, to be the ***tensor algebra*** over V with multiplication $ab \equiv a \otimes b$.

  The degree of an element $a_1 \otimes a_2 \otimes ... \otimes a_i$ with $a_j \in V$ for all j is $|a_1 \otimes a_2 \otimes ... \otimes a_i| = \sum_{j=1}^{i} |a_j|$.

- The ***free commutative graded algebra*** ΛV on a graded vector space V is TV/I, where I is the ideal generated by all elements of the form $a \otimes b - (-1)^{|a||b|} \cdot b \otimes a$, with $a, b \in V$. The equivalence class of an element $a_1 \otimes a_2 \otimes ... \otimes a_i$ with $a_j \in V$ for all j is denoted $a_1 \wedge a_2 \wedge ... \wedge a_i$.

  If $\{v_i\}_{i\in\Omega}$ is a basis for V, we say it is a ***basis*** for ΛV and we write $\Lambda V = \Lambda(\{v_i\}_{i\in\Omega})$.

  If there exists a differential $d : \Lambda V \to \Lambda V$ (of degree +1) which is a derivation then ΛV is a commutative cochain algebra and we denote it by $(\Lambda V, d)$.

  Let $a_1 \wedge a_2 \wedge ... \wedge a_i \in \Lambda V$ with $a_j \in V$ for all j, then its ***word length*** is i, and $\Lambda^i V$ is the linear span of all elements of word length i.

  We write $\Lambda^+ V \equiv \Lambda^{\geq 1} V$.

- If (A, d) is a cca, its ***homology algebra*** is the graded algebra

$$H(A, d) \equiv \mathrm{Ker}\, d / \mathrm{Im}\, d.$$

- A cca (A, d) is ***connected*** if $H^0(A, d) = \mathbb{Q}$ and it is ***simply connected*** if it is connected and $H^1(A, d) = 0$.

- Let $\phi : (A, d) \to (B, d)$ be a morphism of cca's. If the induced morphism $H(\phi) : H(A, d) \to H(B, d)$ is an isomorphism, we say that ϕ is a ***quasi-isomorphism***.

- Let $\phi, \psi : (A, d) \to (B, d)$ be two morphisms of cca's. They are ***homotopic as chain complexes*** if there exists a linear map $h : A \to B$ of degree 1 such that $\phi(a) - \psi(a) = h(da) + dh(a)$ for all $a \in A$. We write $\phi \cong \psi$.

- If there exists a chain of quasi-isomorphisms

$$(A, d) \longleftarrow (C(0), d) \longrightarrow (C(1), d) \longleftarrow ... \longleftarrow (C(k), d) \longrightarrow (B, d)$$

  i.e. a ***weak equivalence***, between the cca's (A, d) and (B, d), we say that they are ***weakly equivalent***.

There exist pull-back objects in the category of commutative cochain algebras:

Definition. Let $\phi : (B, d) \to (A, d)$ and $\phi' : (B', d) \to (A, d)$ be cca morphisms. We call ***fibre product*** of ϕ and ϕ' the cca

$$(B \times_A B', d) \equiv \{(b, b') \in B \times B' | \phi(b) = \phi(b')\}.$$

One can easily verify that a fibre product of cca's possesses the pull-back property.

2.2.3 Simplicial objects

Definitions.

- Let $\mathfrak{C}$ be a category and K be a sequence of objects $\{K_n\}_{n\geq 0}$ and of morphisms $\{\partial_i : K_{n+1} \to K_n,\ 0 \leq i \leq n+1;\ s_j : K_n \to K_{n+1},\ 0 \leq j \leq n\}_{n\geq 0}$. We say that K is a ***simplicial object with values in*** $\mathfrak{C}$ if the morphisms in K satisfy the identities:

$$\begin{array}{rcll} \partial_i \partial_j & = & \partial_{j-1}\partial_i & \text{if } i < j, \\ s_i s_j & = & s_{j+1} s_i & \text{if } i \leq j \\ \partial_i s_j & = & \begin{cases} s_{j-1}\partial_i & \text{if } i < j, \\ id & \text{if } i = j \text{ or } i = j+1, \\ s_j \partial_{i-1} & \text{if } i > j+1. \end{cases} \end{array}$$

- Let K and L be simplicial objects in $\mathfrak{C}$ and let $\phi = \{\phi_n : K_n \to L_n\}_{n\geq 0}$ be a sequence of morphisms in $\mathfrak{C}$. If the ϕ_n commute with the ∂_i and the s_j, we say that ϕ is a ***simplicial morphism*** between K and L and we write $\phi : K \to L$.

2.3 Equivalence of categories

We give here a short description of the result of Sullivan [] stating that the functor A_{PL} realizes a natural equivalence between the category of rational homotopy types of simply connected spaces X with $H^*(X;\mathbb{Q})$ of finite type, and the category of weak equivalence classes of simply connected commutative cochain algebras (A,d) with $H(A,d)$ of finite type. We do not define the simplicial commutative cochain algebra (A_{PL},d) precisely since this information will not be necessary for our purposes. It is enough to know that $(A_{PL})_0 = \mathbb{Q}$. For more details see [FHT01]. However we need to describe the construction $A(K)$ that allows to mix a simplicial complex K and a simplicial cochain algebra A in order to obtain a cochain algebra $(A(K),d)$.

Definition. Let K be a simplicial set and $A = \{(A_n, d_n)\}_{n\geq 0}$ be a simplicial cochain algebra. We define a cochain algebra $(A(K),d)$ in the following way:

- $A(K)$ is a sequence $\{A^p(K)\}_{p\geq 0}$ of vector spaces $A^p(K)$, whose elements are the simplicial set morphisms from K to $A^p = \{A^p{}_n\}_{n\geq 0}$
- If $\phi, \psi \in A^p(K)$, $\sigma \in K_n$ and $\lambda \in \mathbb{Q}$ then

$$(\phi+\psi)(\sigma) \equiv \phi(\sigma) + \psi(\sigma), \qquad (\lambda\cdot\phi)(\sigma) \equiv \lambda \cdot \phi(\sigma),$$
$$(d\phi)(\sigma) \equiv d(\phi(\sigma)), \qquad (\phi\cdot\psi)(\sigma) \equiv \phi(\sigma)\cdot\psi(\sigma).$$

If we fix the simplicial cochain algebra A, taking for example A_{PL}, we obtain a contravariant functor, also called A_{PL}. Since the algebra A_{PL} is commutative, the image of any simplicial set by the functor A_{PL} is going to be a commutative cochain algebra. The image under this functor of a morphism of simplicial sets $f : K \to L$ is a morphism $A_{PL}(f) : (A_{PL}(L),d) \to (A_{PL}(K),d)$ such that, if $\phi \in A^p_{PL}(L)$ and $\sigma \in K_n$ then

$$[A_{PL}(f)(\phi)](\sigma) = \phi(f(\sigma)).$$

We are most interested by applying A_{PL} to the simplicial set of singular simplices $S_*(X)$ of a topological space X, and we write $A(X) \equiv A(S_*(X))$. It can be shown, for example in [FHT01], that A_{PL} induces a natural equivalence between the category of rational homotopy types of simply connected spaces X with $H^*(X;\mathbb{Q})$ of finite type, and the category of weak equivalence classes of commutative cochain algebras (A,d) with $H(A,d)$ of finite type.

Moreover it can be shown that there exists a natural isomorphism of graded algebras

$$H^*(X;\mathbb{Q}) \xrightarrow{\cong} H(A_{PL}(X),d).$$

2.4 Sullivan models

Up to now we have not described any way to actually build the cca $(A_{PL}(X), d)$ corresponding to a topological space X. A fundamental result allows to restrict our study to a class of simple cochain algebras:

Definitions.

- A ***Sullivan commutative cochain algebra***, or Sullivan algebra, is a commutative cochain algebra of the form $(\Lambda V, d)$, with
  - $V = \{V^p\}_{p\geq 1}$;
  - $V = \bigcup_{k=0}^{\infty} V(k)$, where $V(0) \subset V(1) \subset ...$ is an increasing sequence of graded subspaces such that
$$d = 0 \quad \text{in } V(0)$$
$$d : V(k) \longrightarrow \Lambda V(k-1) \quad \text{for all } k \geq 1.$$
- Let (A, d) be a cca, $(\Lambda V, d)$ be a Sullivan cca, and $\phi : (\Lambda V, d) \xrightarrow{\simeq} (A, d)$ be a quasi-isomorphism, then we say that ϕ is a ***Sullivan model*** for (A, d).
- Let X be a path connected topological space, then a Sullivan model for $(A_{PL}(X), d)$ is called a ***Sullivan model for*** X.

Actually we can even restrict to a class smaller than Sullivan algebras: the class of *minimal* Sullivan algebras:

Definition. Let $(\Lambda V, d)$ be a Sullivan cca. If $\operatorname{Im} d \subset \Lambda^+ V \cdot \Lambda^+ V$ we say that $(\Lambda V, d)$ is ***minimal***.

Here are precisely stated the two fundamental theorems:

Theorem 2.4.1 *Let (A, d) be a cca with $H^0(A, d) = \mathbb{Q}$, then there exists a Sullivan model*
$$\phi : (\Lambda V, d) \xrightarrow{\simeq} (A, d).$$

Theorem 2.4.2 *Let (A, d) be a cca with $H^0(A, d) = \mathbb{Q}$, then there exists a minimal Sullivan model*
$$\phi : (\Lambda V, d) \xrightarrow{\simeq} (A, d),$$
which is unique up to isomorphism.

We describe an inductive construction of the minimal model of a simply connected cca (A, d), taken from [FHT01], p. 138. It will be useful when applying our main theorem in chapter 7.

- Compute $H^2(A, d)$ and choose a vector space V^2 with basis $< v_j >_{j\in J}$ isomorphic to it, with $|v_j| = 2$ for all $j \in J$. It is then easy to build a cca morphism $m_2 : (\Lambda V^2, 0) \to (A, d)$ by sending each v_j to a cocycle of A representing a base vector of $H^2(A)$. Therefore $H^2(m_2) : V^2 \xrightarrow{\cong} H^2(A)$. Moreover $H^1(m_2)$ is an isomorphism because $H^1(A) = 0$ and $H^3(m_2)$ is injective because there are no elements of degree 3 in ΛV^2.
- Our induction hypothesis is that there exists a cca $(\Lambda V^{\leq k}, d)$ and a morphism of cca's $m_k : (\Lambda V^{\leq k}, d) \to (A, d)$ such that $H^i(m_k)$ is an isomorphism for $0 \leq i \leq k$ and $H^{k+1}(m_k)$ is injective.

- We extend m_k to m_{k+1}:

  There are cocycles $\{a_\alpha\}_{\alpha\in K} \subset A$ and $\{z_\beta\}_{\beta\in L} \subset (\Lambda V^{\leq k})^{k+2}$ such that

  $$H^{k+1}(A) = \operatorname{Im} H^{k+1}(m_k) \oplus \bigoplus_{\alpha\in K} \mathbb{Q}\cdot[a_\alpha] \quad \text{and} \quad \operatorname{Ker} H^{k+2}(m_k) = \bigoplus_{\beta\in L} \mathbb{Q}\cdot[z_\beta].$$

  This means that for all $\beta \in L$ there exists a $b_\beta \in A$ such that $m_k(z_\beta) = db_\beta$.

  We choose a vector space V^{k+1} with basis $\{v'_\alpha, v''_\beta\}_{\alpha\in K,\beta\in L}$ and all elements of degree $k+1$ and take $V^{\leq k+1} \equiv V^{\leq k} \oplus V^{k+1}$. We extend the derivation d in $\Lambda V^{\leq k}$ to a derivation in $\Lambda V^{\leq k+1}$ by setting

  $$dv'_\alpha \equiv 0 \quad \text{and} \quad dv''_\beta \equiv z_\beta.$$

  Since d has odd degree, d^2 is also a derivation. Moreover $d^2 = 0$ in V^{k+1} and in $\Lambda V^{\leq k}$ and therefore $d^2 = 0$. Moreover we extend m_k to a morphism of graded algebras $m_{k+1} : \Lambda V^{\leq k+1} \to A$ by setting

  $$m_{k+1}(v'\alpha) \equiv a_\alpha \quad \text{and} \quad m_{k+1}(v''_\beta) \equiv b_\beta.$$

  Since $m_{k+1}d = dm_{k+1}$ in V^{k+1} and in $\Lambda V^{\leq k}$, we have $m_{k+1}d = dm_{k+1}$ everywhere.

  It is clear that $H^{k+1}(m_{k+1})$ is surjective by construction, and therefore is an isomorphism; and that $H^{k+2}(m_{k+1})$ is injective by construction.

Let us now turn to maps and morphisms. First of all it is necessary to introduce a notion of homotopy for morphisms of cca's:

Definitions.

- Let $(\Lambda(t, dt), d)$ denote the free cca with basis $\{t, dt\}$, such that $|t| = 0$, $|dt| = 1$ and $d(t) \equiv dt$. Notice that this cca is not a Sullivan algebra since it contains basis elements of degree 0.

- We denote by ϵ_0, respectively ϵ_1 the augmentations $(\Lambda(t, dt), d) \to (\mathbb{Q}, 0)$ defined by $\epsilon_0(t) = 0$, respectively $\epsilon_1(t) = 1$.

- Let $\phi_0, \phi_1 : (\Lambda V, d) \to (A, d)$ be two morphisms of cca, where $(\Lambda V, d)$ is a Sullivan algebra. If there exists a morphism

  $$\Psi : (\Lambda V, d) \longrightarrow (A, d) \otimes (\Lambda(t, dt), d)$$

  such that $(id \cdot \epsilon_i) \circ \Psi = \phi_i$ for $i = 0, 1$, we say that ϕ_0 and ϕ_1 are ***homotopic*** and that Ψ is a ***homotopy***. We write $\phi_0 \simeq \phi_1$.

It turns out that homotopy is an equivalence relation for morphisms of cca's as it is for continuous maps of topological spaces, and that our functor A_{PL} preserves homotopy:

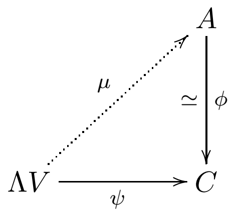
Proposition 2.4.3 *Let $f_0, f_1 : X \to Y$ be homotopic maps, $(\Lambda V, d)$ be a Sullivan algebra and $\psi : (\Lambda V, d) \to (A_{PL}(Y), d)$ be a cca morphism, then*

$$A_{PL}(f_0) \circ \psi \quad \textit{is homotopic to} \quad A_{PL}(f_1) \circ \psi.$$

Moreover if two morphisms $\phi_0, \phi_1 : (\Lambda V, d) \to (A, d)$ from a Sullivan algebra $(\Lambda V, d)$ are homotopic then $H(\phi_0) = H(\phi_1)$.

A very useful result is called the ***lifting lemma***:

Lemma 2.4.4 *Consider the following* **solid** *diagram of cca morphisms.*



*We suppose that ϕ is a surjective quasi-isomorphism and that $(\Lambda V, d)$ is a Sullivan cca, then there exists a cca morphism $\mu : (\Lambda V, d) \to (A, d)$ such that the diagram commutes exactly. We say that μ is a **lift of ψ through** ϕ.*

A similar result with weaker hypothesis allows to associate a unique homotopy class of cca morphisms between Sullivan algebras when given a cca morphism:

Proposition 2.4.5 *Consider the solid diagram from the previous lemma, where this time ϕ is a quasi-isomorphism which is not necessarily surjective, and $(\Lambda V, d)$ is a Sullivan cca. There exists a unique homotopy class of cca morphisms $\mu : (\Lambda V, d) \to (A, d)$ such that the diagram commutes up to homotopy.*

This leads to the

Definition. Consider a cca morphism $\eta : (A, d) \to (B, d)$ and Sullivan models $(\Lambda V, d) \xrightarrow{\alpha} (A, d)$, $(\Lambda W, d) \xrightarrow{\beta} (B, d)$. Since β is a quasi-isomorphism and $(\Lambda V, d)$ is a Sullivan cca, there exists a unique homotopy class of morphisms $\mu : (\Lambda V, d) \to (\Lambda W, d)$ such that $\beta \circ \mu \simeq \eta \circ \alpha$. We call any element of this class a ***Sullivan representative for*** η.

In the end it can be proven that if one restricts its study to simply connected CW-complexes with rational homology of finite type on the one hand and to simply connected Sullivan algebras of finite type on the other hand, there is a bijection between *rational homotopy types* and *isomorphism classes of minimal Sullivan algebras*. There is also a bijection between *homotopy classes of continuous maps of rational spaces* and *homotoy classes of morphisms of Sullivan algebras*.

2.5 Relative Sullivan algebras

There is an alternative way to model cca morphisms than using Sullivan representatives: using relative Sullivan algebras. They have the advantage to model morphisms between algebras with first-degree homology different from zero, and of being good models for fibrations (of topological spaces). Moreover it has been proved that if they are chosen as *cofibrations*, the category of cca's is a *closed model category*.

Definitions.

- A ***relative Sullivan algebra*** (or relative Sullivan cca) is a commutative cochain algebra of the form $(B \otimes \Lambda V, d)$ such that
  - $(B \otimes 1, d)$ is a sub cochain algebra with $H^0(B \otimes 1, d) = \mathbb{Q}$. It is called the ***base algebra*** of $(B \otimes \Lambda V, d)$;
  - $1 \otimes V = \{1 \otimes V^p\}_{p \geq 1}$;

– $1 \otimes V = \bigcup_{k=0}^{\infty}(1 \otimes V(k))$, with $(1 \otimes V(0)) \subset (1 \otimes V(1)) \subset \ldots$ an increasing sequence of graded subspaces such that

$$d : 1 \otimes V(0) \to B \otimes 1 \quad \text{and} \quad d : 1 \otimes V(k) \to B \otimes \Lambda V(k-1) \text{ for } k \geq 1.$$

To simplify the notation we identify thereafter $B \otimes 1 = B$ and $1 \otimes \Lambda V = \Lambda V$.

- Let $\phi : (B, d) \to (C, d)$ be a cca morphism with $H^0(B, d) = \mathbb{Q}$. A ***Sullivan model*** for ϕ is a cca quasi-isomorphism

$$m : (B \otimes \Lambda V, d) \xrightarrow{\simeq} (C, d)$$

such that $(B \otimes \Lambda V, d)$ is a relative Sullivan cca with base algebra (B, d) and $m|_B = \phi$.

In the case of relative Sullivan algebras there also exists a subclass of *minimal* algebras which model cca's in a unique way.

Definitions.

- A relative Sullivan algebra $(B \otimes \Lambda V, d)$ is ***minimal*** if

$$\operatorname{Im} d \subset B^+ \otimes \Lambda V + B \otimes \Lambda^{\geq 2} V.$$

- A ***minimal Sullivan model*** is a Sullivan model $(B \otimes \Lambda V, d) \xrightarrow{\simeq} (C, d)$ such that $(B \otimes \Lambda V, d)$ is minimal.

To state the results about existence and unicity of (minimal) Sullivan models we first need to define the notion of relative homotopy.

Definition. Suppose $\phi_0, \phi_1 : (B \otimes \Lambda V, d) \to (C, d)$ are two cca morphisms, with $(B \otimes \Lambda V, d)$ a relative Sullivan algebra and $\phi_0|_{(B,d)} = \phi_1|_{(B,d)} = \alpha : (B, d) \to (A, d)$. If there is a morphism

$$\Psi : (B \otimes \Lambda V, d) \to (A, d) \otimes \Lambda(t, dt)$$

such that $(id \cdot \epsilon_i) \circ \Psi = \phi_i$ for $i = 0, 1$ and $\Psi|_{(B,d)} = \alpha$ we say that ϕ_0 and ϕ_1 are ***homotopic rel*** B ($\phi_0 \cong \phi_1$ rel B). The morphism Ψ is called a ***relative homotopy from*** ϕ_0 to ϕ_1.

Proposition 2.5.1 *Let $\phi : (B, d) \to (C, d)$ be a cca morphism such that $H^0(B, d) = \mathbb{Q} = H^0(C)$ and $H^1(\phi)$ is injective, then ϕ has a minimal Sullivan model*

$$m : (B \otimes \Lambda V, d) \xrightarrow{\simeq} (C, d).$$

If $m' : (B \otimes \Lambda V', d) \xrightarrow{\simeq} (C, d)$ is another minimal model for ϕ then there exists an isomorphism

$$\alpha : (B \otimes \Lambda V, d) \xrightarrow{\cong} (B \otimes \Lambda V', d)$$

restricting to id_B, and such that $m'\alpha \simeq m$ rel B.

The lifting lemma can also be generalized to relative Sullivan algebras:

Proposition 2.5.2 *Let the following solid diagram be a commutative diagram of cca morphisms*

$$\begin{array}{ccc} B & \xrightarrow{\alpha} & A \\ {\scriptstyle i}\downarrow & \overset{\phi}{\nearrow} & \downarrow{\scriptstyle \eta}\,\simeq \\ B\otimes\Lambda V & \xrightarrow[\psi]{} & C \end{array}$$

where i is the base inclusion of a relative Sullivan algebra.

- *If η is a surjective quasi-isomorphism, there exists a morphism $\phi : (B \otimes \Lambda V, d) \to (A, d)$ such that $\phi \circ i = \alpha$ (we say that ϕ* ***extends*** *α) and $\eta \circ \phi = \psi$ (we say that ϕ lifts ψ).*
- *If η is a not necessarily surjective quasi-isomorphism, there exists a unique homotopy class* $\mathrm{rel}\, B$ *of morphisms $\phi : (B \otimes \Lambda V, d) \to (A, d)$ such that*

$$\phi|_B = \alpha \qquad \textit{and} \qquad \eta \circ \phi \cong \psi \,\mathrm{rel}\, B.$$

It is possible to introduce push-outs of commutative cochain algebra morphisms if at least one of the morphisms is the base injection of a relative Sullivan algebra.

Definition. If $(B \otimes \Lambda V, d)$ is a relative Sullivan algebra and $\psi : (B, d) \to (B', d)$ is a cca morphism such that $H^0(B') = \mathbb{Q}$, then we call ***push-out*** of $(B \otimes \Lambda V, d)$ along ψ the relative Sullivan algebra:

$$(B' \otimes \Lambda V, d) \equiv (B', d) \otimes_{(B,d)} (B \otimes \Lambda V, d).$$

with base $(B', d.)$

The name *push-out* is justified: there exists a commutative diagram of cca morphisms

$$\begin{array}{ccc} B & \xrightarrow{inj_B} & B\otimes\Lambda V \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \psi\circ id} \\ B' & \xrightarrow[inj_{B'}]{} & B'\otimes\Lambda V, \end{array}$$

with inj_B, $inj_{B'}$ the injections of the bases B, B' respectively.

Moreover if there are cca morphisms $\alpha : (B \otimes \Lambda V, d) \to (A, d)$ and $\beta : (B', d) \to (A, d)$ such that $\alpha \circ inj_B = \beta \circ \psi$ then there exists a cca morphism

$$\Phi : (B' \otimes \Lambda V, d) \to (A, d)$$

such that $\Phi \circ (\psi \otimes id) = \alpha$ and $\Phi \circ inj_{B'} = \beta$.

2.6 Modeling adjunction spaces and pull-backs

When given Sullivan models for several topological spaces and continuous maps between them one can try to build Sullivan models, or simply commutative models, for new spaces and maps constructed using the previous ones, while working exclusively in the cca category. This works particularly well when one tries to model adjunction spaces, the homotopy fibre of a map or a pull-back of fibrations.

2.6.1 Modeling adjunction spaces

Definition. Let (Z, B) be a relative CW-complex with the inclusion $j : B \to Z$, and let $f : B \to X$ be a continuous map with B, Z and X path connected. We call ***adjunction space of*** f ***and*** j the space

$$X \cup_f Z = \frac{X \amalg Z}{b \sim f(b)\, \forall b \in B}.$$

Moreover we denote by $j_X : X \to X \cup_f Z$ the inclusion of the relative CW-complex $(X \cup_f Z, X)$ and by $f_Z : Z \to X \cup_f Z$ the inclusion of Z into $X \sqcup Z$ followed by the projection onto $X \cup_f Z$.

According to the previous section it is possible to find Sullivan models

$$m_Z : (\Lambda W_Z, d) \xrightarrow{\simeq} (A_{PL}(Z), d)\,, \qquad m_B : (\Lambda W_B, d) \xrightarrow{\simeq} (A_{PL}(B), d)\,,$$

$$m_X : (\Lambda V, d) \xrightarrow{\simeq} (A_{PL}(X), d)$$

and Sullivan representatives $\phi_f : (\Lambda V, d) \to (\Lambda W_B, d)$ for f and $\phi_j : (\Lambda W_Z, d) \to (\Lambda W_B, d)$ for j.

Note that it is always possible to build a cca (A, d) such that $H(A, d) = \mathbb{Q}$ and a surjective morphism $h : (A, d) \to (\Lambda W_B, d)$. Take for example $A \equiv \Lambda(W_B \oplus \widetilde{W}_B)$ where $\widetilde{W}_B \equiv \{\tilde{w} | w \in W_B\}$, and $d : W_B \to \widetilde{W}_B$ with $dw = \tilde{w}$ for all $w \in W_B$. We then define $h(w) \equiv w$ for all $w \in W_B$ and $h(\tilde{w}) \equiv dw$ for all $\tilde{w} \in \widetilde{W}$.

Proposition 2.6.1 *The fibre product of ϕ_f and $\phi_j \circ h$,*

$$(\Lambda V \times_{(\Lambda W_B, d)} (\Lambda W_Z \otimes A), d)$$

is a commutative model for the adjunction space $X \cup_f Z$. Moreover if either ϕ_j or ϕ_f is surjective the fibre product of ϕ_f and ϕ_j is already a commutative model for $X \cup_f Z$.

Remark. Let us recall that an adjunction space is a push-out in the category of topological spaces. It is interesting to note that it is modeled by a pull-back in the cca category.

2.6.2 Modeling homotopy fibres and pull-backs

Before considering pull-backs we must determine the general look of Sullivan models of (Serre) fibrations.

Proposition 2.6.2 *Let us consider a Serre fibration $p : E \to B$ of path-connected spaces, with path-connected fibres, with the inclusion of the fibre at $* \in B$ being the map $j : F \to E$, then $H^1(A_{PL}(p))$ is injective and therefore there exists a (minimal) Sullivan model*

$$m : (A_{PL}(B) \otimes \Lambda V, d) \xrightarrow{\simeq} (A_{PL}(E), d).$$

Definition. Let $p : E \to B$ be a Serre fibration of path-connected spaces, with path-connected fibres, with the fibre inclusion at $*$ being $j : F \to E$ and a Sullivan model for p being $m : (A_{PL}(B) \otimes \Lambda V, d) \xrightarrow{\simeq} (A_{PL}(E), d)$. Using the augmentation $\epsilon : (A_{PL}(B), d) \to (\mathbb{Q}, 0)$ we define the ***fibre of the model at*** $*$:

$$(\Lambda V, \bar{d}) \equiv (\mathbb{Q}, 0) \otimes_{(A_{PL}(B),d)} (A_{PL}(B) \otimes \Lambda V, d)$$

The following theorem shows that if given a Sullivan model for a Serre fibration it is easy to construct a Sullivan model for its fibre. More generally one can build a Sullivan model for the homotopy fibre of any map whose Sullivan model is known, under mild hypothesis.

Theorem 2.6.3

- *Suppose $p : E \to B$ is a Serre fibration with notations as in the previous definition. If B is simply connected and one of $H_*(B;\mathbb{Q})$, $H_*(F;\mathbb{Q})$ has finite type, then there exists a quasi-isomorphism*

$$\overline{m} : (\Lambda V, \bar{d}) \xrightarrow{\simeq} (A_{PL}(F), d)$$

making the following diagram of cca morphisms commute:

$$\begin{array}{ccc} (A_{PL}(B) \otimes \Lambda V, d) & \xrightarrow{\epsilon \cdot id} & (\Lambda V, \bar{d}) \\ {\scriptstyle m}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\downarrow{\scriptstyle \overline{m}} \\ (A_{PL}(E), d) & \xrightarrow[A_{PL}(j)]{} & (A_{PL}(F), d). \end{array}$$

- *Let $f : X \to Y$ be a continuous map with X path-connected, Y simply connected and $H(Y;\mathbb{Q})$ of finite type, then f has a Sullivan model*

$$m : (A_{PL}(Y) \otimes \Lambda V, d) \xrightarrow{\simeq} (A_{PL}(X), d)$$

and the fibre $(\Lambda V, \bar{d})$ at a given point y_0 is a Sullivan model of the homotopy fibre of f.

One can also replace $(A_{PL}(B), d)$ by one of its Sullivan models $m_B : (\Lambda V_B, d) \xrightarrow{\simeq} (A_{PL}(B), d)$ and obtain Sullivan models

$$m' : (\Lambda V_B \otimes \Lambda V, d) \xrightarrow{\simeq} (A_{PL}(E), d), \qquad \text{and} \qquad \overline{m}' : (\Lambda V, \bar{d}) \xrightarrow{\simeq} (A_{PL}(F), d).$$

We have just seen that one can find a Sullivan model for the fibre of a Serre fibration p when given a Sullivan model of p. It is actually also possible to find a Sullivan model for the total space and a Sullivan model for p when given Sullivan models of the fibre and of the base space.

Proposition 2.6.4 *Let $p : E \to B$ be a Serre fibration between path-connected spaces with path-connected fibre F at $* \in B$. Let B be simply connected, and let one of $H_*(F;\mathbb{Q})$, $H_*(Y;\mathbb{Q})$ have finite type. Suppose that $m_B : (\Lambda V_B, d) \xrightarrow{\simeq} (A_{PL}(B), d)$ is a Sullivan model for B, that there exist a relative Sullivan algebra $(\Lambda V_B \otimes \Lambda W, d)$ and a cca morphism $n : (\Lambda V_B \otimes \Lambda W, d) \to (A_{PL}(E), d)$ which restricts to $A_{PL}(p) \circ m_B$ in $(\Lambda V_B, d)$.*

Then there exists a morphism $\overline{n} : (\Lambda W, \bar{d}) \to (A_{PL}(F), d)$ such that $\overline{n} \circ \epsilon \cdot id = A_{PL}(j) \circ n$. Moreover if $\overline{n}$ is a quasi-isomorphism then so is n, therefore n is a Sullivan model for E.

We can now consider pull-backs of Serre fibrations.

Proposition 2.6.5 *Suppose the following diagram of continuous maps is commutative:*

$$\begin{CD} E' @>g>> E \\ @Vp'VV @VVpV \\ B' @>>f> B \end{CD}$$

where p and p' are Serre fibrations, E and E' are path connected and B and B' are simply connected. We suppose that f is a pointed map and let $\bar{g} : F' \to F$ denote the restriction of g to the fibre of p' at $*$. We assume that one of $H_*(F;\mathbb{Q})$, $H_*(B;\mathbb{Q})$ and one of $H_*(F';\mathbb{Q})$, $H_*(B';\mathbb{Q})$ has finite type.

Let us choose the following Sullivan models for B, B', p and p':

$$\begin{array}{ll} m_B : (\Lambda V_B, d) \to (A_{PL}(B), d); & m_{B'} : (\Lambda V_{B'}, d) \to (A_{PL}(B'), d); \\ m_E : (\Lambda V_B \otimes \Lambda V, d) \to (A_{PL}(E), d); & m_{E'} : (\Lambda V_{B'} \otimes \Lambda V', d) \to (A_{PL}(E'), d). \end{array}$$

If $H^*(\bar{g}) : H^*(F;\mathbb{Q}) \to H^*(\overline{F};\mathbb{Q})$ is an isomorphism, in particular if the previous diagram is a pull-back, then there exists a Sullivan model for E' of the form

$$(\Lambda V_{B'} \otimes \Lambda V, d) \equiv (\Lambda V_{B'}, d) \otimes_{(\Lambda V_B, d)} (\Lambda V_B \otimes \Lambda V, d) \xrightarrow{\simeq} (A_{PL}(E'), d)$$

REMARKS.

- Notice that a pull-back of topological fibrations is modeled by a push-out of cca morphisms.
- Recall from corollary 1.2.3 that the standard homotopy pull-back of two maps f and g is homotopic to the pull-back of f and of the fibration associated to g. Therefore to model a homotopic pull-back it is sufficient to be able to model pull-backs.

2.7 Closed model categories

Commutative cochain algebras are an example of closed model categories: categories with three distinguished classes of maps (weak equivalences, fibrations and cofibrations) which are fit to support a homology theory. Cca's therefore possess nice properties which we need in chapter 6. Before defining closed model categories more precisely we give a useful definition:

Definition. Let $\mathfrak{C}$ be a category and $f : X \to Y$, $g : A \to B$ be morphisms in $\mathfrak{C}$. We say that f is a retract of g if there exists a commutative diagram of morphisms in $\mathfrak{C}$

$$\begin{CD} X @>>> A @>>> X \\ @VfVV @VgVV @VfVV \\ Y @>>> B @>>> Y \end{CD}$$

in which the horizontal maps compose to the identity.

We can now proceed with the main definition.

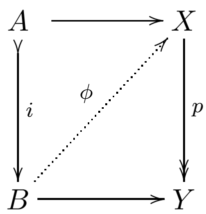
Definition. A ***closed model category*** consists of a category $\mathfrak{C}$ together with three classes of morphisms in $\mathfrak{C}$ called respectively ***fibrations*** (we write $X \twoheadrightarrow Y$), ***cofibrations*** (we write $X \rightarrowtail Y$) and ***weak equivalences*** (we write $X \xrightarrow{\simeq} Y$), satisfying:

CM1 $\mathfrak{C}$ is closed under finite direct and inverse limits.

CM2 Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms in $\mathfrak{C}$. If any two of f, g and $g \circ f$ are weak equivalences, so is the third.

CM3 If f is a retract of g, and g is a weak equivalence, fibration, or cofibration, so is f.

CM4 Given a solid arrow diagram



where i is a cofibration and p is a fibration, if either i or p is a weak equivalence, then there exists a morphism ϕ, represented as a dotted arrow on the previous diagram, letting the whole diagram commute. We say that p ***has the (right) lifting property with respect to*** i, and that i ***has the left lifting property with respect to*** p.

CM5 Any morphism f can be factored as $f = p \circ i$ where

- i is a cofibration and a weak equivalence and p is a fibration;
- i is a cofibration and p is a fibration and a weak equivalence.

2.7.1 Commutative cochain algebras

It is shown in [BG76] that the category of commutative cochain algebras is a closed model category if we choose

- as weak equivalences the quasi-isomorphisms,
- as fibrations all surjective morphisms,
- as cofibrations the inclusions of the base of a relative Sullivan algebra.

In section 2.6 we used commutative cochain algebras to model adjunction spaces, for example attaching cells, and the fibre of a (Serre) fibration. It is interesting to note that to do so we needed surjective morphisms and (relative) Sullivan algebras, which are fibrations and cofibrations in the cca category.

2.7.2 Topological spaces

The category of topological spaces can also be given a closed model category structure by choosing:

- as weak equivalences the weak homotopy equivalences,
- as fibrations the Serre fibrations,
- as cofibrations all maps which have the left lifting property with respect to each map which is both a Serre fibration and a weak homotopy equivalence.

Equivalently a map $f : X \to Y$ is a cofibration if it is a retract of a map $X \to Z$ in which Z is obtained from X by attaching cells.

There exists another closed model category structure on topological spaces with

- as weak equivalences the homotopy equivalences,
- as fibrations the fibrations of spaces,
- as cofibrations the inclusions of NDR pairs.