Relative homotopy invariants of the type of the Lusternik–Schnirelmann category

Relative Homotopie–Invarianten des Types der Kategorie von Lusternik–Schnirelmann

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Eine beim Fachbereich
Mathematik und Informatik
der FREIEN UNIVERSITÄT BERLIN
eingereichte Dissertation

2002

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Datum der Disputation 9. Mai 2003

à Laure et Yvan

Acknowledgements

My thanks go first of all to my director Hans Scheerer, who kindly welcomed me in Berlin as a guest, accepted to supervise my thesis and managed to improve seriously my knowledge of German by letting me discuss mathematics - and any other topics - in this language. I am most grateful for his trust that I would not abandon my thesis even after moving away from Berlin and for his continual guidance throughout my work through long discussions or most recently by email.

There are also three organizations which I must thank gratefully for providing me with money throughout my thesis. Money not only buys food and shelter, it also gives recognition to one's work, which is crucial in mathematics where one sometimes feels isolated from the rest of society by a wall of theorems. Knowing that I was or had been supported was a great encouragement to keep on doing research, especially after my daughter was born. I therefore thank the Freie Universität Berlin for letting me work as an assistant to students during eleven months, thereby also contribuing to my attachment to the place. My acknowledgements go also to the Université de Lausanne for providing me with a scholarship (bourse de perfectionnement et de recherche) for the year 98/99 and to the Société Académique Vaudoise for awarding me the Bourse Félix Bonjour so I could finish my thesis.

There are of course many other people who contributed to this work, either professionally or personally, and I cannot cite them all. Nevertheless I would like to acknowledge most gratefully a few of them. First of all there is Yves Félix, who invited me in Louvain and thereby greatly contributed in solving a problem which had been hindering my work for weeks. Working with him and his students was a great pleasure and a source of lasting motivation and friendships.

Coming back to Berlin I give my warmest thanks to Irasema Sarmiento for sharing my office and offering me her friendship. I also thank the whole topology group and every member of the after-lunch-coffee club who provided serious and less serious discussions.

My thanks go also to Dominique Arlettaz, my former supervisor during my diploma, who contaminated me with his passion for mathematics and remembered to invite me to the wonderful 1999 Algebraic Topology conference in Arollaz, which he organized together with Kathryn Hess.

Finally I must thank my daughter Laure, who accepted to sleep for two hours (almost) every afternoon during one year, in order to give me some time to finish writing my thesis, and my husband Yvan who took time from his research to provide me with articles and books, sometimes from libraries far away, and gave very helpful computer support.

Last but not least I thank my parents for their continuous support and trust.

Contents

Acknowledgements **7**

Introduction **11**

1 A few tools **13**
1.1 Introduction . . . 13
1.2 Homotopy push-outs and homotopy pull-backs . . . 13
1.3 Joins . . . 19

2 Rational homotopy **21**
2.1 Introduction . . . 21
2.2 Definitions . . . 22
2.2.1 Rational homotopy type . . . 22
2.2.2 Commutative cochain algebras . . . 22
2.2.3 Simplicial objects . . . 23
2.3 Equivalence of categories . . . 24
2.4 Sullivan models . . . 25
2.5 Relative Sullivan algebras . . . 27
2.6 Modeling adjunction spaces and pull-backs . . . 29
2.6.1 Modeling adjunction spaces . . . 30
2.6.2 Modeling homotopy fibres and pull-backs . . . 30
2.7 Closed model categories . . . 32
2.7.1 Commutative cochain algebras . . . 33
2.7.2 Topological spaces . . . 34

3 Classical LS-category **35**
3.1 Introduction . . . 35
3.2 Absolute LS-category and cone-length . . . 35
3.3 Relative LS-category and cone-length . . . 37

4 Product formulas **47**
4.1 Introduction . . . 47
4.2 Product formula for relative cone-length . . . 47
4.3 Product formulas for relative categories . . . 52

5 Rational LS-category **59**
5.1 Introduction . . . 59
5.2 Absolute case . . . 59
5.3 Relative case: definitions . . . 61
5.4 Main theorem . . . 63

6 Proof of the main theorem **65**
6.1 Introduction . . . 65
6.2 Rational cojoin . . . 65
6.3 Ganea algebras . . . 68
6.4 First part of the proof . . . 69
6.4.1 Special case: absolute rational category . . . 74
6.5 Second part of the proof . . . 74

7 Applications **79**
7.1 Introduction . . . 79
7.2 Two elementary examples . . . 79
7.2.1 Example 1 . . . 79
7.2.2 Example 2 . . . 80
7.3 Inclusion of a fibre . . . 81
7.4 Euler class . . . 86

8 Sectional category **91**
8.1 Introduction . . . 91
8.2 Classical sectional category . . . 91
8.3 Rational sectional category . . . 94
8.4 Proof of the Theorem . . . 95

Anhänge **103**

A Zusammenfassung **103**

B Lebenslauf **105**

Introduction

Among the numerous homotopy invariants the *category of Lusternik-Schnirelmann*, or *LS-category*, of a topological space has aroused much interest since its definition in 1934 [LS34]. For example it was shown that it is related to another invariant: the *cone-length* of a space [Fox41], [Gan67], [Cor95]. Moreover LS-category can be extended to continuous maps in three different ways [Fox41], [Fad85], thus generating the *F-category*, the *R-category* and the *LS-category* of a map, which are analogous to the *sectional category* of a fibration [Sch66]. Finally Félix and Halperin [FH83] gave a new dimension to the LS-category by transferring it into the context of rational homotopy theory: they gave a method to compute its rationalization directly in the category of *commutative cochain algebras* (in short: cca's). They also rationalized the F-category of a map.

In this thesis we are particularly interested in relative invariants of the type of the LS-category, such as F-category, R-category, LS-category, sectional category and cone-length of a map. In chapter 1 we introduce a few tools which are very useful to define the various relative categories: *homotopy push-outs*, *homotopy pull-backs* and *joins*. Then we give a brief description of rational homotopy theory in chapter 2: we state the equivalence of categories underlying it which links topological spaces and commutative cochain algebras (in short: cca's). We also define (relative) *Sullivan algebras*, which are particularly nice to deal with, and can be used as building blocks when modelizing some topological constructions such as joins.

Chapter 3 is devoted on the one hand to a description of the original LS-category and cone-length. In particular we give three equivalent definitions of the LS-category: in terms of coverings, of *fat wedges* and of *Ganea maps*, constructed by taking consecutive joins. We also give bounds for the LS-category and the cone-length of a product of spaces. On the other hand we introduce the F-category, the R-category and the LS-category of maps, giving for each of them three equivalent definitions, as well as the *cone-length of a map* [Mar98].

In chapter 4 we find a bound for the cone-length of a product of maps and use it to obtain bounds for the F-category, the R-category and the LS-category of a product of maps.

Chapter 5 contains a summary of part of Félix and Halperin's paper [FH83] giving a rationalization of the absolute LS-category and of the F-category and their characterization directly in the rational context. We then introduce a rationalization of the R-category and the relative LS-category and we state our main theorem, allowing to compute them directly in the cca setting: we use any Sullivan model of the morphism f to construct new morphisms π_m with target space $\mathfrak{F}_m$, $m \geq 0$. The category of f depends then on the existence of a homotopy retract for some π_m. We give a proof of this assertion in chapter 6 by defining *Ganea algebras* $\mathfrak{G}_m$ and *Ganea morphisms* $\mathfrak{g}_m$, $m \geq 0$ modelling Ganea spaces and maps, and then by building (homotopy) commutative diagrams involving π_m and $\mathfrak{g}_m$, $m \geq 0$, which relate the existence of a homotopy retract for π_m to the existence of a homotopy retract for $\mathfrak{g}_m$.

Some applications of the main theorem are given in chapter 7: we show that the R-category can take up any value, and we simplify our main result in case the map being considered is the inclusion of a fibre. Moreover we prove that the rational relative category of a spherical fibration does not depend only on the order of its Euler class as it is the case for its rational sectional category.

Finally we devote our last chapter to the study of a new homotopy invariant: the *sectional category of a sequence of maps*, which generalizes both the sectional category of a fibration and the R-category. In this case as for the classical LS-category we give three equivalent definitions in terms of coverings, of generalized fat wedges and of generalized Ganea spaces. Moreover we rationalize the new invariant and prove a theorem allowing its direct computation in the rational setting.