Chapter 1

A few tools

1.1 Introduction

One of the definitions of LS-category is given in terms of Ganea spaces and Ganea maps, which can be constructed as consecutive joins of some chosen maps. Joins are in turn made up of a homotopy pull-back followed by a homotopy push-out. In section 1.2 we introduce homotopy push-outs and homotopy pull-backs in the sense of Mather [Mat76], and give a few important properties. Then in section we define joins and state Doeraene's two "join theorems".

1.2 Homotopy push-outs and homotopy pull-backs

Let us restrict our study to the category of topological spaces, while keeping in mind that everything can be easily translated in a "pointed" context. To work on a clear basis we must define *homotopy commutative diagrams*:

Definition. Consider a diagram W of continuous maps together with collection A of homotopies between maps or composites of maps in W. It is called ***homotopy commutative*** if

- for any two composites $f_1 \circ f_2 \circ ... \circ f_n : X \to Y$ and $g_1 \circ g_2 \circ ... \circ g_k : X \to Y$ of maps f_i, $g_j \in W$, $1 \leq i \leq n$, $1 \leq j \leq m$ with same source and target spaces, there exists a homotopy $H \in A$ between them;
- if it is possible to build two different homotopies H and G between two maps f and g by mixing sums and composites of the homotopies in A, then there exists a homotopy relative to (f, g) between H and G.

We can now recall the definition of fibrations and Serre fibrations, as well as the definition of cofibrations and NDR-pairs.

Definition. We consider a commutative diagram of topological spaces:

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ Z & \xrightarrow[g]{} & Y. \end{array}$$

- The map $p : X \to Y$ is a ***Serre fibration*** if for any $A = K \times \{0\}$ and $Z = K \times I$ with K a CW-complex there exists a continuous map $r : Z \to X$ such that $p \circ r = g$ and $r \circ i = f$, i.e. p has the lifting property with respect to $(K \times I, K \times \{0\})$.
- The map $p : X \to Y$ is a ***fibration*** if it has the lifting property with respect to any $(K \times I, K \times \{0\})$ with K a topological space.

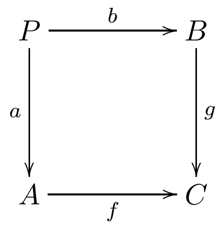
Definitions.

- Let $A \subset X$ be topological spaces, such that for any map $f : X \to Y$ and any homotopy $H : A \times I \to Y$ with $H(a, 0) = f(a)$ for all $a \in A$ there exists an extension $G : X \times I \to Y$ such that $G(a, t) = H(a, t)$ for all $a \in A$, $t \in I$ and $G(x, 0) = f(x)$ for all $x \in X$, then the pair (X, A) is called a ***cofibration*** and is said to have the ***homotopy extension property***
- If the pair (X, A) is a cofibration and A is closed in X then it is called an ***NDR-pair***.

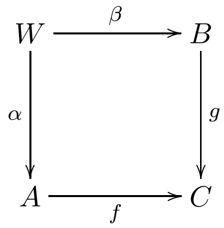
In the category of topological spaces pull-backs and push-outs do not always exist, unless one of the maps involved is a fibration, respectively a cofibration. We therefore introduce *homotopy* pull-backs and *homotopy* pushouts, which possess the pull-back, respectively the push-out, property "up to homotopy".

Definitions.

- Let us consider a homotopy commutative diagram

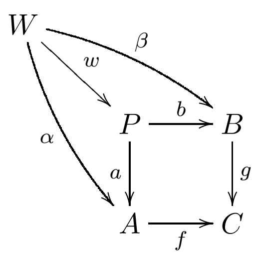


and a homotopy $H : P \times I \to C$ between $g \circ b$ and $f \circ a$. Together they are called a ***homotopy pull-back*** when for any homotopy commutative diagram



equipped with a homotopy G between $g \circ \beta$ and $f \circ \alpha$ we have:

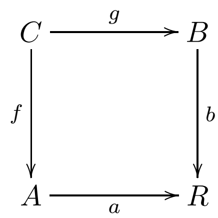
1. there exists a map $w : W \to P$ (we say ***whisker map***) and homotopies $L : \alpha \simeq a \circ w$, $K : \beta \simeq b \circ w$ such that the following diagram

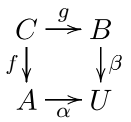


with the homotopies H, G, K, L is homotopy commutative, i.e. $g \circ K + H \circ w + f \circ L \simeq G$ relative to $(g \circ \beta, f \circ \alpha)$;

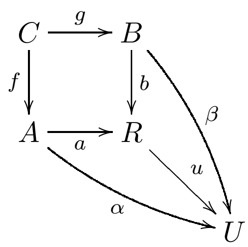
2. if there exists another map $w' : W \to P$ and homotopies $L' : \alpha \simeq a \circ w'$, $K' : \beta \simeq b \circ w'$ such that the previous diagram homotopy commutes when one replaces w with w', L with L' and K with K', then there exists a homotopy $M : w \simeq w'$ such that the previous diagram homotopy commutes when one adds to it the map w' and the homotopies M, L', K', i.e. $K + b \circ M \simeq K'$ relative to $(\beta, b \circ w')$ and $a \circ M + L' \simeq L$ relative to $(\alpha, a \circ w)$.

- To obtain the notion of *homotopy push-out* one must simply "dualize", i.e. reverse all arrows in, the notion of homotopy pull-back, as follows: a ***homotopy push-out*** is a diagram such as the following one:



together with a homotopy $H : C \times I \to R$ between $a \circ f$ and $b \circ g$, such that for any other diagram  and homotopy G between $\alpha \circ f$ and $\beta \circ g$,

1. there exists a ***whisker map*** $u : R \to U$ and homotopies $L : \alpha \simeq u \circ a$, $K : \beta \simeq u \circ b$ making the diagram

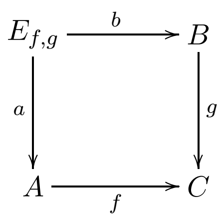


together with the homotopies H, G, K, L, homotopy commutative;

2. if there exists another map $u : R \to U'$ and homotopies $L' : \alpha \simeq a \circ u'$, $K' : \beta \simeq b \circ u'$ such that the previous diagram homotopy commutes when one replaces u with u', L with L' and K with K', then there exists a homotopy $M : u \simeq u'$ such that the previous diagram homotopy commutes when one adds to it the map u' and the homotopies M, L', K'.

Homotopy pull-backs and homotopy push-outs do exist. We can actually construct at least one homotopy pull-back for any two maps with the same target space and one homotopy push-out for any two maps with the same source space, as follows:

- A ***standard homotopy pull-back*** is a homotopy commutative diagram



where

$$E_{f,g} \equiv \{(x,\omega,y) \in A \times C^I \times B \,|\, f(x) = \omega(0) \,\text{and}\, g(y) = \omega(1)\},$$

$a(x,\omega,y) = x$, $b(x,\omega,y) = b$ for all $(x,\omega,y) \in E_{f,g}$ and there is a homotopy $H : E_{f,g} \times I \to C$, $H((x,\omega,y),t) \equiv w(t)$. It is easy to verify that such a homotopy commutative diagram is a homotopy pull-back.

- There is the dual notion of ***standard homotopy push-out***, which is a homotopy commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{g} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle b} \\ A & \xrightarrow[a]{} & Z_{f,g} \end{array}$$

and a homotopy $K : C \times I \to Z_{f,g}$ such that

$$Z_{f,g} \equiv \frac{A \sqcup (C \times I) \sqcup B}{f(c) \sim (c,0) \text{ and } g(c) \sim (c,1)\ \forall c \in C},$$

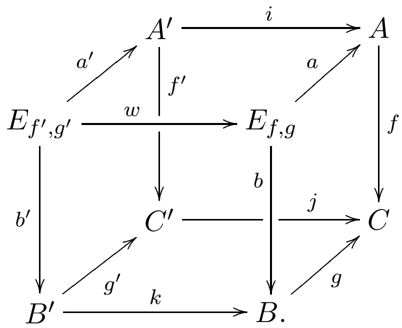
$a(x) = [x]$, $b(y) = [y]$ for all $x \in A$, $y \in B$, and $K(c,t) = [c,t]$. Any standard homotopy push-out is a homotopy push-out.

It is interesting to note that the standard homotopy pull-back and push-out constructions are functorial.

Lemma 1.2.1 *Suppose there exists a homotopy commutative diagram of spaces and maps*

$$\begin{array}{ccc} A' & \xrightarrow{i} & A \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ C' & \xrightarrow{j} & C \\ {\scriptstyle g'}\uparrow & & \uparrow{\scriptstyle g} \\ B' & \xrightarrow{k} & B. \end{array}$$

Let $\begin{array}{ccc} E_{f,g} & \xrightarrow{b} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$ *, respectively* $\begin{array}{ccc} E_{f',g'} & \xrightarrow{b'} & B \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle g'} \\ A & \xrightarrow[f']{} & C \end{array}$ *be the standard homotopy pull-backs of the maps* f*,* g*, respectively* f'*,* g'*, then there exists a map* $w : E_{f',g'} \to E_{f,g}$ *such that the following diagram commutes up to homotopy:*

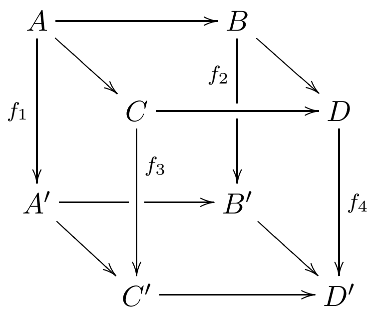


PROOF. We simply build the whisker map w. □

Of course the correspondent dualized lemma for homotopy push-outs is also true.

It is important to notice that if one replaces one of the original maps of a homotopy pull-back by its associated fibration in the standard way and uses it to construct a pull-back, then it is easy to show, using the following lemma, that there exists a homotopy equivalence between the pull-back and the homotopy pull-back.

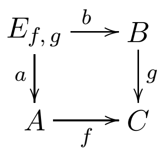
Lemma 1.2.2 *Suppose the following square is a homotopy commutative diagram*



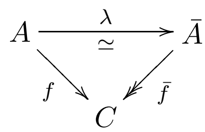
If the top and bottom squares are homotopy pull-backs and f_2, f_3, f_4 are homotopy equivalences, then f_1 is also a homotopy equivalence.

In our case we have:

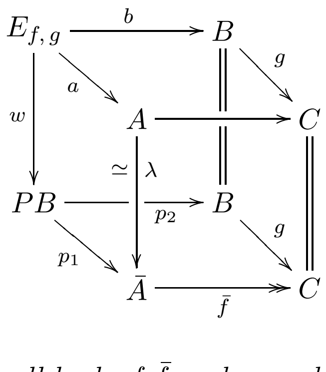
Corollary 1.2.3 *Let*



be a standard homotopy pull-back, and



be a commutative diagram, where $\bar{f}$ is a fibration and λ is a homotopy equivalence, then the following diagram is homotopy commutative

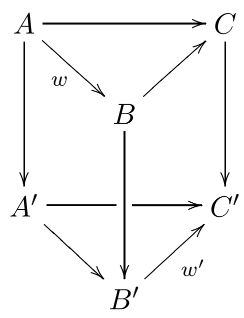


where the bottom square is the pull-back of $\bar{f}$ and g, and w is a whisker map. Moreover, w is a homotopy equivalence.

Analogously one can replace a map by its associated cofibration and show that the push-out and the homotopy push-out are homotopic.

We now state two classical results about homotopy push-outs and pull-backs.

Lemma 1.2.4 (Prism lemma) *Let us consider a homotopy commutative diagram:*



- *If the right face is a homotopy pull-back and w is the whisker map, then the left face is a homotopy pull-back if and only if the back face is a homotopy pull-back.*
- *If the left face is a homotopy push-out and w' is the whisker map, then the right face is a homotopy push-out if and only if the back face is a homotopy push-out.*

Before stating the next lemma we need the notion of *fibration sequence* and *cofibration sequence*.

Definitions.

- Let $f : A \to X$ be a map between spaces A, X. By its ***homotopy cofibre*** $C(f)$ we mean the space $X \cup_f CA$, where CA is the cone over A. The obvious inclusion $X \to C(f)$ is then a cofibration.
- A sequence $A \xrightarrow{f} X \to C$ is called a ***cofibration sequence*** if C is the homotopy cofibre of f and $X \to C$ is the obvious inclusion.
- Let X be a pointed space. The ***Moore path space*** of X is

$$PX \equiv \{(\gamma, l) \in X^{[0,\infty)} \times [0,\infty) \,|\, \gamma(t) = \gamma(l) = *, \forall t \geq l\}$$

and the ***Moore loop space*** of X is

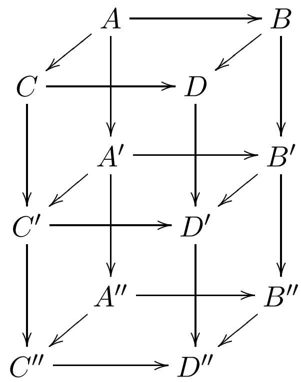
$$\Omega X \equiv \{(\gamma, l) \in PX \,|\, \gamma(0) = *\}.$$

Lemma 1.2.5 *If X is pointed and path connected, then the map $p : PX \to X$, defined as $p(\gamma, l) \equiv \gamma(0)$ is a fibration with fibre ΩX. It is called the* ***Moore path space fibration*** *for X.*

Definitions.

- Let $f : X \to Y$ be a map between pointed path connected spaces. By its ***homotopy fibre*** $F(f)$ we mean the space $X \times_Y PY$ resulting from the pull-back of f and the Moore path space fibration $p : PY \to Y$.
- A sequence $F \xrightarrow{p} X \xrightarrow{f} Y$ is called a ***fibration sequence*** if F is the homotopy fibre of f and p is the projection on the first factor.

Lemma 1.2.6 (Four fibrations lemma) *Let the following diagram be homotopy commutative*



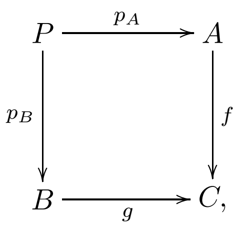
where the two squares $A'B'D'C'$, $A''B''D''C''$ are homotopy pull-backs, $B \to B' \to B''$, $C \to C' \to C''$ and $D \to D' \to D''$ are fibration sequences and $A \to A'$, $A' \to A''$ are the whisker maps induced by the diagram, then the square $ABDC$ is a homotopy pull-back if and only if $A \to A' \to A''$ is a fibration sequence.

Again the dual lemma is true. It can be obtained by reversing all arrows, replacing pull-backs by push-outs and fibration sequences by cofibration sequences.

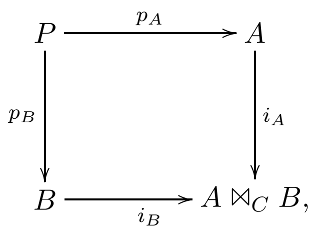
1.3 Joins

To construct Ganea maps and spaces like in the following chapter it is necessary to take a homotopy pull-back of two maps, followed by a homotopy push-out of the two maps that were obtained in the process. We can choose standard homotopy pull-backs and push-outs to obtain a well-determined induced map and call this operation a join.

Definition. Let $f : A \to C$ and $g : B \to C$ be two maps. We first build their standard homotopy pull-back



where p_A and p_B are the projection on A and B respectively. We then take the standard homotopy push-out of p_A and p_B



where i_A, i_B are the inclusions. The whisker map $A \bowtie_C B \longrightarrow C$ is called the ***join of** f **and** g* and is denoted $f \bowtie g$. As for the space $A \bowtie_C B$ it is called the ***join of** A **and** B **over** C*.

REMARK. The join of two maps $X \to \{*\}$, $X' \to \{*\}$ whose target space is $\{*\}$ is usually called the ***join of spaces*** X ***and*** X' and is denoted $X * X'$.

Dually one could define the cojoin, but we are not going to need it when dealing with topological spaces. However we are going to define a *rational cojoin* in the category of commutative cochain algebras (see 6.2). One can verify that the join operation is transitive; we therefore omit any parenthesis hereafter.

We now proceed with the statement of the two *join theorems* from Doeraene (see [Doe98]) which show that the join of homotopy pull-backs is a pull-back and the join of a homotopy pull-back and a homotopy push-out is a homotopy push-out.

Theorem 1.3.1 (Join theorem I) *Let us consider a homotopy commutative diagram made up of two homotopy pull-backs:*

$$\begin{array}{ccccc} A & \xrightarrow{f} & C & \xleftarrow{g} & B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} \\ A' & \xrightarrow[f']{} & C' & \xleftarrow[g']{} & B'. \end{array}$$

Then there exists a homotopy pull-back

$$\begin{array}{ccc} A \bowtie_C B & \xrightarrow{f \bowtie g} & C \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \gamma} \\ A' \bowtie_{C'} B' & \xrightarrow[f' \bowtie g']{} & C'. \end{array}$$

Moreover there exist two homotopy pull-backs

$$\begin{array}{ccc} A & \longrightarrow & A \bowtie_C B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \phi} \\ A' & \longrightarrow & A' \bowtie_{C'} B' \end{array} \qquad \begin{array}{ccc} B & \longrightarrow & A \bowtie_C B \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \phi} \\ B' & \longrightarrow & A' \bowtie_{C'} B'. \end{array}$$

Theorem 1.3.2 (Join theorem II) *Let the following diagram be homotopy commutative*

$$\begin{array}{ccccc} A & \longrightarrow & C & \longleftarrow & B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} \\ A' & \longrightarrow & C' & \longleftarrow & B' \end{array}$$

wth the left square being a homotopy push-out and the right square a homotopy pull-back, then there exist two homotopy push-outs

$$\begin{array}{ccc} A \bowtie_C B & \longrightarrow & C \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \gamma} \\ A' \bowtie_{C'} B' & \longrightarrow & C' \end{array} \qquad \begin{array}{ccc} A & \longrightarrow & A \bowtie_C B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \phi} \\ A' & \longrightarrow & A' \bowtie_{C'} B' \end{array}$$