SPATIO-TEMPORAL FEEDBACK CONTROL OF PARTIAL DIFFERENTIAL EQUATIONS

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Spatio-temporal feedback control of partial differential equations

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"*Felix qui potuit rerum cognoscere causas.*"

Publius Vergilius Maro (70 v.Chr. - 19 v.Chr.)

Spatio-temporal feedback control of partial differential equations

Chapter 1

Introduction

In this dissertation we will extend and apply the concept of noninvasive time-delayed feedback control to partial differential equations. More specifically, we introduce new *spatio-temporal* control terms and apply them to scalar reaction-diffusion equations.

The aim of the introductory chapter is to introduce the main concepts of the dissertation: In Section 1.1 we give a brief exposition of noninvasive time-delayed feedback control, better known as Pyragas control, as well as some of its extensions. In Section 1.2 we formulate the main goal of this dissertation and introduce the notion of *control triples* to describe the structure of the new control terms. Section 1.3 is devoted to the study of scalar reaction-diffusion equations. We focus on the symmetry-properties of equilibria and periodic orbits. In Section 1.4 we introduce the new spatio-temporal control terms, by establishing the relation between the abstract control triple and the properties of the equilibria and periodic orbits. In Section 1.5 we briefly introduce Hill's equation as the fundamental tool of the proofs. In the last section of this chapter we present a grasshopper's guide and give an outline of this thesis.

1.1. Noninvasive time-delayed feedback control

In their ground-breaking work from 1990 on chaos control [55], Ott, Grebogi and Yorke consider the following question: *"... how can one obtain* *improved performance and a desired time-periodic motion by making only small time-dependent pertubations in an accessible system parameter?"*

A particularly successful answer to this question was given by Kestutis Pyragas in his work from 1992 [58], using time delay to introduce a continuous control term which vanishes on the desired periodic orbit. We call such a control term *noninvasive*. For a dynamical system which is given by the ordinary differential equation $\dot{z}(t) = f(z(t))$, $z \in \mathbb{R}^n$, the controlled system is then described by

$$\dot{z}(t) = f(z(t)) + k\big(z(t) - z(t-\tau)\big). \tag{1.1}$$

Here $\tau > 0$ is the time delay, and $k \in \mathbb{R}$ is the weight of the control term, which is usually called the *feedback gain*. Sometimes, the feedback gain is given by a matrix $k \in \mathbb{R}^{n\times n}$.

The control term introduced by Pyragas uses the difference between the delayed state $z(t-\tau)$ and the current state $z(t)$ of the system. Frequently, the time delay τ is chosen to be an integer multiple of the period p of the periodic orbit $z^*(t)$ of the uncontrolled system $\dot{z}(t) = f(z(t))$. In this case, the control vanishes on the orbit itself, and $z^*(t)$ is also a solution of the controlled system. Thus, the control does not change the periodic orbit itself, it only changes its stability properties, i.e., it is noninvasive. Pyragas control can therefore be used to make unstable objects visible without changing them. It is normally used to stabilize unstable periodic orbits as well as unstable steady states [12, 34, 80]. In the latter case, the time delay τ can be chosen arbitrarily.

The Pyragas control method is one of the most used feedback control schemes today. The original paper from 1992 [58] has been cited more then 3300 times (June 2016). The main advantage, also for experimentalists, is given by the fact that one does not need to know anything about the periodic orbit besides its period. In particular, it is a model-independent control scheme and no expensive calculations are needed for its implementation.

As a consequence, Pyragas control has been applied effectively to atomic force microscopes, for which it increases the resolution of an image that would otherwise be reduced by irregular cantilever oscillations [79], and to helicopters carrying heavy suspended loads, for which it has been patented in 2012 [54]. Time-delayed feedback control has also been successfully applied to walking control of robots [73]. Further experimental realizations include optical control of semiconductor lasers [62, 63], chaos control in the enzymatic peroxidase-oxidase reaction [42] and the Belousov-Zhabotinsky reaction [65]. See the survey paper by Pyragas [59] and the references therein for an overview of experimental realizations.

The success of Pyragas control has been verified for a large number of theoretical models as well. Examples include spiral break-up in cardiac tissues [60], flow alignment in sheared liquid crystals [71], stabilization of synchrony in networks of coupled Stuart-Landau oscillators [10, 11], stabilization of periodic orbits in delay equations [17], control of quantum systems [38], entanglement control in quantum networks [29], and synchronization in neural systems [33].

Besides the original control scheme, various extensions and modifications have been proposed. A widely implemented modification was introduced by Socolar, Sukow and Gauthier in 1994 [70] and it is now called *extended time-delayed feedback control*. In contrast to the original control scheme (1.1), the authors use not only one, but multiple delayed feedback signals weighted by a memory parameter. They are then able to stabilize periodic orbits with arbitrarily large Floquet multipliers, which is not possible with standard Pyragas control. Another modification was proposed by Kittel, Parisi and Pyragas in 1995 [39], where a self-adaptive time delay is introduced. For this control scheme, no preliminary information of the unstable periodic orbit is necessary, since its period is determined during the experiment. This adaptive method also has the advantage of being robust under drifts of system parameters.

So far, rigorous mathematical conditions for the success of Pyragas control are mostly limited to ordinary differential equations. For example, stabilization has been proven to be successful for a subcritical Stuart-Landau oscillator, i.e., near Hopf bifurcation, by Fiedler et al. in 2007 [15]. An experimental confirmation was achieved by optical feedback control of a semiconductor laser [63]. This last example has refuted the so-called *"odd-number limitation"*, which was correctly proven for non-autonomous systems in 1997 [52], see also Chapter 2 for a short discussion. In a footnote, Nakajima formulated the conjecture that the odd-number limitation also holds in the autonomous case and this was subsequently often wrongly cited as a proven fact. A corrected version of the odd-number limitation for autonomous equations has been presented by Hooton and Amann in 2012 [32].

In an attempt to overcome the odd-number limitation, Nakajima and Ueda introduced in 1998 what they called *half-period delayed feedback control* [53]. See Fiedler et al. [16] from 2010 for a more detailed analytical investigation of a half-period feedback scheme. The main idea is to use only a fraction of the period as time delay, and not the full period. In 2011 this idea has been extended to include equivariance of the system in the control scheme, see for example Schneider for a case study of three coupled oscillators [66, 67]. For the more general results see Schneider [68] or Postlethwaite et al. [56]. The equivariant approach is successful for periodic orbits with a prescribed spatio-temporal pattern, which is reflected in the control scheme. Accordingly, it is particularly useful for a *selective* stabilization of periodic orbits. Equivariant Pyragas control can also be used to stabilize unstable periodic orbits in networks, where it is usually difficult to target specific periodic orbits. For a detailed study of a system with dihedral group symmetry, see Schneider and Bosewitz [69]. Furthermore, the equivariant control scheme does not suffer from any general upper limit on the unstable Floquet multiplier, as Bosewitz has shown in 2014 [7]. This is in contrast to the original method by Pyragas, for which Fiedler [14] has proved an upper bound in 2008.

Even though many applications and extensions of Pyragas control have been proposed since 1992, surprisingly few publications consider the *spatial* properties of *partial* differential equations for control. Many examples where only time delay is used can be found in [3–5, 23, 26, 40, 64, 76]. A first attempt to use *space* as well as *time* was proposed by Lu et al. in 1996 [43], but there and in subsequent publications [50, 57] spatial modifications and time delay are only used separately.

Combinations of spatial and temporal delay have not been used so far, and thus, the possibilities of feedback control for partial differential equations are far from exhausted. One could even say that we have not even started to appreciate fully the power of delayed feedback control for spatially extended systems.

The spirit of Pyragas control guides us through this thesis. We keep in mind the main concept – namely that the control is noninvasive. We set out to explore the concept of noninvasive control in the domain of partial differential equations, using both *space* and *time* for control. In Section 1.2 we state the main goal of this thesis and also explain the general structure of the new *spatio-temporal* control terms, before applying this concept to scalar reaction-diffusion equations.

1.2. Main goal and introduction of the control triple

It is the main goal of this dissertation to introduce a new concept of spatio-temporal feedback control for partial differential equations.

For this new concept, which employs both time and space for control, we also want to offer the first systematic investigation in the context of scalar reaction-diffusion equations. More precisely, we want to selectively stabilize unstable equilibria and periodic orbits of scalar reaction-diffusion equations by using new types of *spatio-temporal* feedback control.

The new control terms satisfy one condition: They are noninvasive on the desired equilibrium or periodic orbit in the sense of Pyragas control. In other words, the control vanishes when the target orbit is reached.

We might be tempted to use the method by Pyragas directly, as it has been done previously [3–5, 23, 26, 40, 64, 76]. We claim, however, that this is impossible for our scalar reaction-diffusion equations, see Theorem 2.2. We therefore need to develop a more successful strategy to control spatio-temporal patterns in partial differential equations.

In this section we explain the general structure of our new feedback control terms. This control structure should be applicable to many partial differential equations, and in particular to those with symmetry.

The general idea, as already used by Pyragas [58], is to use differences between output signals and "transformed" output signals. These differences vanish on the desired orbit, that is, they are noninvasive. In the case of Pyragas control, "transformed" means "time-delayed". In our case, we use a more general concept which we explain below.

Before we describe the new control terms let us ask the following question: Which accessible system parameters can be modified and used for stabilization?

Following in the footsteps of Pyragas, who addresses the system parameter **time**, we can, in the context of partial differential equations use both system parameters, **space** x and **time** t, for the construction of the new control terms.

Also the **output signal** u of the system can be seen as an accessible system parameter, as it has been used previously in the context of equivariant Pyragas control [56, 67–69]. See also Chapter 3 for a comparison of our results and those of equivariant Pyragas control.

In total, we propose to introduce the notion of **control triples** to describe the transformation of the output signal:

$$\big(\textbf{output signal, space, time}\big)$$

We then construct the feedback control as follows: We consider *noninvasive* differences of the current output signal $u(x,t)$ and the transformed output signal $\tilde{u}\left(\tilde{x},\tilde{t}\,\right)$, where the control triple indicates the transformation in each of the three system parameters: output signal $u \mapsto \tilde{u}$, space $x \mapsto \tilde{x}$, and time $t \mapsto \tilde{t}$.

A *control term* is then defined by a *fixed control triple* and a *variable feedback gain* k, where k is either a scalar or a matrix. For simplicity, we consider constant feedback gains throughout this thesis.

As a first example, let us consider an *equilibrium* of some arbitrary partial differential equation, that is, a time-independent solution. In this case, it is feasible to use differences of output signals at different moments of *time*. The transformations in the output signal u and the space x simplify to the identity. Then

$$k\big(u(x,t) - u(x,t-\tau)\big), \tag{1.2}$$

$k \in \mathbb{R}$, is a noninvasive control term for all time delays $\tau > 0$. However, we are not limited to a fixed time delay τ; any transformation of time can be used for noninvasive control if it can be experimentally realized.

In the case of time periodic orbits with minimal period p, the time delay is fixed to an integer multiple of the period p.

While control terms of this type are the obvious application of Pyragas control to partial differential equations, we will prove in this thesis that they do not succeed in the case of scalar-reaction diffusion equations.

Similarly, for any *spatially periodic* equilibrium with period Φ, we are permitted to construct control triples where the non-identity transformation is in space (fixed "spatial delay" Φ or integer multiples of Φ) as well as in time (arbitrary "temporal delay" $\tau \geq 0$). Thus, a noninvasive control term takes the form

$$k\big(u(x,t) - u(x-\Phi,t-\tau)\big). \tag{1.3}$$

To the authors knowledge, such feedback control terms mixing spatial and temporal delay have not been investigated before. We apply such control terms to frozen waves in the context of scalar reaction-diffusion equations.

Consider next plane *waves* of the form $u(x,t) = A\exp(\mathrm{i}\kappa \cdot x - \mathrm{i}ct)$, where $x \in \mathbb{R}^n$, $A \in \mathbb{R}$ is the amplitude, $\kappa \in \mathbb{R}^n$ is the wave vector and $c \in \mathbb{R}$ is the wave speed. Then

$$k\big(u(x,t) - \exp(\mathrm{i}\kappa \cdot a - \mathrm{i}c\tau)\, u(x-a,t-\tau)\big), \tag{1.4}$$

$a \in \mathbb{R}^n$, is a noninvasive control term. In addition to the transformations in space and time, we also transform the output by a multiplication with $\exp(\mathrm{i}\kappa \cdot a - \mathrm{i}c\tau)$.

In *equivariant systems* we find more elaborate spatio-temporal patterns. Equivariance is usually described in terms of groups, and, as a first step towards the construction of suitable control terms, we can find a description of the pattern in terms of group theory [13, 21, 22]. In the equivariant setting we interpret the transformations of the output signal, space, and time as (linear) group actions.

We emphasize that all the described constructions of the control triple do *not* depend on specific equations. Therefore, the control triple method is easily applicable to many partial differential equations. In the following we construct explicit control terms for equilibria and periodic orbits in the case of scalar reaction-diffusion equations. We use simple properties of the respective orbits as well as their description in terms of symmetry groups. Having constructed the new control terms, it is of interest to know which control terms stabilize the target orbit successfully and which limitations of spatio-temporal feedback control can occur.

1.3. Scalar reaction-diffusion equations

In this dissertation the main area of application of the new control terms are scalar reaction-diffusion equations. We therefore give a brief introduction to the subject and state the specific conditions on the equations we consider in this thesis. We then search for equilibria and periodic orbits, and also discuss their symmetry properties. This allows us to state specific new control terms in Section 1.4.

A large variety of physical and biological concepts can be modeled by reaction-diffusion systems. Most prominently, many model systems describing pattern formation fall into the category of reaction-diffusion systems. Examples are the equations studied by Turing [74], the Belousov-Zhabotinsky reaction [77, 81], or systems which describe patterns of animal skin [51].

Reaction-diffusion systems are *semilinear parabolic partial differential equations*, see for example the book by Henry [30] as a standard reference.

In this thesis we consider *scalar reaction-diffusion equations* including a linear advection term cu_x,

$$u_t = u_{xx} + f(u) - cu_x, \tag{1.5}$$

$u \in \mathbb{R}$, $x \in S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$, $t > 0$, with periodic boundary conditions:

$$u(0,t) = u(2\pi,t),\ u_x(0,t) = u_x(2\pi,t) \text{ for all } t > 0. \tag{1.6}$$

The function $f : \mathbb{R} \to \mathbb{R}$ is real analytic. This assumption on f is not essential for control, but we restrict f to be analytic in order not to lose ourselves in technical difficulties. The real parameter c is called the *wave speed.*

The initial-value problem associated to (1.5) generates a local semiflow on the Sobolev space $\mathbb{X} = H^s(S^1)$ for $s > 3/2$,

$$\begin{aligned} \Phi_t : H^s(S^1) &\longrightarrow H^s(S^1), \\ u_0 &\longmapsto \Phi_t(u_0) := u(t,\cdot), \end{aligned} \tag{1.7}$$

where $t \geq 0$, and $u(t,x)$ denotes the maximal solution of (1.5) with initial condition $u(0,\cdot) = u_0$ [2, 19]. By the Sobolev embedding theorem, $\mathbb{X}$ embeds into $C^1(S^1)$ [1]. The Sobolev norm of $H^s(S^1)$ is given by

$$\|u\|_{\mathbb{X}} = \sum_{k \geq 0} (1 + k^2)^s \left(a_k^2 + b_k^2\right), \tag{1.8}$$

where

$$u(x) = \sum_{k \geq 0} \left(a_k \cos(kx) + b_k \sin(kx)\right), \tag{1.9}$$

using Fourier expansion and the periodic boundary conditions (1.6). We assume that f is *dissipative*, i.e., that there exists a large fixed ball in $\mathbb{X}$ such that any solution eventually enters this ball and stays there for all later times. Then solutions of equation (1.5) exist globally and we can even study its global attractor, which consists of equilibria, periodic orbits and the heteroclinic connections between those orbits [2, 18]. A sufficient condition for f to be dissipative is the following: f is bounded from above and $f(u) \cdot u < 0$ for all large $|u|$, [18]. Note however that this condition is not necessary for our results. For instance, we consider linear examples in Chapters 2 and 3.

In order to establish successful control schemes of scalar reaction-diffusion equations, we take a specific point of view: We investigate the equivariance of equation (1.5).

Equation (1.5) is rotationally symmetric, more precisely, it is S^1-equivariant with respect to a shift R_θ in the x-variable,

$$R_\theta : \mathbb{X} \to \mathbb{X}, \quad (R_\theta u_0)(x) := u_0(x + \theta), \tag{1.10}$$

$\theta \in S^1$. The S^1-equivariance holds, since the nonlinearity f does not depend explicitly on the space variable x.

Let us first search for solutions which are time-independent, i.e., equilibria. *Equilibria* $\mathcal{U}(x,t)$ of (1.5) are characterized by $\mathcal{U}_t \equiv 0$, and hence, they are 2π-periodic solutions of the ordinary differential equation

$$0 = \mathcal{U}_{xx} + f(\mathcal{U}) - c\mathcal{U}_x. \tag{1.11}$$

An equilibrium which additionally fulfills $R_\theta \mathcal{U} = \mathcal{U}$ for all $\theta \in S^1$ is called *homogeneous*. Homogeneous equilibria fulfill additionally $\mathcal{U}_x \equiv 0$ and consequently $f(\mathcal{U}) = 0$.

All other equilibria $\mathcal{U}$ are called *non-homogeneous* equilibria or *frozen waves*. Such equilibria can only occur for $c = 0$. In the special case of the additional symmetry $\mathcal{U}(-x,t) \equiv \mathcal{U}(x,t)$ and $\mathcal{U}_x(0,t) \equiv 0$, $t \geq 0$, we call the frozen wave a *standing wave*. Alternatively, we call a frozen wave a *twisted standing wave* if the additional symmetry $\mathcal{U}(-x,t) \equiv -\mathcal{U}(x,t)$ and $\mathcal{U}(0,t) \equiv 0$, $t \geq 0$, holds.

Furthermore, we find *relative equilibria* $\mathcal{U}(x,t)$ with respect to the group action of the equivariance group S^1. We call these relative equilibria *rotating waves of speed* $c \neq 0$ if

$$(\Phi_t \mathcal{U})(x) = \mathcal{U}(x - ct) = (R_{-ct}\mathcal{U})(x). \tag{1.12}$$

Rotating waves $\mathcal{U}(x-ct)$ are 2π-periodic solutions of the ordinary differential equation

$$0 = \mathcal{U}_{zz} + f(\mathcal{U}), \tag{1.13}$$

where $z = x - ct$ are *co-rotating coordinates*. The same equation also holds in the case $c = 0$, i.e., for frozen waves. Equation (1.13) is Hamiltonian, and we can therefore describe $\mathcal{U}$ as the motion of a point in a potential field with energy conservation. In theory, we can find the solutions with fixed energy E analytically via the relation

$$\mathcal{U}_z = \pm\sqrt{2(E - F(\mathcal{U}))}, \tag{1.14}$$

where F is the *potential*, $F'(\mathcal{U}) = f(\mathcal{U})$. Only for certain energy values E we find indeed periodic solutions with period 2π (where 2π is not necessarily the minimal period). A well known exception is the harmonic oscillator, which is given by $F(\mathcal{U}) = \frac{1}{2}\mathcal{U}^2$. Here we find 2π-periodic solutions for all $E > 0$. A sketch of an arbitrary potential F and energy values which yield $2\pi/n$-periodic solutions, and hence rotating or frozen waves, can be found in Figure 1.1. For simplicity, the higher order terms ensuring that f is dissipative are not included in the sketch.

The rotating waves are *periodic orbits* unless the wave speed is $c = 0$, in which case they correspond to *frozen waves*, i.e., to a non-homogeneous equilibrium. Both rotating and frozen waves occur in circles given by the group orbits $\{R_\theta \mathcal{U} \mid \theta \in S^1\}$.

It was proven by Angenent and Fiedler [2] and by Matano [47] that all periodic orbits of (1.5) are indeed rotating waves. In other words, no other periodic orbits besides rotating waves occur.

In Chapter 4 we consider the stability properties of rotating and frozen waves. In particular, we review a result by Angenent and Fiedler [2] which tells us that *all waves are unstable*. Therefore, rotating and frozen waves are ideal candidates for the application of our new spatio-temporal control terms.

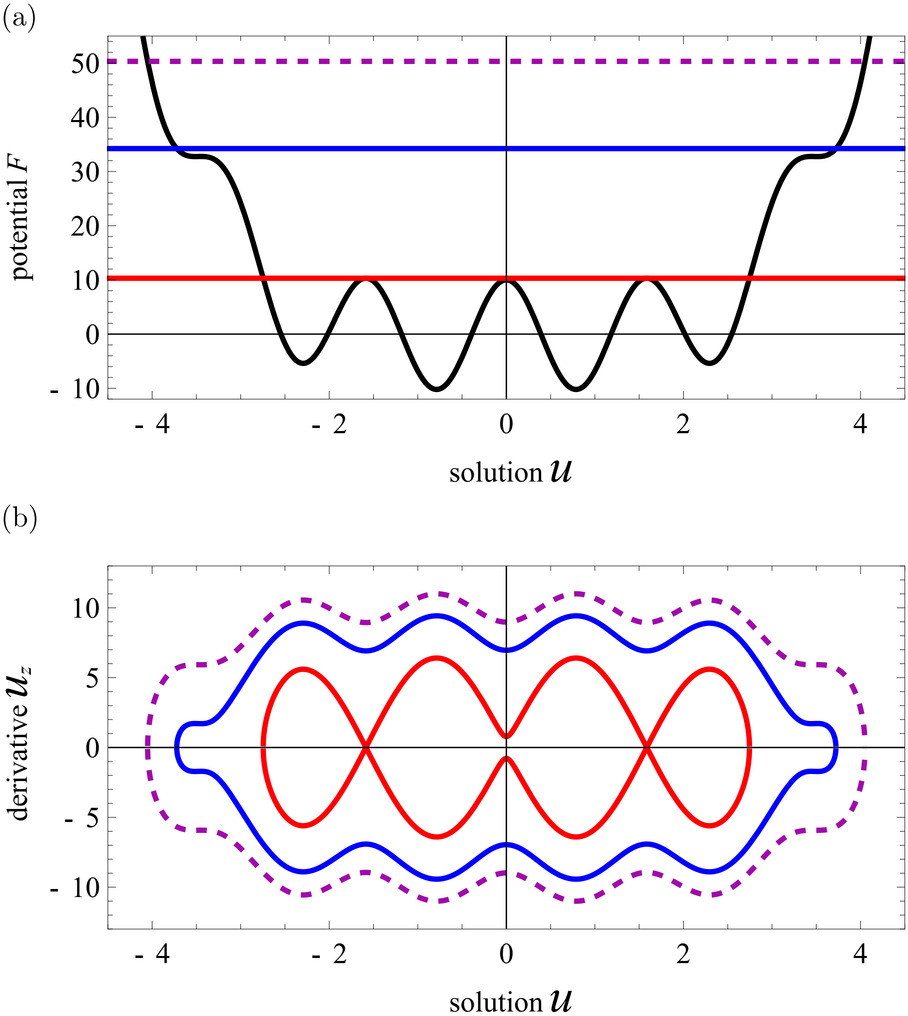


Figure 1.1.: (a) Hamiltonian potential $F(\mathcal{U})$ (black) for an odd nonlinearity $f(\mathcal{U}) = -f(-\mathcal{U})$ with energy levels corresponding to a 2π-periodic solution (red), a π-periodic solution (blue), and a $2\pi/3$-periodic solution (dashed violet). (b) Corresponding solutions in the phase-space $(\mathcal{U}, \mathcal{U}_z)$. This is a zoom-in to the interesting region of the hamiltonian potential; the higher order terms yielding dissipativity cannot be seen.

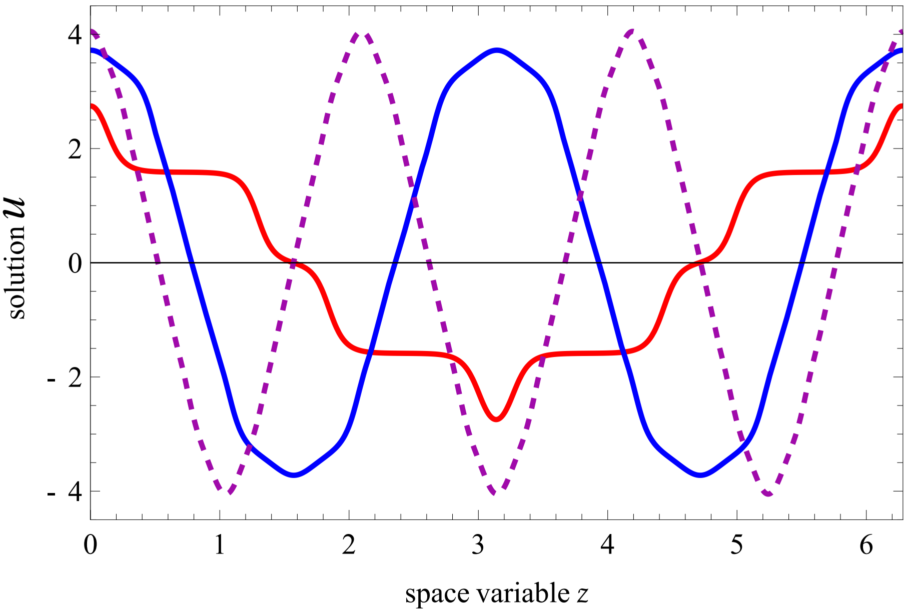


Figure 1.2.: Solutions $\mathcal{U}(z)$ from Fig. 1.1 (b), same color scheme. Note the rotational shift-symmetry for $2\pi/n$-periodic solutions, $n = 1, 2, 3$: $\mathcal{U}(z) = -\mathcal{U}(z - \pi/n)$, as well as the twisted reflection symmetry: $\mathcal{U}(\pi/n + z) = -\mathcal{U}(\pi/n - z)$ and the reflection $\mathcal{U}(0 + z) = \mathcal{U}(0 - z)$.

Throughout, except for the linear case, we assume that homogeneous equilibria are hyperbolic. Furthermore, we assume that the frozen and rotating waves are hyperbolic in the following sense: The trivial characteristic multiplier is also the only one on the unit circle. Again, we exclude only the linear examples from this assumption.

Depending on the nonlinearity $f(\mathcal{U})$, there are further symmetries of the rotating and frozen waves. Let us consider odd f, i.e., $f(\mathcal{U}) = -f(-\mathcal{U})$. Then the potential $F(\mathcal{U})$, with $F'(\mathcal{U}) = f(\mathcal{U})$, is an even function. Consequently, if $\mathcal{U}(z)$ is a solution of equation (1.13), then $-\mathcal{U}(z)$ is also a solution. These solutions, $\mathcal{U}(z)$ and $-\mathcal{U}(z)$, may coincide as sets. If so, the solutions are phase-shifted by half the period, i.e., we find solutions of the form $\mathcal{U}(z) = -\mathcal{U}(z - \pi/n)$.

Additionally, frozen waves with this symmetry are also *reflection symmetric*, in the sense that they are standing or twisted standing waves, depending of the reference point. Let us first consider a reference point z^* with $\mathcal{U}(z^*) = 0$. We then obtain a twisted standing wave $\mathcal{U}(z^* + z) = -\mathcal{U}(z^* - z)$. Note that $z^* = l\pi/n$, $l \in \mathbb{Z}$, l odd, for periodic orbits with minimal period $2\pi/n$. Let us now consider reference points $\hat{z}$ such that $\mathcal{U}(\hat{z})$ is a global maximum (or minimum) of the wave. In this case, we obtain a standing wave of the form $\mathcal{U}(\hat{z} + z) = \mathcal{U}(\hat{z} - z)$. Note that $\hat{z} = 2l\pi/n$, $l \in \mathbb{Z}$, for periodic orbits with minimal period $2\pi/n$.

See Figure 1.2 for example solutions with all these properties. In the following section we find new noninvasive control terms using this knowledge of scalar reaction-diffusion equations and their periodic or stationary solutions.

1.4. Using the control triple to find noninvasive control terms

Let us now find specific control terms for our model equation

$$u_t = u_{xx} + f(u) - cu_x. \tag{1.15}$$

Applying control terms of Pyragas type yields the following controlled equation

$$u_t = u_{xx} + f(u) - cu_x + k\big(u - u(x, t - \tau)\big), \tag{1.16}$$

where $k \in \mathbb{R}$ is the feedback gain and $\tau > 0$ is the time delay. Here τ is arbitrary for equilibria, and $\tau = np$, $n \in \mathbb{N}$, for periodic orbits with minimal period p.

Our new, more general control terms follow the control triple structure

$$\big(\textbf{output signal, space, time}\big).$$

Note that we also allow transformations in space and in the output signal in contrast to Pyragas control.

For our scalar reaction-diffusion equations on the circle, we propose two different types of control.

The *control schemes of rotation type* combine a **scalar multiplication of the output signal**, **rotations in space**, which we interpret as a spatial delay, and a **time delay**. In general, the controlled equation takes the following form:

$$u_t = u_{xx} + f(u) - cu_x + k\big(u - \Psi u(x - \xi, t - \tau)\big), \tag{1.17}$$

where $k, \Psi, \in \mathbb{R}$, $\xi \in S^1$, and $\tau \geq 0$. Here, and throughout the thesis, k is the feedback gain, which decides the sign as well as the amplitude of the control. In most cases, the parameter Ψ (transformation of the output) only takes the values $\Psi = \pm 1$. An exception is the homogeneous zero equilibrium, where Ψ can take any value $\Psi \in \mathbb{R}$. We call the parameter ξ the *spatial delay*, and τ the *temporal delay.* All three parameters Ψ, ξ and τ are fixed parameters, which should be chosen a priori. The feedback gain is a variable parameter, it is chosen a posteriori to guarantee stabilization for a fixed control triple (Ψ, ξ, τ).

Let us discuss a few special cases: We saw in the previous section that all periodic orbits are indeed rotating waves of the form $u(x, t) = \mathcal{U}(x - ct)$. A time shift by $-\tau$ has then the same effect on the wave as a spatial rotation by $+c\tau$, and the controlled equation is of the form

$$u_t = u_{xx} + f(u) - cu_x + k\big(u - u(x - c\tau, t - \tau)\big). \tag{1.18}$$

Here we use a *temporal delay* $\tau > 0$, and, if the speed c of the wave is nonzero, also a *spatial delay* $c\tau$. Furthermore, no transformation of the output is needed, i.e., $\Psi = 1$. The control term is clearly noninvasive on rotating waves of speed c. We consider this control term in detail in Chapter 2, and it turns out that this control term is not suitable for stabilization.

Remark. The control term proposed in equation (1.18) in fact contains the control of Pyragas type as a special case: Equations (1.1) and (1.18) coincide for $c = 0$. For $c \neq 0$, the control terms are the same only for $c\tau = 2\pi n$, $n \in \mathbb{N}$, since $x \in S^1$. In all other cases, equation (1.18) is more general than equation (1.1).

Consider next f odd and rotating or frozen waves with odd symmetry $\mathcal{U}(z) = -\mathcal{U}(z - m\pi/n)$, $m \in \mathbb{Z}$ is odd, and where $2\pi/n$ is the minimal spatial period. For such odd waves, the controlled equation can take the form

$$u_t = u_{xx} + f(u) - cu_x + k\big(u - (-1)\,u(x - \xi, t - \tau)\big), \tag{1.19}$$

with the following condition relating the spatial delay ξ and the temporal delay τ:

$$\xi - c\tau = m\pi/n, \qquad m \in \mathbb{Z} \text{ odd}. \tag{1.20}$$

Note that both the spatial delay ξ and the temporal delay τ can be chosen zero, independently. Thus, we can construct control terms which use only temporal delay or use only spatial delay. However, the case $\xi = \tau = 0$ is not allowed, since m is odd. It is important to note that $\Psi = -1$: It is the reason that the proposed control scheme successfully stabilizes rotating and frozen waves. We will discuss the corresponding results in Chapter 3.

Last, consider the case of homogeneous equilibria. Here we have to distinguish between those equilibria which take a fixed, non-zero value and those equilibria which take the value zero. In the case of homogeneous non-zero equilibria, controlled equations are of the form

$$u_t = u_{xx} + f(u) - cu_x + k\big(u - u(x - \xi, t - \tau)\big). \tag{1.21}$$

It is obvious that the control-term is noninvasive on any homogeneous equilibrium for any spatial delay ξ and any temporal delay τ. The parameter Ψ is 1, similar to the case of the rotating waves. It turns out that we cannot stabilize homogeneous equilibria with control terms of this form; see Chapter 2 for further discussion.

In the case of the homogeneous zero equilibrium, any real parameter Ψ can be chosen:

$$u_t = u_{xx} + f(u) - cu_x + k\big(u - \Psi u(x - \xi, t - \tau)\big). \tag{1.22}$$

In particular $\Psi = 0$ is also an acceptable choice, and we can always find control terms which guarantee stabilization, see Chapter 3.

The *control schemes of reflection type* combine a **scalar multiplication of the output signal** and **reflections in space** with **time delay**. For such control terms, we only stabilize equilibria and we therefore restrict to the case $c = 0$.

Consider both homogeneous and non-homogeneous equilibria with the even reflection-symmetry $\mathcal{U}(x) = \mathcal{U}(-x)$ around a reference point $\hat{x}$ (standing waves). Without loss of generality, $\hat{x} = 0$. Then the controlled equation is of the general form

$$u_t = u_{xx} + f(u) + k\big(u - u(-x, t - \tau)\big). \tag{1.23}$$

Here the transformation of the output signal is again a multiplication by $\Psi = 1$. Transformation of space is a reflection around the reference point $\hat{x} = 0$. Transformation in time is an arbitrary time delay $\tau \geq 0$.

Finally, consider twisted standing waves, i.e., equilibria with odd reflection symmetry $\mathcal{U}(x) = -\mathcal{U}(-x)$, and the homogeneous zero equilibrium. In these two cases, the controlled equation is of the form

$$u_t = u_{xx} + f(u) + k\big(u - \Psi u(-x, t - \tau)\big), \tag{1.24}$$

where the transformation of the output signal $\Psi \in \mathbb{R}$ can take any real value for the homogeneous zero equilibrium, and $\Psi = -1$ for any other twisted standing wave. We will discuss the corresponding results in Chapter 3.

For control schemes of reflection type, we do not consider rotating waves, since they would imply controls which combine rotations and reflections in space. Such *control schemes of mixed type* are beyond the scope of this thesis. See Section 8.3 for a short outlook.

In the remaining chapters of the thesis, we either demonstrate the success of the control terms introduced in this chapter, or we find general limitations on the control terms.

1.5. Main tool of the proofs: Hill's equation

In this section we introduce the main tool of the proofs in this thesis: Hill's equation. It is our goal to prove that our new spatio-temporal control terms, as stated in the previous section, indeed successfully stabilize equilibria and periodic orbits. We now explain briefly how the question of successful stabilization leads us to Hill's equation. A detailed plan of the proofs follows in Chapter 4.

To determine whether stabilization by our new spatio-temporal control terms is successful or not, we need to answer the following question: Which local stability do the equilibria and periodic orbits have in the controlled system?

To prove local asymptotic stability, it suffices to prove linear stability [78]. We therefore consider the linear variational equation, where we linearize around the respective equilibrium or periodic orbit and solve this linearized equation by an exponential Ansatz.

We obtain linear *ordinary* differential equations with a periodic coefficient, possibly *with delay*. We call the delay a *spatio-temporal* delay, since it is neither the temporal nor the spatial delay of the control terms, but a mixture of them. The *eigenvalues* λ, which we need to prove stabilization, correspond to a *parameter* in the ordinary (delay) differential equation.

For linear stability, it is a necessary and sufficient condition that all eigenvalues are found in the left half of the complex plane, i.e., that all eigenvalues have negative real part. The eigenfunctions corresponding to the eigenvalues are 2π-periodic solutions of the ordinary (delay) differential equations.

Therefore, we enter the field of Floquet analysis, searching for periodic solutions of homogeneous linear ordinary (delay) differential equations of second order with real periodic coefficients. These equations are a delayed form of *Hill's equation* ($0 = g_{xx}+(-\lambda+Q(x))g$, where $g : \mathbb{R} \to \mathbb{R}$, $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$, and $Q(x)$ is a 2π-periodic coefficient).

We can therefore rephrase our original question of stability of equilibria and periodic orbits in the controlled system to the following question:

For which parameter values do there exist periodic solutions of the modified Hill's equation with spatio-temporal delay?

In this way, and combined with symmetry aspects, Hill's equation becomes the main tool of the proofs. In light of its central role in this thesis, it seems appropriate to introduce Hill's equation shortly.

Hill's equation is named after George W. Hill who considered this equation in 1877 in an investigation on lunar stability [31]. The equations of Hill's type are closely related to the more widely known equations of Sturm-Liouville type, first investigated by Sturm in 1836 [72].

For a broad introduction to Hill's equation, see for example the book by Magnus and Winkler [45]. Important special cases of Hill's equation are Mathieu's equation [48] and Meissner's equation [49], which only consider special periodic coefficients.

Many applications of Hill's equation besides lunar motion have been discovered since Lyapunov first established its general importance for stability problems in 1907 [44]. Most notably, the one-dimensional Schrödinger equation of an electron in a crystal is of Hill's type [8]. Furthermore, Hill's equation also features prominently in the study of periodic solutions of the Korteweg-deVries equation [41]. Applications of the more special Mathieu equation include vibrations in an elliptic drum, the inverted pendulum, the radio frequency quadruple, particle traps and many more. See for example the review paper by Ruby [61] for an overview.

So far, there exist few results for Hill's equation with delay. First interesting results were established by Insperger in 2002 [35], and by Insperger and Stépán [36], who consider Hill's equation with delay in the context of mechanical applications. In this context, the proofs in Chapters 5 and 6 can also be read solely in light of new results on the delayed Hill's equation.

1.6. Grasshopper's guide and outline of the thesis

We begin with a grasshopper's guide: Carefully and completely reading Chapter 1 is indispensable for the understanding of this thesis. The main new results can be found in Chapters 2 and 3. Readers who do not want to dwell on the proofs (Chapters 4, 5, and 6) may continue directly with the application of the control triple method in Chapter 7 and the conclusion in Chapter 8.

In the following we give a short outline of each of the remaining chapters of this thesis.

In Chapter 2 we attempt to stabilize equilibria and periodic orbits by Pyragas control and slight generalizations. However, Pyragas control fails its stabilization task. We illustrate the failure of the control mechanism with a simple linear example. Furthermore, we compare these new results on the control of partial differential equations to the odd-number limitation, a well-known control limitation for ordinary differential equations.

In Chapter 3 we state the main new results on control terms using the control triple method, for control schemes of both rotation and reflection type. We point out the difference in the stabilization mechanisms using the same example equation as in Chapter 2 for illustration. We also compare our new results to results for ordinary differential equations, including the stabilization of an unstable focus and stabilization via equivariant Pyragas control.

Preliminaries of the proofs are discussed in Chapter 4. We first analyze the stability of rotating and frozen waves in the uncontrolled equation. Next, we collect useful properties of Hill's equation in the special case that it derives from a scalar reaction-diffusion equation. We also give a short introduction to Floquet theory for the delayed Hill's equation and to partial delay differential equations. Finally, we calculate the linear variational equations for our controlled equation and explain the procedure of the proofs.

In CHAPTER 5 we prove stabilization for control schemes of rotation type. Note that this proof includes the success of the control triple method (Chapter 3) as well as the failure of Pyragas control (Chapter 2). The proof is divided into four steps, depending on the form of the linear variational equation. The main tool of the proof is Hill's equation. It is combined with symmetry properties of the frozen or rotating waves.

In CHAPTER 6 we prove stabilization for control schemes of reflection type. Here we can re-use many of the results from Chapter 5 concerning Hill's equation, and we therefore lay more importance on the symmetry aspects.

In CHAPTER 7 we apply the control triple method to a specific reaction-diffusion equation, namely the Chafee-Infante equation. To achieve stabilization, we first introduce the Chafee-Infante equation with some key properties. First, we treat the homogeneous equilibria where we include a short bifurcation analysis. Second, we discuss the stabilization of frozen waves where we compare the control types of rotation and reflection type.

Finally, in CHAPTER 8, we conclude this thesis. First, we give an overview of the aims, methods, and results. Second, we discuss our results in the general framework of time-delayed feedback control. Last, we give an outlook on further research on the control triple method.

Chapter 2

Failure of Pyragas control

In this chapter we study time-delayed feedback control of equilibria and waves in scalar reaction-diffusion equations. We focus on Pyragas control with generalizations and discuss why these control schemes fail in the case of scalar reaction-diffusion equations. The limitation of Pyragas control is the main result of this chapter.

In Section 2.1 we state the corresponding theorems: Noninvasive control schemes similar to Pyragas control fail to stabilize rotating waves, see Theorem 2.1. As a corollary and as the main result of this chapter, we find that Pyragas control fails to stabilize periodic orbits in scalar reaction-diffusion equations, see Corollary 2.2. Moreover, the stabilization of equilibria, both homogeneous and non-homogeneous, fails. We attempt to use Pyragas control or slightly more general control terms, see Corollary 2.3 and Theorem 2.4. Theorem 2.5 tells us under which conditions Pyragas control *destabilizes* homogeneous equilibria which are stable in the uncontrolled equation.

In Section 2.2 we illustrate why Pyragas control fails using a simple linear example for which we can calculate the spectrum explicitly.

Finally, in Section 2.3 we compare our results to well-known restrictions of Pyragas control from the theory of ordinary differential equations.

2.1. Results

In this section we consider scalar reaction-diffusion equations of the form

$$u_t = u_{xx} + f(u) - cu_x, \tag{2.1}$$

with all the assumptions stated in Section 1.3. Remember that all periodic orbits of minimal period $p > 0$ are also rotating waves of the form $\mathcal{U}(x - ct)$ with speed $c \in \mathbb{R}$. For this type of equation it is known that all rotating and frozen waves are unstable.

The instability persists if noninvasive control terms of the form $k\big(u - u(x - c\tau, t - \tau)\big)$ are added:

Theorem 2.1 (Failure of control for rotating waves)**.** *Consider a periodic orbit, i.e., a rotating wave $\mathcal{U}(x-ct)$, of the scalar reaction-diffusion equation $u_t = u_{xx} + f(u) - cu_x$, with periodic boundary conditions and the assumptions from Section 1.3.*

Then the rotating wave $\mathcal{U}(x-ct)$ is also unstable in the equation including noninvasive control,

$$u_t = u_{xx} + f(u) - cu_x + k\big(u - u(x - c\tau, t - \tau)\big), \tag{2.2}$$

for any feedback gain $k \in \mathbb{R}$ and any time delay $\tau > 0$.

As the most important consequence and main result of this chapter, we conclude that Pyragas control fails to stabilize periodic orbits in scalar reaction-diffusion equations:

Corollary 2.2 (Failure of Pyragas control for periodic orbits)**.** *Consider a time-periodic orbit with minimal period p of the scalar reaction-diffusion equation $u_t = u_{xx} + f(u) - cu_x$, with periodic boundary conditions and the assumptions from Section 1.3.*

Then the periodic orbit is also unstable in the equation including Pyragas control,

$$u_t = u_{xx} + f(u) - cu_x + k\big(u - u(x, t - np)\big), \tag{2.3}$$

for any feedback gain $k \in \mathbb{R}$ and any $n \in \mathbb{N}$.

Note that Corollary 2.2 is included in Theorem 2.1 for a time delay $\tau = 2\pi n/c$, $n \in \mathbb{N}$.

As a further corollary of Theorem 2.1, we find that Pyragas control also fails to stabilize unstable equilibria which occur in the case $c = 0$:

Corollary 2.3 (Failure of Pyragas control for equilibria)**.** *Consider an unstable equilibrium $\mathcal{U}(x)$, homogeneous or non-homogeneous, of the scalar reaction-diffusion equation $u_t = u_{xx} + f(u)$, with periodic boundary conditions and the assumptions from Section 1.3.*

Then the equilibrium $\mathcal{U}(x)$ is also unstable in the equation including Pyragas control,

$$u_t = u_{xx} + f(u) + k\big(u - u(x, t - \tau)\big), \tag{2.4}$$

for any feedback gain $k \in \mathbb{R}$ and any time delay $\tau > 0$.

Let us now consider unstable homogeneous equilibria, only. In analogy to Theorem 2.1, the failure of Pyragas control persists also for control terms using arbitrary spatial delay for homogeneous equilibria:

Theorem 2.4 (Failure of control for homogeneous equilibria)**.** *Consider an unstable* homogeneous *equilibrium $\mathcal{U}$ of the scalar reaction-diffusion equation $u_t = u_{xx} + f(u) - cu_x$, with periodic boundary conditions and the assumptions from Section 1.3.*

Then the homogeneous equilibrium $\mathcal{U}$ is also unstable in the equation including Pyragas control,

$$u_t = u_{xx} + f(u) + k\big(u - u(x - \xi, t - \tau)\big), \tag{2.5}$$

for any feedback gain $k \in \mathbb{R}$, any spatial delay $\xi \in S^1$, and any time delay $\tau > 0$.

The only stable objects in scalar reaction-diffusion equations are homogeneous equilibria. Pyragas control destabilizes those objects for large enough feedback gains:

Theorem 2.5 (Pyragas destabilization of homogeneous equilibria)**.** *Consider a* stable homogeneous *equilibrium* $\mathcal{U}$ *of the scalar reaction-diffusion equation* $u_t = u_{xx} + f(u) - cu_x$*, with periodic boundary conditions and the assumptions from Section 1.3. Fix a time delay* $\tau > 0$ *and a spatial delay* $\xi \in S^1$*.*

Then there exists a feedback gain $k^*(\tau) \in \mathbb{R}$*, such that for all* $k > k^*(\tau)$ *the homogeneous equilibrium* $\mathcal{U}$ *is unstable in the equation including Pyragas control,*

$$u_t = u_{xx} + f(u) - cu_x + k\big(u - u(x - \xi, t - \tau)\big). \tag{2.6}$$

All theorems from this section are proven in Chapter 5.

The failure of Pyragas control in the stabilization of equilibria and waves provides us with information of two kinds: First, any experimental or numerical stabilization is bound to fail and therefore, we do not need to try. Second, as we will see in the following section, there is a simple reason for the failure of Pyragas control. We can therefore try to circumvent this problem, once recognized, and create new, successful control terms. In this thesis the new control terms are constructed using the control triple as introduced in Section 1.2. They stabilize the equilibria and waves successfully, see the results in Chapter 3.

2.2. An example of Pyragas control

In this section we illustrate with a simple linear example why Pyragas control does not stabilize the target equilibria and rotating waves. This is *only an illustration and no proof.* The general proof of Theorems 2.1–2.5 follows in Chapter 5.

As an example, consider the following linear reaction-diffusion equation:

$$u_t = u_{xx} + u. \tag{2.7}$$

Before we apply Pyragas control, let us shortly analyze equation (2.7). No periodic orbits exist, since no advection term is present. We therefore search for equilibria, which are characterized by $u_t \equiv 0$. Equilibria are hence 2π-periodic solutions of the ordinary differential equation $0 = u_{xx} + u$. We conclude that all equilibria are frozen waves of the form $A\sin(x+\theta)$, $\theta \in S^1$, where $A \in \mathbb{R}$ is the amplitude of the frozen wave.

We introduce a control term of the form $k\big(u - u(x,t-\tau)\big)$. Then the equation including Pyragas control takes the form

$$u_t = u_{xx} + u + k\big(u - u(x,t-\tau)\big), \tag{2.8}$$

with variable feedback gain $k \in \mathbb{R}$, and arbitrary but fixed time delay $\tau > 0$.

From Corollary 2.3 it is known that we cannot find any feedback gain $k \in \mathbb{R}$ such that the control stabilizes the frozen waves of the form $A\sin(x+\theta)$, for all time delays $\tau > 0$.

In this section we calculate explicitly the eigenvalues of the frozen waves to determine their stability. We will see that there is always at least one real positive eigenvalue, hence stabilization is impossible.

Having chosen a linear system, no linearization is necessary, and we can solve equation (2.8) directly via the Ansatz $u(x,t) = g(x)e^{\lambda t}$. A formal justification of the exponential Ansatz can be found in Chapter 4. The $\lambda \in \mathbb{C}$ are the eigenvalues (for simplicity, we calculate with complex λ, it should be clear, however, that all solutions are real). We then obtain the following ordinary differential equation, which includes the eigenvalues λ as a parameter:

$$\lambda g = g_{xx} + g + k\left(g - ge^{-\lambda\tau}\right). \tag{2.9}$$

The eigenfunctions are the 2π-periodic solutions of equation (2.9). To find the 2π-periodic solutions and hence the eigenvalues λ, it is useful to rearrange equation (2.9) in the following way:

$$0 = g_{xx} + \left(-\lambda + 1 + k - ke^{-\lambda\tau}\right) g. \tag{2.10}$$

This is the well known pendulum equation with parameters λ, τ, and k. Furthermore, it is also an equation of Hill's type, where the periodic coefficient $-\lambda + 1 + k - ke^{-\lambda\tau}$ is a constant.

The 2π-periodic solutions of equation (2.10) are given explicitly by the trigonometric functions $\sin(Nx)$ and $\cos(Nx)$, $N = 0, 1, 2, \ldots$ We therefore obtain the following set of characteristic equations:

$$0 = -N^2 - \lambda + 1 + k - ke^{-\lambda\tau}, \qquad N = 0, 1, 2, \ldots \tag{2.11}$$

We next split equation (2.11) into real and imaginary part, where we use the notation $\lambda = \mu + \mathrm{i}\nu$:

$$\mu = -N^2 + 1 + k\left(1 - e^{-\mu\tau}\cos(\nu\tau)\right), \qquad N = 0, 1, 2, \ldots, \tag{2.12}$$

$$\nu = -ke^{-\mu\tau}\sin(\nu\tau). \tag{2.13}$$

At this point we simplify our illustration to real eigenvalues $\lambda = \mu$. This is justified, since the second equation (2.13) is always fulfilled if we choose $\nu = 0$. (Note, however, that complex conjugated eigenvalues do exist!) To prove instability, it suffices to show the existence of at least one real and strictly positive eigenvalue.

Therefore, we search for the corresponding real eigenvalues μ determined by equation (2.12) where we have simplified $\cos(\nu\tau) = 1$:

$$\mu = -N^2 + 1 + k\left(1 - e^{-\mu\tau}\right), \qquad N = 0, 1, 2, \ldots \tag{2.14}$$

Using this equation, we can calculate directly which feedback gain k has to be applied to obtain a given real eigenvalue μ:

$$k_N(\mu) = \frac{\mu - 1 + N^2}{1 - e^{-\mu\tau}}, \qquad N = 0, 1, 2, \ldots \tag{2.15}$$

Now we are almost done: It is sufficient to consider the case $N = 0$, which is depicted in red in Figure 2.1.

For $N = 0$ we find a pole at $\mu = 0$ because the denominator $1 - e^{-\mu\tau}$ is zero. Therefore, it is impossible for the real eigenvalues μ to cross zero for a finite feedback gain k.

Moreover, we can indeed find real positive eigenvalues μ for every feedback gain $k \in \mathbb{R}$. Observe that for $\mu \searrow 0$, k_0 approaches $-\infty$, while in the limit $\mu \to +\infty$, we find that k_0 approaches $+\infty$. In between, $k_0(\mu)$ is continuous and strictly monotonically increasing. Hence, $k_0 : (0, \infty) \to \mathbb{R}$

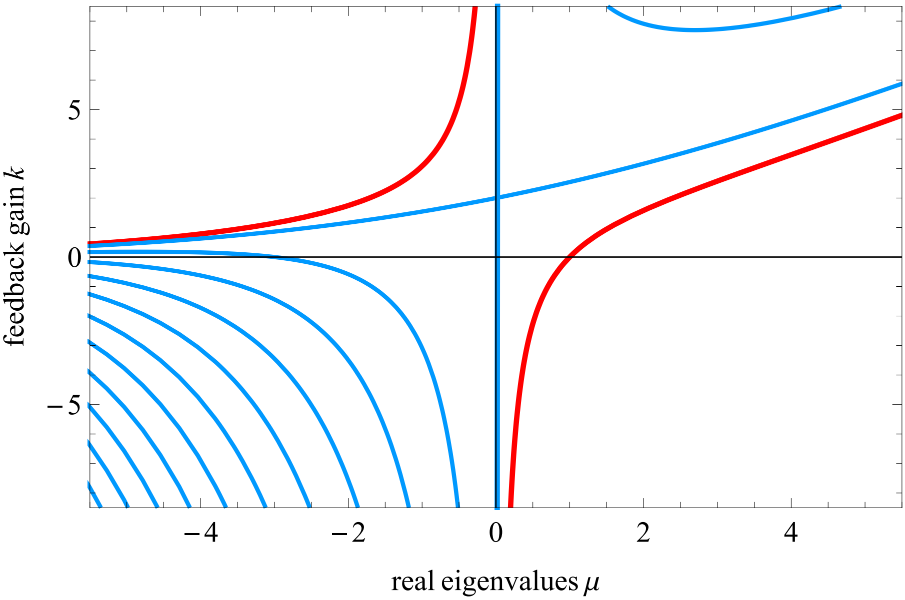


Figure 2.1.: The feedback gain k (vertical axis) plotted versus the real eigenvalues μ (horizontal axis). The time delay is $\tau = 0.5$. Note that for every feedback gain $k \in \mathbb{R}$ we obtain strictly positive eigenvalues for $N = 0$ (red curve), which excludes stabilization. The curves for $N \geq 1$ are blue.

is a bijective function. We can conclude that there exists a positive real eigenvalue μ for all feedback gains $k \in \mathbb{R}$.

Therefore, *stabilization is indeed impossible* and it must be our topmost priority to avoid poles at $\mu = 0$ as in equation (2.15).

Remark. For $N = 1$ we find a trivial eigenvalue $\mu = 0$ of multiplicity two for all feedback gains k: All eigenfunctions are frozen waves of the form $A\sin(x+\theta)$, $A \in \mathbb{R}$, $\theta \in [0, 2\pi]$. These eigenvalues $\mu = 0$ reflect both the S^1-equivariance (as in the parameter θ) and the linearity (as in the parameter A) of equation (2.7). In nonlinear equations generally only one trivial eigenvalue at $\mu = 0$ persists, which reflects the S^1-equivariance.

2.3. Comparison with ordinary differential equations

In this section we compare the limitations that we have just discovered to well-known restrictions of Pyragas control for ordinary differential equations.

This is important for two reasons: First, we want to put our results in perspective with well-known results on Pyragas control to emphasize their importance in the general context of feedback control. Our results are not stand-alone results, they fit into the larger framework of the few analytical results on Pyragas control which exist up to date. Second, we want to see the close connection between the odd-number limitation and our results – the reason of failure of Pyragas control is the same in both cases, even though the odd-number limitation has been formulated for non-autonomous ordinary differential equations and we consider reaction-diffusion equations. This is an important step on the path to the ultimate goal to find conditions for the success and failure of Pyragas control for general dynamical systems.

Before we turn our discussion to the odd-number limitation, which is the main objective of this section, let us briefly mention that there exists an interesting approach by Just et al. [37]. They discuss under which conditions periodic orbits in non-autonomous ordinary differential equations can be stabilized by Pyragas control. Their results may be summarized as *"only orbits with a finite torsion can be stabilized"* [37]. "Finite torsion" can be translated to complex, non-real eigenvalues. In our case, we have at least one real, positive eigenvalue, excluding stabilization from the point of view taken by Just et al. However, since no precise formulation is presented by Just et al., a detailed comparison of our results with theirs is difficult.

Similar restrictions on eigenvalues are known for non-autonomous ordinary differential equations, subject to control of Pyragas type. This restriction is called the *odd-number limitation*. It is not our purpose to enter the interesting historical discussion related to the odd-number limitation. For a short summary and some remarks, see Section 1.1.

For the purpose of comparison, let us cite the odd-number limitation for non-autonomous equations, as it was correctly stated and proven by Nakajima in [52]. We assume that $f(u,t)$ is p-periodic with respect to t, and that there exists a time $t^* < p$ such that $f(u,0) \neq f(u,t^*)$.

Theorem 2.6 (Odd-number limitation, Nakajima [52])**.** *If the linear variational equation of the* non-autonomous *ordinary differential equation*

$$\dot{u}(t) = f(u(t),t), \quad (u \in \mathbb{R}^n) \tag{2.16}$$

about the target hyperbolic *unstable periodic orbit $\mathcal{U}(t)$ with period p,*

$$\dot{v}(t) = Df\big(\mathcal{U}(t),t\big)\, v(t), \tag{2.17}$$

has an odd number of real Floquet multipliers greater than unity, then the unstable periodic orbit $\mathcal{U}(t)$ can never be stabilized by the delayed feedback control

$$\dot{u}(t) = f(u(t),t) + k\big(u(t) - u(t-p)\big) \tag{2.18}$$

for any value of the feedback gain k, where $k \in \mathbb{R}^{n\times n}$.

Most importantly, the characteristic equation of hyperbolic orbits in non-autonomous systems does not possess any Floquet multiplier on the unit circle. Furthermore, it is assumed that the number of real Floquet multipliers greater than unity is odd, hence the name "odd-number limitation".

Let us relate our simple example to the odd-number limitation. For the Floquet exponents these conditions mean that the characteristic equation does not possess any purely imaginary eigenvalues and that the number of real, strictly positive Floquet exponents is odd. The Floquet exponents correspond to the eigenvalues λ which we have calculated explicitly in the previous section for our simple example.

In the example from Section 2.2, we have an infinite number of independent characteristic equations of the form (without control)

$$\lambda = -N^2 + 1, \qquad N = 0, 1, 2, \ldots, \tag{2.19}$$

which are simple for $N = 0$ and double for $N \geq 1$. If only one of these equations fulfills the conditions of the odd-number limitation, then stabilization is ruled out by the odd-number limitation.

Since the eigenvalue for $N = 0$ is simple, real, and positive ($\lambda = 1$), the conditions of the odd-number limitation are indeed fulfilled. Therefore, we cannot hope to stabilize the frozen waves from the example in Section 2.2. This is in accordance with our general Theorems 2.1 – 2.4.

We will see in Chapter 4 that scalar reaction-diffusion equations of the form $u_t = u_{xx} + f(u) - cu_x$ indeed always possess exactly one isolated real and positive eigenvalue. Furthermore, we will prove in Chapter 5 that stabilization by Pyragas control is indeed always impossible, as already suggested (but not rigorously proven) by the odd-number limitation.

Chapter 3

Success of the control triple method

In this chapter we control equilibria and waves in scalar reaction-diffusion equations. We present results for two new types of spatio-temporal control terms, which have been introduced in Section 1.4 using the control triple method. We distinguish between control schemes of rotation and reflection type. Both control types successfully stabilize equilibria and waves.

In Section 3.1 we state the main result of this thesis – the success of the control triple method. In Theorem 3.1 we state the conditions under which a control term of rotation type successfully stabilizes frozen and rotating waves in scalar reaction-diffusion equations. We find that a sign change in the output signal successfully stabilizes the frozen and rotating waves. This is achieved by a spatio-temporal delay corresponding to half the spatial period. In contrast, those control terms which use full spatial periods and consequently no sign change of the output signal fail to stabilize equilibria and waves, see Theorem 3.2. In Theorem 3.3 we state conditions under which a control term of rotation type successfully stabilizes the homogeneous zero equilibrium.

We then turn to control terms of reflection control type. Here we distinguish between even and odd waves. We can stabilize odd waves (twisted standing waves), since the reflection in space implies a sign change of the output signal, see Theorem 3.4. However, even waves (standing waves) cannot be stabilized, since the reflection in space gives no sign change of the output signal, see Theorem 3.5. In Theorem 3.6 we stabilize the homogeneous zero equilibrium using reflections in space.

In Section 3.2 we use the same linear example as in Chapter 2 to illustrate the successful control mechanisms for both control types.

Finally, in Section 3.3 we compare our control mechanism to the mechanism of Pyragas control applied to an unstable focus. Here we find a surprising analogy between the time delay of Pyragas control and the spatio-temporal delay of the control triple method. Furthermore, we compare the concept of control triples to the concept of equivariant Pyragas control which has been successfully used for ordinary differential equations with a similar outcome.

3.1. Results

In this section we consider scalar reaction-diffusion equations of the form

$$u_t = u_{xx} + f(u) - cu_x, \tag{3.1}$$

with all the assumptions stated in Section 1.3. We remember that all periodic orbits are rotating waves. The equilibria are either homogeneous equilibria or frozen waves. Both the rotating and frozen waves are always unstable.

We present our results in two subsections, depending on their control type: Results on controls of rotation type are stated in Subsection 3.1.1, while results on controls of reflection type can be found in Subsection 3.1.2.

3.1.1. Control schemes of rotation type

Control schemes of rotation type combine a sign change in the output signal, a rotation in space, and a time delay. The following theorems state the conditions under which the new control terms of rotation type are successful. In particular a transformation $\Psi \neq 1$ of the output signal is crucial for the success of a particular control term.

Theorem 3.1 (Successful stabilization of odd rotating and frozen waves). *Consider a rotating or frozen wave $\mathcal{U}(x-ct) = \mathcal{U}(z)$ with minimal spatial period $2\pi/n$ of the scalar reaction-diffusion equation $u_t = u_{xx} + f(u) - cu_x$, with periodic boundary conditions and the assumptions from Section 1.3. Additionally, assume $f(u) = -f(-u)$ and suppose that the rotating or frozen wave is odd, $\mathcal{U}(z) = -\mathcal{U}(z - \pi/n)$, with unstable dimension $2n - 1$.*

Then there exists a feedback gain $k^* \in \mathbb{R}$ such that the following holds:

For all $k < k^*$, there exists a time delay $\tau^* = \tau^*(k)$ such that the rotating or frozen wave $\mathcal{U}(x - ct) = \mathcal{U}(z)$ is stable in the controlled equation

$$u_t = u_{xx} + f(u) - cu_x + k\big(u - (-1)u(x - \xi, t - \tau)\big), \tag{3.2}$$

where the spatial delay ξ and the temporal delay $\tau < \tau^*$ are related via

$$\xi - c\tau = m\pi/n, \tag{3.3}$$

where m is odd and co-prime to n.

In some special cases such as for linear systems or if all eigenvalues of the uncontrolled equation are known a priori, we can state more precise conditions, on the feedback gain k^* as well as on the time delay τ^*. We will discuss these cases during the proof in Chapter 5.

We have seen in the previous theorem that stabilization is possible whenever the spatial delay ξ and the temporal delay τ are related via $\xi - c\tau = m\pi/n$, where m is odd and co-prime to n. However, control fails whenever m is even:

Theorem 3.2 (Failure of control for rotating and frozen waves). *Consider a rotating or frozen wave $\mathcal{U}(x-ct) = \mathcal{U}(z)$ with minimal spatial period $2\pi/n$ of the scalar reaction-diffusion equation $u_t = u_{xx} + f(u) - cu_x$, with periodic boundary conditions and the assumptions from Section 1.3.*

Then the rotating or frozen wave $\mathcal{U}(x-ct)=\mathcal{U}(z)$ is also unstable in the equation including noninvasive control,

$$u_t = u_{xx} + f(u) - cu_x + k\big(u - u(x-\xi, t-\tau)\big), \tag{3.4}$$

for any feedback gain $k \in \mathbb{R}$, where the spatial delay ξ and the temporal delay τ are related via

$$\xi - c\tau = m\pi/n, \qquad m \in \mathbb{Z},\ m \text{ even}. \tag{3.5}$$

Remark. Theorem 3.2 also contains Pyragas control ($\tau = p$) as a special case.

Using the full control triple, stabilization of the homogeneous zero equilibrium can be successfully achieved. In this rather special case the transformation of the output signal is not just limited to a sign change, but can be a multiplication with any real number.

Theorem 3.3 (Successful stabilization of the zero equilibrium)**.** *Consider the homogeneous zero equilibrium of the scalar reaction-diffusion equation $u_t = u_{xx} + f(u) - cu_x$, with periodic boundary conditions and the assumptions from Section 1.3.*

Choose some real number $\Psi \neq 1$ and a spatial delay $\xi \geq 0$. If the feedback gain $k \in \mathbb{R}$ fulfills the condition

$$k\big(1 - \Psi\cos(\xi N)\big) < N^2 - f'(0) \qquad \text{for all } N \in \mathbb{N}, \tag{3.6}$$

and if the time delay $\tau \geq 0$ is small enough, then the homogeneous zero equilibrium is stable in the equation including control,

$$u_t = u_{xx} + f(u) + k\big(u - \Psi\, u(x-\xi, t-\tau)\big). \tag{3.7}$$

In particular, if $\Psi = 0$, then the zero equilibrium is stable for $k < -f'(0)$.

Remark. In principle, it is possible to stabilize *every* homogeneous equilibrium $\mathcal{U}$, because we can shift it to be the zero equilibrium. To see this, we define $U := u - \mathcal{U}$ and apply control as follows:

$$U_t = U_{xx} + f(U + \mathcal{U}) + k\big(U - \Psi\, U(x-\xi, t-\tau)\big). \tag{3.8}$$

Retransforming $u = U + \mathcal{U}$, and using $\Psi \neq 1$, we obtain

$$u_t = u_{xx} + f(u) + k\big(u - \Psi\, u(x - \xi, t - \tau) - (1 - \Psi)\,\mathcal{U}\big). \tag{3.9}$$

To summarize, the control triple method of rotation type stabilizes equilibria and waves successfully if a transformation of the output signal is included. This can be achieved for a half-period feedback scheme or in the case of the homogeneous zero equilibrium where we use additional information on the equilibrium. A full-period feedback scheme fails stabilization, as already indicated by the limitations of Pyragas control discussed in Chapter 2.

3.1.2. Control schemes of reflection type

Control terms of reflection type combine a sign change in the output signal, a reflection in space and a time delay. Because of the reflection in space, we only consider frozen waves here. A moving reference point and hence a mixture of rotation and reflection would be necessary to tackle rotating waves. See Chapter 8 for a short discussion. The conditions for successful control are stated in the following theorem.

Theorem 3.4 (Successful stabilization of twisted standing waves)**.** *Consider a twisted standing wave $\mathcal{U}(x) = -\mathcal{U}(-x)$ of the scalar reaction-diffusion equation $u_t = u_{xx} + f(u)$, with $f(u) = -f(-u)$, periodic boundary conditions and the assumptions from Section 1.3.*

Suppose that the unstable dimension of the twisted standing wave is exactly one.

If the feedback gain k fulfills

$$k < -\left(\max_{x\in[0,2\pi]} f'(\mathcal{U}(x))\right) \Big/ 2, \tag{3.10}$$

then there exists a time delay $\tau^* = \tau^*(k)$ such that the following holds:

The twisted standing wave $\mathcal{U}(x) = -\mathcal{U}(-x)$ is stable in the equation including control,

$$u_t = u_{xx} + f(u) + k\big(u - (-1)u(-x, t - \tau)\big). \tag{3.11}$$

for all $0 \leq \tau < \tau^*(k)$.

Roughly speaking, Theorem 3.4 asserts that twisted standing waves can be stabilized under certain assumptions. Frozen waves with an even symmetry, i.e., standing waves, however, behave differently and they cannot be stabilized by the same method:

Theorem 3.5 (Failure of stabilization of standing waves)**.** *Consider a standing wave $\mathcal{U}(x) = \mathcal{U}(-x)$ of the scalar reaction-diffusion equation $u_t = u_{xx} + f(u)$, with periodic boundary conditions and the assumptions from Section 1.3.*

Then the standing wave $\mathcal{U}(x)$ is also unstable in the equation including control,

$$u_t = u_{xx} + f(u) + k\big(u - u(-x, t - \tau)\big), \tag{3.12}$$

for any feedback gain $k \in \mathbb{R}$ and any time delay $\tau \geq 0$.

The homogeneous zero equilibrium can be interpreted as a twisted standing wave. This already indicates that stabilization is possible. We can now state the analog of Theorem 3.3.

Theorem 3.6 (Successful stabilization of the zero equilibrium)**.** *Consider the homogeneous zero equilibrium of the scalar reaction-diffusion equation $u_t = u_{xx} + f(u)$, with periodic boundary conditions and the assumptions from Section 1.3.*

Choose some real number $\Psi \neq 1$ and a time delay $\tau \geq 0$.

If the feedback gain $k \in \mathbb{R}$ fulfills the condition

$$k\left(1 - \Psi e^{-\mu\tau}\right) < \mu - f'(0) \qquad \text{for all } \mu > 0, \tag{3.13}$$

as well as the condition

$$|k\Psi\tau| < 1, \tag{3.14}$$

and if the unstable dimension of the zero equilibrium is exactly one, then the homogeneous zero equilibrium is stable in the equation including control,

$$u_t = u_{xx} + f(u) + k\big(u - \Psi u(-x, t-\tau)\big). \tag{3.15}$$

In particular, if $\Psi = 0$*, then the zero equilibrium is stable for* $k < -f'(0)$*.*

Summarizing Theorems 3.1–3.6, we note that a transformation of the output signal is essential for a successful control. In our scalar case transformation is mostly a sign change achieved by an odd symmetry, either by rotations in space or by reflections in space. In the rather special case of the zero equilibrium, more choices are possible.

Note that Pyragas control does not include any transformation of the output signal and only the control triple method introduced in this dissertation takes such a general viewpoint.

3.2. Two examples of the control triple method

In this section we present two examples to illustrate why the control triple method succeeds. Both examples are *only illustrations without proofs*. We emphasize the crucial difference between the two new control schemes and Pyragas control in Chapter 2. The proof for control schemes of rotation type follows in Chapter 5, and the proof for control schemes of reflection type follows in Chapter 6.

3.2.1. Control schemes of rotation type

As in Chapter 2, consider the following linear equation as an example:

$$u_t = u_{xx} + u. \tag{3.16}$$

Previously, we have already analyzed equation (3.16). We found that there do not exist any periodic orbits which are not equilibria and that all equilibria are frozen waves of the form $\mathcal{U}(x) = A\sin(x+\theta)$, $\theta \in S^1$, $A \in \mathbb{R}$. See Chapter 2 for a detailed discussion.

Let us now investigate the stability of the frozen waves $\mathcal{U}(x)$ in the following equation including control:

$$u_t = u_{xx} + u + k\big(u - (-1)\, u(x-\pi, t-\tau)\big). \tag{3.17}$$

We use a sign change of the output signal "(-1)", a spatial delay π, and an arbitrary time delay $\tau \geq 0$. It is straightforward to check that this control term is indeed noninvasive on all frozen waves.

Again, since the equation is linear, no linearization is needed, and we can solve equation (3.17) directly via the Ansatz $u(x,t) = g(x)e^{\lambda t}$. We then obtain the following equation:

$$\lambda g = g_{xx} + g + k\left(g + e^{-\lambda\tau} g(x-\pi)\right). \tag{3.18}$$

Note that the spatial and the temporal delay behave differently: The temporal delay gives an exponential term in λ. In contrast, the spatial delay results in a delay in equation (3.18), making it a delay differential equation. Again, we obtain an equation of Hill's type with a constant coefficient, but this time with an additional delay.

Since the delay differential equation (3.18) is linear, we solve it via an exponential Ansatz, $g(x) = e^{\eta x}$, $\eta \in \mathbb{C}$. For a detailed discussion of the mathematical details at this step see Chapter 5.

Similarly to Chapter 2, we only search for periodic solutions of (not necessarily minimal) period 2π. These solutions are the eigenfunctions and we are interested in the question for which λ there exist such solutions. Solutions of period 2π can only be expected for $\eta = \pm \mathrm{i}N$. As characteristic equations we obtain

$$\lambda = -N^2 + 1 + k\left(1 + e^{-\lambda\tau \pm \mathrm{i}\pi N}\right), \qquad N = 0, 1, 2, \ldots \tag{3.19}$$

We can split equation (3.19) into real and imaginary part, where we use again the convenient notation $\lambda = \mu + \mathrm{i}\nu$:

$$\mu = -N^2 + 1 + k\left(1 + e^{-\mu\tau}\cos(\nu\tau \pm \pi N)\right), \qquad N = 0, 1, 2, \ldots \quad (3.20)$$
$$\nu = ke^{-\mu\tau}\sin(\nu\tau \pm \pi N), \qquad N = 0, 1, 2, \ldots \quad (3.21)$$

Now two cases occur, one for N even and one for N odd, respectively: If N is even, equations (3.20) and (3.21) simplify to

$$\mu = -N^2 + 1 + k\left(1 + e^{-\mu\tau}\cos(\nu\tau)\right), \qquad N = 0, 2, 4, \ldots \quad (3.22)$$
$$\nu = ke^{-\mu\tau}\sin(\nu\tau), \quad (3.23)$$

and if N is odd, equations (3.20) and (3.21) simplify to

$$\mu = -N^2 + 1 + k\left(1 - e^{-\mu\tau}\cos(\nu\tau)\right), \qquad N = 1, 3, 5, \ldots \quad (3.24)$$
$$\nu = -ke^{-\mu\tau}\sin(\nu\tau). \quad (3.25)$$

In both cases the second equation is fulfilled for $\nu = 0$. In analogy to Chapter 2, it is therefore justified to search for the corresponding real eigenvalues μ determined by the first equations. No proof could be finished by only regarding real eigenvalues. However, since the crucial difference to Pyragas control lies *only in the real eigenvalues*, and not in the complex eigenvalues, we leave the discussion of the complex eigenvalues to the general proof in Chapter 5.

The simplified equations for the real eigenvalues μ read

$$\mu = -N^2 + 1 + k\left(1 + e^{-\mu\tau}\right), \qquad N = 0, 2, 4, \ldots \quad (3.26)$$

for N even, and

$$\mu = -N^2 + 1 + k\left(1 - e^{-\mu\tau}\right), \qquad N = 1, 3, 5, \ldots \quad (3.27)$$

for N odd. From these equations we directly calculate which feedback gain $k \in \mathbb{R}$ has to be applied to reach a specific real eigenvalue μ. We obtain the two functions

$$k_N(\mu) = \frac{\mu - 1 + N^2}{1 + e^{-\mu\tau}}, \qquad N = 0, 2, 4, \ldots \quad (3.28)$$

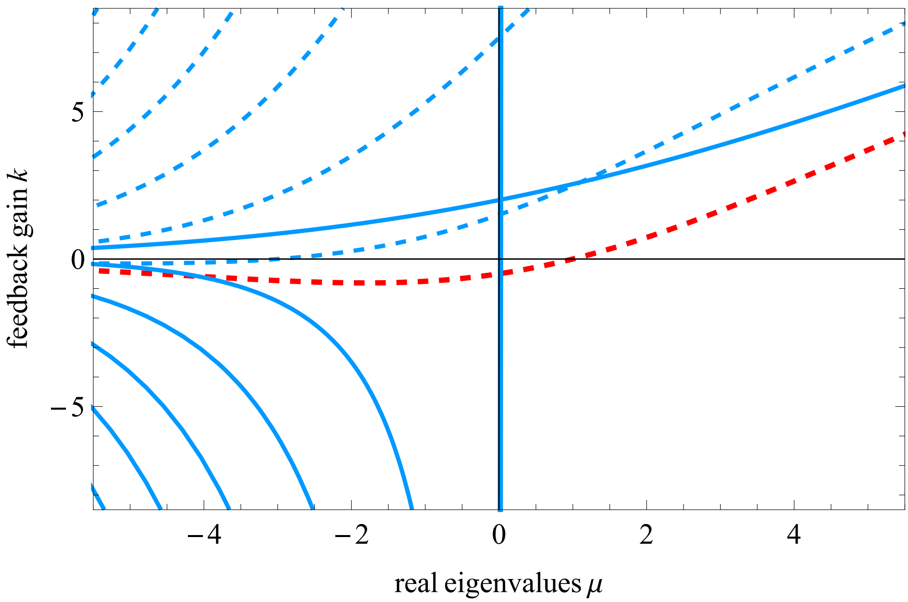


Figure 3.1.: Control of rotation type: The values of the feedback gain k (vertical axis), plotted versus the real eigenvalues μ (horizontal axis). The time delay is $\tau = 0.5$. Note that for $k < -0.5$ all nontrivial real eigenvalues are strictly negative. This is a strong indication that the control is indeed successful. The curve for $N = 0$ is red, while all curves for $N \geq 1$ are blue. Curves corresponding to even N are dashed.

for N even, and

$$k_N(\mu) = \frac{\mu - 1 + N^2}{1 - e^{-\mu\tau}}, \qquad N = 1, 3, 5, \ldots \tag{3.29}$$

for N odd. Note that the "+"-sign in the denominator occurs for N even, and the "−"-sign (as known from Pyragas control) occurs for N odd.

In particular – and this is the crucial difference to the control scheme of Pyragas – the "+"-sign in the denominator occurs for $N = 0$, which was problematic in the Pyragas case.

For the control triple method, the eigenvalue curve for $N = 0$ indeed crosses zero for $k = -0.5$. This curve is red and dashed in Figure 3.1. For $k < -0.5$ all real eigenvalues (except the trivial one) are strictly negative.

Assuming that all complex conjugated eigenvalues also have strictly negative real part (we postpone this to the proof in Chapter 5), we have shown that, indeed, our new control scheme, which uses the full control triple, successfully stabilizes the frozen waves in our example.

Remark. In contrast to Pyragas control, the control is also successful for $\tau = 0$. All eigenvalues are then real and the eigenvalues only cross zero if N is even, simplifying the calculations to

$$k_{N,\,N\,\text{even}}(\mu) = \tfrac{1}{2}\left(\mu - 1 + N^2\right). \tag{3.30}$$

We can conclude that for $\tau = 0$, control is indeed always possible for $k < -0.5$.

3.2.2. Control schemes of reflection type

In this section we consider again the example reaction-diffusion equation

$$u_t = u_{xx} + u. \tag{3.31}$$

Remember that only frozen waves exist and that they are all of the form $\mathcal{U}(x) = A\sin(x+\theta)$, $\theta \in S^1$, $A \in \mathbb{R}$. See Chapter 2 for a detailed discussion.

In this section let us select the frozen wave with $\theta = 0$. Then $\mathcal{U}(x) = A\sin(x) = -A\sin(-x) = -\mathcal{U}(-x)$, i.e., the frozen wave is odd with respect to the reference point at $x = 0$.

For such an odd wave, we consider the following equation including spatio-temporal feedback control of reflection type,

$$u_t = u_{xx} + u + k\big(u - (-1)\,u(-x, t-\tau)\big). \tag{3.32}$$

Again, we have used a sign change in the output signal "(-1)". Furthermore, we have used a reflection in the space coordinate "$(-x)$" and an arbitrary time delay $\tau \geq 0$. Note that the control term $k\big(u-(-1)\,u(-x,t-\tau)\big)$ is noninvasive only on waves which are odd functions with respect to the reference point $x = 0$. The respective waves are $\mathcal{U}(x) = A\sin(x)$ and $\mathcal{U}(x) = A\sin(x+\pi)$.

To calculate the eigenvalues and eigenfunctions, let us solve equation (3.32) by the Ansatz $u(x,t) = g(x)e^{\lambda t}$. We then obtain the following differential equation:

$$\lambda g = g_{xx} + g + k\left(g + g(-x)e^{-\lambda\tau}\right). \tag{3.33}$$

The eigenfunctions are the 2π-periodic solutions of equation (3.33). In Chapter 6 we prove the following: If there exist nontrivial 2π-periodic solutions of equation (3.33), then there also exist nontrivial periodic solutions which are either even or odd. Therefore, we can reduce our search for periodic solutions to the search for odd or even periodic solutions.

Let us first search for even eigenfunctions $g(x) = g(-x)$. We then obtain the ordinary differential equation

$$\lambda g = g_{xx} + g + k\left(g + ge^{-\lambda\tau}\right), \tag{3.34}$$

which is a pendulum equation. It is also an equation of Hill's type with the constant coefficient $-\lambda + 1 + k + ke^{-\lambda\tau}$. The 2π-periodic solutions of equation (3.34) are given explicitly by $\sin(Nx)$ and $\cos(Nx)$, $N \in \mathbb{N}$. However, only solutions of the form $\cos(Nx)$ matter for the Ansatz $g(x) = g(-x)$. As characteristic equations we obtain

$$\lambda = -N^2 + 1 + k\left(1 + e^{-\lambda\tau}\right), \qquad N = 0, 1, 2, \ldots \tag{3.35}$$

Characteristic equations of this form are already known from the previous section. In the same way as before, we calculate the feedback gain $k \in \mathbb{R}$ necessary to obtain a given real eigenvalue μ,

$$k_N(\mu) = \frac{\mu - 1 + N^2}{1 + e^{-\mu\tau}}, \qquad N = 0, 1, 2, \ldots \tag{3.36}$$

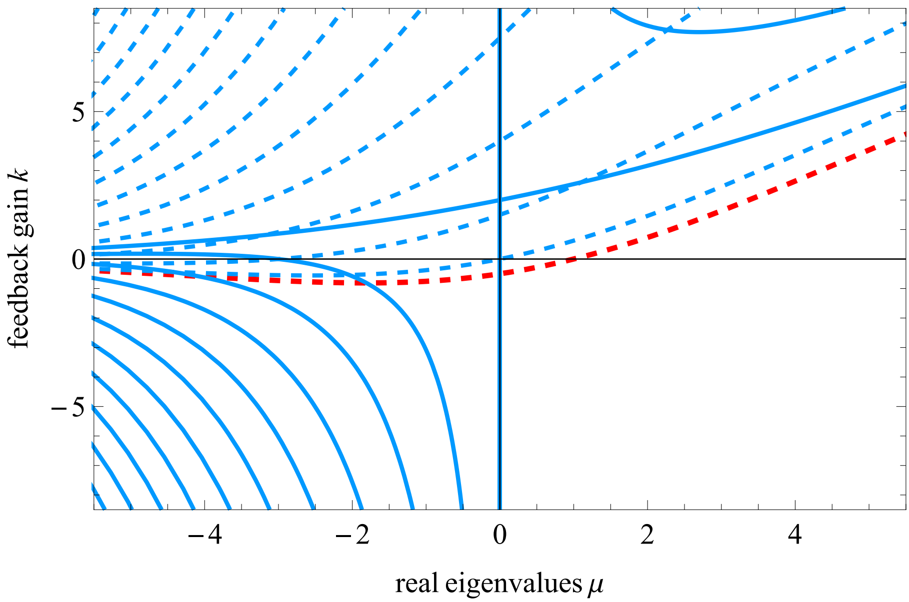


Figure 3.2.: Control of reflection type: The values of the feedback gain k_N (vertical axis), plotted versus the real eigenvalues μ (horizontal axis). The time delay is $\tau = 0.5$. Note that for $k < -0.5$ all real eigenvalues are strictly negative. This is a strong indication that the control is indeed successful. The curve for $N = 0$ is red, while all curves for $N \geq 1$ are blue. The solid lines are eigenvalues belonging to odd eigenfunctions, while the dashed lines give eigenvalues corresponding to even eigenfunctions.

The corresponding curves cross zero for $k_N(0) = (N^2 - 1)/2$. In Figure 3.2 the curves corresponding to even eigenfunctions are dashed. The curve for $N = 0$ is red, the curves for $N \geq 1$ are blue.

We next consider odd eigenfunctions $g(x) = -g(-x)$. In this case, we can simplify equation (3.33) as follows:

$$\lambda g = g_{xx} + g + k\left(g - ge^{-\lambda\tau}\right). \tag{3.37}$$

Again, we have obtained a pendulum equation and the periodic solutions are given explicitly by $\sin(Nx)$ and $\cos(Nx)$, $N \in \mathbb{N}$. Here only solutions of the type $\sin(Nx)$ for $N \geq 1$ matter, since we search for odd solutions.

Plugging in the explicit solutions, we obtain the characteristic equations

$$\lambda = -N^2 + 1 + k\left(1 - e^{-\lambda\tau}\right), \qquad N = 1, 2, 3, \ldots \tag{3.38}$$

Note that the difference between equation (3.35) and equation (3.38) lies only in the sign of the exponential term $\pm e^{-\lambda\tau}$. Again, we calculate for real eigenvalues μ:

$$k_N(\mu) = \frac{\mu - 1 + N^2}{1 - e^{-\mu\tau}}, \qquad N = 1, 2, 3, \ldots \tag{3.39}$$

In Figure 3.2 the corresponding curves for $N \geq 1$ are blue and solid. Most importantly, there does not exist any curve with denominator $1 - e^{-\mu\tau}$ for $N = 0$. Therefore, stabilization is indeed successful for $k < -0.5$. The proof, including information on the complex eigenvalues, will be finished in Chapter 6.

3.3. Comparison with ordinary differential equations

In this section we compare the control triple method to variations of Pyragas control which have been applied in the context of ordinary differential equations. Again, it is possible to see our results in the larger framework of delayed feedback control.

There are close connections between our work and Pyragas control and its variations – the reasons for success and failure of control are the same in the context of equivariant Pyragas control and in the context of Pyragas control of an unstable focus. Therefore, our results add to the few analytical results on Pyragas control, bringing us closer to understanding the control mechanisms and ultimately finding general successful delayed feedback control schemes for all dynamical systems.

3.3.1. Pyragas control of an unstable focus

It may be a surprise that we now compare the control triple method to standard Pyragas control of an unstable focus: The control triple method uses transformations in the output signal, space, and time, while standard Pyragas control only uses time delay. However, there are interesting parallels and the symmetry of the unstable focus combined with a specific time delay achieves the same effect as our control triple. It seems therefore appropriate to discuss shortly the mechanism of Pyragas control of an unstable focus. Here we follow the approach by Hövel and Schöll [34].

Let us consider the equilibrium $u_1 \equiv u_2 \equiv 0$ of the dynamical system

$$\dot{u}_1 = \quad au_1 + bu_2, \tag{3.40}$$

$$\dot{u}_2 = -bu_1 + au_2, \tag{3.41}$$

where a and $b \neq 0$ are real parameters, u_1, $u_2 \in \mathbb{R}$. If $a < 0$ the trivial equilibrium is a stable focus. We do not consider this case here. If $a = 0$, the equilibrium is a center with an eigenfrequency $1/T_0$ determined by the parameter $b \neq 0$ via the period $T_0 = 2\pi/b$. The equilibrium is an unstable focus if $a > 0$, which we assume throughout this section. The eigenfrequency is again determined by $b/(2\pi)$, even though the focus has no closed periodic orbits.

To stabilize the unstable focus, feedback control of Pyragas type is applied as follows:

$$\dot{u}_1 = \quad au_1 + bu_2 + k\big(u_1(t) - u_1(t-\tau)\big), \tag{3.42}$$

$$\dot{u}_2 = -bu_1 + au_2 + k\big(u_1(t) - u_2(t-\tau)\big). \tag{3.43}$$

The feedback gain k is real, and the time delay $\tau > 0$ can be chosen arbitrarily. The control is then noninvasive on the trivial equilibrium $u_1 \equiv u_2 \equiv 0$. In the following, we want to determine for which time delays $\tau > 0$ there exist feedback gains $k \in \mathbb{R}$ such that Pyragas control stabilizes the unstable focus.

Using an exponential Ansatz to determine the stability of the equilibrium $u_1 \equiv u_2 \equiv 0$ in the coupled equations (3.42) and (3.43) including control, we find the characteristic equation

$$\lambda = a \pm \mathrm{i}b + k\left(1 - e^{\lambda\tau}\right). \tag{3.44}$$

Let us split the characteristic equation into real and imaginary part. Using again the notation $\lambda = \mu + \mathrm{i}\nu$ we find

$$\mu = a + k\left(1 - e^{-\mu\tau}\cos(\nu\tau)\right) \tag{3.45}$$

$$\nu = \pm b - ke^{-\mu\tau}\sin(\nu\tau). \tag{3.46}$$

These equations are very familiar to us, compare with equations (2.12) and (2.13) as well as with equations (3.20) and (3.21). The main difference is given by the parameter b. Since $b \neq 0$, we do not find real eigenvalues for $\nu = 0$. However, we find complex conjugate eigenvalues with $\nu = \pm b$ if $\sin(\nu\tau) = 0$. This is the case if $\nu\tau = b\tau = n\pi$, $n \in \mathbb{N}$. It follows that $\tau = n\pi/b$.

Let us first assume that n is even, for simplicity assume that $n = 2$. Then $\tau = T_0$, i.e., the "period". In this case, $\cos(\nu\tau) = 1$. The equation for the real part μ simplifies to

$$\mu = a + k\left(1 - e^{-\mu\tau}\right), \tag{3.47}$$

with $\tau = T_0$. This equation is already known from Pyragas control in Chapter 2, equation (2.14), if we define $a := 1 - N^2$. Therefore, stabilization of the focus is impossible. The same holds for all time delays which are integer multiples of T_0.

Now assume that n is odd, more specifically, assume $n = 1$. In this case, $\tau = T_0/2$, i.e., half the "period", and $\cos(\nu\tau) = -1$. The equation for the real part μ simplifies to

$$\mu = a + k\left(1 + e^{-\mu\tau}\right), \tag{3.48}$$

with $\tau = T_0/2$. This equation is known from the control schemes of rotation type, where we have used a spatial delay of half the period, see Section 3.2.1, equation (3.26). We have seen that stabilization is successful for such characteristic equations.

In Figures 3.3 and 3.4 the green curves give the real parts of the complex eigenvalues λ from the characteristic equation (3.44). The plots are calculated using the well-known Lambert W function. As in [34] we have plotted the eigenvalues for varying time delay but for a fixed feedback gain $k \in \mathbb{R}$. This is a very interesting point of view:

In Figure 3.3 we compare these eigenvalue curves from the controlled focus (green curves) with the curves (2.11) obtained in Chapter 2, i.e., Pyragas control applied to a scalar reaction-diffusion equation (red and orange curves for $N = 0$). To compare, we need to choose $a = 1$. Furthermore, we have chosen $b = \pi$ and therefore $T_0 = 2$. We see that a time delay of a full "period" T_0 is not successful, and the integer multiples $\tau = nT_0$, $n \in \mathbb{N}$, correspond to the local maxima of the eigenvalue curves. Note that the red curve, indicating the failure of Pyragas control for scalar reaction-diffusion equations, and the green curve meet exactly in those local maxima at $\tau = nT_0$.

In Figure 3.4 we compare the eigenvalue curves from the controlled focus to the eigenvalue curves from the example in Section 3.2.1 with a spatial delay of half a spatial period (red and orange curves for $N = 0$). We see that Pyragas control of an unstable focus is successful for a time delay τ which is half the "period" T_0. The eigenvalue curve even reaches a global minimum for this specific half-period time delay. Note that this minimum eigenvalue coincides with the eigenvalue obtained by the control triple method with a spatial delay of half a spatial period (red curve).

Furthermore, note that the red curve, corresponding to the control triple method, always yields smaller or at least equal eigenvalues. The red curve and the green curve meet exactly in the local minima at $\tau = \frac{2n+1}{2}T_0$. In a certain sense, minimizing the eigenvalues, the control triple method can be seen as an optimum control method for this type of characteristic equations.

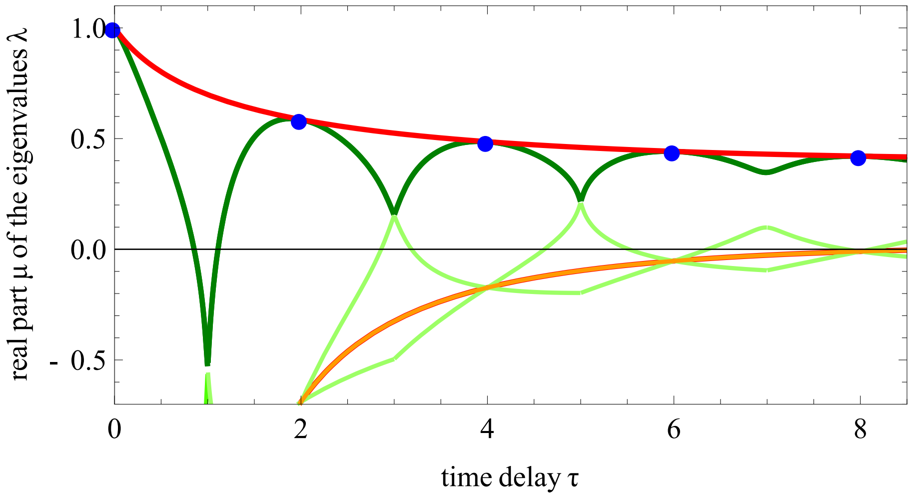


Figure 3.3.: Real parts μ of the eigenvalues λ versus time delay τ for a feedback gain $k = -0.6$, and parameters $a = 1$, $b = \pi$. The green curves are the eigenvalues of a controlled focus with "period" $T_0 = 2$ (dark green: main branch of the Lambert W function, light green: second branch). We compare with the control triple method from Section 3.2.1: The red and orange curves give the eigenvalues of the waves in $u_t = u_{xx} + u$ (only the curves corresponding to $N = 0$). The blue dots at $\tau = 0, 2, 4, 6, 8$ denote the local maxima of μ of the controlled focus. Exactly at these points, the control triple method gives the same eigenvalues.

As an interesting byproduct, Hövel and Schöll [34] found numerically that, after some time, the output signal and the delayed output signal of the focus are in anti-phase if the time delay is chosen to be half the "period" $\tau = T_0/2$. Remarkably, the symmetry which we have constructed in space with spatio-temporal control terms can therefore also occur using only Pyragas control with a very specific time delay!

We have also learned that the important term for stabilization is the term in front of the exponential $e^{-\lambda\tau}$.

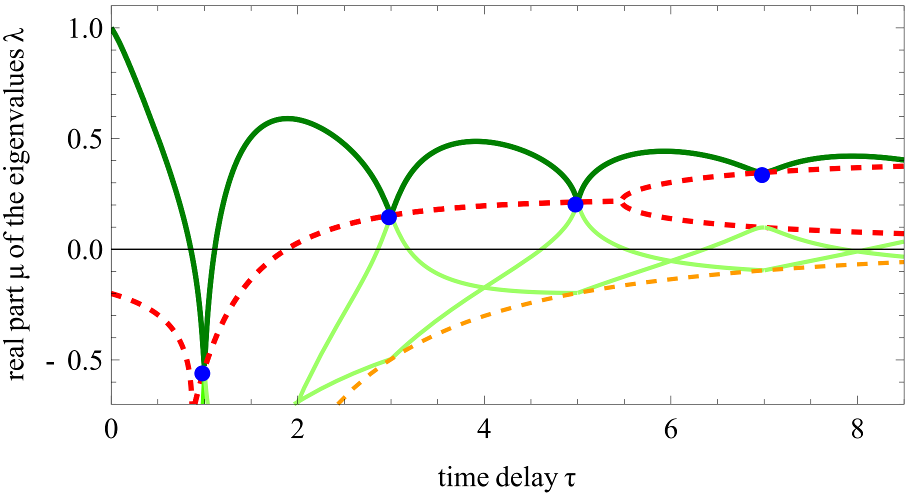


Figure 3.4.: Real parts μ of the eigenvalues λ versus time delay τ for a feedback gain $k = -0.6$, and parameters $a = 1$, $b = \pi$. The green curves are the eigenvalues of a controlled focus with "period" $T_0 = 2$ (dark green: main branch of the Lambert W function, light green: second branch). We compare with the control triple method from Section 3.2.1: The red and orange curves give the eigenvalues of the waves in $u_t = u_{xx} + u$ (only the curves corresponding to $N = 0$). The blue dots at $\tau = 1, 3, 5, 7$ denote the local minima of μ of the controlled focus. Exactly at these points, the control triple method gives the same eigenvalues.

In conclusion, choosing a spatial delay of half the spatial period in scalar reaction-diffusion equations has the same effect as choosing a time delay which is half the "period" T_0 of the unstable focus.

In summary, we have found a surprising analogy between the spatial delay in reaction-diffusion equations and the eigenfrequency of an unstable focus in ordinary differential equations.

3.3.2. Equivariant Pyragas control

In this section we want to compare the new control triple method for partial differential equations with equivariant Pyragas control for ordinary differential equations. No space is present in ordinary differential equations, therefore equivariant Pyragas control only transforms the output signal and time.

The *half-period feedback scheme*, introduced by Nakajima and Ueda in [53], was the first step in the direction of equivariant Pyragas control. It was explored in more detail by Fiedler et al. in [16]. Originally, the purpose of the half-period feedback scheme was to overcome the odd-number limitation [52].

In this section let us concentrate on the half-period feedback scheme by Fiedler et al. [16]. Their scheme is designed for the stabilization of two coupled Stuart-Landau oscillators in anti-phase. We will see that stabilization with the equivariant scheme is successful, while standard Pyragas control fails.

Consider the coupled oscillator system

$$\dot{u}_1 = F(u_1) + a\,(u_2 - u_1) \tag{3.49}$$

$$\dot{u}_2 = F(u_2) + a\,(u_1 - u_2)\,, \tag{3.50}$$

where $u_1,\, u_2 \in \mathbb{R}^2 \cong \mathbb{C}$, with diffusive coupling constant $0 < a < 1/\pi$, and dynamics

$$F(u) = \left(\Lambda + \mathrm{i} + \gamma |u|^2\right) u, \tag{3.51}$$

where $\Lambda \in \mathbb{R}$ is a bifurcation parameter and $\gamma \in \mathbb{C}$ is the spring constant. For all parameter values, there is a trivial zero equilibrium $u_1 \equiv u_2 \equiv 0$. In-phase Hopf bifurcation occurs at $\Lambda = 0$, and anti-phase Hopf bifurcation occurs at $\Lambda = 2a$. To see this, it is convenient to use the coordinates

$$u_+ = \tfrac{1}{2}\,(u_1 + u_2) \qquad \text{(average)} \tag{3.52}$$

$$u_- = \tfrac{1}{2}\,(u_1 - u_2) \qquad \text{(asynchrony).} \tag{3.53}$$

Then the dynamics in the *in-phase subspace* $U_+ := \{(u_+, u_-) \mid u_- = 0\}$ reduces to

$$\dot{u}_+ = F(u_+) = \left(\Lambda + \mathrm{i} + \gamma|u_+|^2\right) u_+, \tag{3.54}$$

i.e., the normal form of an Hopf bifurcation. The Hopf bifurcation is subcritical for $\operatorname{Re}\gamma > 0$, and supercritical for $\operatorname{Re}\gamma < 0$. The period p_+ of the bifurcating in-phase orbit is given explicitly by

$$p_+ = \frac{2\pi}{1 - \Lambda \operatorname{Im}\gamma / \operatorname{Re}\gamma}. \tag{3.55}$$

Conversely, in the *anti-phase subspace* $U_- := \{(u_+, u_-) \mid u_+ = 0\}$ the dynamics reduces to

$$\dot{u}_- = F(u_-) - 2au_- = \left(\Lambda + \mathrm{i} + \gamma|u_-|^2\right) u_- - 2au_-, \tag{3.56}$$

where the normal form of an Hopf bifurcation is shifted by $2a$. Again, the Hopf bifurcation is subcritical for $\operatorname{Re}\gamma > 0$, and supercritical for $\operatorname{Re}\gamma < 0$. The period p_- of the bifurcating anti-phase orbit is given explicitly by

$$p_- = \frac{2\pi}{1 - (\Lambda - 2a) \operatorname{Im}\gamma / \operatorname{Re}\gamma}. \tag{3.57}$$

Both the in-phase and the anti-phase subspace are dynamically invariant subspaces.

Let us now introduce half-period feedback control as in [16] with the aim of stabilizing the anti-phase periodic orbit bifurcating at $\Lambda = 2a$:

$$\dot{u}_1 = F(u_1) + a\,(u_2 - u_1) + k\big(u_2(t-\tau) - u_1\big) \tag{3.58}$$

$$\dot{u}_2 = F(u_2) + a\,(u_1 - u_2) + k\big(u_1(t-\tau) - u_2\big), \tag{3.59}$$

with half period delay $\tau = p_-/2$, and variable feedback gain $k \in \mathbb{C}$.

Instead of discussing in detail the theorems in [16], which establish the existence of stabilization regions, let us shortly explain the stabilization mechanism.

Stabilization near Hopf bifurcation is established as follows: First, we determine a region of feedback gains $k \in \mathbb{C}$ where the zero equilibrium

$u_1 \equiv u_2 \equiv 0$ is stable at the anti-phase Hopf bifurcation point $\Lambda = 2a$. For the purpose of comparing equivariant Pyragas control with the control triple method, this is already sufficient.

However, the actual stabilization region is only a subset of the region of feedback gains where the trivial equilibrium is stable: The feedback gain must be adjusted such that the Hopf bifurcation is supercritical in order to guarantee stabilization.

Therefore, let us now calculate the region where the zero equilibrium is stable at the anti-phase Hopf bifurcation point $\Lambda = 2a$: Linearizing at the zero equilibrium $u_1 \equiv u_2 \equiv 0$ yields

$$\dot{u}_+ = (\Lambda + \mathrm{i})\, u_- + k\big(u_+(t-\tau) - u_+\big), \tag{3.60}$$

$$\dot{u}_- = (\Lambda + \mathrm{i})\, u_- - 2au_- - k\big(u_-(t-\tau) + u_-\big), \tag{3.61}$$

in the coordinates u_+, u_-. Note that the linearized equations decouple in those coordinates. Using an exponential Ansatz $u_\pm(t) = e^{\eta t}$, we obtain the characteristic equations

$$\eta = \Lambda + \mathrm{i} + k\big(e^{\eta\tau} - 1\big), \tag{3.62}$$

$$\eta = \Lambda + \mathrm{i} - 2a - k\big(e^{\eta\tau} + 1\big). \tag{3.63}$$

We plug in $\Lambda = 2a$, and a time delay $\tau = \pi$, which corresponds to half the period of the anti-phase periodic orbit at $\Lambda = 2a$. The characteristic equations simplify to

$$\eta = 2a + \mathrm{i} + k\big(e^{\eta\pi} - 1\big), \tag{3.64}$$

$$\eta = \mathrm{i} - k\big(e^{\eta\pi} + 1\big). \tag{3.65}$$

Without control ($k = 0$), the equilibrium has two unstable eigenvalues $\eta = 2a \pm \mathrm{i}$, and two purely imaginary eigenvalues $\eta = \pm\mathrm{i}$.

We are interested in the stability changes, therefore we set $\eta = \mathrm{i}\omega$ and rearrange as follows:

$$k = k_+(\omega) = \frac{\mathrm{i}\omega - \mathrm{i} - 2a}{e^{\mathrm{i}\pi\omega} - 1}, \tag{3.66}$$

$$k = k_-(\omega) = \frac{\mathrm{i} - \mathrm{i}\omega}{e^{\mathrm{i}\pi\omega} + 1}. \tag{3.67}$$

We obtain two sets of curves in the complex plane ($k \in \mathbb{C}$) determining the boundaries of stability. Crossing the lines increases or decreases the number of unstable eigenvalues by two. For full mathematical details, see [16].

The curves are drawn in Figure 3.5(a). In parentheses, we have denoted the number of eigenvalues of the zero equilibrium with strictly positive real part. Without control, i.e., $k = 0$, there are two unstable eigenvalues.

The colored region has no eigenvalues with strictly positive real part, and the zero equilibrium is stable here. In [16], it is shown that there exists a subset of the colored region for which the anti-phase periodic orbit is stable. Hence, the half-period feedback scheme is successful.

The same process can be carried out for standard Pyragas control as well. For brevity, we only give the main equations here. Let us introduce Pyragas control as follows:

$$\dot{u}_1 = F(u_1) + a\,(u_2 - u_1) + k\big(u_1(t-\tau) - u_1\big), \tag{3.68}$$

$$\dot{u}_2 = F(u_2) + a\,(u_1 - u_2) + k\big(u_2(t-\tau) - u_2\big), \tag{3.69}$$

where $\tau = p_-$, i.e., a full period of the anti-phase periodic orbit. We obtain the characteristic equations

$$\eta = \Lambda + \mathrm{i} + k\big(e^{\eta\tau} - 1\big), \tag{3.70}$$

$$\eta = \Lambda + \mathrm{i} - 2a + k\big(e^{\eta\tau} - 1\big). \tag{3.71}$$

As before, we put $\Lambda = 2a$. We use $\tau = 2\pi$ for standard Pyragas control and search for purely imaginary eigenvalues $\eta = \mathrm{i}\omega$. The curves determining the stability boundaries are given by

$$k = k_+(\omega) = \frac{\mathrm{i}\omega - \mathrm{i} - 2a}{e^{2\pi\mathrm{i}\omega} - 1}, \tag{3.72}$$

$$k = k_-(\omega) = \frac{\mathrm{i}\omega - \mathrm{i}}{e^{2\pi\mathrm{i}\omega} - 1}. \tag{3.73}$$

The corresponding curves are drawn in Figure 3.5(b). No regions of stability can be found. Consequently, Pyragas control fails its stabilization task.

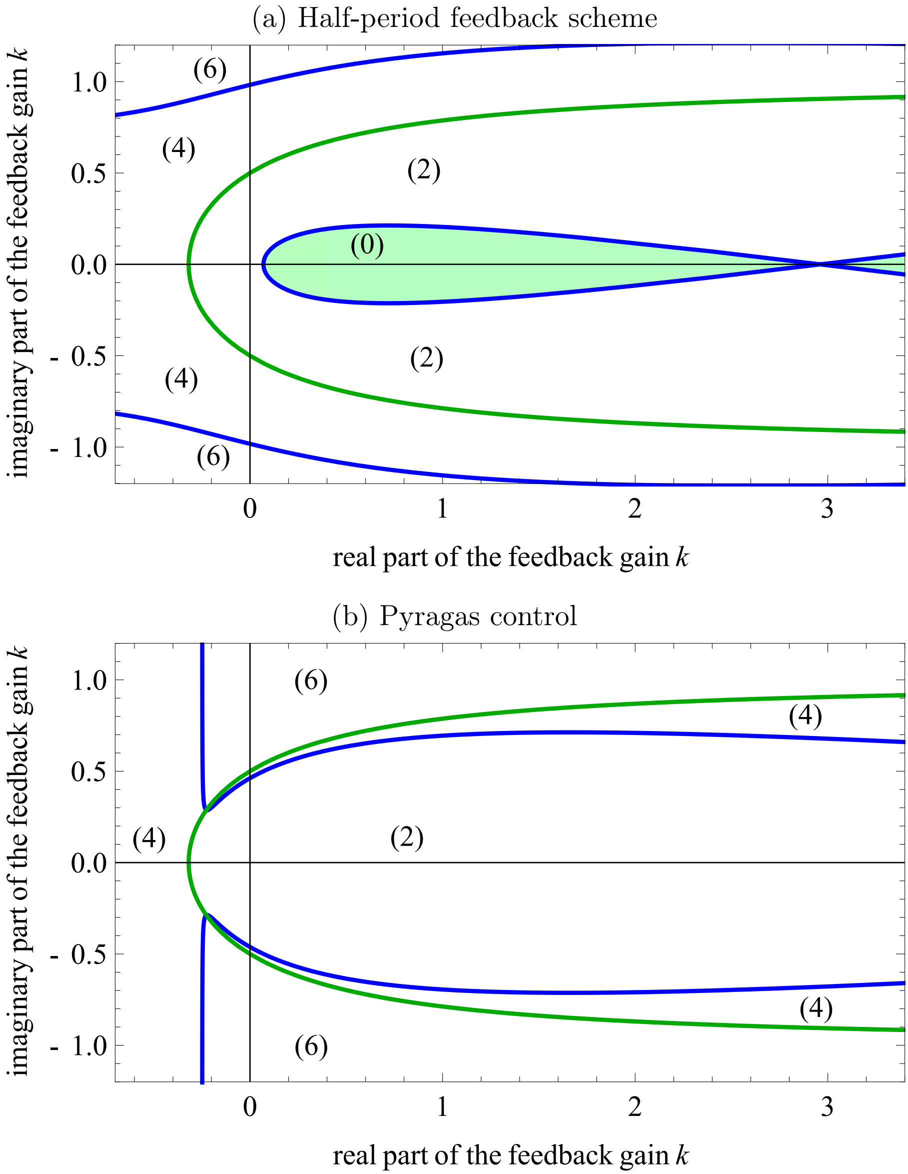


Figure 3.5.: Curves determining the stability boundaries of the zero equilibrium for $a = 0.07$. In parentheses, the number of eigenvalues with strictly positive real part is denoted. (a) Half-period feedback scheme: In the colored region, the zero equilibrium is stable. (b) Pyragas control: No region of stability exists.

Let us summarize: The half-period feedback scheme is successful where standard Pyragas control is not. This is in accordance with our results in Section 3.1.

General equivariant Pyragas control, as treated by [56, 66–69] uses pairs $(h, \Theta(h))$, where h denotes the transformation of the output signal and $\tau = \Theta(h)p$ denotes the time delay. $\Theta(h)$ is not necessarily given by 1/2 but can be any number between 0 and 1, if the equivariance allows it. This is the case for many partial differential equations and we should explore the many possibilities of combining equivariant Pyragas control with the control triple method. Note in particular that there exist further connections between equivariant Pyragas control and the control triple method: The condition m co-prime to n from Theorem 3.1 also features prominently in the stabilization of n oscillators coupled in a bi-directional ring [69].

In conclusion, equivariant Pyragas control can also serve as a model for the control triple method in general equivariant partial differential equations.

Chapter 4

Preliminaries for the proofs

In this chapter we collect some preliminary results for the proofs to follow in Chapter 5 and 6. In Section 4.1 we prove that all rotating and frozen waves are unstable in scalar reaction-diffusion equations. We derive Hill's equation from scalar reaction-diffusion equations. We compile some useful observations on Hill's equation in Section 4.2. In Section 4.3 we turn to Floquet theory for Hill's equation with delay. Section 4.4 is devoted to a short introduction to partial delay differential equations, where we focus on stability analysis. Section 4.5 contains a discussion of the linear variational equations. We also outline the proofs for the two different control types as given in Chapters 5 and 6, respectively.

4.1. Stability analysis of the equation without control

As a preliminary result we first investigate the stability of frozen and rotating waves in scalar reaction-diffusion equations without control. We prove a simplified version of the following theorem:

Theorem 4.1 (Angenent and Fiedler [2]). *All rotating and frozen waves of the scalar reaction-diffusion equation* $u_t = u_{xx} + f(u, u_x)$ *are unstable.*

Remark. Note that frozen and rotating waves are spatially non-homogeneous by definition. Homogeneous equilibria can be stable. We will encounter such an example in Chapter 7, where we will apply the control triple method to the Chafee-Infante equation.

The proof of Theorem 4.1 was given by Angenent and Fiedler [2] in an elegant way, using the zero-number (also called lap-number) theory, see also the work by Matano [46]. However, this theory has so far not been established in the case of partial delay differential equations. Therefore, we aim for an elementary proof, shedding light also on the case which includes the control term. In this proof, after the formal linearization, we combine two properties of the characteristic equation with a specific property of the frozen or rotating wave. However, in order not to lose ourselves in technical difficulties, we only prove Theorem 4.1 in the (simpler, yet nontrivial) case $u_t = u_{xx} + f(u) - cu_x$ for which we apply the control terms.

Proof of Theorem 4.1 in the case $u_t = u_{xx} + f(u) - cu_x$. As in [2], let us first calculate formally the linearized flow related to the local asymptotic stability of frozen and rotating waves. By definition the Fréchet derivative of the flow Φ_t on $\mathbb{X}$ at the wave $\mathcal{U}$ is given as follows:

$$v_2 = d\Phi_t(\mathcal{U})v_1 \tag{4.1}$$

holds if and only if there exists a solution $v_*(s, x)$ of

$$v_s = v_{xx} + cv_x + Q(s, x)v, \tag{4.2}$$

with $x \in S^1$, $0 \leq s \leq t$, and $v_*(0, x) = v_1(x)$, $v_*(t, x) = v_2(x)$. Here the coefficient $Q(s, x)$ is given by:

$$Q(s, x) = f'(\mathcal{U}(s, x)). \tag{4.3}$$

The linear operator $d\Phi_t(\mathcal{U}) : v_1 \mapsto v_2$ is bounded and compact on $\mathbb{X}$ [30]. Using the standard Riesz-Schauder Theorem [1] for compact operators, we can conclude that its spectrum consists of a countable sequence of eigenvalues. Each of them has finite multiplicity. We call the eigenvalues λ_j with $j \in \mathbb{N}_0$.

Let us concentrate on the linearization around a rotating or frozen wave $u(x,t) = \mathcal{U}(x - ct)$. To simplify the stability analysis for rotating waves, we perform a coordinate transformation to a co-rotating frame:

$$z = x - ct, \quad t = t, \tag{4.4}$$

freezing all rotating waves. For convenience, we always refer to the coordinate z from hereon, also in the case of an equilibrium, where $z = x$. In these co-rotating coordinates, the coefficient $Q(z) = f'(\mathcal{U}(z))$ does not depend on time. Note that the function $Q(z)$ is 2π-periodic in z. Having disposed of the time-dependence, it is now sufficient to consider the linearized equation

$$v_t = v_{zz} + Q(z)v. \tag{4.5}$$

To obtain the eigenvalues λ, we solve this linear equation by separation of variables and apply an exponential Ansatz in time,

$$v(z,t) = g(z)e^{\lambda t}. \tag{4.6}$$

We obtain the equation

$$\lambda g = g_{zz} + Q(z)g, \tag{4.7}$$

which is an homogeneous linear ordinary differential equation of second order with real 2π-periodic coefficient $Q(z)$, $z \in \mathbb{R}$. Equation (4.7) is known as *Hill's equation* in the literature. It is named after George W. Hill, who investigated this equation first in 1877 in the context of lunar motion [31]. The general reference on Hill's equation is the book by Magnus and Winkler [45]. Usually, the analysis of linear differential equations with periodic coefficients is done by *Floquet theory*, established by Gaston Floquet in 1883 [20]. Since Hill's equation does not only have periodic coefficients but is also a scalar equation, we rely directly on the many detailed results concerning Hill's equation.

Eigenfunctions are given by those solutions of Hills' equation (4.7) which fulfill the periodic boundary conditions. The values λ for which such periodic solutions exist are the *eigenvalues.* If there exists at least one strictly positive eigenvalue λ, then the wave is linearly unstable.

The question of stability has now been reduced to the following question:

For which parameter values of λ do there exist periodic solutions of Hill's equation?

The first answer to this question was given by Alexander Lyapunov [44] in 1907 and by Otto Haupt in 1914 and 1918 [27, 28]. We present a shortened version of their theorem here.

Theorem 4.2 (Oscillation Theorem, [45])**.** *To every differential equation of Hill's type, $\lambda g = g_{zz} + Q(z)g$, there belongs a monotonically decreasing infinite sequence of real numbers $\{\lambda_n\}_{n\geq 0}$ such that Hill's equation has a solution of period 2π if and only if $\lambda = \lambda_n$. The λ_n satisfy the inequalities*

$$\lambda_0 > \lambda_1 \geq \lambda_2 > \lambda_3 \geq \lambda_4 > \lambda_5 \geq \lambda_6 > \dots \tag{4.8}$$

and also

$$\lim_{n\to\infty} \lambda_n^{-1} = 0. \tag{4.9}$$

For complex values of λ, Hill's equation has no bounded solutions except the trivial zero solution.

For a proof of the Oscillation Theorem, we refer the reader to [45]. Let us mention two important consequences of the Oscillation Theorem: First, all eigenvalues are real. Second, there exists a largest eigenvalue λ_0 and our task is to show that λ_0 is strictly positive.

Having established general properties of the eigenvalues, we now turn to a specific property of rotating and frozen waves: $\mathcal{U}_z$ is an eigenfunction corresponding to the eigenvalue 0 for a wave $\mathcal{U}$. In particular, any non-homogeneous frozen or rotating waves are non-hyperbolic. Since $\mathcal{U}$ is non-constant, $\mathcal{U}_z$ has at least two zeros in $z \in [0, 2\pi]$ (infinitely many for $z \in \mathbb{R}$), or more precisely sign changes, by periodicity. However, it turns out that the eigenfunction corresponding to the largest eigenvalue λ_0 has no zeros:

Theorem 4.3 (Number of zeros 1, [45])**.** *Fix $\lambda \in \mathbb{R}$. Then either all nontrivial solutions of Hill's equation have only a finite number of zeros, or all solutions of Hill's equation have infinitely many zeros.*

Let λ_0 be the largest value of $\lambda \in \mathbb{R}$ for which Hill's equation has a 2π-periodic solution. Then for $\lambda \geq \lambda_0$, all nontrivial solutions have only a finite number of zeros, but for $\lambda < \lambda_0$, every solution has infinitely many zeros.

Remark. Note that this theorem classifies *all* nontrivial solutions of Hill's equation: Either they have no zeros or they have infinitely many zeros. Consequently, this also holds for the periodic solutions. Only one 2π-periodic solution has no zeros, it occurs for λ_0. In the autonomous case, this periodic solution is the constant eigenfunction.

In particular, we can conclude that any eigenfunction which has at least one zero (and thus, by periodicity, infinitely many of them) is not the eigenfunction which belongs to the eigenvalue λ_0.

Thus, there must exist an eigenvalue $\lambda_0 > 0$ which proves the instability of the frozen and rotating waves. This concludes the proof of Theorem 4.1. □

Let us finish this section with a short discussion of stability in the sense of the unstable dimension of a rotating or frozen wave. Here we merely cite the corresponding theorems for brevity.

Theorem 4.4 (Number of zeros 2, [2, 27, 45])**.** *If $\lambda \in \{\lambda_{2n-1}, \lambda_{2n}\}$, then g has exactly $2n$ zeros in $0 \leq x < 2\pi$.*

Remark. The zeros are also simple [2].

While we do not prove the theorem, it can be made plausible when considering the case $Q(z) \equiv Q$, for which a simple Fourier decomposition yields the result. For our purposes, the following corollary, refining Theorem 4.1, is most important:

Corollary 4.5 (Unstable dimensions, [2]). *The unstable dimension, i.e., the number of strictly positive eigenvalues, of a rotating or frozen wave with minimal period* $2\pi/n$ *is either* $2n$ *or* $2n-1$.

The eigenfunction $\mathcal{U}_x$ corresponding to the eigenvalue $\lambda = 0$ has exactly $2n$ zeros. Using Theorem 4.4, the eigenfunction with $2n$ zeros corresponds either to λ_{2n-1} or to λ_{2n}. Note that the eigenvalues are ordered by the Oscillation Theorem. This implies Corollary 4.5.

4.2. Useful observations

In this section we show three properties of the function $Q(z)$ from Hill's equation. These properties are specific to Hill's equation if $Q(z)$ comes from a reaction-diffusion equation. They do not hold in general. Here we assume $Q(z) = f'(\mathcal{U}(z))$, where $\mathcal{U}(z)$ is a rotating or frozen wave (not a homogeneous equilibrium) and f nonlinear.

First, we show that the maximum value of $Q(z)$ is positive, more precisely, $\bar{Q} = \max_{z\in[0,2\pi]} Q(z) \geq \lambda_0 > 0$.

To show this, we first cite the following fact [45]: *If* $-\lambda + Q(z) < 0$ *for all* $z \in [0, 2\pi]$, *then the nontrivial solutions of* $\lambda g = g_{zz} + Q(z)g$ *have only finitely many zeros.*

For a proof of this fact, see either Magnus and Winkler [45], or Chapter 5, Step 3, where we present the proof in a slightly different context.

We have seen in the previous section that a periodic solution, having infinitely many zeros, exists for $\lambda = 0$. Also all nontrivial solutions for $0 \leq \lambda < \lambda_0$ have infinitely many zeros, see Theorem 4.3. We can conclude:

$$\lambda \leq \bar{Q} \text{ for all } 0 \leq \lambda < \lambda_0. \tag{4.10}$$

This proves $\bar{Q} = \max_{z\in[0,2\pi]} Q(z) \geq \lambda_0 > 0$, as claimed above.

Remark. In the other case, $Q(z) \equiv Q \equiv \bar{Q}$, we find $\bar{Q} = \lambda_0$. Note that here λ_0 is not necessarily positive, see Chapter 7 for an example.

Besides positivity, the function $Q(z)$ has other useful properties: Let us consider a scalar reaction-diffusion equation of the form

$$u_t = u_{zz} + f(u), \tag{4.11}$$

where f is an odd function, i.e., $f(u) = -f(-u)$. It follows that f' is an even function. Let us now consider rotating or frozen waves $\mathcal{U}(z)$ with minimal period $\omega = 2\pi/n$ and $\mathcal{U}(z) = -\mathcal{U}(z - \pi/n)$. Then

$$\begin{align} Q(z) &= f'(\mathcal{U}(z)) \tag{4.12}\\ &= f'(-\mathcal{U}(z - \pi/n)) \tag{4.13}\\ &= f'(\mathcal{U}(z - \pi/n)) \tag{4.14}\\ &= Q(z - \pi/n), \tag{4.15}\end{align}$$

i.e., $Q(z)$ is of period $\omega/2 = \pi/n$. Already in the next section, we will see how this property greatly simplifies the stability analysis.

Consider now waves which are odd with respect to the reference point 0, i.e., twisted standing waves: $\mathcal{U}(z) = -\mathcal{U}(-z)$. Then

$$\begin{align} Q(z) &= f'(\mathcal{U}(z)) \tag{4.16}\\ &= f'(-\mathcal{U}(-z)) \tag{4.17}\\ &= f'(\mathcal{U}(-z)) \tag{4.18}\\ &= Q(-z), \tag{4.19}\end{align}$$

i.e., $Q(z)$ is an even function. This property will be crucial to the proof in Chapter 6.

4.3. Floquet theory for Hill's equation with delay

In the proof for control schemes of rotation type, we will also consider Hill's equation with time delay. Therefore, we give a brief exposition of Floquet theory for delay differential equations. We follow the presentation of Hale [24] and Hale and Verduyn Lunel [25]. At least to some extent, Floquet theory for delay differential equations is analogous to Floquet theory for ordinary differential equations.

In this thesis, we focus on the case of Hill's equation with delay, i.e., on equations of the form

$$\lambda g = g_{zz} + Q(z)g + k\big(g - \Psi e^{-\lambda\tau} g(z-\varphi)\big). \tag{4.20}$$

We assume that $Q(z)$ is $\omega/2 = \pi/n$-periodic. Here $\varphi := \xi - c\tau$ is the *spatio-temporal delay.*

Theorem 4.6 (Floquet exponents, [24, 25])**.** *$\eta \in \mathbb{C}$ is a Floquet exponent of equation* (4.20) *if and only if there exists a nonzero solution of equation* (4.20) *of the form*

$$g(z) = p(z)e^{\eta z}, \tag{4.21}$$

where $p(z+\omega) = p(z)$.

Moreover, the following theorem holds true:

Theorem 4.7 (Stability, [24, 25])**.** *The solution $g = 0$ of the equation*

$$\lambda g = g_{zz} + Q(z)g + k\big(g - \Psi e^{-\lambda\tau} g(z-\varphi)\big) \tag{4.22}$$

is asymptotically stable if and only if all Floquet exponents of equation (4.22) *have negative real part. If there exists $\eta \in \mathbb{C}$ with $\operatorname{Re}\eta = 0$, then there are solutions which are bounded for all times.*

However, a complete Floquet decomposition, as known in the case of ordinary differential equations, does not always exist in the case of delay differential equations. An explicit counter-example to such a decomposition is given by the equation $\dot g(t) = (\sin t)g(t-2\pi)$. For this example equation, there exist solutions which converge to zero faster than any exponential, but do not become identically zero after finite time. As a consequence, no 2π-periodic change of variables reduces this equation to an autonomous equation. See [24, 25] for more details.

Usually, the most difficult part of Floquet theory is the calculation of the Floquet exponents. Fortunately, in our special case of Hill's equation with delay, we do not have to deal with the additional difficulty of the delay φ: The spatio-temporal delay $\varphi = m\pi/n$ is an integer multiple of the period $\omega = \pi/n$ of the periodic coefficient $Q(z)$. Therefore, we observe that we can eliminate the time delay as follows:

We calculate the Floquet exponents, using the Ansatz

$$g(z) = p(z)e^{\eta z}, \tag{4.23}$$

as justified by Theorem 4.6, where

$$p(z) = p\left(z + 2\pi\right). \tag{4.24}$$

We then obtain a new equation of the form

$$0 = p_{zz} + 2\eta\, p_z + \left(-\lambda + \eta^2 + Q(z) + k - k\Psi e^{-\lambda\tau - \eta\varphi}\right) p, \tag{4.25}$$

which is an ordinary differential equation for which we can (at least theoretically) determine the Floquet exponents.

4.4. A short introduction to partial delay differential equations

Let us go back to our original problem: Spatio-temporal feedback control of partial differential equations, where we concentrate on scalar reaction-diffusion equations. If we add a delayed control term to the reaction diffusion equation, we obtain a partial *delay* differential equation. In this section we give a very short summary of basic stability results in this context. We follow the presentation of Wu [78], but skip the proofs.

As in Chapter 1, we denote $\mathbb{X} := H^s(S^1)$, $s > 3/2$. Let C denote the Banach space of continuous $\mathbb{X}$-valued functions on the interval $[-\tau, 0]$, together with the supremum norm $\|\cdot\|$, i.e., $C := C([-\tau, 0]; \mathbb{X})$.

We consider the following abstract delay differential equation

$$\tfrac{d}{dt}u = A_T u + F(u, u(t-\tau)). \tag{4.26}$$

Here A_T is the infinitesimal generator of an analytic compact semigroup $\{T(t)\}_{t\geq 0}$. In our case A_T is defined via $A_T\, u = u_{zz}$, and its domain is given by $\mathcal{D}(A_T) = \left\{ u \in C^2([0, 2\pi]) \,\middle|\, u(0) = u(2\pi),\;\; u'(0) = u'(2\pi)\right\}$. We suppose that the operator F satisfies a local Lipschitz condition and we furthermore assume that F is Fréchet differentiable with locally Lipschitz continuous Fréchet derivative. $F : \mathbb{X} \times C \to \mathbb{X}$ also includes the terms with time delay. In our case F includes the reaction term f as well as the control term. In this thesis we only consider a single time delay $\tau > 0$. Wu also treats far more general cases including multiple time delays and distributed time delay.

Consider now the linearized equation

$$\tfrac{d}{dt}u = A_T u + DF_1(\mathcal{U},\mathcal{U})u + DF_2(\mathcal{U},\mathcal{U})u(t-\tau). \tag{4.27}$$

Define $A_U : \mathcal{D}(A_U) \subset C \to C$ as $A_U\phi = \dot{\phi}_t$, with corresponding domain $\mathcal{D}(A_U) = \big\{\phi \,\big|\, \dot{\phi}_t \in C,\; \phi(0) \in \mathcal{D}(A_T),\; \dot{\phi}_t^-(0) = A_T\phi(0) + DF_1(\mathcal{U},\mathcal{U})\phi(0)$ $+DF_2(\mathcal{U},\mathcal{U})\phi(-\tau)\big\}$. Here, and only for the definition of the operator A_U with its domain, we have used the notation $\phi_t(\theta) = \phi(t+\theta)$, $-\tau \leq \theta \leq 0$, as it is often used in the context of delay differential equations. This is not to be confused with the partial derivative with respect to t, as used in partial differential equations and in the rest of this thesis.

We define the corresponding *solution semigroup* $\{U(t)\}_{t\geq 0}$, $U(t) : C \to C$ of the linearized equation by

$$U(t)\phi := u^{\phi}(t+\theta), \qquad -\tau \leq \theta \leq 0, \tag{4.28}$$

where $u^{\phi}(t+\theta)$, $-\tau \leq \theta \leq 0$ denotes the solution of (4.27) with prehistory ϕ in C. $\{U(t)\}_{t\geq 0}$ is a strongly continuous semigroup of bounded linear operators in C. Wu proves that A_U is indeed the infinitesimal generator of $\{U(t)\}_{t\geq 0}$.

To determine stability, we now discuss the spectral properties of A_U. To this aim, we introduce the *characteristic equation*

$$\Delta(\lambda)g = 0, \tag{4.29}$$

where $\Delta(\lambda)$ is an $\mathbb{X}$-valued operator defined by

$$\Delta(\lambda)g = A_T g - \lambda g + DF_1(\mathcal{U},\mathcal{U})g + DF_2(\mathcal{U},\mathcal{U})e^{-\lambda\tau}g. \tag{4.30}$$

We call λ an *eigenvalue* if there exists a function $g \in \mathcal{D}(A_T)\backslash\{0\}$ (i.e., fulfilling the periodic boundary conditions) which solves the characteristic equation $\Delta(\lambda)g = 0$.

The following three theorems give us a good estimate on the spectrum of partial delay differential equations.

Theorem 4.8 (The spectrum is countable, [78]). *For each $t > \tau$, we have*

- *The spectrum $\sigma(U(t))$ is a countable set. It is compact and 0 is the only possible accumulation point. If $\mu \neq 0$ is an element of $\sigma(U(t))$, then μ also belongs to the point spectrum $P\sigma(U(t))$.*
- *$P\sigma(U(t)) = e^{tP\sigma(A_U)}$ plus possibly $\{0\}$.*
- *If $\lambda \in P\sigma(A_U)$, then the generalized eigenspace of λ is finite dimensional.*

In particular, only point spectrum $P\sigma(A_U)$ occurs. It possesses the following properties:

Theorem 4.9 (Upper bound on the spectrum, [78]). *There exists a real number β such that $\operatorname{Re}\lambda \leq \beta$ for all $\lambda \in P\sigma(A_U)$. Moreover, if γ is a given real number, then there exists only a finite number of eigenvalues $\lambda \in P\sigma(A_U)$ such that $\gamma \leq \operatorname{Re}\lambda$.*

The upper bound β translates directly to the linear stability in the following way:

Theorem 4.10 (Linear stability, [78]). *Let β be the smallest real number such that if λ is an eigenvalue, then $\operatorname{Re}\lambda \leq \beta$.*

- *If $\beta < 0$, then for all prehistories $\phi \in C$, $\|U(t)\phi\| \to 0$ as $t \to \infty$.*
- *If $\beta = 0$, then there exists a prehistory $\phi \in C\backslash\{0\}$ such that $\|U(t)\phi\| = \|\phi\|$ for all $t \geq 0$.*
- *If $\beta > 0$, then there exists a prehistory $\phi \in C$ such that $\|U(t)\phi\| \to \infty$ as $t \to \infty$.*

Suppose that the zero solution of the linearized system (4.27) is asymptotically stable. Then the next theorem gives us local (exponential) stability of the equilibrium $\mathcal{U}$ also in the nonlinear equation (4.26).

Theorem 4.11 (Principle of linearized stability, [78]). *There exist* $\varepsilon > 0$, $M \geq 1$ *and* $\alpha > 0$ *such that if* $\|\phi - \mathcal{U}\| < \varepsilon$, *then the solution* u^ϕ *of equation* (4.26) *in* C *with prehistory* ϕ *exists on* $[-\tau, \infty)$ *and*

$$\left\| u^\phi(t) - \mathcal{U} \right\| \leq M e^{-\alpha t} \|\phi - \mathcal{U}\|, \qquad t \geq 0. \tag{4.31}$$

With this background, we are now ready to plan the proof of stability in our controlled scalar reaction-diffusion equation.

4.5. Linear variational equations and steps of the proof

Having proven instability in the case without control and collected general information about partial delay differential equations, we can now proceed with the study of the stabilization regions for the controlled equation. In co-rotating coordinates, where all rotating waves are frozen, the controlled equation of rotation type takes the form

$$u_t = u_{zz} + f(u) + k\big(u - \Psi u(z - \varphi, t - \tau)\big), \tag{4.32}$$

while the equation of reflection type is of the form

$$u_t = u_{zz} + f(u) + k\big(u - \Psi u(-z, t - \tau)\big). \tag{4.33}$$

First, we linearize around the frozen wave $\mathcal{U}(z)$ and obtain the linear variational equations

$$v_t = v_{zz} + Q(z)v + k\big(v - \Psi v(z - \varphi, t - \tau)\big), \tag{4.34}$$

and

$$v_t = v_{zz} + Q(z)v + k\big(v - \Psi v(-z, t - \tau)\big), \tag{4.35}$$

with 2π-periodic coefficient

$$Q(z) = f'(\mathcal{U}(z)). \tag{4.36}$$

For details on the linearization, see Sections 4.1 and 4.4. As discussed in Section 4.4, we obtain the characteristic equations

$$\lambda g = g_{zz} + Q(z)g + k\big(g - \Psi e^{-\lambda\tau} g(z-\varphi)\big), \tag{4.37}$$

and

$$\lambda g = g_{zz} + Q(z)g + k\big(g - \Psi e^{-\lambda\tau} g(-z)\big). \tag{4.38}$$

Note that the stability is determined by an ordinary (delay) differential equation. Equation (4.37) is a delayed version of Hill's equation, which we have studied in Section 4.3.

It is a necessary and sufficient condition that all eigenvalues λ – which need yet to be determined – are located in the left half of the complex plane, i.e., that all eigenvalues have negative real part. Throughout the thesis we denote the complex eigenvalues by $\lambda = \mu + \mathrm{i}\nu$.

We are left with the task of determining the values λ for which 2π-periodic solutions of the characteristic equations exist.

4.5.1. Control schemes of rotation type

In order to learn as many details on the control mechanism as possible, we divide the proof into four parts. In Section 5.1 we discuss the simplest case where we make the following assumptions: The variational equation is autonomous, i.e., $Q(z) \equiv Q$. Furthermore, we assume that the spatio-temporal delay $\varphi = \xi - c\tau$ is zero. Hence, the object of Section 5.1 is an equation of the following form:

$$v_t = v_{zz} + Qu + k\big(v - \Psi v(z, t-\tau)\big). \tag{4.39}$$

The first condition ($Q(z) \equiv Q$) holds automatically if we consider either linear systems or homogeneous equilibria $\mathcal{U}(z) \equiv \mathcal{U}$ (in which case $Q(z) = f'(\mathcal{U}(z)) \equiv f'(\mathcal{U}) \equiv Q$). The second condition ($\varphi = 0$) corresponds to the

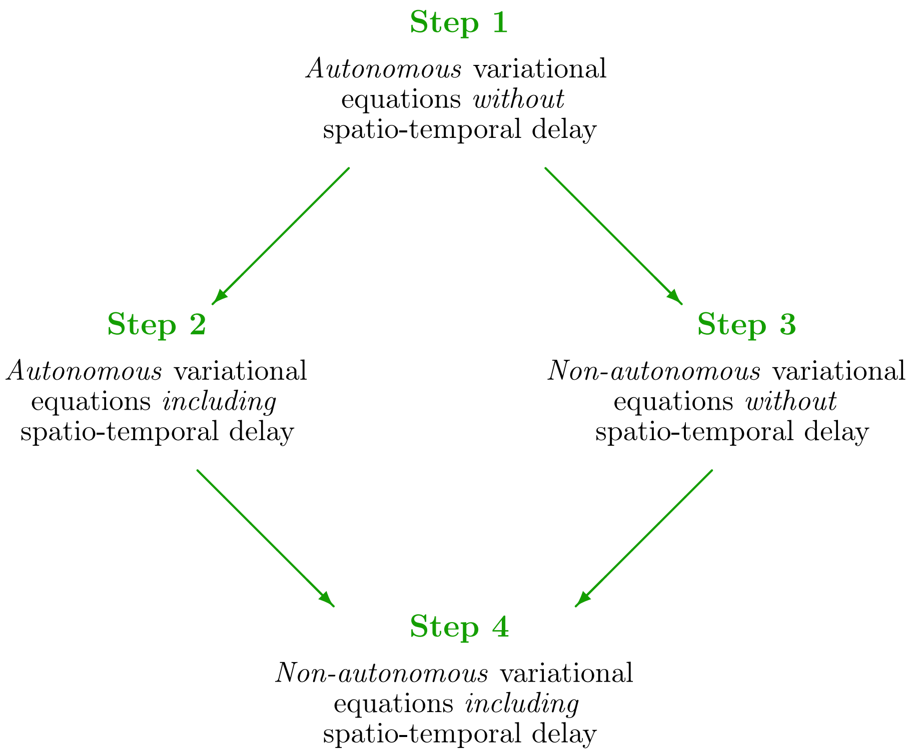


Figure 4.1.: Steps of the proof (control scheme of rotation type).

situation of Pyragas control as discussed in Chapter 2. We can therefore prove many special cases of the theorems in Chapters 2 and 3. The main advantage is that we can obtain explicit results, which we can then successively extend and compare with the (sometimes approximative) results in subsequent steps. Therefore, Step 1 forms the base for the complete proof (also for the control scheme of reflection type) and should be read carefully.

In Step 2 in Section 5.2 we will show how to dispense with the assumption on zero spatio-temporal delay in the autonomous case. We consider linear variational equations of the type

$$v_t = v_{zz} + Qv + k\big(v - \Psi v(z-\varphi, t-\tau)\big), \tag{4.40}$$

with $\varphi \neq 0$. Such equations occur for the control triple method in the case of linear equations or for homogeneous equilibria. We are therefore able to prove the success of the control triple method for the first time (but for linear equations, only) in Step 2. Many lemmata from Step 2 closely resemble those from Step 1. However, there are crucial differences, which we point out in detail, due to the spatio-temporal delay.

We change our viewpoint in Step 3 in Section 5.3: Here we consider non-autonomous linear variational equations, but without spatio-temporal delay φ, similar to Step 1. The variational equation now takes the form

$$v_t = v_{zz} + Q(z)v + k\big(v - \Psi v(z, t-\tau)\big). \tag{4.41}$$

The conditions are chosen such that we can complete the proof of the failure of Pyragas control as in Chapter 2. We strongly use the results from Step 1. It is in this step that Hill's equation features most prominently.

We complete the proofs for the control triple method in Step 4 in Section 5.4. We consider the most general form of the variational equation

$$v_t = v_{zz} + Q(z)v + k\big(v - \Psi v(z-\varphi, t-\tau)\big). \tag{4.42}$$

The results follow quite fast from the previous steps.

4.5.2. Control schemes of reflection type

For the control scheme of reflection type in Chapter 6, we re-use many of the intermediary results from Chapter 5. To be able to apply the results, we first discuss even and odd eigenfunctions. Afterwards, we treat the even and odd eigenfunctions separately. With this preliminary consideration, we prove the successful stabilization of twisted standing waves, the failure of control of standing waves, as well as the successful stabilization of the zero equilibrium. No new arguments are necessary.

Chapter 5

Proof for control schemes of rotation type

In this chapter the control triple (output signal, space, time) is of the following form:

(multiplication Ψ, rotation $\cong$ spatial delay ξ, time delay τ).

It combines a scalar multiplication $\Psi \in \mathbb{R}$ of the output signal, a rotation in space, which we interpret as a spatial delay $\xi \in S^1$, with a time delay $\tau \geq 0$.

The general form of the reaction-diffusion equation including control is

$$u_t = u_{xx} + f(u) - cu_x + k\big(u - \Psi u(x - \xi, t - \tau)\big), \tag{5.1}$$

with periodic boundary conditions and all the assumptions on f as stated in Section 1.3. Here $k \in \mathbb{R}$ is the variable feedback gain.

The results which we prove in this chapter include the failure of Pyragas control for periodic orbits (Corollary 2.2) and equilibria (Corollary 2.3). Furthermore, we also prove the more general results in Theorem 2.1 for the control of rotating waves and in Theorem 2.4 for the control of homogeneous equilibria. The destabilization of homogeneous equilibria is proven as in Theorem 2.5. Moreover, all the results for control schemes of rotation type in Chapter 3 are proven. More precisely, we prove the theorems on the success (Theorem 3.1) and the failure (Theorem 3.2) of control of rotating and frozen waves, as well as the successful stabilization of the zero equilibrium (Theorem 3.3).

Throughout this chapter we fix a scalar reaction-diffusion equation $u_t = u_{xx} + f(u) - cu_x$, an equilibrium $\mathcal{U}(x)$ or a rotating wave $\mathcal{U}(x-ct)$, as well as the control triple given by ξ, τ and Ψ. The only parameter which is varied is the real feedback gain k, and it is our task to determine under which conditions feedback gains k exist, such that the control triple is successful for a specific equilibrium or rotating wave.

For our convenience, we introduce the following notations: We call $\varphi := \xi - c\tau$ the *spatio-temporal delay.* During the proof, we will see that the value φ, combining the spatial and the temporal delay, is often more decisive for stabilization than the single values ξ or τ. We also introduce

$$\bar{Q} := \max_{z\in[0,2\pi]} Q(z) := \max_{z\in[0,2\pi]} f'\big(\mathcal{U}(z)\big), \tag{5.2}$$

as a measure of the instability of a rotating wave or equilibrium $\mathcal{U}(z)$. Note that $Q(z)$ is 2π-periodic. If $f'\big(\mathcal{U}(z)\big) \equiv \bar{Q}$ is a constant, we use the notation $\bar{Q} = Q$ instead.

Recall that in Section 4.5 we found the linear variational equation in co-rotating coordinates $z = x - ct$,

$$v_t = v_{zz} + Q(z)v + k\big(v - \Psi v(z-\varphi, t-\tau)\big). \tag{5.3}$$

During the proof in this chapter, we follow the four steps outlined in Section 4.5. Roughly, the four steps can be described as follows:

Step 1: $Q(z) \equiv Q$, $\varphi = 0$

Step 2: $Q(z) \equiv Q$, φ arbitrary

Step 3: $Q(z)$ arbitrary, $\varphi = 0$

Step 4: $Q(z)$ arbitrary, φ arbitrary

For more details, see Section 4.5.

The proofs in this chapter establish for the first time the existence or nonexistence of successful feedback gains k for a prescribed control term of rotation type.

5.1. Step 1: Autonomous variational equations without spatio-temporal delay

The general linear variational equation in co-rotating coordinates has the form

$$v_t = v_{zz} + Q(z)v + k\big(v - \Psi v(z-\varphi, t-\tau)\big). \tag{5.4}$$

In this section we concentrate on the special case $Q(z) \equiv Q$, $\varphi = 0$, i.e., the variational equation simplifies to

$$v_t = v_{zz} + Qv + k\big(v - \Psi v(z, t-\tau)\big). \tag{5.5}$$

Throughout, the stability of the zero equilibrium in equation (5.5) gives us the stability of the equilibrium or rotating wave.

This section is organized as follows: In Subsection 5.1.1 we state the theorems corresponding to equation (5.5). From these theorems we can already conclude some special cases of the theorems in Chapters 2 and 3. In Subsection 5.1.2 we describe the positions of the eigenvalues in the complex plane. Subsequently, we can derive conditions on the real eigenvalues in Subsection 5.1.3, and on the complex conjugated eigenvalues in Subsection 5.1.4, thereby proving the theorems in Subsection 5.1.1.

5.1.1. Theorems

The first theorem tells us about the failure of Pyragas-like controls (i.e., $\Psi = 1$, $\varphi = 0$) to stabilize the zero equilibrium in the linear variational equation:

Theorem 5.1 (Step 1: Failure of control of the zero equilibrium in the linear variational equation)**.** *Consider the linear variational equation*

$$v_t = v_{zz} + Qv + k\big(v - v(z, t-\tau)\big), \tag{5.6}$$

with periodic boundary conditions, $Q > 0$, real feedback gain k, and positive time delay $\tau \geq 0$.

Then the zero equilibrium is unstable for all feedback gains $k \in \mathbb{R}$.

Remark. Theorem 5.1 implies Theorems 2.2–2.3 for arbitrary waves but linear dynamics $f(u) = Qu$. Furthermore, it implies Theorem 2.4 for homogeneous equilibria and arbitrary dynamics $f(u)$.

Control triples where the transformation of the output signal is $\Psi = -1$, however, are successful:

Theorem 5.2 (Step 1: Successful stabilization of the zero equilibrium in the linear variational equation)**.** *Consider the linear variational equation*

$$v_t = v_{zz} + Qv + k\big(v - (-1)v(z, t - \tau)\big), \tag{5.7}$$

with periodic boundary conditions, $Q > 0$ and positive time delay $\tau \geq 0$.

If the feedback gain $k \in \mathbb{R}$ is chosen in such a way that

$$-1/\tau < k < -Q/2 \tag{5.8}$$

(if such an interval of feedback gains exists), then the zero equilibrium is stable.

More precisely, if $k < -Q/2$, then the zero equilibrium of equation (5.7) *is stable if the time delay τ is strictly smaller than*

$$\tau(k) = \inf_{N<\sqrt{Q},\, N\in\mathbb{N}} \frac{\arccos\left(\frac{k+Q-N^2}{-k}\right)}{\sqrt{k^2 - (Q + k - N^2)^2}}. \tag{5.9}$$

Remark. Theorem 5.1 is a predecessor of Theorem 3.1, but no special case yet, because we restrict ourselves to zero spatio-temporal delay. A first special case of Theorem 3.1 (the linear case) will be proven in Section 5.2.

Let us now state the first result for the stabilization of the zero equilibrium.

Theorem 5.3 (Step 1: Successful stabilization of the zero equilibrium)**.** *Consider the homogeneous zero equilibrium of the linear equation $u_t = u_{zz} + Qu$, with periodic boundary conditions.*

Choose some real number $\Psi \neq 1$ *and a time-delay* $\tau \geq 0$.

If the feedback gain $k \in \mathbb{R}$ *fulfills the condition*

$$k\left(1 - \Psi e^{-\mu\tau}\right) < \mu - Q \qquad \text{for all } \mu > 0, \tag{5.10}$$

as well as the condition

$$|k\Psi\tau| > 1, \tag{5.11}$$

then the homogeneous zero equilibrium is stable in the equation including control,

$$u_t = u_{zz} + Qu + k\big(u - \Psi u(z, t - \tau)\big). \tag{5.12}$$

In particular, if $\Psi = 0$, *then the zero equilibrium is stable for* $k < -Q$.

Remark. Theorem 5.3 implies Theorem 3.3 in the case $\xi = 0$.

We also prove the following theorem on the destabilization of homogeneous equilibria:

Theorem 5.4 (Step 1: Pyragas destabilization of homogeneous equilibria)**.** *Consider a* stable *homogeneous equilibrium* $\mathcal{U}$ *of the equation*

$$u_t = u_{zz} + Qu. \tag{5.13}$$

Choose some time delay $\tau > 0$, *then we can find* $k^* \in \mathbb{R}$ *such that for all feedback gains* $k > k^*$, $\mathcal{U}$ *is unstable in the equation including Pyragas control,*

$$u_t = u_{zz} + Qu + k\big(u - u(z, t - \tau)\big). \tag{5.14}$$

Remark. Theorem 5.4 implies Theorem 2.5 in the case $\xi = 0$.

We will prove Theorems 5.1–5.4 in Sections 5.1.2–5.1.4, starting with determining the positions of the eigenvalues, such that we can derive conditions of the real as well as on the complex conjugated eigenvalues next.

5.1.2. Positions of the eigenvalues

Let us now investigate the linear variational equation of the form

$$v_t = v_{zz} + Qv + k\big(v - \Psi v(z, t-\tau)\big), \tag{5.15}$$

in order to prove Theorems 5.1–5.4. We solve equation (5.15) via separation of variables and an exponential Ansatz in the time-variable t, $u(z,t) = g(z)e^{\lambda t}$, $\lambda \in \mathbb{C}$. Note that λ might be complex, but that all solutions are real. In this case, we have the solution $u(z,t) = g(z)(e^{\lambda t}+e^{\bar{\lambda} t})$ instead, where $\bar{\lambda}$ denotes the complex conjugate of λ. For simplicity, we calculate with λ complex.

The $\lambda \in \mathbb{C}$ are the eigenvalues which carry the stability information of the equilibrium or wave. Via the Ansatz $u(z,t) = g(z)e^{\lambda t}$, we obtain an ordinary differential equation of the form

$$\lambda g = g_{zz} + Qg + k\left(g - \Psi e^{-\lambda\tau} g\right), \tag{5.16}$$

i.e., a second-order equation of Hill's type with constant coefficient $-\lambda + Q + k\left(1 - \Psi e^{\lambda\tau}\right)$. To find the eigenvalues λ, we need to find those values λ such that equation (5.16) has 2π-periodic solutions (compare with Chapter 4). Since equation (5.16) is a pendulum equation (but with rather complicated and complex coefficients), we can find the periodic solutions and the corresponding eigenvalues implicitly:

Lemma 5.5 (Positions of the eigenvalues)**.** *The eigenvalues $\lambda = \mu + \mathrm{i}\nu$ such that there exist 2π-periodic solutions of the equation*

$$\lambda g = g_{zz} + Qg + k\left(g - \Psi e^{-\lambda\tau} g\right), \tag{5.17}$$

are either real, or they can, for each $N \in \mathbb{N}_0$, be determined as crossings of the two curves

$$\nu(\mu) = \pm\frac{1}{\tau}\arccos\left(\frac{-\mu + Q + k - N^2}{k\,\Psi\,e^{-\mu\tau}}\right) \pm \frac{2\pi n}{\tau},\ \ n \in \mathbb{N}_0, \tag{5.18}$$

$$\mu(\nu) = -\frac{1}{\tau}\log\left(\frac{\nu}{k\,\Psi\sin(\nu\tau)}\right), \tag{5.19}$$

in the complex plane.

In particular, for $\tau = 0$, all eigenvalues λ are real.

Remark. Note that the curve $\mu(\nu)$, which gives the real part μ of the eigenvalue λ depending on its imaginary part ν, does not depend on either Q or N. This property is useful for finding conditions on the complex conjugated eigenvalues.

Proof. The proof of Lemma 5.5 is straightforward. Rewrite equation (5.16) in a more convenient way to see the linear pendulum structure:

$$0 = g_{zz} + \left(-\lambda + Q + k - k\,\Psi e^{-\lambda\tau}\right) g. \tag{5.20}$$

Let us solve equation (5.20) by an exponential Ansatz $g(z) = \exp(\eta z)$. Here the new eigenvalue $\eta \in \mathbb{C}$ (not to be confused with the eigenvalue λ of the original system) is given by

$$\eta = \pm\sqrt{\lambda - Q - k + k\,\Psi e^{-\lambda\tau}}. \tag{5.21}$$

The periodic boundary conditions have to be fulfilled which is the case if and only if $\eta = \pm \mathrm{i}N$, with $N \in \mathbb{N}_0$. We are left with the task of determining λ from the transcendental equation

$$-N^2 = \lambda - Q - k + k\Psi e^{-\lambda\tau}. \tag{5.22}$$

To analyze the values of λ fulfilling this equation, let us split λ into its real and imaginary part $\lambda = \mu + \mathrm{i}\nu$:

$$\mu = Q + k - k\Psi e^{-\mu\tau}\cos(\nu\tau) - N^2 \tag{5.23}$$
$$\nu = k\Psi e^{-\mu\tau}\sin(\nu\tau). \tag{5.24}$$

Note that the imaginary part ν appears only once in the first equation, while the real part μ appears only once in the second equation. Rearranging, we obtain the two explicit formulas in Lemma 5.5. An eigenvalue λ must fulfill both equations and is therefore characterized by the crossings of the two curves $\nu(\mu)$ and $\mu(\nu)$. In the case $\tau = 0$, the equations simplify to

$$\mu = Q + k - k\Psi - N^2 \tag{5.25}$$
$$\nu = 0. \tag{5.26}$$

Hence, all eigenvalues are real if the time delay τ is zero. This completes the proof of Lemma 5.5. □

For better visualization, we have drawn example curves from Lemma 5.5 in Figures 5.1–5.4. In Figure 5.1 we see an example of the failure of Pyragas control, while Figure 5.2 gives us a hint that those control triples which use $\Psi = -1$ as transformation of the output signal might be successful. Figure 5.3 is again for $\Psi = -1$, but here at the limit of successful control, i.e., we choose the feedback gain k in such a way that the eigenvalue with the largest real part just crosses the imaginary axis, in this case at zero. In Figure 5.4 we illustrate what happens if the feedback parameter k has the wrong sign.

5.1.3. Conditions on the real eigenvalues

Having found the positions of the eigenvalues λ in the complex plane, we now establish conditions on the real eigenvalues to be negative.

Lemma 5.6 (Zero crossings of the real eigenvalues)**.** *For the real eigenvalues μ such that there exist 2π-periodic solutions of the equation*

$$\mu g = g_{zz} + Qg + k\left(g - \Psi e^{-\mu\tau} g\right), \tag{5.27}$$

the following properties hold:

If the inequality

$$-\mu + Q + k - k\Psi e^{-\mu\tau} < 0 \tag{5.28}$$

holds for all $\mu > 0$, then there does not exist any 2π-periodic solution for real positive values $\mu > 0$ of equation (5.27)*.*

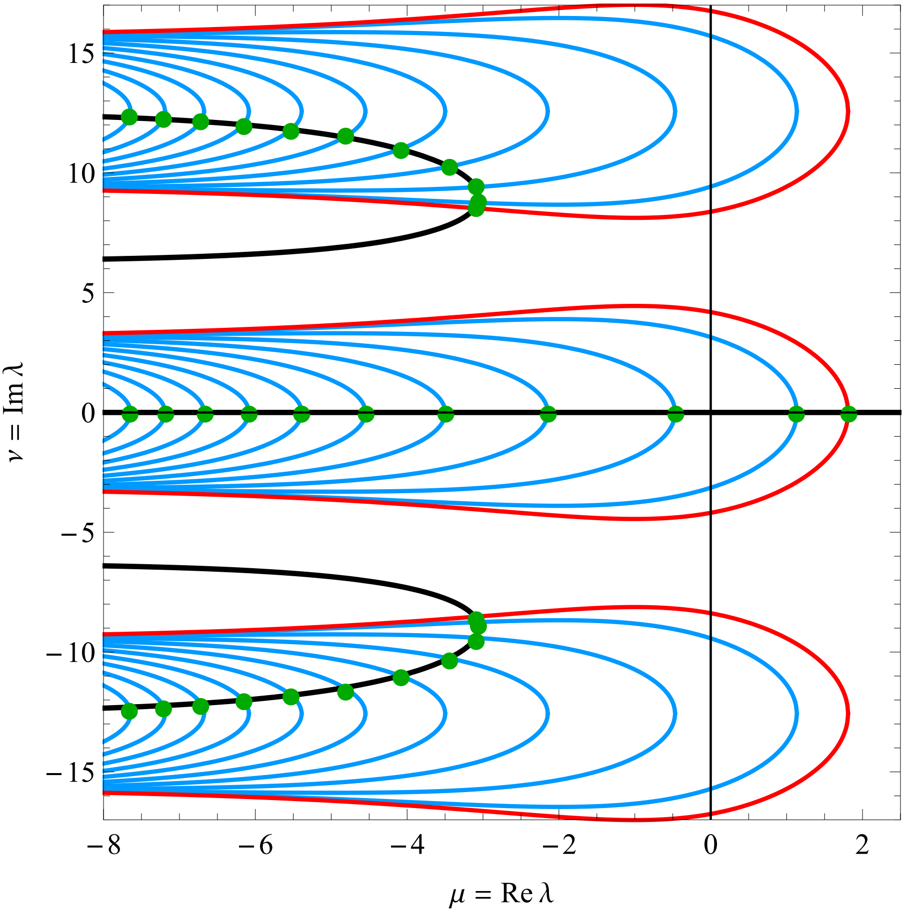


Figure 5.1.: Failure of Pyragas control: Positions of the eigenvalues (green dots) for a fixed feedback gain $k = -2$. Here $Q = 3$. The control triple is defined by $\Psi = 1$, $\varphi = \xi - c\tau = 0$, and $\tau = 0.5$. The curve $\mu(\nu)$ is drawn in black, while $\nu(\mu)$ is drawn in red for $N = 0$ and in blue for all $N \geq 1$. Note that eigenvalues can also occur on the real axis.

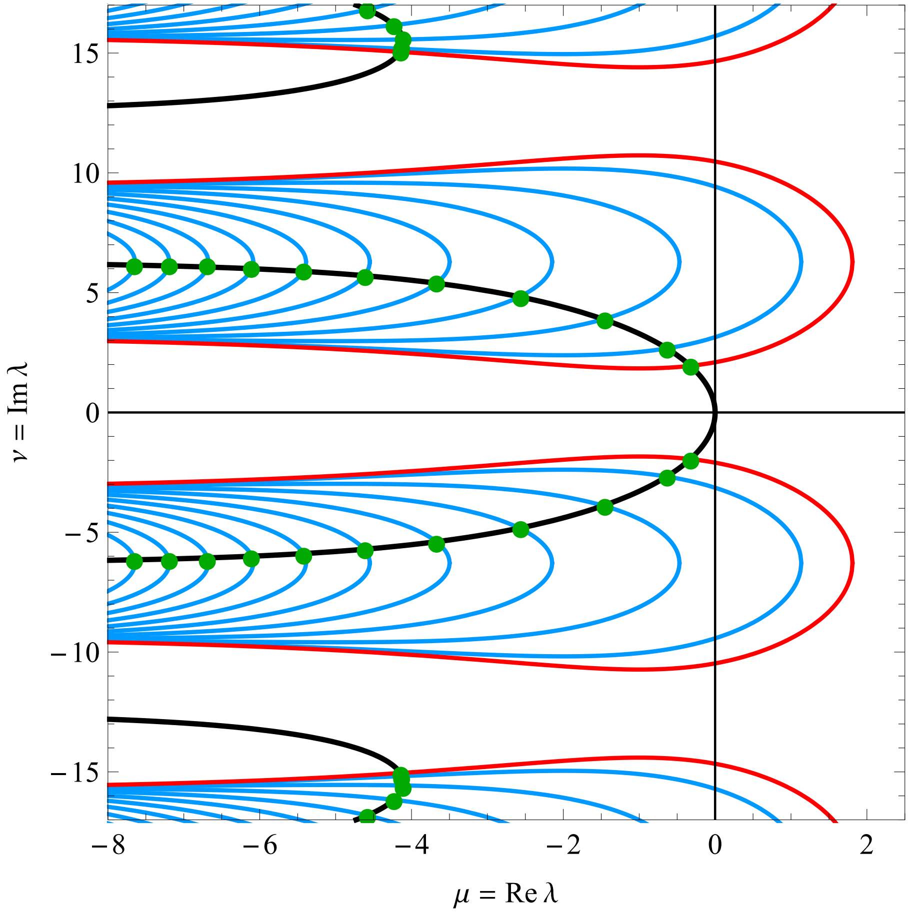


Figure 5.2.: Successful control with $\Psi = -1$: Positions of the eigenvalues (green dots) for a fixed feedback gain $k = -2$. Here $Q = 3$. The control triple is defined by $\Psi = -1$, $\varphi = \xi - c\tau = 0$, and $\tau = 0.5$. The curve $\mu(\nu)$ is drawn in black, while $\nu(\mu)$ is drawn in red for $N = 0$ and in blue for all $N \geq 1$. There are no real eigenvalues.

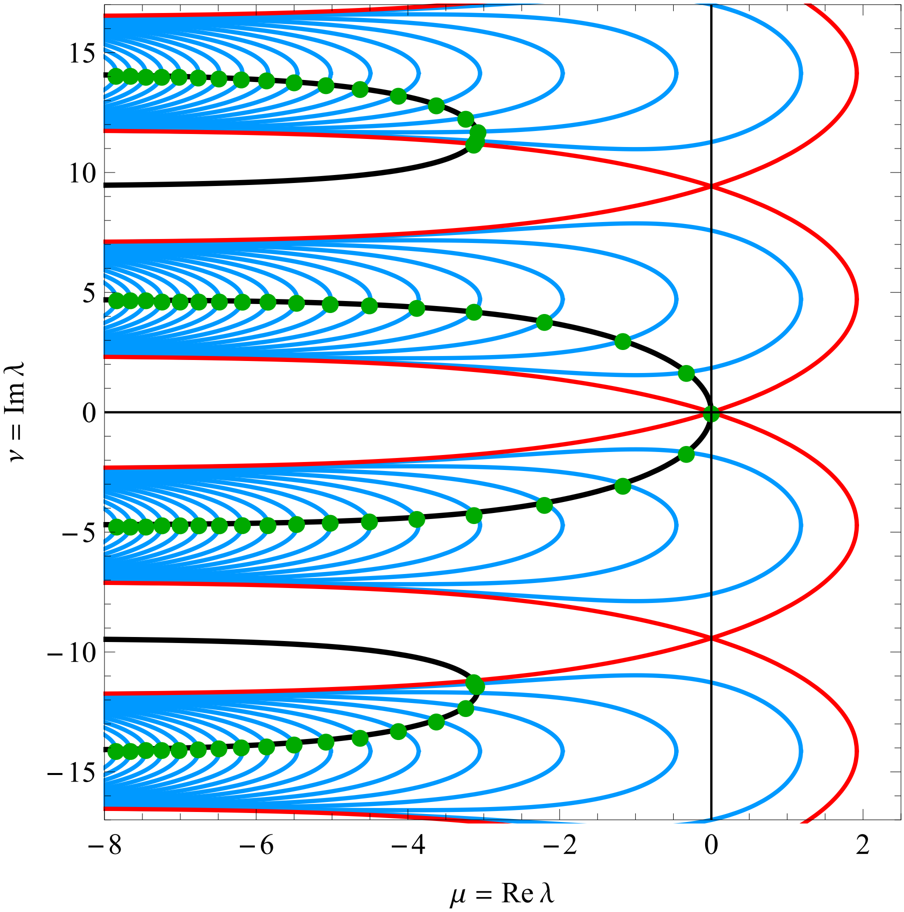


Figure 5.3.: Limit of successful control: Positions of the eigenvalues (green dots) for a fixed feedback gain $k = -1.5$. Here $Q = 3$. The control triple is defined by $\Psi = -1$, $\varphi = \xi - c\tau = 0$, and $\tau = 2/3$. The curve $\mu(\nu)$ is drawn in black, while $\nu(\mu)$ is drawn in red for $N = 0$ and in blue for all $N \geq 1$.

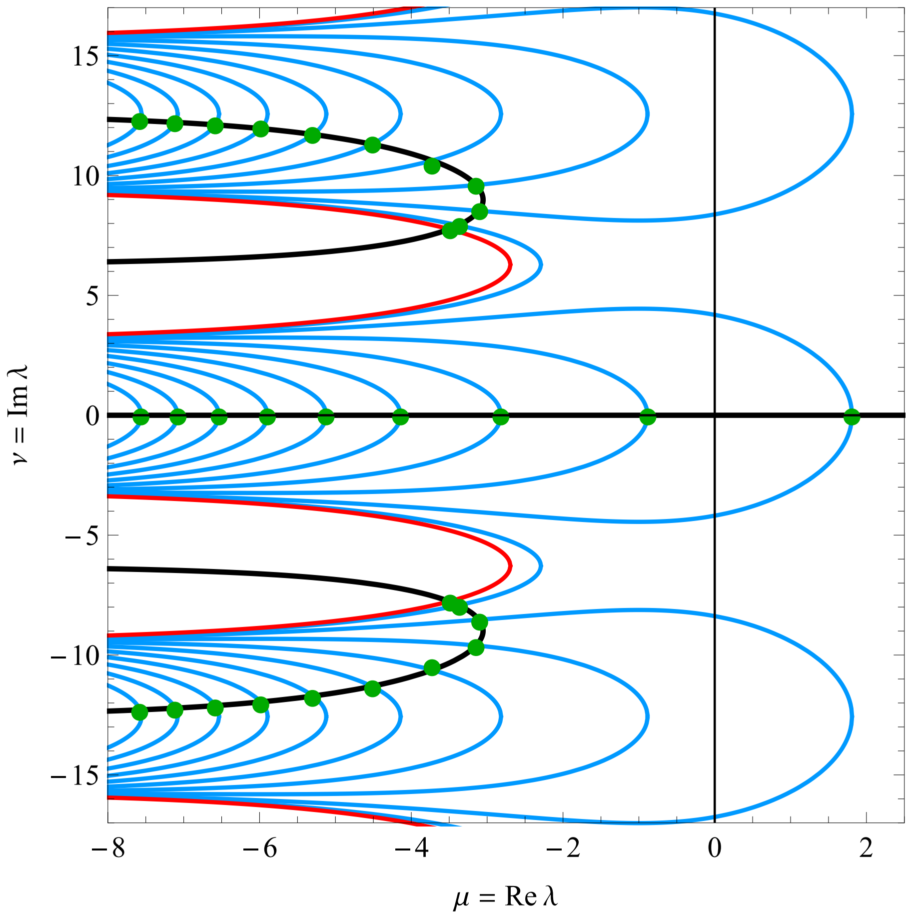


Figure 5.4.: Wrong sign of the feedback gain: Positions of the eigenvalues (green dots) for a fixed feedback gain $k = +2$. Here $Q = 3$. The control triple is defined by $\Psi = -1$, $\varphi = \xi - c\tau = 0$, and $\tau = 2/3$. The curve $\mu(\nu)$ is drawn in black, while $\nu(\mu)$ is drawn in red for $N = 0$ and in blue for all $N \geq 1$.

If $\Psi \neq 1$, then there exists a sequence of feedback gains $\{k_N\}_{N\in\mathbb{N}}$,

$$k_N = \frac{Q - N^2}{\Psi - 1}, \tag{5.29}$$

for which the eigenvalue μ crosses zero if the feedback gain $k \in \mathbb{R}$ is increased: The crossing of the real eigenvalues μ is from negative to positive if the inequality

$$1 - \Psi\left(1 + \tau(N^2 - Q)\right) > 0 \tag{5.30}$$

is fulfilled. If $1 - \Psi\left(1 + \tau(N^2 - Q)\right) < 0$, the crossing is from positive to negative. No other eigenvalues $\mu = 0$ occur.

In the case $\Psi = 1$, an eigenvalue $\mu = 0$ occurs for all feedback gains $k \in \mathbb{R}$ if such an eigenvalue exists in the equation without control, i.e., for $k = 0$. No eigenvalues cross zero except for $Q = N^2$, where the zero-crossing occurs at

$$k = 1/\tau. \tag{5.31}$$

From these rather technical conditions, we derive the following corollary, where we find simple conditions for the cases $\Psi = 1$ (e.g., Pyragas control) and $\Psi = -1$ (control triple method with a sign change of the output signal).

Corollary 5.7 (Conditions on the real eigenvalues)**.** *For the real eigenvalues μ such that there exist 2π-periodic solutions of the equation*

$$\mu g = g_{zz} + Qg + k\left(g - \Psi e^{-\mu\tau} g\right), \tag{5.32}$$

$Q > 0$, the following properties hold:

If $\Psi = +1$, then there exist positive eigenvalues μ for all time delays $\tau \geq 0$ and for all feedback gains $k \in \mathbb{R}$.

If $\Psi = -1$, the feedback gain k fulfills the condition $k < -Q/2$, and the time delay is bounded by $\tau < 2/Q$, then all real eigenvalues μ are negative.

Remark. Corollary 5.7 proves Theorem 5.1.

Proof. We begin by proving Corollary 5.7 from Lemma 5.6. In the case $\Psi = +1$, there exists at least one strictly positive eigenvalue $\mu = Q$ without control, i.e., if the feedback gain k is zero. Since no eigenvalue crossing occurs for $N = 0$, the corresponding eigenvalue stays positive.

In the case $\Psi = -1$, there exists a sequence of feedback gains $\{k_N\}_{N \in \mathbb{N}}$, $k_N = -\left(Q - N^2\right)/2$, for which the eigenvalue μ crosses zero if the feedback gain $k \in \mathbb{R}$ is increased. This sequence is monotonically increasing in N, with the smallest value being $k_0 = -Q/2$. It remains to check that the crossings at the k_N for $N^2 \leq Q$ are in the correct direction, i.e., from positive to negative, for $\tau < 2/Q$. The following inequality needs to be fulfilled:

$$1 - \Psi\left(1 + \tau(N^2 - Q)\right) < 0. \tag{5.33}$$

With $\Psi = -1$, we obtain

$$2 + \tau(N^2 - Q) < 0, \tag{5.34}$$

which we can simplify to

$$\tau < 2/(Q - N^2). \tag{5.35}$$

For $N^2 \leq Q$, we therefore obtain $\tau < 2/Q$, which was our assumption. □

Let us next prove Lemma 5.6.

Proof. Consider the value $D \in \mathbb{R}$ defined as

$$D := -\mu + Q + k - k\Psi e^{-\mu\tau}, \tag{5.36}$$

such that equation (5.45) takes the form

$$g_{zz} + Dg = 0. \tag{5.37}$$

This is a simple pendulum equation, and 2π-periodic solutions must fulfill $D = N^2$. Hence, no 2π-periodic solutions and therefore no eigenvalues for $\mu > 0$ can occur if $D < 0$. This proves the first claim of Lemma 5.6.

Starting with the condition $D = -\mu + Q + k - k\Psi e^{-\mu\tau} = N^2$, we obtain for the feedback gain k as a function of the real eigenvalue μ:

$$k(\mu) = \frac{\mu - Q + N^2}{1 - \Psi e^{-\mu\tau}}. \tag{5.38}$$

Hence, we obtain the feedback gains k_N for which an eigenvalue $\mu = 0$ occurs in the case $\Psi \neq 1$:

$$k_N = \frac{Q - N^2}{\Psi - 1}. \tag{5.39}$$

In the case $\Psi = 1$, k_N is infinite, except for $Q = N^2$. Using L'Hôpital's rule, we obtain:

$$\lim_{\mu\to 0} k(\mu) = \lim_{\mu\to 0} \frac{\mu - Q + N^2}{1 - e^{-\mu\tau}} = \lim_{\mu\to 0} \frac{1}{\tau e^{-\mu\tau}} = \frac{1}{\tau}. \tag{5.40}$$

Let us now calculate in which direction the real eigenvalue μ crosses zero. Differentiating equation (5.38) with respect to μ at $\mu = 0$ (for $\Psi \neq 1$) yields

$$k'(\mu) = \frac{1 - \Psi e^{-\mu\tau}\left(1 + \tau(\mu + N^2 - Q)\right)}{(1 - \Psi e^{-\mu\tau})^2}. \tag{5.41}$$

At $\mu = 0$ we find the slope of the feedback gain as

$$k'(0) = \frac{1 - \Psi\left(1 + \tau(N^2 - Q)\right)}{(1 - \Psi)^2}. \tag{5.42}$$

This proves the second claim of Lemma 5.6.

Last, let us consider the case $\Psi = 1$. It is obvious that the above calculations do not apply in this case, since we cannot divide through zero. Once again, we start with the condition

$$D = -\mu + Q + k - k\Psi e^{-\mu\tau} = N^2. \tag{5.43}$$

Plugging in $\mu = 0$, we obtain the condition

$$D = Q + k - k = Q = N^2, \tag{5.44}$$

i.e., an eigenvalue $\mu = 0$ exists if it exists in the equation without control ($k = 0$). In this case, it exists for all feedback gains $k \in \mathbb{R}$. This concludes the proof of the lemma. □

The following lemma on the existence of positive eigenvalues proves the Pyragas-destabilization of stable homogeneous equilibria.

Lemma 5.8 (Existence of positive real eigenvalues)**.** *For the real eigenvalues μ such that there exist 2π-periodic solutions of the equation*

$$\mu g = g_{zz} + Qg + k\left(g - e^{-\mu\tau} g\right), \tag{5.45}$$

$Q < 0$*, the following properties hold:*

There exist real eigenvalues μ which are implicitly determined by the feedback gain k,

$$k(\mu) = \frac{\mu - Q}{1 - e^{-\mu\tau}}. \tag{5.46}$$

$k(\mu)$ is a continuous function with the following asymptotic behavior:

$$\lim_{\mu \searrow 0} k(\mu) = +\infty \quad \text{exponentially},$$
$$\lim_{\mu \to +\infty} k(\mu) = +\infty \quad \text{linearly}.$$

Since $k(1) < \infty$, it follows that there exists a pair (k^*, μ^*) such that $k^* < \infty$, k^* minimal, $\mu^* > 0$, and $k^* = k(\mu^*)$.

Furthermore, for all $k > k^*$, there exists at least one positive eigenvalue μ determined via the relation (5.46).

Remark. Lemma 5.8 proves Theorem 5.4.

Proof. This lemma can easily be proven by explicit calculation: As above, the eigenvalues μ have to fulfill the condition

$$D = -\mu + Q + k - ke^{-\mu\tau} = N^2. \tag{5.47}$$

Let us concentrate on the case $N = 0$ (existence is enough for our purposes here). We obtain

$$k(\mu) = \frac{\mu - Q}{1 - e^{-\mu\tau}}, \tag{5.48}$$

as claimed above. The limits $\lim_{\mu \searrow 0} k(\mu) = +\infty$ and $\lim_{\mu \to +\infty} k(\mu) = +\infty$, as well as the other conclusions are obvious from the formula (5.46). □

5.1.4. Conditions on the complex conjugated eigenvalues

In addition to the conditions on the real eigenvalues, we also establish conditions on the complex conjugated eigenvalues. As we have seen in Chapter 4, no complex conjugated eigenvalues exist in the reaction-diffusion equation without control. In the controlled case, however, we obtain complex conjugated eigenvalues if the time delay is nonzero, see also Lemma 5.5 for the positions of the eigenvalues.

Lemma 5.9 (Conditions on the complex conjugated eigenvalues)**.** *If the time delay $\tau = 0$ is zero, then all eigenvalues are real.*

Now fix a time delay $\tau > 0$. If the feedback gain k fulfills the inequality

$$|k| < 1/(|\Psi|\tau), \tag{5.49}$$

then all complex conjugated eigenvalues have negative real part.

Conversely, now fix a feedback gain $k \in \mathbb{R}\backslash\{0\}$. If the time delay τ fulfills $\tau < \tau^*(k)$, and if $\tau^*(k) > 0$,

$$\tau^*(k) = \min_{N^2<Q,\, N\in\mathbb{N}} \frac{\arccos\left(\frac{k+Q-N^2}{k\Psi}\right)}{\sqrt{k^2\Psi^2 - (Q+k-N^2)^2}}, \qquad \text{for } \Psi k > 0, \tag{5.50}$$

$$\tau^*(k) = \min_{N^2<Q,\, N\in\mathbb{N}} \frac{\arccos\left(\frac{k+Q-N^2}{|k\Psi|}\right)+\pi}{\sqrt{k^2\Psi^2 - (Q+k-N^2)^2}}, \qquad \text{for } \Psi k < 0, \tag{5.51}$$

then all pairs of complex conjugated eigenvalues have negative real part. This threshold on the time delay τ is sharp.

Now consider the case $|\Psi| = 1$.
If $N^2 < Q$, no complex conjugated eigenvalues cross the imaginary axis for $k > (N^2 - Q)/2$.
If $N^2 > Q$, no complex conjugated eigenvalues cross the imaginary axis for $k < (N^2 - Q)/2$.

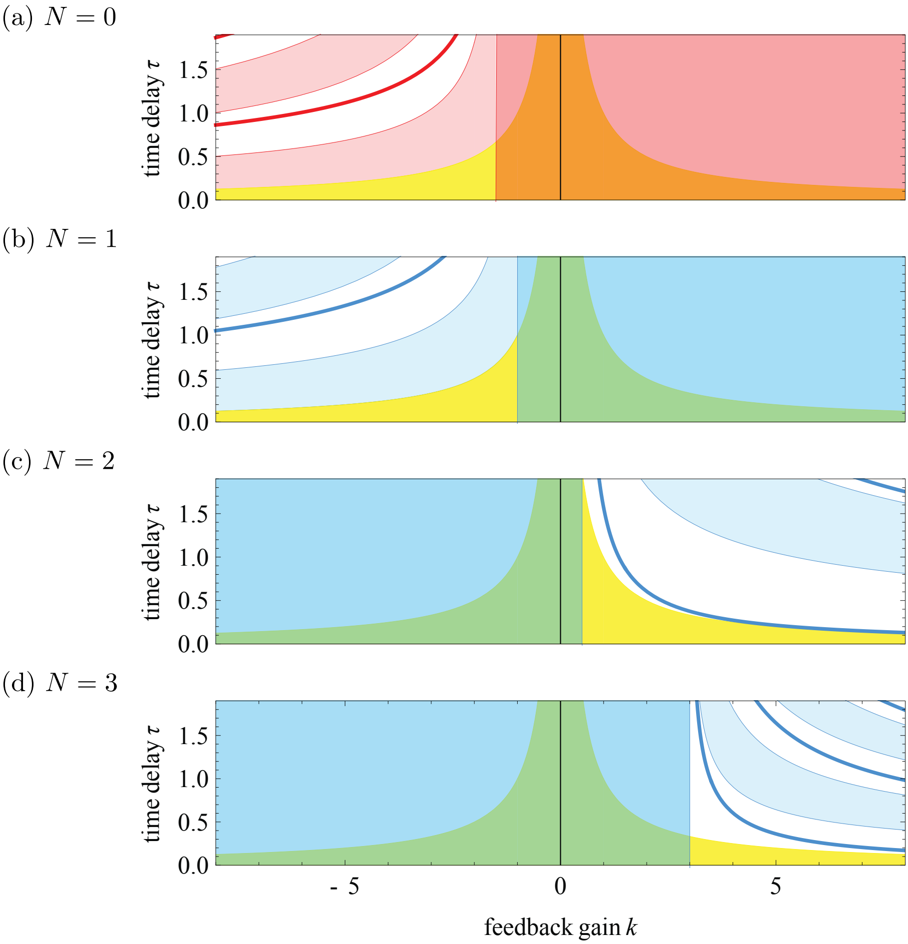


Figure 5.5.: Illustration of Lemma 5.9 for parameters $Q = 3$ and **multiplication** $\Psi = 1$ for $N = 0, 1, 2, 3$: If the time delay τ is zero, then all eigenvalues are real. In the yellow region in the background, all complex conjugated eigenvalues have negative real part, since $|k| < 1/\tau$. Note that this region is independent of N. The solid thick curves give the sharp upper bound $\tau^*(k)$ on the time delay, depending on the parameter N. The curves are red for $N = 0$ and blue for $N = 1, 2, 3$. The regions of nonexistence of the eigenvalues are shaded in light red and light blue. No eigenvalue crossings occur for $k > (N^2 - Q)/2$, ($N = 0, 1$, i.e., for $N^2 < Q$), and $k < (N^2 - Q)/2$, ($N \geq 2$, i.e., for $N^2 > Q$), respectively (darker red and darker blue shaded regions).

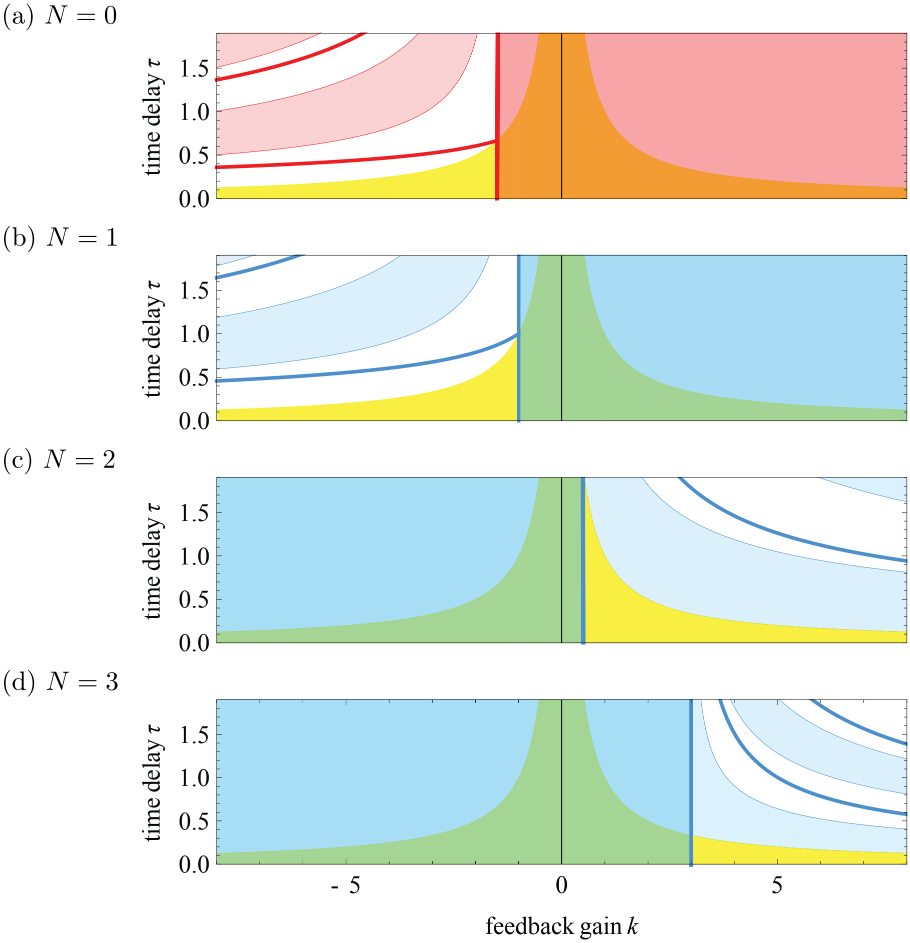


Figure 5.6.: Illustration of Lemma 5.9 for parameters $Q = 3$ and **multiplication** $\Psi = -1$ for $N = 0, 1, 2, 3$: If the time delay τ is zero, then all eigenvalues are real. In the yellow region in the background, all complex conjugated eigenvalues have negative real part, since $|k| < 1/\tau$. Note that this region is independent of N. The solid thick curves give the sharp upper bound $\tau^*(k)$ on the time delay, depending on the parameter N. The curves are red for $N = 0$ and blue for $N = 1, 2, 3$. The regions of nonexistence of the eigenvalues are shaded in light red and light blue. No eigenvalue crossings occur for $k > (N^2 - Q)/2$, ($N = 0, 1$, i.e., for $N^2 < Q$), and $k < (N^2 - Q)/2$, ($N \geq 2$, i.e., for $N^2 > Q$), respectively (darker red and darker blue shaded regions).

Remark. (a) We will only use the threshold (5.49) in the remaining parts of the thesis. Though this threshold is not sharp, it has several advantages over the sharp threshold (5.50): First of all, it is really easy to check. Second, this threshold is independent of Q and N, therefore it also persists in the non-autonomous case.
(b) Together with Corollary 5.7 or Lemma 5.6, Lemma 5.9 implies Theorem 5.2.

Proof. In order to prove the first estimate, we use Lemma 5.5 on the positions of the eigenvalues. There we already concluded that no complex conjugated eigenvalues occur if the time delay τ is zero. We now use the formula

$$\mu(\nu) = -\frac{1}{\tau}\log\left(\frac{\nu}{k\Psi\sin(\nu\tau)}\right) \tag{5.52}$$

from Lemma 5.5. Hence, all eigenvalues have negative real part μ if

$$\left|\frac{\nu}{k\Psi\sin(\nu\tau)}\right| > 1, \tag{5.53}$$

in the case of existence (note that eigenvalues do not necessarily need to exist due to the logarithm, but the conclusion remains the same). A global minimum of $|\nu/(k\Psi\sin(\nu\tau))|$ is obtained for $\nu = 0$. Hence, we find the condition

$$\left|\frac{1}{k\Psi\tau}\right| > 1. \tag{5.54}$$

This proves the second claim of the lemma and gives a non-sharp threshold on either the feedback gain k or the time delay τ.

Moreover, we can find the sharp threshold on the time delay τ in the following way: We search for purely imaginary eigenvalues with $\mu = 0$ and ν arbitrary. Starting with equation (5.22) from the proof of Lemma 5.5, we find the following simplified characteristic equations:

$$N^2 = Q + k\left(1 - \Psi\cos(\nu\tau)\right), \tag{5.55}$$

$$\nu = k\Psi\sin(\nu\tau). \tag{5.56}$$

Rearranging and quadrating the equations yields:

$$\left(Q - N^2 + k\right)^2 = k^2\Psi^2 \cos^2(\nu\tau), \tag{5.57}$$

$$\nu^2 = k^2\Psi^2 \sin^2(\nu\tau). \tag{5.58}$$

We add those two equations and, using the trigonometric equality $\cos^2\theta + \sin^2\theta = 1$ ($\forall\theta \in [0, 2\pi]$), we obtain

$$\left(Q - N^2 + k\right)^2 + \nu^2 = k^2\Psi^2. \tag{5.59}$$

This is a quadratic equation in ν which can be solved explicitly,

$$\nu = \pm\sqrt{k^2\Psi^2 - \left(Q - N^2 + k\right)^2}. \tag{5.60}$$

Above, we have seen that pairs of complex conjugated eigenvalues are determined by

$$\mu(\nu) = -\frac{1}{\tau}\log\left(\frac{\nu}{k\Psi\sin(\nu\tau)}\right). \tag{5.61}$$

Hence, they only exist in those regions where $k\Psi\nu\sin(\nu\tau)$ is positive. In the case $k\Psi > 0$, eigenvalues may exist for $2m\pi \leq |\nu|(2m+1)\pi$, $m \in \mathbb{N}_0$, while in the case $k\Psi < 0$, eigenvalues may exist for $(2m+1)\pi \leq |\nu|(2m+2)\pi$, $m \in \mathbb{N}_0$, as well as on the real line. The regions of nonexistence of the eigenvalues are shaded in light colors in Figures 5.5 and 5.6. We can rearrange the first of the characteristic equations and use the expression for ν to obtain

$$\tau^*(k) = \frac{\arccos\left(\frac{k+Q-N^2}{k\Psi}\right) + 2\pi n}{\sqrt{k^2\Psi^2 - \left(k + Q - N^2\right)^2}}, \qquad \text{for } \Psi k > 0, \tag{5.62}$$

$$\tau^*(k) = \frac{\arccos\left(\frac{k+Q-N^2}{|k\Psi|}\right) + \pi + 2\pi n}{\sqrt{k^2\Psi^2 - \left(k + Q - N^2\right)^2}}, \qquad \text{for } \Psi k > 0. \tag{5.63}$$

Obviously, the minimum value is obtained for $n = 0$. The two different formulas originate from the existence regions in ν. Since we have determined the time delay τ for *all* purely imaginary and non-real eigenvalues, the threshold is sharp. This completes the proof of the lemma. □

In this way, we have proven Theorems 5.1–5.4. We will permanently use those results in the following sections, where we first add the spatio-temporal delay and then later also consider non-autonomous variational equations.

5.2. Step 2: Autonomous variational equations including spatio-temporal delay

Let us go back to the general linear variational equation in co-rotating coordinates:

$$v_t = v_{zz} + Q(z)v + k\big(v - \Psi v(z-\varphi, t-\tau)\big). \tag{5.64}$$

In this section we include the spatio-temporal delay $\varphi \neq 0$. We concentrate again on the autonomous case $Q(z) \equiv Q$. The variational equation simplifies to

$$v_t = v_{zz} + Qv + k\big(v - \Psi v(z-\varphi, t-\tau)\big). \tag{5.65}$$

Again, the stability of the equilibrium or wave is determined by the stability of the zero equilibrium in equation (5.65).

This section is organized parallel to Step 1: The theorems corresponding to equation (5.65) are stated in Subsection 5.2.1. We next investigate the positions of the eigenvalues in Subsection 5.2.2. Conditions on the real eigenvalues are stated and proved in Subsection 5.2.3, while the conditions on the complex conjugated eigenvalues can be found in Subsection 5.2.4.

5.2.1. Theorems

The first theorem of Step 2 tells us about the success of the control triple method:

Theorem 5.10 (Step 2: Successful stabilization of the zero equilibrium in the linear variational equation)**.** *Consider the linear variational equation*

$$v_t = v_{zz} + Qv + k\big(v - (-1)v(z-\varphi, t-\tau)\big), \tag{5.66}$$

with $Q > 0$*, periodic boundary conditions and the assumptions from Section 1.3. Suppose that the unstable dimension is exactly* $2n-1$*,* $n \geq 1$*. The spatio-temporal delay is fixed to* $\varphi = \xi - c\tau = m\pi/n$*, where* m *is odd and co-prime to* n*. The time delay is given by* $\tau \geq 0$*.*

Then there exists a feedback gain $k^* \in \mathbb{R}$*,*

$$k^* = \min\left\{ \frac{N^2 - Q}{1+\cos(\varphi N)} \,\middle|\, N \in \mathbb{N},\ 0 \leq N < \sqrt{Q} \right\}, \tag{5.67}$$

such that the following holds:

For all feedback gains $k < k^*$*, there exists a time delay* $\tau^* = \tau^*(k)$ *such that the zero equilibrium is stable in equation* (5.66) *for all time delays* $\tau < \tau^*$*.*

Remark. Theorem 5.10 implies Theorem 3.1, i.e., the successful stabilization of rotating and frozen waves using the control triple method, in the case of linear dynamics $f(u) = Qu$.

However, the control fails again if no transformation of the output signal is present:

Theorem 5.11 (Step 2: Failure of control of the zero equilibrium in the linear variational equation)**.** *Consider the linear variational equation*

$$v_t = v_{zz} + Qv + k\big(v - v(z-\varphi, t-\tau)\big), \tag{5.68}$$

with $Q > 0$*, periodic boundary conditions and the assumptions from Section 1.3.*

Then the zero equilibrium of equation (5.68) *is unstable for any time delay* τ*, any spatio-temporal delay* φ *and any feedback gain* k*.*

Remark. Theorem 5.11 implies Theorem 3.2 in the case of linear dynamics $f(u) = Qu$. Furthermore, Theorem 5.11 implies Theorem 2.4 on Pyragas control of homogeneous equilibria.

We next state yet another result on the stabilization of the zero equilibrium:

Theorem 5.12 (Step 2: Successful stabilization of the zero equilibrium)**.** *Consider the linear variational equation*

$$v_t = v_{zz} + Qv + k\big(v - \Psi v(z - \varphi, t - \tau)\big), \tag{5.69}$$

with periodic boundary conditions and the assumptions from Section 1.3.

Choose some real number $\Psi \neq 1$ and a time delay $\tau \geq 0$. If the feedback gain $k \in \mathbb{R}$ fulfills the condition

$$k\left(1 - \Psi e^{-\mu\tau}\right) < \mu - Q \qquad \text{for all } \mu > 0, \tag{5.70}$$

as well as the condition

$$|k\Psi| < \tau, \tag{5.71}$$

and if the spatio-temporal delay $\varphi \geq 0$ is small enough, then the homogeneous zero equilibrium of equation (5.69) *is stable. In particular, if $\Psi = 0$, then the zero equilibrium is stable for $k < -Q$.*

Remark. Theorem 5.12 implies Theorem 3.3, in fact, the two theorems are almost identical.

We will prove Theorems 5.10–5.12 in the following three subsections.

5.2.2. Positions of the eigenvalues

In this section we consider linear variational equations of the form

$$v_t = v_{zz} + Qv + k\big(v - \Psi v(z - \varphi, t - \tau)\big), \tag{5.72}$$

in order to prove Theorems 5.10–5.12. Similarly to Step 1, we solve equation (5.72) via separation of variables and an exponential Ansatz in the time-variable t, $u(z,t) = g(z)e^{\lambda t}$, $\lambda \in \mathbb{C}$. As before, the $\lambda \in \mathbb{C}$ are the eigenvalues of the equilibrium or wave.

Carrying out the Ansatz $u(z,t) = g(z)e^{\lambda t}$, we obtain an ordinary *delay* differential equation of the form

$$\lambda g = g_{zz} + Qg + k\big(g - \Psi g(z-\varphi)e^{\lambda\tau}\big); \tag{5.73}$$

a second-order equation of Hill's type. In contrast to Step 1, the equation contains the spatio-temporal delay φ as a delay. This significantly complicates the stability analysis in comparison to Step 1.

Recall that we need to find those parameters λ such that equation (5.73) has 2π-periodic solutions.

Let us first consider the case $\tau = 0$, i.e., no time delay. In this case, (5.73) simplifies to

$$\lambda g = g_{zz} + Qg + k\big(g - \Psi g(z-\varphi)\big), \tag{5.74}$$

i.e., the parameter λ does not occur exponentially. Since equation (5.74) is linear, we solve it via the exponential Ansatz $g(z) = \exp(\eta z)$, $\eta \in \mathbb{C}$. The periodic boundary conditions need to be fulfilled, hence $\eta = \pm \mathrm{i}N$, $N \in \mathbb{N}_0$. As a result, we obtain characteristic equations of the form

$$\lambda = -N^2 + Q + k - k\Psi e^{\pm \mathrm{i}\varphi N}. \tag{5.75}$$

Splitting this equation into real and imaginary part, we obtain the eigenvalues $\lambda = \mu + \mathrm{i}\nu$ explicitly:

$$\mu = -N^2 + Q + k - k\Psi\cos(\varphi N), \tag{5.76}$$

$$\nu = \mp k\Psi\sin(\varphi N). \tag{5.77}$$

Thus, in the case where the temporal delay vanishes, it is easy to check whether all eigenvalues have negative real part. Indeed, the eigenvalues are negative if N is large enough, compared to the (usually positive) parameter Q.

Summarizing, we obtain the following lemma on the positions of the eigenvalues for zero time delay τ:

Lemma 5.13 (Step 2: Positions of the eigenvalues for zero time delay)**.** *The eigenvalues $\lambda = \mu + \mathrm{i}\nu$ such that there exist 2π-periodic solutions of the equation*

$$\lambda g = g_{zz} + Qg + k\big(g - \Psi g(z - \varphi)\big) \tag{5.78}$$

are explicitly given by the following formulas:

$$\mu = -N^2 + Q + k - k\Psi \cos(\varphi N) \qquad \text{and} \tag{5.79}$$

$$\nu = \pm k\Psi \sin(\varphi N), \qquad \text{for } N = 0, 1, 2, \dots \tag{5.80}$$

Consider the case $Q > 0$ and $\Psi = 1$. Then the strictly positive real eigenvalue $\lambda = Q$ exists for all feedback gains k.

Now consider the case $Q > 0$ and $\Psi = -1$. If there exists $N \in \mathbb{N}_0$ with $N < \sqrt{Q}$ such that $\varphi N/\pi$ is an odd integer, then there exists a strictly positive real eigenvalue $\lambda = -N^2 + Q$ for all feedback gains k. If such an N does not exist, then there exists a $k^* \in \mathbb{R}$ such that all eigenvalues have negative real part for all feedback gains $k < k^*$.

In Figure 5.7 we have plotted the real part μ of the eigenvalue λ versus the feedback gain k, illustrating the results of the above lemma.

The case of non-zero time delay requires a more thorough investigation. Starting with the following lemma, which gives the position of the eigenvalues implicitly, we will continue our investigation in the two following sections.

Lemma 5.14 (Step 2: Positions of the eigenvalues)**.** *The eigenvalues $\lambda = \mu + \mathrm{i}\nu$ such that there exist 2π-periodic solutions of the equation*

$$\lambda g = g_{zz} + Qg + k\big(g - \Psi e^{-\lambda\tau} g(z - \varphi)\big), \tag{5.81}$$

are implicitly given by the crossings of the two curves

$$\nu(\mu) = \pm\frac{1}{\tau} \arccos\left(\frac{-\mu + Q + k - N^2}{k\Psi e^{-\mu\tau}}\right) \mp \frac{\varphi N - 2\pi n}{\tau}, \;\; n \in \mathbb{N}_0, \tag{5.82}$$

$$\mu(\pm\nu) = -\frac{1}{\tau} \log\left(\frac{\nu}{k\,\Psi \sin(\nu\tau \pm \varphi N)}\right), \tag{5.83}$$

in the complex plane.

Proof. The proof is mostly analogous to the proof of Lemma 5.5: Using the exponential Ansatz $g(z) = \exp(\eta z)$, and looking only for solutions which fulfill the periodic boundary conditions (i.e., $\eta = \pm \mathrm{i}N$), we obtain that the eigenvalues λ must fulfill the following two equations:

$$-N^2 + Q + k - \lambda = k\Psi e^{-\lambda\tau + \mathrm{i}\varphi N}, \tag{5.84}$$

$$-N^2 + Q + k - \lambda = k\Psi e^{-\lambda\tau - \mathrm{i}\varphi N}. \tag{5.85}$$

Splitting $\lambda = \mu + \mathrm{i}\nu$ into real and imaginary part and rearranging equations (5.84) and (5.85) for the real and imaginary part separately, we obtain the curves as claimed above. □

Already at this stage, let us formulate an important consequence of Lemma 5.14.

Corollary 5.15 (Step 2: Including spatio-temporal delay, failure of control)**.** *If the zero equilibrium is unstable in the linear variational equation of the form*

$$v_t = v_{zz} + Qv + k\big(v - \Psi v(z, t - \tau)\big), \tag{5.86}$$

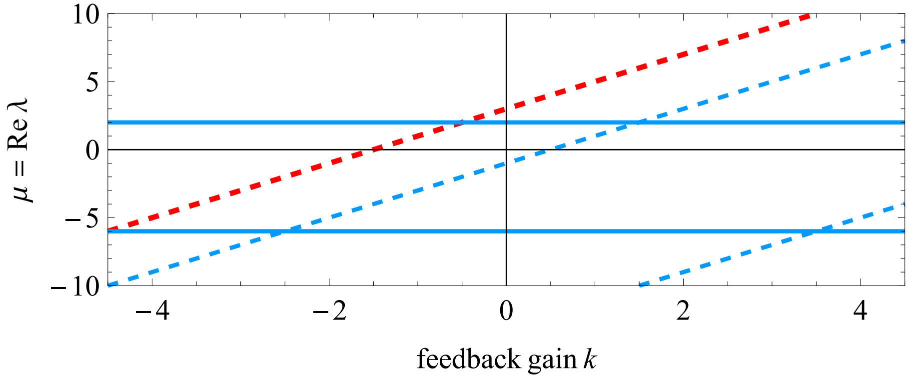
then it is also unstable in the linear variational equation including arbitrary spatio-temporal delay $\varphi \in \mathbb{R}$:

$$v_t = v_{zz} + Qv + k\big(v - \Psi v(z - \varphi, t - \tau)\big). \tag{5.87}$$

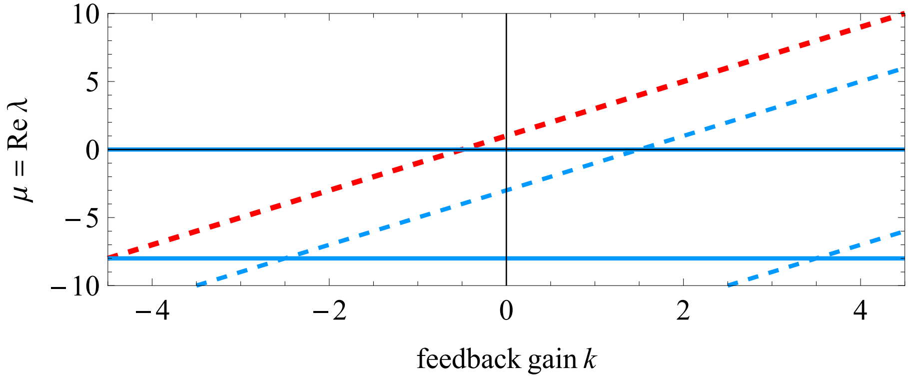
Remark. Corollary 5.15 together with Theorem 5.1 implies Theorem 5.11 on the failure of Pyragas control in the linear case. Furthermore, Corollary 5.15 together with Theorem 5.4 implies Theorem 2.5 on the destabilization of homogeneous equilibria via Pyragas control for an arbitrary spatial delay ξ.

Proof. In the case $N = 0$, the eigenvalues λ determined by Lemma 5.5 ($\varphi = 0$) coincide with the curves determined by Lemma 5.14. □

a) $\varphi = \pi$, $Q = 3$, stabilization fails



b) $\varphi = \pi$, $Q = 1$, stabilization succeeds



b) $\varphi = \pi/2$, $Q = 3$, stabilization succeeds

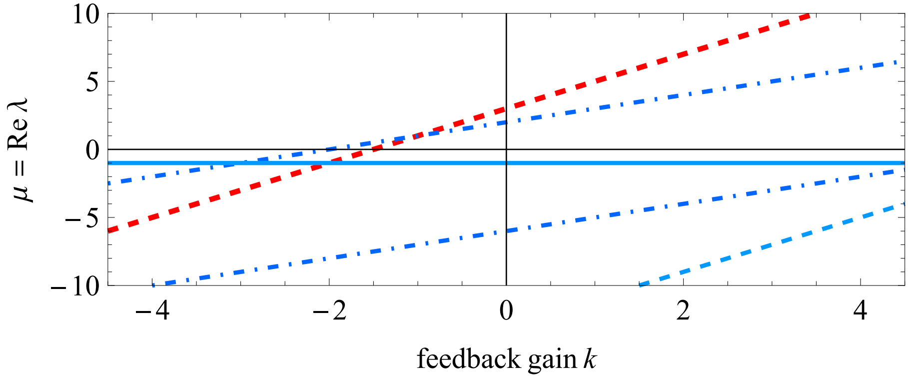


Figure 5.7.: Real part μ of the eigenvalues λ versus the feedback gain k. The time delay τ is zero. Curves for $\cos(\varphi N) = 1$ are dashed, curves for $\cos(\varphi N) = -1$ are solid, and curves for $\cos(\varphi N) = 0$ are dot-dashed.

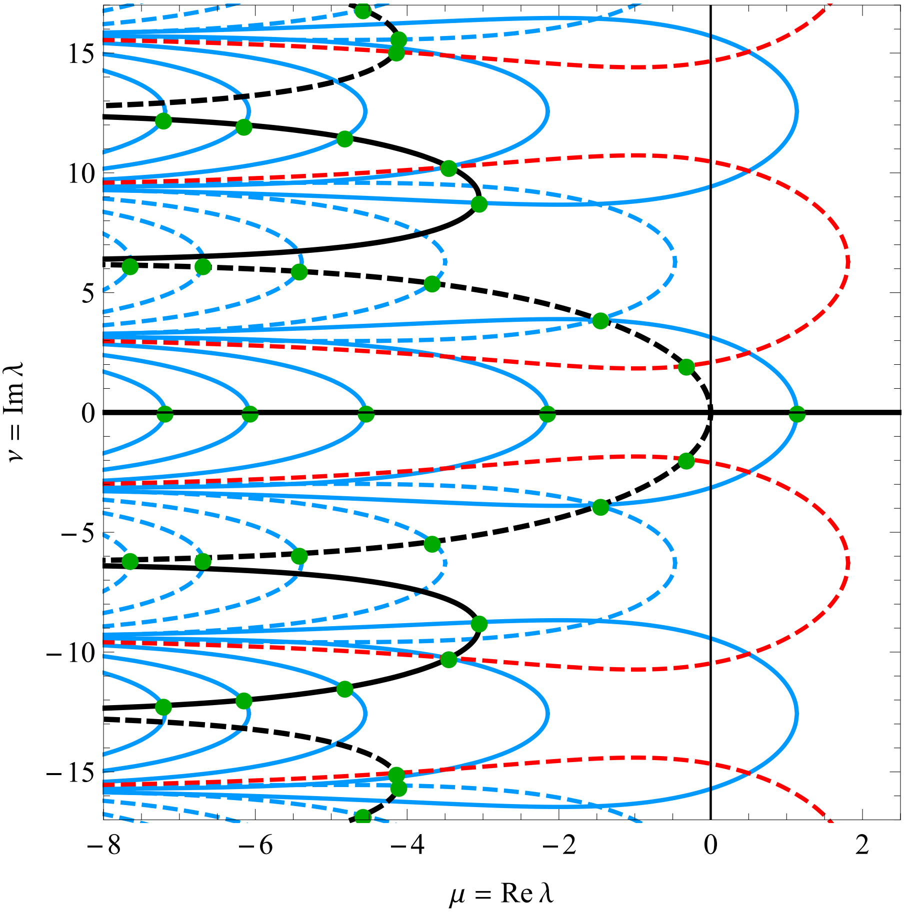


Figure 5.8.: Control triple method, failure: Positions of the eigenvalues (green dots) for a fixed feedback gain $k = -2$. Here $Q = 3$. The control triple is defined by $\Psi = -1$, $\varphi = \xi - c\tau = \pi$, and $\tau = 0.5$. The curve $\mu(\nu)$ is drawn in black, while $\nu(\mu)$ is drawn in red for $N = 0$ and in blue for all $N > 0$. Curves for even N are dashed, curves for odd N are solid.

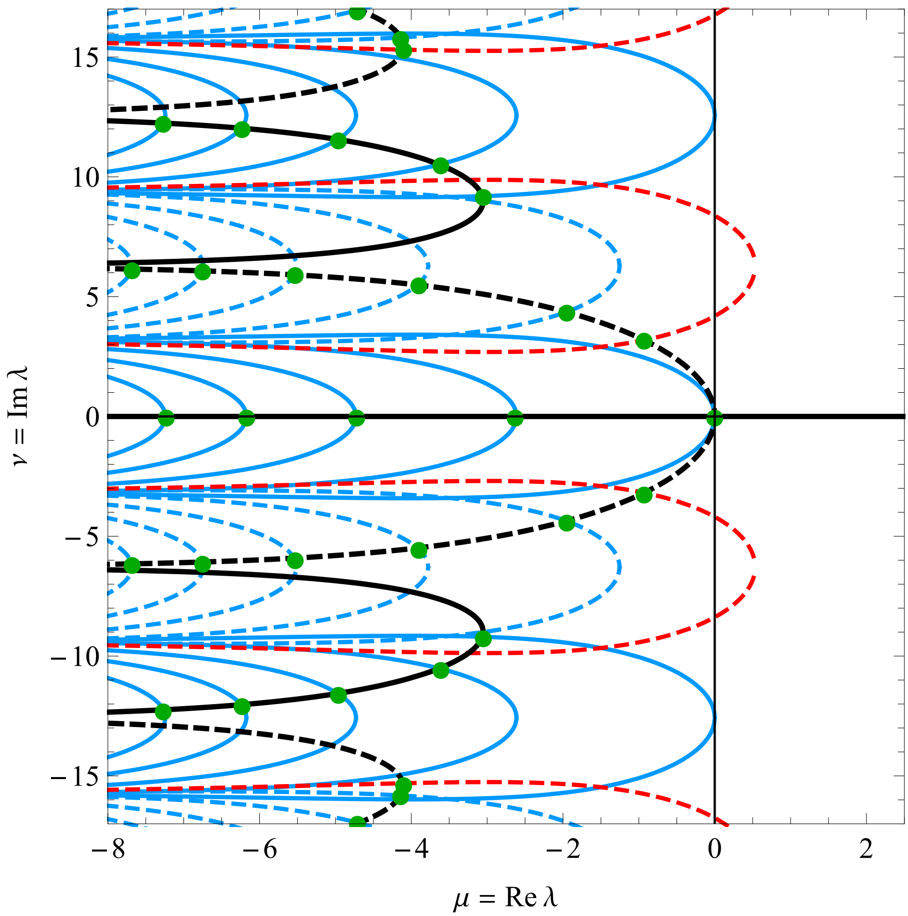


Figure 5.9.: Control triple method, success: Positions of the eigenvalues (green dots) for a fixed feedback gain $k = -2$. Here $Q = 1$. The control triple is defined by $\Psi = -1$, $\varphi = \xi - c\tau = \pi$, and $\tau = 0.5$. The curve $\mu(\nu)$ is drawn in black, while $\nu(\mu)$ is drawn in red for $N = 0$ and in blue for all $N > 0$. Curves for even N are dashed, curves for odd N are solid.

5.2.3. Conditions on the real eigenvalues

Similarly to Step 1, we now establish necessary and sufficient conditions for the real eigenvalues to be negative.

Lemma 5.16 (Step 2: Zero crossings of the real eigenvalues)**.** *Real eigenvalues μ, such that there exist 2π-periodic solutions of the equation*

$$\mu g = g_{zz} + Qg + k\left(g - \Psi e^{-\mu\tau} g(z-\varphi)\right), \tag{5.88}$$

have to fulfill the equations

$$\mu = -N^2 + Q + k - k\Psi e^{\mu\tau}\cos(\varphi N), \qquad N \in \mathbb{N}, \tag{5.89}$$

and they only exist for those N which fulfill $\sin(\varphi N) = 0$.

If they exist and additionally fulfill the inequalities

$$-\mu - N^2 + Q + k - k\Psi e^{-\mu\tau}\cos(\varphi N) < 0, \qquad N \in \mathbb{N}, \tag{5.90}$$

for all $\mu > 0$, then there does not exist any 2π-periodic solution for real positive values $\mu > 0$ of equation (5.88).

If additionally $\Psi\cos(\varphi N) \neq 1$, then there exists a feedback gain k_N,

$$k_N = \frac{Q - N^2}{\Psi\cos(\varphi N) - 1}, \tag{5.91}$$

for which the eigenvalue μ crosses zero if the feedback gain $k \in \mathbb{R}$ is increased through k_N: The crossing of the real eigenvalues μ is from negative to positive if the inequality

$$1 - \Psi\cos(\varphi N)\left(1 + \tau(N^2 - Q)\right) > 0 \tag{5.92}$$

is fulfilled. If $1 - \Psi\cos(\varphi N)\left(1 + \tau(N^2 - Q)\right) < 0$, the crossing is from positive to negative. No other eigenvalues $\mu = 0$ occur.

In the case $\Psi\cos(\varphi N) = 1$, an eigenvalue $\mu = 0$ occurs for all feedback gains $k \in \mathbb{R}$ if such an eigenvalue exists in the equation without control, i.e., for $k = 0$.

The proof of Lemma 5.16 is analogous to the proof of Lemma 5.6, therefore we do not repeat it here.

As a further simple consequence, we find the same simple conditions for the cases $\Psi = 1$ (e.g., Pyragas control) and $\Psi = -1$ (control triple method with a sign change of the output signal) as in Corollary 5.7. For completeness, let us include the conditions here.

Corollary 5.17 (Step 2: Conditions on the real eigenvalues)**.** *For the real eigenvalues μ, such that there exist 2π-periodic solutions of the equation*

$$\mu g = g_{zz} + Qg + k\left(g - \Psi e^{-\mu\tau} g(z-\varphi)\right), \tag{5.93}$$

$Q > 0$, the following properties hold:

If $\Psi = +1$, then there exist positive eigenvalues μ for all time delays $\tau \geq 0$ and for all feedback gains $k \in \mathbb{R}$.

If $\Psi = -1$, the feedback gain k fulfills the condition $k < -Q/2$, and the time delay is bounded by $\tau < 2/Q$, then all real eigenvalues μ are negative.

Remark. The proof of Corollary 5.17 is trivial if you remember $\cos(\varphi N) = \pm 1$. Corollary 5.17 proves Theorem 5.11 once more.

5.2.4. Conditions on the complex conjugated eigenvalues

In the last subsection of Step 2, we establish conditions on the complex conjugated eigenvalues.

Let us first consider the simpler case $\tau = 0$. In this case, we found that the real part μ of the complex conjugated eigenvalues is given by

$$\mu = -N^2 + Q + k - k\Psi\cos(\varphi N), \qquad N = 0, 1, 2, \ldots \tag{5.94}$$

It is therefore easy to check if all eigenvalues have negative real part, even for the general case $\Psi \in \mathbb{R}$. Additionally, we can find the crossings of the imaginary axis for feedback gains

$$k_N = \frac{N^2 - Q}{1 - \Psi \cos(\varphi N)}, \qquad N = 0, 1, 2, \dots \tag{5.95}$$

Note that we exclude $\cos(\varphi N) = \pm 1$, since the corresponding eigenvalues would be real.

However, in the case of $\tau > 0$, we have to work a little bit more. In this case, let us look only for purely imaginary eigenvalues $\lambda = \mathrm{i}\nu$. We obtain the equations (note the $\pm$)

$$-N^2 + Q + k - \mathrm{i}\nu = k\Psi e^{-\mathrm{i}\nu\tau \pm \mathrm{i}\varphi N}. \tag{5.96}$$

Again, we split these equations into real and imaginary part so that we obtain

$$0 = -N^2 + Q + k - k\Psi \cos(\nu\tau \mp \varphi N), \tag{5.97}$$

$$\nu = k\Psi \sin(\nu\tau \mp \varphi N). \tag{5.98}$$

Taking the square of both equations, adding them, and rearranging yields a quadratic equation in the imaginary part ν of the eigenvalues λ:

$$\nu^2 = k^2\Psi^2 - (-N^2 + Q + k)^2. \tag{5.99}$$

Going back to the first equation (5.97), we have to solve for τ, which only appears once in this equation.We find

$$\frac{-N^2 + Q + k}{k\Psi} = \cos(\nu\tau \mp \varphi N), \tag{5.100}$$

$$\arccos\left(\frac{-N^2 + Q + k}{k\Psi}\right) = \nu\tau \mp \varphi N, \tag{5.101}$$

and finally

$$\tau^*(k) = \frac{\arccos\left(\frac{-N^2+Q+k}{k\Psi}\right) \pm \varphi N}{\sqrt{k^2\Psi^2 - (-N^2 + Q + k)^2}}. \tag{5.102}$$

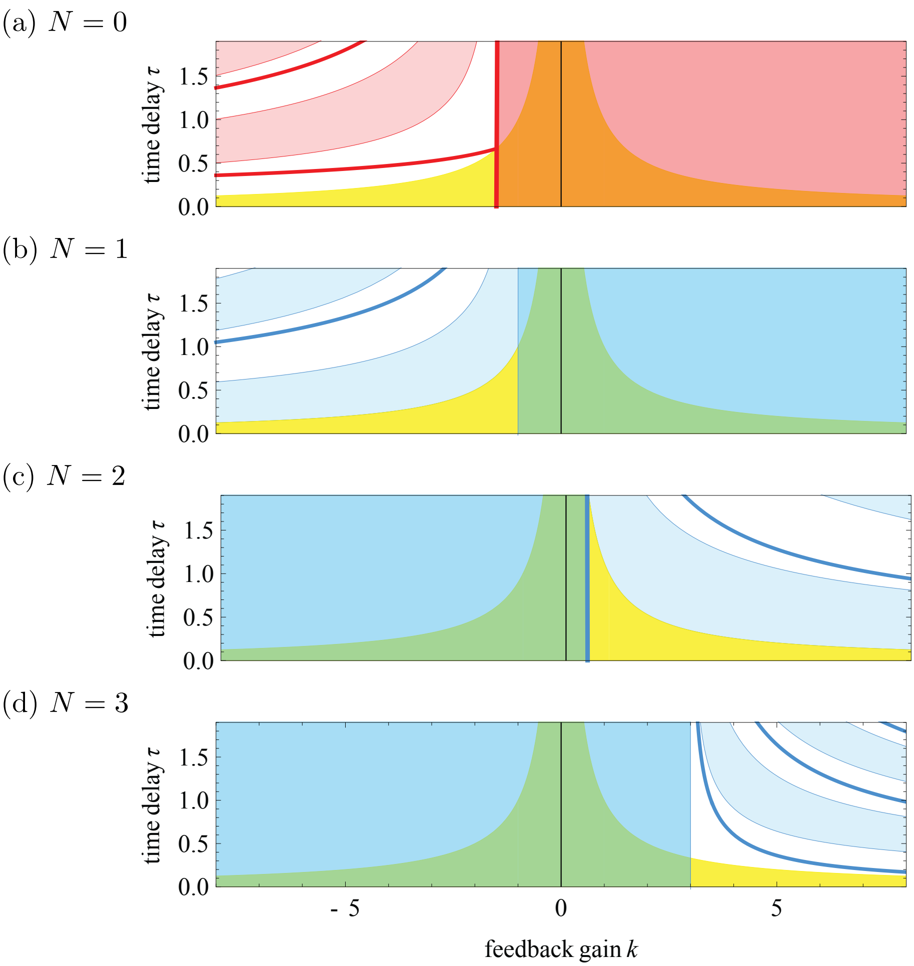


Figure 5.10.: Illustration of Lemma 5.18 for parameters $Q = 3$, multiplication $\Psi = -1$, and $\varphi = \pi$ for $N = 0, 1, 2, 3$: If the time delay τ is zero, then all eigenvalues are real. In the yellow region in the background, all complex conjugated eigenvalues have negative real part, since $|k| < 1/\tau$. Note that this region is independent of N. The solid thick curves give the sharp upper bound $\tau^*(k)$ on the time delay, depending on the parameter N. The curves are red for $N = 0$ and blue for $N = 1, 2, 3$. The regions of nonexistence of the eigenvalues are shaded in light red and light blue. No eigenvalue crossings occur for $k > (N^2 - Q)/2$, ($N = 0, 1$, i.e., for $N^2 < Q$), and $k < (N^2 - Q)/2$, ($N \geq 2$, i.e., for $N^2 > Q$), respectively (darker red and darker blue shaded regions).

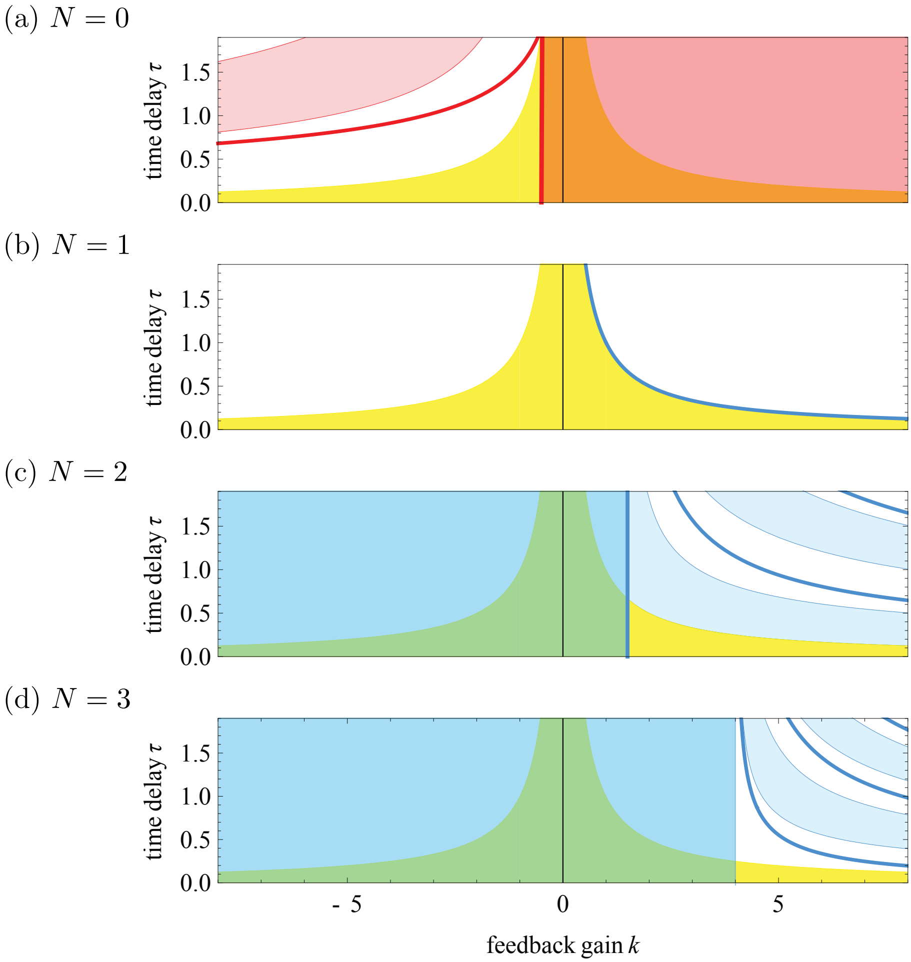


Figure 5.11.: Illustration of Lemma 5.18 for parameters $Q = 1$, multiplication $\Psi = -1$, and $\varphi = \pi$ for $N = 0, 1, 2, 3$: If the time delay τ is zero, then all eigenvalues are real. In the yellow region in the background, all complex conjugated eigenvalues have negative real part, since $|k| < 1/\tau$. Note that this region is independent of N. The solid thick curves give the sharp upper bound $\tau^*(k)$ on the time delay, depending on the parameter N. The curves are red for $N = 0$ and blue for $N = 1, 2, 3$. The regions of nonexistence of the eigenvalues are shaded in light red and light blue. No eigenvalue crossings occur for $k > (N^2 - Q)/2$, ($N = 0$, i.e., for $N^2 < Q$), and $k < (N^2 - Q)/2$, ($N \geq 2$, i.e., for $N^2 > Q$), respectively (darker red and darker blue shaded regions).

Summarizing, we find the following lemma:

Lemma 5.18 (Step 2: Conditions on the complex conjugates eigenvalues)**.** *If the complex conjugated eigenvalue $\lambda = \mu + \mathrm{i}\nu$ has negative real part for zero time delay $\tau = 0$, i.e., if $\mu = -N^2+Q+k-k\Psi\cos(\varphi N) < 0$, then this eigenvalue also has negative real part for all time delays $0 \leq \tau < \tau^*(k)$,*

$$\tau^*(k) = \min_{N\in\mathbb{N}} \frac{\arccos\left(\frac{-N^2+Q+k}{k\Psi}\right) \pm \varphi N}{\sqrt{k^2\Psi^2 - (-N^2+Q+k)^2}}. \tag{5.103}$$

In the case $\varphi = \pi$, Lemma 5.9 holds with slight modifications: If the time delay $\tau = 0$ is zero, then all eigenvalues are real.

Now fix a time delay $\tau > 0$. If the feedback gain k fulfills the inequality

$$|k| < 1/(|\Psi|\tau), \tag{5.104}$$

then all complex conjugated eigenvalues have negative real part.

Conversely, now fix a feedback gain $k \in \mathbb{R}\backslash\{0\}$ and $N \in \mathbb{N}$ even. If the time delay τ fulfills $\tau < \tau^*(k)$, and if $\tau^*(k) > 0$,

$$\tau^*(k) = \min_{N^2<Q,\, N\in\mathbb{N}} \frac{\arccos\left(\frac{k+Q-N^2}{k\Psi}\right)}{\sqrt{k^2\Psi^2 - (Q+k-N^2)^2}}, \qquad \text{for } \Psi k > 0, \tag{5.105}$$

$$\tau^*(k) = \min_{N^2<Q,\, N\in\mathbb{N}} \frac{\arccos\left(\frac{k+Q-N^2}{|k\Psi|}\right) + \pi}{\sqrt{k^2\Psi^2 - (Q+k-N^2)^2}}, \qquad \text{for } \Psi k < 0, \tag{5.106}$$

then all pairs of complex conjugated eigenvalues have negative real part.

Now consider $N \in \mathbb{N}$ odd. If the time delay τ fulfills $\tau < \tau^*(k)$, and if

$\tau^*(k) > 0$,

$$\tau^*(k) = \min_{N^2<Q,\, N\in\mathbb{N}} \frac{\arccos\left(\frac{k+Q-N^2}{k\Psi}\right)}{\sqrt{k^2\Psi^2 - (Q+k-N^2)^2}}, \qquad \text{for } \Psi k < 0, \quad (5.107)$$

$$\tau^*(k) = \min_{N^2<Q,\, N\in\mathbb{N}} \frac{\arccos\left(\frac{k+Q-N^2}{|k\Psi|}\right)+\pi}{\sqrt{k^2\Psi^2 - (Q+k-N^2)^2}}, \qquad \text{for } \Psi k > 0, \quad (5.108)$$

then all pairs of complex conjugated eigenvalues have negative real part.

These thresholds on the time delay τ are sharp.

Now consider the case $|\Psi| = 1$.
If $N^2 < Q$, no complex conjugated eigenvalues cross the imaginary axis for $k > (N^2 - Q)/2$.
If $N^2 > Q$, no complex conjugated eigenvalues cross the imaginary axis for $k < (N^2 - Q)/2$.

5.3. Step 3: Non-autonomous variational equations without spatio-temporal delay

In Step 3 let us once more go back to the general linear variational equation in co-rotating coordinates:

$$v_t = v_{zz} + Q(z)v + k\big(v - \Psi v(z-\varphi, t-\tau)\big). \qquad (5.109)$$

In this section we consider the case where $Q(z)$ is a non-constant function depending on z. However, we simplify and assume that the spatio-temporal delay φ is zero, to obtain a linear variational equation of the form

$$v_t = v_{zz} + Q(z)v + k\big(v - \Psi v(z, t-\tau)\big). \qquad (5.110)$$

Analogously to Step 1 and Step 2, the stability of the equilibrium or rotating wave is given by the stability of the zero equilibrium of equation (5.110).

This section is organized as follows: The theorems corresponding to equation (5.110) are stated in Subsection 5.3.1. From these theorems we conclude the remaining results of Chapter 2 and some partial results for Chapter 3. These results are proved via finding the positions of the eigenvalues in Subsection 5.3.2, and subsequently concluding conditions on the real eigenvalues in Subsection 5.3.3, as well as conditions on the complex conjugated eigenvalues in Subsection 5.3.4.

5.3.1. Theorems

We can now formulate the failure of Pyragas-like controls to stabilize the zero equilibrium in the linear variational equation:

Theorem 5.19 (Step 3: Failure of control of the zero equilibrium in the linear variational equation)**.** *Consider the linear variational equation*

$$v_t = v_{zz} + Q(z)v + k\big(v - v(z, t - \tau)\big), \tag{5.111}$$

with periodic boundary conditions, positive time delay $\tau > 0$, and real feedback gain k. Assume that there exists at least one strictly positive eigenvalue $\lambda_* > 0$ in the case without control ($k = 0$).

Then the zero equilibrium is unstable for all feedback gains $k \in \mathbb{R}$.

Remark. Theorem 5.19 implies Theorems 2.1–2.3.

In contrast to Pyragas control, the control triple method successfully stabilizes the zero equilibrium if the transformation of the output signal is $\Psi = -1$. This is stated in the following theorem:

Theorem 5.20 (Step 3: Successful stabilization of the zero equilibrium in the linear variational equation)**.** *Consider the linear variational equation*

$$v_t = v_{zz} + Q(z)v + k\big(v - (-1)v(z, t - \tau)\big), \tag{5.112}$$

with periodic boundary conditions, $\bar{Q} = \max_{z\in[0,2\pi]} Q(z) > 0$, and positive time delay $\tau > 0$.

If the feedback gain $k \in \mathbb{R}$ *is chosen in such a way that*

$$-1/\tau < k < -\bar{Q}/2 \tag{5.113}$$

(if such an interval of feedback gains exists), then the zero equilibrium is stable.

Remark. (a) Theorem 5.20 does not yet prove Theorem 3.1, since no spatio-temporal delay is present, but it gives us a hint that it is true.
(b) Theorem 5.20 is analogous to Theorem 5.2, but more general. The sharp upper bound on the time delay τ, in Theorem 5.2, $\tau^*(k)$, depends explicitly on the assumption that $Q(z) \equiv Q$ is a constant which we do not assume here. The condition $-1/\tau < k$ still holds if we drop the assumption on $Q(z)$.

In addition to the sufficient conditions given in the previous theorem, we also establish a necessary condition for successful stabilization:

Theorem 5.21 (Step 3: Necessary condition for the control of the zero equilibrium in the linear variational equation)**.** *Consider the linear variational equation*

$$v_t = v_{zz} + Q(z)v + k\big(v - (-1)u(z, t-\tau)\big), \tag{5.114}$$

with periodic boundary conditions, $\bar{Q} = \max_{z\in[0,2\pi]} Q(z) > 0$*, real feedback gain* k*, and positive time delay* $\tau > 0$*.*

Then the zero equilibrium is unstable for all feedback gains

$$k > -\mathcal{Q}/2 := -\left(\frac{1}{2\pi}\int_0^{2\pi} Q(z)dz\right)\Big/\,2. \tag{5.115}$$

Remark. The necessary and sufficient conditions coincide only in the case $\mathcal{Q} = \bar{Q}$. This only occurs under the additional assumption $Q(x) \equiv Q$, which we have discussed in detail in Step 1.

We divide the proof of Theorems 5.19–5.21 into three parts: In the first part, Subsection 5.3.2, we prove a result on the positions of the eigenvalues. We focus on the differences to the analogous result in Step 1. In

Subsection 5.3.3 we find conditions on the real eigenvalues. In the last part, Subsection 5.3.4, we give an estimate which describes when the complex conjugated eigenvalues are in the left half of the complex plane, i.e., have negative real part.

5.3.2. Positions of the eigenvalues

To find the positions of the eigenvalues, let us investigate the linear variational equation of the form

$$v_t = v_{zz} + Q(z)v + k\big(v - \Psi v(z, t-\tau)\big). \tag{5.116}$$

Similarly to Steps 1 and 2, we solve equation (5.116) via separation of variables and an exponential Ansatz, $v(z,t) = g(z)e^{\lambda t}$, $\lambda \in \mathbb{C}$. As before, the $\lambda \in \mathbb{C}$ are the eigenvalues which carry the stability information of the equilibrium or wave. We obtain an ordinary differential equation of the form

$$\lambda g = g_{zz} + Q(z)g + k\left(1 - \Psi e^{-\lambda\tau}\right)g, \tag{5.117}$$

i.e., a second-order equation of Hill's type with periodic boundary conditions. The coefficient $-\lambda + Q(z) + k\left(1 - \Psi e^{-\lambda\tau}\right)$ is non-constant in z, which is in contrast to both Step 1 and Step 2. Here lies the additional difficulty of Step 3.

To find the eigenvalues λ, we need to find those values λ such that equation (5.117) has 2π-periodic solutions (compare with Steps 1, 2, and Chapter 4).

Lemma 5.22 (Step 3: Positions of the eigenvalues)**.** *The eigenvalues $\lambda = \mu + \mathrm{i}\nu$, such that there exist 2π-periodic solutions of Hill's equation*

$$\lambda g = g_{zz} + Qg + k\left(1 - \Psi e^{-\lambda\tau}\right)g, \tag{5.118}$$

are either real, or they lie on the curve

$$\mu(\nu) = -\frac{1}{\tau}\log\left(\frac{\nu}{k\Psi\sin(\nu\tau)}\right) \tag{5.119}$$

in the complex plane. In particular, for $\tau = 0$, all eigenvalues λ are real.

Proof. The proof of Lemma 5.22 is straightforward. We split equation (5.118) into its real and imaginary part:

$$0 = g_{zz} + \left(-\lambda + Q(z) + k - k\Psi e^{-\mu\tau}\cos(\nu\tau)\right) g, \tag{5.120}$$
$$0 = \left(-\nu + k\Psi e^{-\mu\tau}\sin(\nu\tau)\right) g. \tag{5.121}$$

The second equation yields

$$\nu = k\Psi e^{-\mu\tau}\sin(\nu\tau). \tag{5.122}$$

In the case of zero time delay, $\tau = 0$, it follows $\nu = 0$, and hence all eigenvalues are real. For non-zero time delay, the curve

$$\mu(\nu) = -\frac{1}{\tau}\log\left(\frac{\nu}{k\Psi\sin(\nu\tau)}\right) \tag{5.123}$$

follows immediately. This completes the proof of Lemma 5.22. □

5.3.3. Conditions on the real eigenvalues

In this section we establish conditions on the real eigenvalues. In comparison to Step 1, they are complicated by the non-constant function $Q(z)$. However, in the first lemma, which proves Theorem 5.19, there exists an elegant way to circumvent this problem:

Lemma 5.23 (Step 3: Existence of strictly positive eigenvalues)**.** *Suppose there exists a real eigenvalue $\lambda_* > 0$ such that $0 = g_{zz} + (-\lambda_* + Q(z))g$ has a 2π-periodic solution.*

Fix an arbitrary feedback gain $k \in \mathbb{R}$, and any time delay $\tau > 0$.

Then there exists at least one real $\mu > 0$ such that

$$0 = g_{zz} + \left(-\mu + Q(z) + k - ke^{-\mu\tau}\right) g \tag{5.124}$$

also has a periodic solution for this λ.

The proof of Lemma 5.23 is straightforward, because we have already done the relevant calculation in Chapter 2:

Proof. We know that $0 = g_{zz} + (-\lambda_* + Q(z))g$ has a 2π-periodic solution. Let us therefore search for $\mu > 0$ such that

$$-\lambda_* = -\mu + k - ke^{-\mu\tau}. \tag{5.125}$$

We calculate explicitly which feedback gain has to be applied to obtain a given real eigenvalue μ:

$$k(\mu) = \frac{\nu - \lambda_*}{1 - e^{-\mu\tau}}. \tag{5.126}$$

Following the discussion of the example in Chapter 2, where we have obtained the same formula for $\lambda_* = 1$, we conclude that $k : (0, \infty) \to \mathbb{R}$ is bijective and that there exists a positive real eigenvalue μ for all real feedback gains k. □

In the following two lemmata, we have to find approximations of the function $Q(z)$ to obtain the results. Since we cannot longer determine the eigenvalue crossings through zero explicitly, we derive sufficient and necessary conditions separately.

Lemma 5.24 (Step 3: Sufficient condition on the real eigenvalues)**.** *For the real eigenvalues μ, such that there exist 2π-periodic solutions of Hill's equation*

$$\mu g = g_{zz} + Q(z)g + k\left(1 - \Psi e^{-\mu\tau}\right) g, \tag{5.127}$$

the following holds:

For a fixed feedback gain k, all real eigenvalues μ are negative if

$$-\mu + \bar{Q} + k - k\Psi e^{-\mu\tau} < 0 \tag{5.128}$$

for all $\mu > 0$, where $\bar{Q} = \max_{z\in[0,2\pi]} Q(z)$.

Proof. Here we mainly follow the proof of Theorem 2.1 from the book on Hill's equation by Magnus and Winkler [45], page 14. Denote $D(z) = \mu - Q(z) - k + \Psi ke^{-\mu\tau}$ and choose the feedback gain k_* such that $D(z) > 0$ for all z and for all $\mu \geq 0$. Consider the ordinary differential equation

$$g_{zz}(z) = D(z)g(z). \tag{5.129}$$

We now consider the two normalized solutions, which are defined by the initial conditions

$$g_1(0) = 1, \quad g_1'(0) = 0, \tag{5.130}$$

$$g_2(0) = 0, \quad g_2'(0) = 1. \tag{5.131}$$

It is clear that the two normalized solutions are linearly independent and that any solution can be constructed as a linear combination of these two solutions. We therefore only need to show for these two specific solutions that they are strictly monotonically increasing. More precisely, we show $g_1'(z) > 0$ for all $z > 0$ as well as $g_2'(z) > 0$ for all $z > 0$. This excludes the option of periodic solutions and therefore the existence of eigenvalues $\lambda \geq 0$. Consider now the first normalized solution g_1 with the initial conditions $g_1(0) = 1$ and $g_1'(0) = 0$. Since $D(0) > 0$, it follows that $g_1''(0) > 0$, which implies $g_1'(z) > 0$ for all sufficiently small positive z. We can therefore conclude that, should there exist any ε such that $g_1'(\varepsilon) = 0$, then ε is positive and bounded away from zero.

It remains to show that such a lower bound ε does not exist. Suppose now that we have indeed found $\varepsilon > 0$ such that $g_1'(\varepsilon) = 0$. $g_1(z)$ solves the differential equation (5.129) for all z and we can multiply the equation by $g_1(z)$ and integrate from 0 to ε. We then obtain

$$\left(g_1'(\varepsilon)\right)^2 = \int_0^\varepsilon D(z) g_1(z) g_1'(z)\, dz. \tag{5.132}$$

The left hand side is zero by assumption. On the right hand side, however, the integrand is strictly positive for all $0 \leq z < \varepsilon$. Hence, the term on the right hand side is strictly positive. This yields a contradiction.

Almost the same considerations show that $g_2(z) > 0$ for all $z > 0$. Here $g_2'(0) = 1 > 0$ by assumption, therefore we find the same contradiction. We can conclude that no periodic solutions exist. □

Lemma 5.25 (Step 3: Necessary condition on the real eigenvalues)**.** *For the real eigenvalues μ such that there exist 2π-periodic solutions of Hill's equation*

$$\mu g = g_{zz} + Q(z) g + k\left(1 - \Psi e^{-\mu\tau}\right) g, \tag{5.133}$$

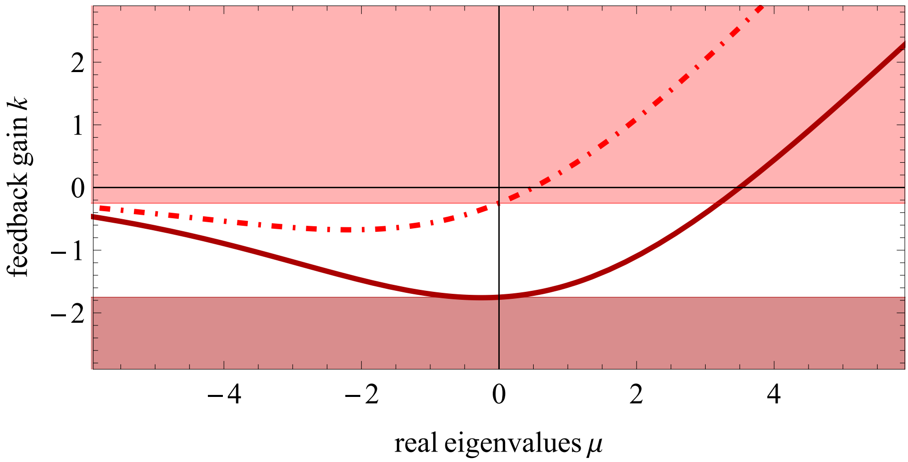


Figure 5.12.: Comparison of the sufficient condition from Lemma 5.24 (solid dark red eigenvalue curve, crosses zero at $-\bar{Q}/2$; in the bottom region, shaded in darker red, no positive real eigenvalues exist) with the necessary condition from Lemma 5.25 (dot-dashed light red eigenvalue curve, crosses zero at $-\mathcal{Q}/2$; in the top region, shaded in lighter red, strictly positive real eigenvalues exist). Parameters: $\Psi = -1$, $\tau = 0.5$, $\bar{Q} = 3.5$, $\mathcal{Q} = 0.5$.

the following holds:

If for a fixed feedback gain $k \in \mathbb{R}$ there exists at least one $\mu > 0$ such that

$$-\mu + \mathcal{Q} + k - k\Psi e^{-\mu\tau} > 0, \tag{5.134}$$

then there always exists at least one real, strictly positive eigenvalue μ.

Proof. We rewrite equation (5.133) in the form

$$0 = g_{zz} + \left(C + \tilde{Q}(z)\right) g, \tag{5.135}$$

where

$$C := -\mu + \mathcal{Q} + k - ke^{-\mu\tau}, \qquad \text{and} \tag{5.136}$$

$$\tilde{Q}(z) := Q(z) - \mathcal{Q}. \tag{5.137}$$

Note that $\int_0^{2\pi} \tilde{Q}(z)\,dz = 0$. Let us for a moment interpret C as a free parameter. Then for equation (5.135), the following statement holds:

The smallest value C_0 of C for which Hill's equation (5.135) *has a 2π-periodic solution is not positive, and $C_0 = 0$ if and only if $\tilde{Q}(z) \equiv 0$.*

While this statement was first proven by Borg in 1946 [6], we follow here the elegant proof by Ungar from 1961 [75], as explained in the book by Magnus and Winkler [45].

To prove the statement, remember that we have shortly investigated the oscillation properties of Hill's equation (5.135) in Chapter 4. Therefore, we know that the periodic solution which belongs to C_0 has no zeros. Let us call this solution $g_*(z)$. Without loss of generality we assume $g_*(z) > 0$ for all z. Consider now the differentiable and 2π-periodic function

$$h(z) = \frac{d}{dz} \log g_*(z). \tag{5.138}$$

Then

$$h'(z) = \left(\frac{g_*'(z)}{g_*(z)}\right)', \tag{5.139}$$

and hence $h(z)$ fulfills the *Riccati equation*

$$h'(z) + h^2(z) = -C_0 + \tilde{Q}(z), \tag{5.140}$$

which is of first order, but nonlinear. We integrate both sides from 0 to 2π, and we obtain

$$\int_0^{2\pi} h^2(z)\,dz = -2\pi C_0. \tag{5.141}$$

Remember that $\int_0^{2\pi} Q(z)\,dz = 0$ by definition. Furthermore, $\int_0^{2\pi} h'(z)\,dz = 0$ by periodicity of $h(z)$. We can therefore conclude that C_0 is non-negative. Moreover, $C_0 = 0$ if and only if $h \equiv 0$. This can only occur if $g_*(z) \equiv \gamma \neq 0$ is a constant. Hence, γ fulfills equation (5.135) for $C = C_0 = 0$,

$$0 = \left(0 + \tilde{Q}(z)\right)\gamma, \tag{5.142}$$

and we conclude $\tilde{Q}(z) = 0$. This proves the statement.

Hence, we obtain $C > 0$ as a sufficient condition for the existence of positive eigenvalues, as stated in the lemma. □

5.3.4. Conditions on the complex conjugated eigenvalues

In this section we investigate the behavior of the complex conjugated eigenvalues. This is relatively simple, as our condition does not depend on $Q(z)$. Therefore, the relevant work has already been done in Step 1, and we are ready to state the following lemma:

Lemma 5.26 (Step 3: Conditions on the complex conjugated eigenvalues)**.** *For the eigenvalues $\lambda = \mu + \mathrm{i}\nu$ such that there exist 2π-periodic solutions of the equation*

$$\lambda g = g_{zz} + Q(z)g + k\left(1 - \Psi e^{-\lambda\tau}\right)g, \tag{5.143}$$

the following holds:

If the time delay τ is zero, then all eigenvalues are real. Now fix a time delay $\tau > 0$. If the feedback gain k fulfills the inequality

$$|k| < 1/\left(|\Psi|\tau\right), \tag{5.144}$$

then all pairs of complex conjugated eigenvalues have negative real part.

Proof. In Lemma 5.22 we have shown that the complex conjugated eigenvalues $\lambda = \mu + \mathrm{i}\nu$, such that there exist 2π-periodic solutions of the equation

$$\lambda g = g_{zz} + Q(z)g + k\left(1 - \Psi e^{-\lambda\tau}\right)g, \tag{5.145}$$

lie on the curve

$$\mu(\nu) = -\frac{1}{\tau} \log \left(\frac{\nu}{k\Psi \sin(\nu\tau)} \right). \tag{5.146}$$

We have already investigated this curve in Step 1, therefore we do not repeat the proof here. □

5.4. Step 4: Non-autonomous variational equations including spatio-temporal delay

In Step 4 we finally consider the general form of the linear variational equation in co-rotating coordinates,

$$v_t = v_{zz} + Q(z)v + k\big(v - \Psi v(z - \varphi, t - \tau)\big), \tag{5.147}$$

in order to finish the remaining proofs of the theorems in Chapter 3. Here $Q(z) = f'(\mathcal{U}(z))$. If the rotating wave has minimal period $2\pi/n$, then also $Q(z)$ has period $2\pi/n$. If, in addition, the rotating wave is odd, then the minimal period of $Q(z)$ is given by π/n, as we have seen in Chapter 4. In this section we can restrict our attention to $\Psi = \pm 1$, since the theorems on the stabilization of the homogeneous zero equilibrium have already been proven in Steps 1 and 2.

Again, we investigate the stability of the zero equilibrium of equation (5.147) to determine the stability of the frozen or rotating waves.

We organize this section as follows: The theorems corresponding to equation (5.147) are stated in Subsection 5.4.1. These results prove the remaining theorems from Chapter 3 for control schemes of rotation type. To prove these results, we first find the positions of the eigenvalues in Subsection 5.4.2, and proceed with conditions on the real eigenvalues in Subsection 5.4.3, as well as with conditions on the complex conjugated eigenvalues in Subsection 5.4.4.

5.4.1. Theorems

The following theorem proves the success of the control triple method in the general case. Note the conditions on $Q(z) = f'(\mathcal{U}(z))$ and the spatio-temporal delay.

Theorem 5.27 (Step 4: Successful control of the zero equilibrium in the linear variational equation)**.** *Consider the linear variational equation*

$$v_t = v_{xx} + Q(z)v + k\big(v - (-1)v(z - \varphi, t - \tau)\big), \qquad (5.148)$$

with periodic boundary conditions, where $Q(z)$ has minimal period π/n, $\bar{Q} > 0$, the spatio-temporal delay is given by $\varphi = \xi - c\tau = m\pi/n$, m odd and co-prime to n, and $\tau > 0$. Assume that there exist exactly $2n-1$ real, strictly positive eigenvalues in the case without control, i.e., for $k = 0$.

Then there exists a feedback gain $k^* \in \mathbb{R}$ such that the following holds:

For all $k < k^*$ there exists a time delay $\tau^* = \tau^*(k)$ such that the zero equilibrium of equation (5.148) *is stable for all time delays $\tau < \tau^*$.*

Remark. Theorem 5.27 proves Theorem 3.1 on the success of the control triple method.

In contrast, if the conditions on $Q(z)$ and the spatio-temporal delay φ are changed slightly (double minimal period of $Q(z)$, and φ an even integer multiple of that minimal period), then control always fails:

Theorem 5.28 (Step 4: Failure of control of the zero equilibrium in the linear variational equation)**.** *Consider the linear variational equation*

$$v_t = v_{xx} + Q(z)v + k\big(v - v(z - \varphi, t - \tau)\big), \qquad (5.149)$$

with periodic boundary conditions, where $Q(z)$ has minimal period $2\pi/n$, the spatio-temporal delay is given by $\varphi = \xi - c\tau = m\pi/n$, m even, $\tau > 0$, and linearly unstable zero equilibrium for $k = 0$.

Then the zero equilibrium of equation (5.149) *is unstable for all feedback gains $k \in \mathbb{R}$.*

Remark. Theorem 5.28 proves Theorem 3.2.

We prove Theorems 5.27 and 5.28 in the following three subsections. We use many of the results from Steps 1–3.

5.4.2. Positions of the eigenvalues

Solving the linear variational equation (5.147) by an exponential Ansatz $v(z,t) = g(z)e^{\lambda t}$, we obtain the following equation of Hill's type with delay:

$$\lambda g = g_{zz} + Q(z)g + k\left(g - \Psi e^{-\lambda\tau} g(z-\varphi)\right). \tag{5.150}$$

First, note that the spatio-temporal delay φ is an integer multiple of the minimal period of $Q(z)$. Second, we only need to consider Floquet solutions of equation (5.150), since our current interest is limited to the periodic solutions and their corresponding eigenvalues. We therefore make the Ansatz

$$g(z) = p(z)e^{\eta z}, \qquad \text{with} \quad p(z) = p\,(z + 2\pi)\,. \tag{5.151}$$

Remark. Suppose there were 2π-periodic solutions $g^*(z)$ which were not given by Floquet solutions. Then, these solutions could be written as $g^*(z) = g^*(z)e^{0t} = p(z)e^{0t}$. They are hence Floquet solutions with Floquet exponent 0. This yields a contradiction. We can conclude that we can indeed find every periodic solution, and hence each eigenfunction, by a Floquet Ansatz.

We then obtain another equation of Hill's type,

$$0 = p_{zz} + 2\eta\, p_z + \left(-\lambda + \eta^2 + Q(z) + k - k\Psi e^{-\lambda\tau - \eta\varphi}\right) p, \tag{5.152}$$

but without delay φ; the delay has become an exponential term. We underline that this step is only possible because the spatio-temporal delay φ is an integer multiple of the period of $Q(z)$, see Chapter 4 and [24, 25].

The Floquet exponent η can only take integer values on the imaginary axis, $\eta = \pm \mathrm{i}N$, in order to yield periodic solutions $g(z) = p(z)e^{\eta z} = p(z)e^{\pm \mathrm{i}Nz}$. We obtain

$$0 = p_{zz} \pm 2\mathrm{i}N\, p_z + \left(-\lambda - N^2 + Q(z) + k - k\Psi e^{-\lambda\tau \mp \mathrm{i}\varphi N}\right) p. \quad (5.153)$$

Let us next transform this equation to the standard form of Hill's equation, inverting the coordinate transformation $g(z) = e^{\pm \mathrm{i}Nz}p(z)$ (see [45]) to obtain

$$0 = g_{zz} + \left(-\lambda + Q(z) + k - k\Psi e^{-\lambda\tau \mp \mathrm{i}\varphi N}\right) g. \quad (5.154)$$

This small detour has eliminated the spatio-temporal delay and is therefore crucial for the analysis of periodic solutions.

Similarly to Step 2, let us first consider the case of zero time delay, $\tau = 0$. We obtain the simplified equation

$$0 = g_{zz} + \left(-\lambda + Q(z) + k - k\Psi e^{\mp \mathrm{i}\varphi N}\right) g. \quad (5.155)$$

Let us split this equation into real and imaginary part:

$$0 = g_{zz} + (-\mu + Q(z) + k - k\Psi\cos(\varphi N))\, g \quad (5.156)$$
$$0 = (-\nu \pm k\Psi\sin(\varphi N))\, g. \quad (5.157)$$

We easily conclude the following lemma:

Lemma 5.29 (Step 4: Positions of the eigenvalues for zero time delay)**.** *The eigenvalues $\lambda = \mu + \mathrm{i}\nu$, such that there exist 2π-periodic solutions of the equation*

$$0 = g_{zz} - \lambda g + Q(z)g + k\big(g - \Psi g(z - \varphi)\big), \quad (5.158)$$

have imaginary part

$$\nu = \pm k\Psi\sin(\varphi N) \qquad \textit{for} \quad N = 0, 1, 2, \ldots \quad (5.159)$$

In particular, real eigenvalues exist only if $\varphi N = 0 \mod \pi$.

In contrast to Step 2, the real part μ of the eigenvalues λ cannot be determined explicitly. We will derive the conditions on the real part of the eigenvalues in Subsection 5.4.3.

Let us now turn to the case of nonzero time delay τ.

Lemma 5.30 (Step 4: Positions of the eigenvalues)**.** *The eigenvalues $\lambda = \mu + \mathrm{i}\nu$, such that there exist 2π-periodic solutions of the equation*

$$\lambda g = g_{zz} + Q(z)g + k\big(g - \Psi e^{-\lambda\tau} g(z-\varphi)\big), \tag{5.160}$$

lie on the curves

$$\mu(\pm\nu) = -\frac{1}{\tau}\log\left(\frac{\nu}{k\Psi\sin(\nu\tau \pm \varphi N)}\right) \qquad \text{for } N = 0,1,2,\dots \tag{5.161}$$

in the complex plane.

Proof. We split equation (5.154) into real and imaginary part to obtain (the imaginary part)

$$\nu g = k\Psi e^{-\mu\tau}\sin(-\nu\tau \pm \varphi N)g. \tag{5.162}$$

Rearranging this equation for μ, as well as complex conjugation, yields the desired result. □

In the following two subsections, we determine more specific conditions on the eigenvalues to guarantee stability.

5.4.3. Conditions on the real eigenvalues

Let us now turn to conditions on the real eigenvalues. Similarly to Step 3, it is impossible to determine the eigenvalue crossings through zero explicitly, and we can only derive necessary and sufficient conditions on the real eigenvalues to be negative. Let us start by simplifying equation (5.154) for real $\lambda = \mu$, i.e., $\nu = 0$. We obtain

$$0 = g_{zz} + \left(-\mu + Q(z) + k - k\Psi e^{-\mu\tau \mp \mathrm{i}\varphi N}\right) g, \tag{5.163}$$

which we can split into real and imaginary part:

$$0 = g_{zz} + \left(-\mu + Q(z) + k - k\Psi e^{-\mu\tau}\cos(\varphi N)\right) g, \tag{5.164}$$
$$0 = \pm k\Psi e^{-\mu\tau}\sin(\varphi N) g. \tag{5.165}$$

As in the previous section, we note that real eigenvalues only occur for such N where $\varphi N = 0 \mod \pi$. It follows that $\cos(\varphi N) = \pm 1$, $\sin(\varphi N) = 0$, and the equations simplify to a single equation which reads

$$0 = g_{zz} + \left(-\mu + Q(z) + k \mp k\Psi e^{-\mu\tau}\right) g. \tag{5.166}$$

Hence, we are back to a slightly modified version of Step 3.

Let us first consider the case $N = 0$.

In the case $\Psi = -1$, we can then guarantee stabilization by Lemma 5.24 (or exclude stabilization by Lemma 5.25) from Step 3, and nothing is left to prove.

In the case $\Psi = 1$, we can exclude stabilization by Lemma 5.23 from Step 3, finishing the proof of Theorem 5.28.

We are therefore left with the case $\Psi = -1$, $N > 0$, and $\varphi = m\pi/n$, m odd. Let us now consider $N > 0$ minimal such that $\cos(\varphi N) = -1$, i.e., N minimal and non-zero such that $\varphi N = (mN\pi)/n = 0 \mod \pi$. By construction of the control, mN/n is not an odd integer for any $N < n$.

From the case without control, we know that there exist exactly two eigenfunctions with period $2\pi/n$, where, by hyperbolicity, exactly one eigenfunction belongs to the eigenvalue zero.

We can treat this eigenvalue with the aid of Lemma 5.6 (we use it for the case $Q = N^2$, which corresponds to an eigenvalue zero). We found that the eigenvalue zero persists if control is introduced. Furthermore, we found that real eigenvalues, which appear because of the control term, can only cross zero at $k = 1/\tau$, and in particular, no zero crossings occur for negative feedback gains k. Note that, in this case, we can make this explicit statement about the eigenvalue crossings only because the trivial eigenvalue 0 is known.

Since we have assumed that the dimension of the unstable manifold is exactly $2n-1$, we know that the second eigenfunction with period $2\pi/n$ has a strictly negative eigenvalue, we call it λ_*. It follows that all real new eigenvalues μ associated to this eigenvalue fulfill

$$-\lambda_* = -\mu + k - ke^{-\mu\tau}, \tag{5.167}$$

which we can rearrange to read

$$k(\mu) = \frac{\mu - \lambda_*}{1 - e^{-\mu\tau}}. \tag{5.168}$$

We find that $k(\mu)$ is continuous except for $\mu = 0$, which is the only pole. Furthermore, we find the following limiting values:

$$\lim_{\mu \to -\infty} k(\mu) = 0 \qquad \text{(from above)} \tag{5.169}$$

$$\lim_{\mu \nearrow 0} k(\mu) = -\infty \qquad \text{(exponentially)} \tag{5.170}$$

$$\lim_{\mu \searrow 0} k(\mu) = +\infty \qquad \text{(exponentially)} \tag{5.171}$$

$$\lim_{\mu \to +\infty} k(\mu) = +\infty \qquad \text{(linearly).} \tag{5.172}$$

Since $k(1) < \infty$ and the function k is continuous for positive μ, we can conclude that there exists a $k^* \in \mathbb{R}$ such that, for all feedback gains $k < k^*$, no strictly positive eigenvalues exist. Since the case $k = 0$ yields no strictly positive eigenvalues and the local minimum of $k(\mu)$, $\mu \in (0, \infty)$, is also the global minimum, we can therefore conclude that any restriction gives at worst $k < 0$ as a sufficient condition, which is already included in the restriction $k < \bar{Q}$ from above.

The general case $N > 0$ such that $\cos(\varphi N) = -1$ follows analogously. Here we assume the existence of any $\lambda_{**} < \lambda_*$, and it is easy to see that this imposes no new restrictions on the feedback gain k.

Let us summarize our results in the following lemma:

Lemma 5.31 (Step 4: Conditions on the real eigenvalues)**.** *Consider the real eigenvalues such that there exist 2π-periodic solutions of the delayed equation of Hill's type,*

$$0 = g_{zz} - \mu g + Q(z)g + k\big(g - (-1)e^{-\mu\tau}g(z-\varphi)\big), \tag{5.173}$$

where $Q(z)$ has minimal period π/n, $\bar{Q}\max_{z\in[0,2\pi]} Q(z) > 0$, the spatio-temporal delay is given by $\varphi = \xi - c\tau = m\pi/n$, m odd and co-prime to n, and $\tau > 0$. Assume that there exist exactly $2n-1$ real positive eigenvalues in the case without control, i.e., for $k = 0$.

For a fixed feedback gain k, all real eigenvalues μ are negative if

$$k < -\bar{Q}/2. \tag{5.174}$$

Remark. We can also replace the condition $k < \bar{Q}/2$ by the condition $k < -\lambda_0/2$, where λ_0 is the largest eigenvalue in the equation without control, if this eigenvalue is known explicitly.

5.4.4. Conditions on the complex conjugated eigenvalues

Here we only need to consider the case $\Psi = -1$. Thus, we investigate the following equations of Hill's type:

$$0 = g_{zz} + \left(-\lambda + Q(z) + k + ke^{-\lambda\tau\mp iN\varphi}\right)g, \qquad N = 0,1,2,\ldots \tag{5.175}$$

First, we consider once more the case of zero time delay for which we can easily conclude the following: If $\bar{Q}+k+k\cos(\varphi N) < 0$, then stabilization is successful. Note that $\cos(\varphi N) = \pm 1$ is excluded, because this would yield real eigenvalues, which we have already discussed in the previous section. If known, we can replace $\bar{Q}$ by the eigenvalues $\lambda_N^{1,2}$, i.e., the two eigenvalues belonging to the eigenfunctions having exactly n zeros in the case without control.

As a sufficient condition for successful stabilization we find

$$k < \min\left\{ -\frac{\lambda_N^{1,2}}{1+\cos(\varphi N)} \,\middle|\, N < n \right\}. \tag{5.176}$$

This can always be achieved, since the denominator is strictly bounded away from zero. Note the similarity to the condition in Step 2: We have simply replaced $Q - N^2$ by $\lambda_N^{1,2}$.

To complete the proof in the case of nonzero time delay, we invoke Lemma 5.18 where we replace again $Q - N^2$ by $\lambda_N^{1,2}$. If these eigenvalues are not known explicitly, the upper bound $\bar{Q}$ is sufficient, though the threshold is not sharp.

We have now proved all results from Chapter 2 as well as the results for control schemes of rotation type from Chapter 3.

Chapter 6

Proof for control schemes of reflection type

In this chapter the control triple (output signal, space, time) is of the following form:

$$(\text{multiplication } \Psi, \text{ reflection } \kappa, \text{ time delay } \tau).$$

In other words, the control triple for schemes of reflection type combines a scalar multiplication $\Psi \in \mathbb{R}$ of the output signal and a reflection in space $\kappa : x \mapsto -x$ with a time delay $\tau \geq 0$.

In this case, the reaction-diffusion equation including control takes the form

$$u_t = u_{xx} + f(u) + k\big(u - \Psi u(-x, t - \tau)\big), \tag{6.1}$$

where $k \in \mathbb{R}$ is the variable feedback gain. We assume periodic boundary conditions and the assumptions on f from Section 1.3. Note that the time delay τ can be chosen arbitrarily, since we only consider frozen waves.

In this chapter we prove all the results on control schemes of reflection type from Chapter 3. More specifically, we prove Theorem 3.4 on the successful stabilization of odd frozen waves, Theorem 3.5 on the failed stabilization of even frozen waves, as well as Theorem 3.6 on the successful stabilization of the zero equilibrium.

We fix a scalar reaction-diffusion equation $u_t = u_{xx} + f(u)$, a frozen wave $\mathcal{U}(x)$, as well as a control triple given by the reflection κ, a time delay τ and a scalar multiplication Ψ of the output signal. Again, the only parameter which is varied is the feedback gain $k \in \mathbb{R}$. We want to determine if a feedback gain k exists such that the frozen wave is stable. In the case of existence, we would like to get an estimate on the stabilization region in the feedback gain k.

The linear variational equation is given by

$$v_t = v_{xx} + Q(x)v + k\big(v - \Psi v(-x, t-\tau)\big), \tag{6.2}$$

where $Q(x) = f'(\mathcal{U}(x))$. As in Chapters 4 and 5, we solve the above equation by an exponential Ansatz $v(x,t) = g(x)e^{\lambda t}$, for which we obtain an equation of the form

$$\lambda g = g_{xx} + Q(x)g + k\left(g - \Psi\, e^{-\lambda\tau} g(-x)\right). \tag{6.3}$$

In the following section we will show, using a simple argument by Magnus and Winkler [45], that we can restrict the search of eigenfunctions to even and odd eigenfunctions, respectively.

6.1. Even and odd eigenfunctions

In Section 4.2 we have seen that for twisted standing waves $\mathcal{U}(x) = -\mathcal{U}(-x)$ the function $Q(x)$ is even, $Q(x) = Q(-x)$. The same holds for standing waves of the type $\mathcal{U}(x) = \mathcal{U}(-x)$, in which case it is trivial that $Q(x) = Q(-x)$. These simple observations are the key ingredients to the proof in this chapter. As we will see in the following lemma, we can conclude that all eigenfunctions are either odd or even functions. Interpreted correctly, as we will see in the following section, we re-use many of the results from Chapter 5 and arrive quite quickly at our desired results.

Lemma 6.1 (adapted from Magnus and Winkler [45])**.** *Consider the ordinary differential equation*

$$\lambda g = g_{xx} + Q(x)g + k\left(g - \Psi\, e^{-\lambda\tau} g(-x)\right) \tag{6.4}$$

and assume that $Q(x)$ is even, $Q(x) = Q(-x)$. Let $g_1(x)$ and $g_2(x)$ be the normalized solutions defined by the initial conditions

$$g_1(0) = 1, \quad g_1'(0) = 0, \quad \text{and} \quad g_2(0) = 0, \quad g_2'(0) = 1. \tag{6.5}$$

Whenever a nontrivial solution of period 2π exists, there also exists such a solution which is either odd or even. Therefore, these solutions are necessarily multiples of one of the normalized solutions unless all solutions are periodic with period 2π.

Proof (Magnus and Winkler [45]). Let $g^*(x)$ be any global solution of equation (6.4). Then also $g^*(-x)$ is a solution. By the initial conditions, the normalized solution $g_1(x)$ is even and the other normalized solution $g_2(x)$ is odd. Consider now a solution $g_p(x)$ of period 2π. Then

$$g_3(x) := g_p(x) + g_p(-x), \qquad g_4(x) := g_p(x) - g_p(-x) \tag{6.6}$$

are also 2π-periodic solutions by linearity. Then g_3 is even and g_4 is odd. Unless $g_p \equiv 0$, not both solutions can be identically zero. Therefore, we can conclude that if there exist any solutions with period 2π, then there also exists such a solution which is even or odd, and hence a multiple of one of the normalized solutions. □

We can therefore restrict our search to eigenfunctions which are either odd and fulfill $g(-x) = -g(x)$, or even, fulfilling $g(-x) = g(x)$. Even eigenfunctions fulfill the equation

$$\lambda g = g_{xx} + Q(x)g + k\left(g - \Psi e^{-\lambda\tau} g\right), \tag{6.7}$$

while odd eigenfunctions fulfill the equation

$$\lambda g = g_{xx} + Q(x)g + k\left(g + \Psi e^{-\lambda\tau} g\right). \tag{6.8}$$

Thus, we can use all the results from Chapter 5, Steps 1 and 3. In the case of odd eigenfunctions, we have to replace Ψ by $-\Psi$.

Before we go on to conclude the final results, let us observe the following: There is only one eigenfunction belonging to an unstable eigenvalue, since we have assumed that the unstable dimension is exactly one. By Section 4.1 we know that the eigenfunction belonging to the largest eigenvalue has no zeros. It must therefore be an even function.

Furthermore, the eigenfunction belonging to the eigenvalue zero, $\mathcal{U}_x$, is odd if the wave $\mathcal{U}$ is even, and even if the wave $\mathcal{U}$ is odd.

6.2. Conclusions

In this section we collect the necessary material to prove Theorems 3.4–3.6. For better readability, there is one subsection for the concluding remarks of each theorem.

6.2.1. Successful stabilization of twisted standing waves

Roughly speaking, the first theorem of this section tells us that control schemes of reflection type can stabilize the zero equilibrium in the linear variational equation, if the reflection symmetry is odd and the unstable dimension is exactly one:

Theorem 6.2 (Step 1: Successful stabilization of the zero equilibrium via odd reflections)**.** *Consider the linear variational equation*

$$v_t = v_{xx} + Q(x)v + k\big(v - (-1)v(-x, t - \tau)\big), \tag{6.9}$$

with periodic boundary conditions, $\bar{Q} > 0$, and real feedback gain k. Assume that there exists exactly one eigenvalue with strictly positive real part if $k = 0$.

If the feedback gain k fulfills $k < -\bar{Q}/2$, then there exists a time delay $\tau^* = \tau^*(k)$ such that the zero equilibrium of equation (6.9) *is stable for all* $0 \leq \tau < \tau^*(k)$.

Remark. This theorem proves Theorem 3.4 on the success of control schemes of reflection type for twisted standing waves.

Proof. Using the fact that we only need to consider odd or even eigenfunctions, we investigate the following equation for even eigenfunctions,

$$\lambda g = g_{xx} + Q(x)g + k\left(g + e^{-\lambda\tau} g\right), \tag{6.10}$$

and the following equation for odd eigenfunctions,

$$\lambda g = g_{xx} + Q(x)g + k\left(g - e^{-\lambda\tau} g\right). \tag{6.11}$$

Equation (6.10) can be treated with Lemma 5.24 from Step 3 of the previous chapter to guarantee strictly negative real eigenvalues for $k < \bar{Q}/2$ belonging to even eigenfunctions. Note that this includes the only unstable eigenvalue as well as the eigenvalue zero. The complex conjugated eigenvalues are taken care of by Lemma 5.9, where we replace $Q - N^2$ by the eigenvalues $\lambda_N^{1,2}$ and choose $\Psi = -1$.

All eigenvalues fulfilling equation (6.11) in the case without control ($k = 0$) are strictly negative, we fix an arbitrary eigenvalue $\lambda_* < 0$. All real new eigenvalues emerging from this eigenvalue can be found as solutions to the equation

$$-\lambda_* = -\mu + k - ke^{-\mu\tau}. \tag{6.12}$$

We have studied this equation in detail in Subsection 5.4.3, therefore we will not repeat the analysis here. We conclude that all real eigenvalues μ are negative for $k < 0$. The complex conjugated eigenvalues associated to equation (6.11) are taken care of by Lemma 5.9 once more: We replace $Q - N^2$ by the eigenvalues $\lambda_N^{1,2}$ and choose $\Psi = 1$, completing the proof of Theorem 6.2. □

6.2.2. Failed control of standing waves

If we use an even reflection symmetry, the control fails to stabilize the zero equilibrium in the linear variational equation:

Theorem 6.3 (Step 1: Failure of control of the zero equilibrium via odd reflections)**.** *Consider the linear variational equation*

$$v_t = v_{xx} + Q(x)v + k\big(v - v(-x, t-\tau)\big), \tag{6.13}$$

with periodic boundary conditions, real feedback gain k, and arbitrary but fixed time delay $\tau \geq 0$.

Then the zero equilibrium of equation (6.13) *is unstable for all feedback gains $k \in \mathbb{R}$.*

Remark. This theorem proves Theorem 3.5 for standing waves.

Proof. It suffices to show that there exists at least one eigenvalue with with strictly positive real part. Without control, i.e., $k = 0$, this eigenvalue belongs to an even eigenfunction because the eigenfunction has no zeros. Let us therefore restrict our attention to even eigenfunctions. As we have seen in the previous section, even eigenfunctions fulfill the equation

$$\lambda g = g_{xx} + Q(x)g + k\left(g - e^{-\lambda\tau} g\right), \tag{6.14}$$

thus, we can invoke Lemma 5.23, telling us that there indeed exists an eigenvalue with strictly positive real part, which proves the theorem. □

6.2.3. Successful stabilization of the zero equilibrium

The homogeneous zero equilibrium has both an even and an odd reflection symmetry, which indicates that stabilization is possible, using Theorem 6.2. However, more is possible, as we see in the following theorem:

Theorem 6.4 (Step 1: Successful stabilization of the zero equilibrium)**.** *Consider the homogeneous zero equilibrium of the linear equation $v_t = v_{xx} + Qv$, with periodic boundary conditions.*

Choose some real number $\Psi \neq 1$ and a time delay $\tau \geq 0$.

If the feedback gain $k \in \mathbb{R}$ fulfills the conditions

$$k\left(1 - \Psi e^{-\mu\tau}\right) < \mu - Q \qquad \text{for all } \mu > 0, \tag{6.15}$$

$$k\left(1 + \Psi e^{-\mu\tau}\right) < \mu \qquad \text{for all } \mu > 0, \tag{6.16}$$

as well as the condition

$$|k\Psi\tau| < 1, \tag{6.17}$$

and if the unstable dimension of the zero equilibrium is exactly one, then the homogeneous zero equilibrium is stable in the equation including control,

$$v_t = u_{xx} + Qv + k\big(v - \Psi v(-x, t-\tau)\big). \tag{6.18}$$

In particular, if $\Psi = 0$, then the zero equilibrium is stable for $k < -Q$.

Remark. Theorem 6.4 proves Theorem 3.6, in fact, the two theorems are almost identical.

Proof. Real eigenvalues associated to even eigenfunctions can be treated with Lemma 5.6 from Step 1, Chapter 5. The same lemma can be invoked for real eigenvalues associated to odd eigenfunctions, but with $Q = 0$ and Ψ replaced by $-\Psi$, this yields the second condition. We treat the complex conjugated eigenvalues with Lemma 5.9, also from Step 1, Chapter 5. There we obtained the condition $|k\Psi\tau| < 1$, independent of Q and the sign of Ψ, completing the proof of Lemma 6.4. □

In this way, we have completed the proofs for control types of reflection type quickly, using a few key lemmata from Steps 1 and 3 from Chapter 5. The main aspects for successful control are the same for control schemes of rotation and reflection type. Therefore, also in reaction-diffusion systems with more elaborate symmetries, we can expect similar conditions. To apply our results directly to such systems, it would be useful to construct controls in such a way that the variational equations decouple in a suitable coordinate system.

Chapter 7

Applying the control triple method

The aim of this chapter is to apply the new control triple method to a specific scalar reaction-diffusion equation; the Chafee-Infante equation. Two main questions arise: First, which noninvasive control terms can we use in the control triple method? Second, how can we decide whether a specific control term is successful?

We proceed as follows: In Section 7.1 we give a short introduction to the Chafee-Infante equation. In Section 7.2 we apply the control triple method to the homogeneous equilibria. We include a bifurcation analysis in order to better understand the control mechanism. We also compare the control schemes of rotation and reflection type. In Section 7.3 we apply the control triple method to the frozen waves of the Chafee-Infante equation. We focus on comparing different control schemes of both the rotation and the reflection type. We give a short summary of this chapter in Section 7.4.

7.1. The Chafee-Infante equation

The Chafee-Infante equation [9] is a scalar reaction-diffusion equation of the form

$$u_t = u_{xx} + \alpha u \left(1 - u^2\right), \tag{7.1}$$

where $\alpha \in \mathbb{R}$ is a bifurcation parameter. The reaction term $f_\alpha(u) = \alpha u(1 - u^2)$ is nonlinear (more precisely: cubic), but no advection term of the type u_x is present. For every $\alpha > 0$, we find exactly three homogeneous equilibria: $\mathcal{U} \equiv 0$ and $\mathcal{U} \equiv \pm 1$. No periodic orbits occur. The frozen waves $\mathcal{U}(x)$ are given by the 2π-periodic solutions of the ordinary differential equation

$$0 = \mathcal{U}_{xx} + \alpha \mathcal{U} \left(1 - \mathcal{U}^2\right). \tag{7.2}$$

For our purposes, it is not necessary to know the frozen waves explicitly. Instead, it suffices to collect a few key properties, which are easily noted. First, the frozen waves all lie between the two homogeneous equilibria $\mathcal{U} \equiv \pm 1$ and they oscillate around the third homogeneous equilibrium $\mathcal{U} \equiv 0$. In particular, we can conclude $-1 < \mathcal{U}(x) < 1$ for all $x \in [0, 2\pi]$. Moreover, we can conclude that there exists $x_0 \in [0, 2\pi]$ such that $\mathcal{U}(x_0) = 0$. Furthermore, note that the nonlinearity $f_\alpha(\mathcal{U}) = a\mathcal{U}\left(1 - \mathcal{U}^2\right)$ is odd. Hence, all frozen waves $\mathcal{U}(x)$ of period $2\pi/n$ additionally fulfill $\mathcal{U}(x) = -\mathcal{U}(x - \pi/n)$, as well as $\mathcal{U}(x_0 + x) = -\mathcal{U}(x_0 - x)$ where $\mathcal{U}(x_0) = 0$. See also the discussion in Chapter 1.

Equation (7.2) is an Hamiltonian equation with potential

$$F(\mathcal{U}) = \alpha \left(\tfrac{1}{2}\mathcal{U}^2 - \tfrac{1}{4}\mathcal{U}^4\right), \tag{7.3}$$

and Hamiltonian

$$H(\mathcal{U}, \mathcal{U}_x) = \tfrac{1}{2}\mathcal{U}_x + \alpha \left(\tfrac{1}{2}\mathcal{U}^2 - \tfrac{1}{4}\mathcal{U}^4\right). \tag{7.4}$$

Its solutions with fixed energy E are determined by

$$\mathcal{U}_x = \pm\sqrt{2\,(E - F(\mathcal{U}))}. \tag{7.5}$$

The potential together with the energy levels that yield solutions of minimal periods 2π (red), π (blue), and $2\pi/3$ (dashed violet) are drawn in Figure 7.1. The corresponding solutions can be found in Figure 7.2 with the same color scheme.

The linear variational equation at any frozen wave $\mathcal{U}(x)$ takes the form

$$v_t = v_{xx} + \alpha \left(1 - 3\mathcal{U}(x)^2\right) v, \tag{7.6}$$

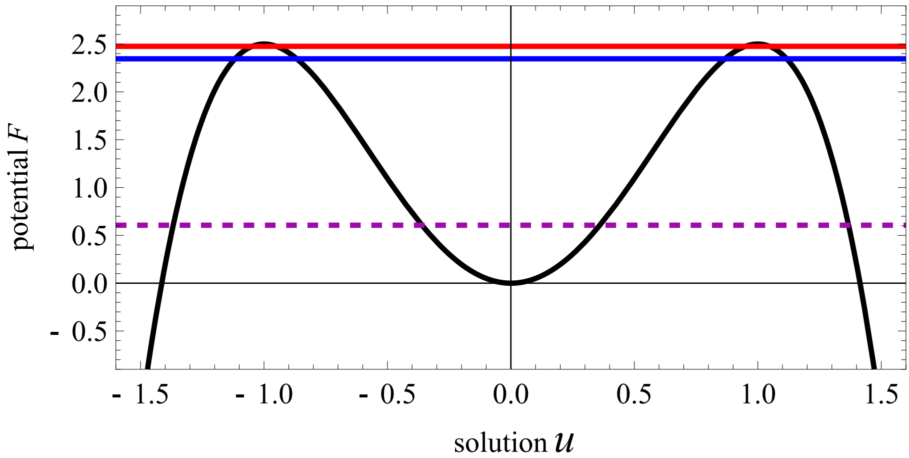


Figure 7.1.: Hamiltonian potential $F(\mathcal{U})$ (black) versus $\mathcal{U}$ for $\alpha = 10$. The red, blue, and dashed violet lines indicate the energy levels for which frozen waves of period 2π, π, and $2\pi/3$ occur, respectively.

hence we find the function $Q(x) = f'_\alpha(\mathcal{U}(x)) = \alpha\left(1 - 3\mathcal{U}^2(x)\right)$.

Let us continue by investigating the stability of the homogeneous equilibria. For the equilibria $\mathcal{U} \equiv \pm 1$ we find $v_t = v_{xx} - 2\alpha v$, and we can therefore find the eigenvalues directly as $\lambda = -2\alpha - k^2$, $k \in \mathbb{N}_0$, where all eigenvalues except for the first have multiplicity 2. We conclude that the equilibria $\mathcal{U} \equiv \pm 1$ are both stable for all parameters $\alpha > 0$.

For the equilibrium $\mathcal{U} \equiv 0$, we find the linear variational equation $v_t = v_{xx} + \alpha v$. We can therefore calculate the eigenvalues easily as $\lambda = \alpha - k^2$, $k \in \mathbb{N}_0$, where all eigenvalues except for $\lambda = a$ have multiplicity 2. In particular, we can conclude that the equilibrium $\mathcal{U} \equiv 0$ is unstable for all $\alpha > 0$. Bifurcations occur for $\alpha = n^2$, $n \in \mathbb{N}$, where frozen waves of period $2\pi/n$ bifurcate in direction of increasing α. The unstable dimension, i.e., the number of eigenvalues with strictly positive real part, of $\mathcal{U} \equiv 0$ is given by $2n - 1$ for $\alpha \in \left((n-1)^2, n^2\right]$. The frozen waves inherit this instability, thus their unstable dimension is either $2n - 1$ or

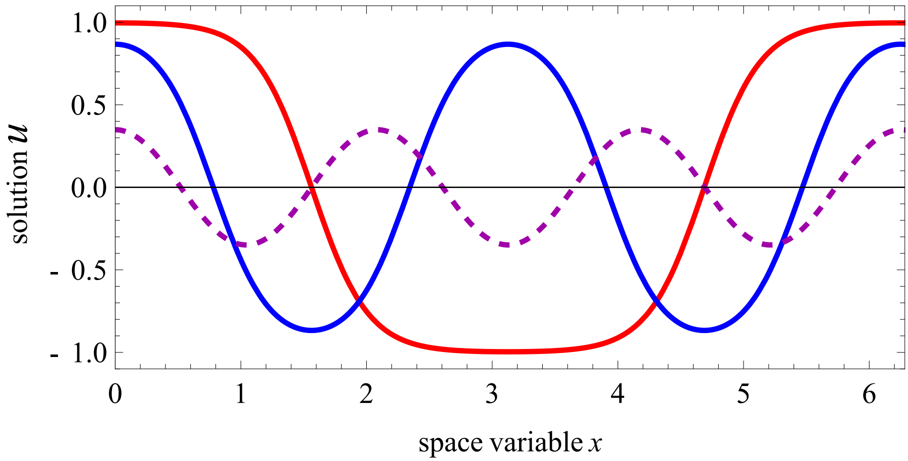


Figure 7.2.: Frozen waves $\mathcal{U}(x)$ for $a = 10$ (red for period 2π, blue for period π, dashed violet for period $2\pi/3$).

$2n$. The unstable dimension is determined more precisely in relation with the *period map*. Here the period map is defined as the minimal period $T(\beta, 0)$ of a solution of (7.2) with initial conditions $\mathcal{U}(0) = \beta$, $\mathcal{U}_x(0) = 0$, see for example [18]. The period map together with the periods 2π, π, and $2\pi/3$ are shown in Figure 7.3.

Theorem 7.1 (Unstable dimension versus period map [18])**.** *The sign of*

$$\partial_\beta T(\beta, 0)|_{\beta=0} \neq 0 \tag{7.7}$$

decides the unstable dimension of a hyperbolic frozen wave $\mathcal{U}$ of minimal period $2\pi/n$ in the Chafee-Infante equation to be given by

$$i(\mathcal{U}) = 2n - 1 \iff \partial_\beta T(\beta, 0)|_{\beta=0} > 0, \tag{7.8}$$

$$i(\mathcal{U}) = 2n \iff \partial_\beta T(\beta, 0)|_{\beta=0} < 0, \tag{7.9}$$

where $i(\mathcal{U})$ denotes the unstable dimension (Morse index) of the wave $\mathcal{U}$.

Remark. (a) The actual theorem in [18] holds for more general equations than the Chafee-Infante equation.

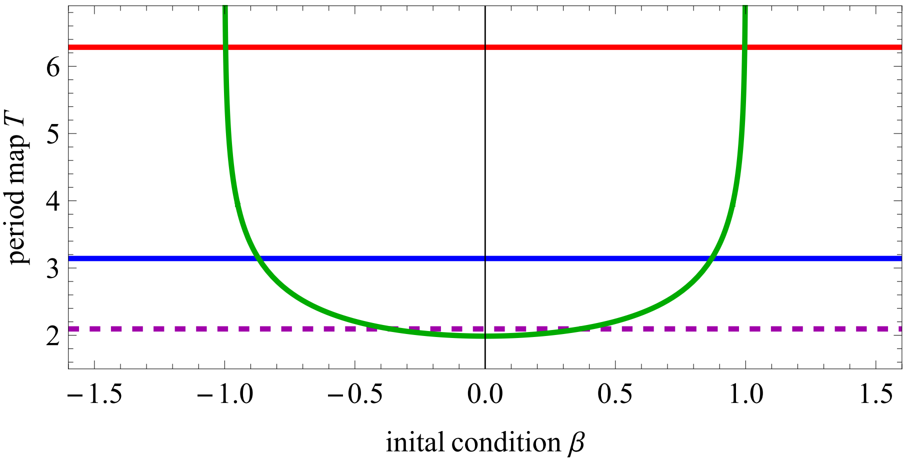


Figure 7.3.: Period map $T(\beta, 0)$ (green) for $\alpha = 10$. The horizontal lines indicate periods 2π (red), π (blue), and $2\pi/3$ (dashed violet).

(b) Throughout this thesis (except for the linear case) we have assumed that the rotating and frozen waves are hyperbolic. In fact, they are hyperbolic if and only if $\partial_\beta T(\alpha, 0)|_{\beta=0} \neq 0$ [18].

For the Chafee-Infante equation, the period increases with amplitude. Therefore, all the frozen waves of the Chafee-Infante equation are hyperbolic. Hence, we can conclude that the unstable dimension of all frozen waves is $2n - 1$. Consequently, the waves fulfill the requirements of Theorem 3.1.

7.2. Control of homogeneous equilibria

In this section we add control terms of rotation and reflection type to the Chafee-Infante equation. It is our aim to observe the bifurcations which lead to the stabilization of the zero equilibrium $\mathcal{U} \equiv 0$ and to the destabilization of the homogeneous equilibria $\mathcal{U} \equiv \pm 1$. We consider the following equation including control of rotation type:

$$u_t = u_{xx} + \alpha u \left(1 - u^2\right) + k\big(u - \Psi u(x - \xi, t - \tau)\big), \tag{7.10}$$

where $k \in \mathbb{R}$, $\xi \in S^1$, $\tau \geq 0$ and $\Psi \in \mathbb{R}$ for the stabilization of the zero equilibrium, or $\Psi = 1$ for the destabilization of the equilibria $\mathcal{U} \equiv \pm 1$. Moreover, consider the following equation including control of reflection type:

$$u_t = u_{xx} + \alpha u \left(1 - u^2\right) + k\big(u - \Psi u(-x, t - \tau)\big). \tag{7.11}$$

We search for homogeneous equilibria induced by the control term. For both control types, all homogeneous equilibria fulfill the equation

$$0 = \alpha \mathcal{U} \left(1 - \mathcal{U}^2\right) + k\big(\mathcal{U} - \Psi \mathcal{U}\big). \tag{7.12}$$

Note that $\mathcal{U} \equiv 0$ is always a solution. The other homogeneous equilibria are given by

$$\mathcal{U} \equiv \pm\sqrt{1 + k(1 - \Psi)/\alpha}. \tag{7.13}$$

First, consider the case $\Psi = 1$. Note that in this case, the homogeneous equilibria $\mathcal{U} \equiv \pm 1$ remain unchanged for all feedback gains k. This corresponds to the fact that the control is noninvasive on these equilibria, which it is not for arbitrary Ψ.

Let us compare the bifurcation of equilibria to the condition from Theorem 3.3 for the stabilization of the equilibrium $\mathcal{U} \equiv 0$. For $\Psi \neq 1$ we find a supercritical pitchfork bifurcation at $k = -\alpha/(1 - \Psi)$. Here two equilibria branch from the zero equilibrium, they exist for all feedback gains $k > -\alpha/(1 - \Psi)$. Their absolute value $|\mathcal{U}|$ increases with increasing k. For $k = 0$, the equilibria coincide with the known equilibria $\mathcal{U} \equiv \pm 1$ from the uncontrolled equation. Let us compare the location of the pitchfork bifurcation to the conditions for stabilization from Theorem 3.3. As a necessary condition, we found

$$k\left(1 - \Psi \cos(\xi N)\right) < N^2 - f'_\alpha(0) \qquad \text{for all} \quad N \in \mathbb{N}, \tag{7.14}$$

where $f'_\alpha(0) = \alpha\left(1 - 3 \cdot 0^2\right) = \alpha$. In particular, the condition must hold for $N = 0$:

$$k\left(1 - \Psi\right) \leq -\alpha, \tag{7.15}$$

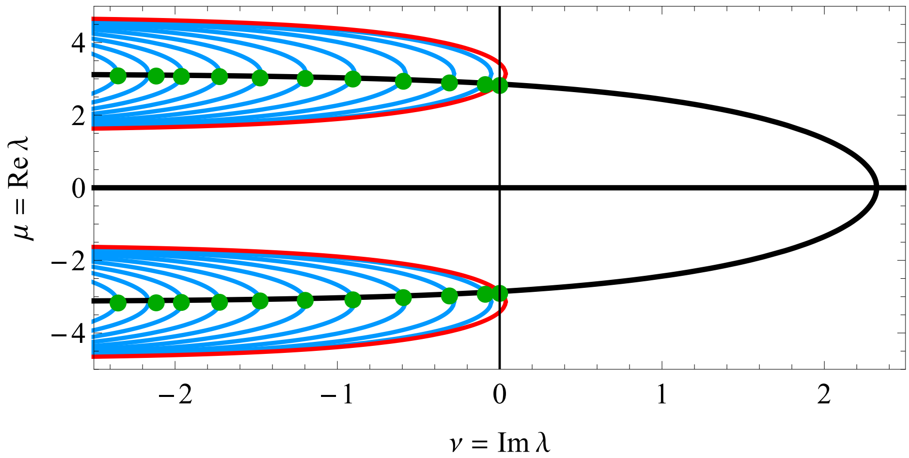


Figure 7.4.: Destabilization of the homogeneous equilibria $\mathcal{U} \equiv \pm 1$: Positions of the eigenvalues (green dots) for a fixed feedback gain $k = 10.2$. Here $\alpha = 10$, and the control triple is given by $\Psi = 1$, $\xi = 0$, and $\tau = 1.0$. The curve $\mu(\nu)$ is drawn in black, while $\nu(\mu)$ is drawn in red for $N = 0$ and in blue for all $N \geq 1$. Compare Step 1, Chapter 5.

which is indeed fulfilled for all feedback gains $k \leq -\alpha/(1-\Psi)$. We see that the condition from Theorem 3.3 is sharp for the stabilization of the zero equilibrium in the Chafee-Infante equation.

Let us next investigate shortly the destabilization of the equilibria $\mathcal{U} \equiv \pm 1$, where we simplify to the case $\xi = 0$, and, by noninvasiveness, $\Psi = 1$. Here we can exclude bifurcation through stationary orbits by the following argument: All homogeneous equilibria and frozen waves $\mathcal{U}(x)$, including those which are induced by the control term, are 2π-periodic solutions of the equation

$$0 = \mathcal{U}_{xx} + \alpha \mathcal{U}\left(1 - \mathcal{U}^2\right) + k\big(\mathcal{U} - \mathcal{U}\big). \tag{7.16}$$

The last term cancels and no frozen waves are induced by the control term. Therefore, any bifurcating solution must be non-stationary. This is confirmed by the direct calculation of the eigenvalues (compare Step

1, Chapter 5), where a destabilizing Hopf bifurcation occurs. See Figure 7.4 for illustration.

In conclusion, destabilization of the equilibria $\mathcal{U} \equiv \pm 1$ using $\Psi = 1$ cannot occur via Pitchfork bifurcation, since no other homogeneous equilibria occur.

7.3. Control of frozen waves

In this section we compare the success of the control schemes of rotation and reflection type for the stabilization of the frozen waves of the Chafee-Infante equation, see Theorems 3.1 and 3.4. Let us start by investigating control terms of rotation type:

$$u_t = u_{xx} + \alpha u \left(1 - u^2\right) + k\big(u - (-1)u(x - m\pi/n, t - \tau)\big), \qquad (7.17)$$

where $k \in \mathbb{R}$ is the feedback gain and $\tau \geq 0$ is the time delay. The spatial delay is given by $\xi = m\pi/n$. Note that the spatial delay ξ is the same as the spatio-temporal delay φ; $\varphi = \xi - c\tau = \xi - 0\tau = \xi$. This is only the case for frozen waves where the wave speed c is zero. The corresponding linear variational equation is given by

$$v_t = v_{xx} + \alpha \left(1 - 3\mathcal{U}(x)^2\right) v + k\big(v - (-1)v(x - m\pi/n, t - \tau)\big). \qquad (7.18)$$

To obtain an estimate on the feedback gain k in order to guarantee a successful stabilization, we determine the parameter $\bar{Q} = \max_{x\in[0,2\pi]} Q(x) = \alpha(1-3\mathcal{U}(x)^2)$. The maximum of this function can thus be computed in a surprisingly easy way: It is given by $\bar{Q} = \alpha > 0$, as we find the maximum of the function Q at those x_0 with $\mathcal{U}(x_0) = 0$. Thus, the maximum is given by the value at the homogeneous equilibrium $\mathcal{U} \equiv 0$ around which the frozen wave $\mathcal{U}(x)$ oscillates. In particular, $\bar{Q}$ does not depend on a specific frozen wave.

We now consider the three cases corresponding to the minimal periods 2π, π, and $\pi/2$ separately.

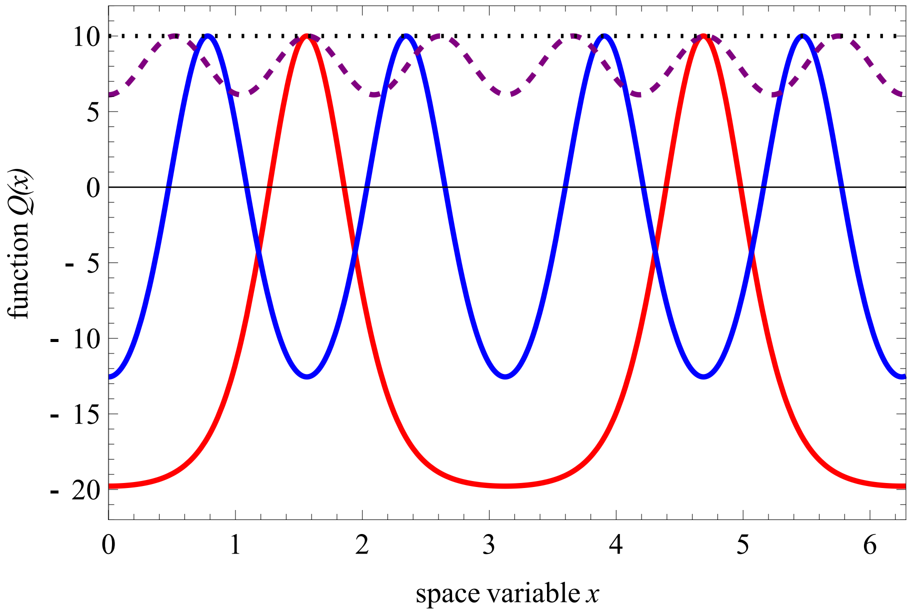


Figure 7.5.: The function $Q(x)$ for the frozen waves $\mathcal{U}(x)$ (red for period 2π, blue for period π, dashed violet for period $2\pi/3$) for $a = 10$. Note that $\bar{Q} = a = 10$ for all waves (black dotted line).

For the wave with minimal period 2π, we find $n = 1$ (the minimal period is $2\pi/n$). By Theorem 7.1, the unstable dimension is given by $2n - 1 = 2 \cdot 1 - 1 = 1$. We have to choose m odd and co-prime to $n = 1$, and hence, we can select any odd m. By 2π-periodicity in x, $m = 1$ is the only relevant possibility. Therefore, $\xi = \varphi = \pi$. From Lemma 5.31, we can conclude that all real eigenvalues are negative if the feedback gain k fulfills

$$k < -\bar{Q}/2 = -\alpha/2. \tag{7.19}$$

Conditions on the complex conjugated eigenvalues can be found in Step 3, Chapter 5:

$$|k| < 1/\tau. \tag{7.20}$$

Hence, stabilization can always be achieved for a small enough time delay.

Let us next consider the frozen wave with minimal period π. Here we have $n = 2$, and the unstable dimension is $2n - 1 = 2 \cdot 2 - 1 = 3$ by Theorem 7.1. Again, we have to choose m odd and co-prime to $n = 2$, and we can hence select any odd m. The relevant spatio-temporal delay φ thus takes the values $\pi/2$ and $3\pi/2$. From Lemma 5.31, we conclude that all real eigenvalues are negative if the feedback gain k fulfills

$$k < -\bar{Q}/2 = -\alpha/2. \tag{7.21}$$

Furthermore, in Subsection 5.4.4, we found as a sufficient condition for the complex conjugated eigenvalues to have negative real part:

$$k < \min_{N<2} \left\{ -\frac{\lambda_N^{1,2}}{1 + \cos(\varphi N)} \right\}, \tag{7.22}$$

if the time delay is not too large. Here the $\lambda_N^{1,2}$ are the two eigenvalues which correspond to the two eigenfunctions with exactly N zeros. In the case $N = 0$, only one such eigenfunction exists, and consequently, only one eigenvalue, which we call λ_0 in the sequel. The $\lambda_N^{1,2}$ are bound from above by α, see also Step 3, Chapter 5. We find $\cos(\varphi \cdot 0) = 1$, $\cos(\varphi \cdot 1) = 0$. We can therefore conclude that $k < -\alpha$ is a sufficient condition for stabilization if the time delay is small enough. Note that the feedback gain necessary for stabilization does not depend on m. However, this is an exceptional case, and therefore, we next consider the case $n = 4$, which is the smallest n such that m influences the stability conditions.

Let us now consider the wave with minimal period $\pi/2$. Here $n = 4$, and the corresponding unstable dimension is given by $2n - 1 = 2 \cdot 4 - 1 = 7$. Again, we have to choose m odd and co-prime to $n = 4$, and we can hence use any odd m again. The relevant values of the spatio-temporal delay φ thus take the values $\pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$. Here $\varphi = \pi/4$ is equivalent to $\varphi = 7\pi/4$, and $\varphi = 3\pi/4$ is equivalent to $\varphi = 5\pi/4$. This is due to the fact that $\cos(\theta) = \cos(2\pi - \theta)$ for all $\theta \in [0, 2\pi]$. Once more, if the sufficient condition

$$k < \min_{N<4} \left\{ -\frac{\lambda_N^{1,2}}{1 + \cos(\varphi N)} \right\} \tag{7.23}$$

is fulfilled, then the complex conjugated eigenvalues have negative real part for small enough time delay, see Subsection 5.4.4. Let us calculate the values of the cosine-function for $\varphi = \pi/4$ and $\varphi = 3\pi/4$:

$$\cos(0\frac{\pi}{4}) = 1; \quad \cos(1\frac{\pi}{4}) = +\frac{1}{\sqrt{2}}; \quad \cos(2\frac{\pi}{4}) = 0; \quad \cos(3\frac{\pi}{4}) = -\frac{1}{\sqrt{2}},$$
$$\cos(0\frac{\pi}{4}) = 1; \quad \cos(1\frac{3\pi}{4}) = -\frac{1}{\sqrt{2}}; \quad \cos(2\frac{3\pi}{4}) = 0; \quad \cos(3\frac{3\pi}{4}) = +\frac{1}{\sqrt{2}}.$$

In the case $m = 1$, we obtain the conditions

$$k < -\lambda_0 \cdot 1/2, \tag{7.24}$$
$$k < -\lambda_1^{1,2} \cdot (2 - \sqrt{2}), \tag{7.25}$$
$$k < -\lambda_2^{1,2} \cdot 1, \tag{7.26}$$
$$k < -\lambda_3^{1,2} \cdot (2 + \sqrt{2}). \tag{7.27}$$

In contrast, in the case $m = 3$, we obtain the conditions

$$k < -\lambda_0 \cdot 1/2, \tag{7.28}$$
$$k < -\lambda_1^{1,2} \cdot (2 + \sqrt{2}), \tag{7.29}$$
$$k < -\lambda_2^{1,2} \cdot 1, \tag{7.30}$$
$$k < -\lambda_3^{1,2} \cdot (2 - \sqrt{2}). \tag{7.31}$$

This conditions are not equivalent, and as the most important point of this short analysis, we conclude that the parameter m matters in fact for successful control! We see that $m = 1$ and $m = 3$ are equivalent from the viewpoint of equivariance, but they are *not equivalent from the viewpoint of control theory.* Since we need to distinguish clearly between the concepts of equivariance and control, we use the control triple notation for our new spatio-temporal feedback control schemes.

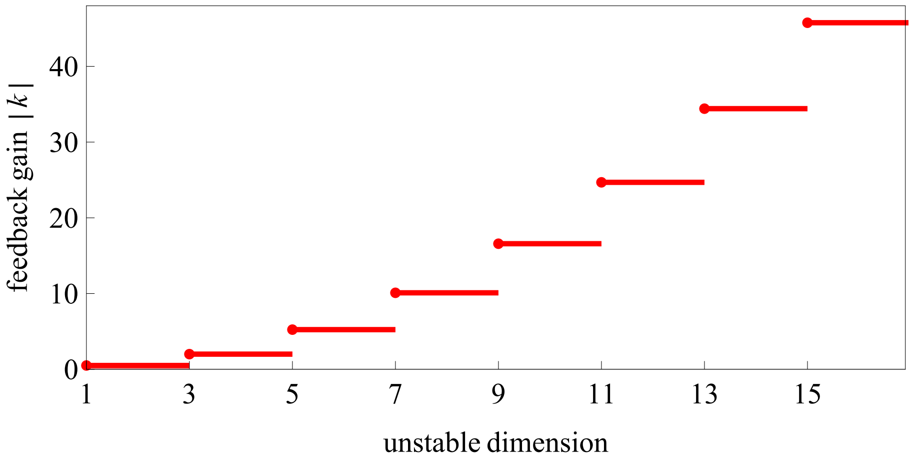


Figure 7.6.: Feedback gain versus unstable dimension, $|k| > \bar{Q}/(1 + \cos(\pi/n))$. We have scaled $\bar{Q} = 1$. Stabilization can be guaranteed for (negative) feedback gains with absolute value larger than the depicted threshold. Note the quadratic asymptotic behavior $|k| \approx 2\bar{Q}n^2/\pi^2$.

Note that we can indeed find feedback gains k such that stabilization is possible for arbitrary $n \in \mathbb{N}$ for controls of rotation type. In Figure 7.6 we have sketched the "worst case" feedback gain k, i.e., the feedback gain k sufficient for stabilization if only the parameter $\bar{Q}$ is known. In this case, the feedback strength increases quadratically with the unstable dimension, asymptotically for large unstable dimensions $2n - 1$. To see this, set all $\lambda_N^{1,2} = \bar{Q}$. The minimum value of $-\frac{\lambda_N^{1,2}}{1+\cos(\varphi N)}$ is obtained if $\cos(\varphi N)$ is closest to 1, i.e., $\cos(\pi/n)$ (this value is actually obtained by some N). We approximate $\cos(\pi/n) \approx 1 - \pi^2/(2n^2)$ to obtain the approximation

$$k \approx -2\bar{Q}n^2/\pi^2. \tag{7.32}$$

Thus, indeed, we can stabilize arbitrarily high unstable dimension, and the feedback gain grows at most quadratically.

To finish this chapter, we shortly consider controls of reflection type,

$$u_t = u_{xx} + \alpha u \left(1 - u^2\right) + k\big(u - (-1)u(-x, t - \tau)\big). \tag{7.33}$$

Here the stabilization of the frozen wave is only possible if the unstable dimension is exactly one, see Theorem 3.4 and the proof in Chapter 6. Only the wave with minimal period 2π has unstable dimension 1. Then the wave is stable for $k < \bar{Q}/2 = -\alpha/2$ and a small enough time delay.

7.4. Summary

First, we have investigated the Chafee-Infante equation, which has three homogeneous equilibria, two stable equilibria and one unstable zero equilibrium. Frozen waves with n zeros bifurcate from the zero equilibrium, their unstable dimension is $2n - 1$.

We have then successfully applied the control triple method to the stabilization of the zero equilibrium, and we have seen that the rotation type and the reflection type yield the same result. A bifurcation analysis has shown that the conditions of stabilization from Theorem 3.3 (Successful stabilization of the zero equilibrium) are sharp. Destabilization of the homogeneous equilibria occurs via bifurcation of non-stationary orbits.

Last, the success of the control method has also been confirmed for the frozen waves. Using the reflection type, we can stabilize the wave with exactly one unstable dimension. This is in contrast to the control schemes of rotation type, where we have seen that we can stabilize arbitrarily high unstable dimensions. Moreover, we have seen that the necessary feedback gain grows at most quadratically, asymptotically for large unstable dimensions $2n - 1$. Finally, we want to draw attention to the important difference between the equivariant description and the control triple method. We have seen an explicit example where two equivalent equivariant descriptions inspire two different control triples and also yield different control results. This shows the strong need for a separate notation for control purposes, fulfilled by the control triple method.

Chapter 8

Conclusion

To conclude this thesis, we comment on the control triple method from a variety of perspectives. We present a short overview of our aims, methods, and results in Section 8.1. In Section 8.2 we discuss the control triple method in a larger framework. In Section 8.3 we indicate open problems and give an outlook on further research.

8.1. Overview

It is the main goal of this dissertation to introduce a new concept of spatio-temporal feedback control for partial differential equations. To this aim, we introduce the notion of a *control triple* which replaces the mere delaying of time, as used by Pyragas [58] and many others. The control triple defines how we transform *output signal*, *space*, and *time* in the control term such that the control term is noninvasive on the desired unstable orbit. The control triple method aims to stabilize an unstable orbit.

This new Ansatz extends the Pyragas control scheme, and it is especially well suited for the control of partial differential equations. It incorporates the spatio-temporal patterns of the equilibria and periodic orbits into the control term. The control triple method is model-independent.

Having introduced the control triple method, which should serve as a general method for the control of partial differential equations, we conduct a systematic investigation of the control triple method in the context of scalar reaction-diffusion equations on the circle. We distinguish two types of control terms: the rotation type and the reflection type.

For *control schemes of rotation type* we show that spatio-temporal delays of *half* the spatial period combined with a small time delay and a sign change in the output signal are successful in the stabilization of equilibria and periodic orbits. In contrast, those control terms which use a *full* spatial period, and consequently no sign change of the output signal, fail their task of stabilization for every time delay. As a corollary, these results include the failure of Pyragas control for scalar reaction-diffusion equations.

For *control schemes of reflection type*, we prove the stabilization for those frozen waves with an *odd* reflection symmetry (twisted standing waves), but not for those with an *even* reflection symmetry (standing waves). Here again, control is successful only if a sign change of the output signal is incorporated into the control triple.

For both control types, we succeed in stabilizing the zero equilibrium by choosing an arbitrary scalar multiplication as transformation of the output signal, rotations or reflections in space, and a small time delay.

The proof of stabilization for control schemes of rotation type uses a modified version of Hill's equation with spatio-temporal delay. We observe an amazing interplay of eigenfunctions, oscillation numbers (zero numbers), delay (both spatial and temporal), and symmetry: The spatio-temporal delay is an integer multiple of the intrinsic period of Hill's equation. This fact allows us to conduct a detailed stability analysis and prove the results on the success of the control triple method and the failure of Pyragas control. The proof of stabilization for control schemes of reflection type follows directly from this proof if it is applied to the even and odd eigenfunctions separately.

The application of the control triple method to the Chafee-Infante equation confirms that, indeed, only a few key properties of the frozen waves and homogeneous equilibria are necessary to construct valid and successful control triples. We can select and stabilize any frozen wave occurring in the Chafee-Infante equation. The absolute value of the feedback gain necessary for stabilization grows at most quadratically with the unstable dimension. In particular, note that we are able to stabilize waves with an arbitrarily high unstable dimension.

8.2. Discussion

The aim of this discussion is to take a step back and discuss and comment our results in the general framework of time-delayed feedback control.

Due to the combination of time delay, which results in an infinite-dimensional equation, and the need for concrete results in control theory, only few analytical results on Pyragas control and related control schemes have been obtained up to date. As all results in this thesis are analytical in nature, they expand our knowledge on the mechanisms of time-delayed feedback control and it is indispensable to connect the new results to the few facts which are known from the literature. In this way, the control triple method as well as the investigations in the context of scalar reaction-diffusion equations are an important step towards the ultimate goal of understanding the underlying principles of feedback control.

To underline these general principles, we have compared our results for *partial* differential equations to results for *ordinary* differential equations. We have found a number of surprisingly close connections on a deeper level: Comparing with the *odd-number limitation* for non-autonomous ordinary differential equations in Chapter 2, we found that a similar mechanism also prevents stabilization via Pyragas control in scalar reaction-diffusion equations. Comparing with *Pyragas control of an unstable focus* in Chapter 3, we found a stunning analogy between spatial delay in reaction-diffusion equations and the eigenfrequency of the unstable focus. Half-period feedback schemes are successful in both cases, while full-period feedback schemes are not. The same holds if we compare the control triple method to *equivariant Pyragas control*, see Chapter 3. The equivariant approach holds many possibilities of applying the control triple methods to general partial differential equations.

In most situations, the presence of a time delay in a dynamical system is seen as a burden, as it greatly increases the dimensionality and the complexity of a dynamical system. In this thesis, however, with the concept of the control triple, we use delay as a *tool* to achieve our goals. For control types of rotation type, we have even introduced a *spatial delay*, thereby allowing stabilization to succeed (remember that Pyragas control, where only time delay is present, fails as a stabilization method). In Chapter 3 we have proven that a cleverly chosen combination of spatial and temporal delays renders stabilization possible.

We can also interpret the spatial and temporal delays as additional parameters which allow us to observe interesting dynamics. In particular, note the difference between reaction-diffusion equations with and without delay: Without delay, *all* rotating and frozen waves in reaction-diffusion equations are unstable, but with spatio-temporal delay, we have shown the existence of stable waves by explicit construction.

Going far beyond the passive description of symmetry, we actively make use of symmetry concepts such as invariance and equivariance of solutions under certain group actions. Note, however, that the concept of control and the concept of symmetry are distinct. This is also the reason why we have opted to introduce the control triple notation rather than use established notation in terms of groups: In the group notation, certain group elements are seen as equivalent if they describe the same symmetry. However, if used as a control term, they yield different results, see the discussion in Chapter 7. Therefore, we use the control triple notation which distinguishes between these descriptions. In short: The control properties of the group theoretical description might differ even if the symmetry does not. This justifies the new control triple notation, designed explicitly for control purposes. Moreover, the control triple method is not limited to equivariant systems: already trivial orbits such as zero equilibria yield a large variety of possible control triples, even if no symmetry is present in that particular system. Therefore, we do not want to limit ourselves with the group notation.

In this dissertation, we have unified the different concepts of control, delay, symmetry, and partial differential equations to present a new concept of spatio-temporal feedback control.

8.3. Outlook

In this thesis, we have successfully introduced the control triple method for control of partial differential equations. We are now in a situation to encourage further investigations in the new research area of spatio-temporal feedback control for partial differential equations.

This section is organized as follows: First we give an outlook on the control triple method as a tool for general partial differential equations und we shortly discuss real-life applications. In the remainder of this section we return to the control triple method for scalar reaction-diffusion equations: We discuss two open problems which follow directly from the results in Chapters 2 and 3 and we present four different options to extend the control triple method.

Let us start with a general outlook on future research and applications: The control triple method is not limited to scalar reaction-diffusion equations – quite the contrary: It is designed to provide a tool for **general partial differential equations**. In the introductory chapter, we have already seen how we could – in general – tackle arbitrary equilibria, equilibria with spatial patterns such as periodicity, and plane waves which occur in many physical systems. Many other examples, including spiral waves and traveling waves, could be added to this list. It will be particularly interesting to apply the control triple method to higher dimensional domains, where we are not limited anymore to rotations and reflections as on the circle. Furthermore, systems of partial differential equations provide opportunities to use matrices as linear transformations of the output signal instead of scalar multiplications.

Concerning **real-life applications**, where the shape of the wave is unknown, a self-adapting spatial delay, similar to [39], would in principle be possible. Note that for partial differential equations we need to adapt the spatial delay, and not the time delay as for ordinary differential equations.

Some **open problems** on the control triple method follow directly from the results in Chapters 2 and 3 for scalar reaction-diffusion equations. In particular, we want to draw attention to the following two open problems:

To discuss the first open problem, let us go back to our results about the success of the control triple method in Chapter 3. We have seen that we can stabilize waves with unstable dimension $2n-1$ under certain requirements. However, we cannot stabilize waves with unstable dimension $2n$ by the same argument we use to prove the failure of Pyragas control: The $2n$-th simple positive eigenvalue cannot cross zero, since $\cos(\varphi N)\Psi = \cos(m\pi/n \cdot n)(-1) = (-1)^2 = 1$ (remember that m is odd). Previously a similar problem was overcome in the control of a single Stuart-Landau oscillator using a complex feedback gain k, see Fiedler et al. [15]. Unfortunately, complex feedback gains are no option for real, one-dimensional equations. Therefore, it is not at all clear which form such a control term would take.

As for the second open problem, we believe that the failure of Pyragas control as proven in Chapter 2 does not persist for complex u. This conjecture is motivated by the comparison of the control triple method with Pyragas control of an unstable focus in Chapter 3. The characteristic equation of the eigenvalues $\lambda \in \mathbb{C}$ of the unstable focus is given by

$$\lambda = a \pm \mathrm{i}b + k\big(1 - e^{-\lambda\tau}\big), \tag{8.1}$$

where both a and b are real parameters. Here, $\pm\mathrm{i}b$ is the difference to real u. As we have seen, $b \neq 0$ destroys real eigenvalues and makes them complex conjugates. Therefore, we cannot exclude stabilization by the odd-number limitation. We can expect a similar mechanism for complex u and complex function $Q(z)$.

In this outlook we present **four different options to extend the**

control triple method which are each of quite a different nature. As we will see, the control triple method leaves ample space for possible extensions.

In this thesis, we have strictly separated control terms of reflection and rotation type. But why not mix them? Such **control schemes of mixed type** would then be of the form

$$u_t = u_{xx} + f(u) - cu_x + k\big(u - \Psi_1\Psi_2 u(-x-\xi, t-\tau)\big), \tag{8.2}$$

where both

$$u_t = u_{xx} + f(u) - cu_x + k\big(u - \Psi_1 u(x-\xi, t-\tau)\big), \tag{8.3}$$

and

$$u_t = u_{xx} + f(u) - cu_x + k\big(u - \Psi_2 u(-x, t-\tau)\big), \tag{8.4}$$

are valid equations of rotation and reflection type, respectively. Do the control regions become larger if the two control schemes are mixed or do they vanish? How many unstable dimensions can be stabilized by such mixed control schemes? Such questions can expected to be answered using analytical methods similar to those in Chapters 5 and 6.

The control triple method in its most general form is not restricted to *constant* transformations of output signal, space, and time. Therefore, it can in principle be used for the stabilization of any rotating wave. We propose to extend the control triple method by **control schemes of co-rotating type**:

$$u_t = u_{xx} + f(u) - cu_x + k\big(u - \Psi(x-ct)\, u(x-\xi, t-\tau)\big). \tag{8.5}$$

Here both the spatial delay ξ and the time delay τ take arbitrary values and they do not need to be related in any way. Then Ψ is not necessarily a unique function of $x - ct$ which guarantees non-invasiveness of the control. For a rotating wave, the function $\Psi(x - ct)$ rotates along with the periodic orbit, as is indicated by the argument $x - ct$. The function Ψ is 2π-periodic in $x - ct$. In particular, if we chose $\xi = c\tau$, then $\Psi(x - ct) \equiv 1$ follows immediately, and we obtain the control terms inspired directly by Pyragas control. The same holds for frozen waves with $c = 0$. Control schemes of co-rotating type can be applied to any rotating or frozen wave. However, we have to pay a heavy price for the generalization to a non-constant function $\Psi(x - ct)$: It is necessary to have almost complete knowledge of the periodic orbit. Therefore, it can be expected that the control schemes of co-rotating type can be used in few situations only, however, with a high likelihood of stabilization.

In real life applications, distributed delays are a common feature. Therefore, we propose to include this phenomenon into the control triple method by considering additive control terms of the form

$$k\left(u - \frac{1}{2\pi T}\int_0^T \int_0^{2\pi} \Xi(\xi)\Theta(\tau)\,\Psi(\xi,\tau)\,u(x-\xi,t-\tau)\,\mathrm{d}\xi\mathrm{d}\tau\right), \qquad (8.6)$$

which we call **control schemes of distributed type**. We distribute the control both over space, with corresponding kernel $\Xi(\xi)$, and over time, with kernel $\Theta(\tau)$ and maximum time delay $0 \leq T \leq \infty$. Note that we might have to choose different output transformations $\Psi(\xi,\tau)$ depending on the spatial delay ξ and the temporal delay τ. The kernels satisfy

$$\frac{1}{2\pi}\int_0^{2\pi} \Xi(\xi)\,\mathrm{d}\xi = 1, \qquad (8.7)$$

as well as

$$\frac{1}{T}\int_0^{T} \Theta(\tau)\,\mathrm{d}\tau = 1, \qquad (8.8)$$

to guarantee noninvasiveness. The control schemes discussed in this thesis correspond to Dirac kernels Ξ and Θ. Multiple discrete delays are also included in this control scheme, as well as extended feedback control similar to [70]. Such control terms are very hard to tackle analytically and we therefore suggest numerical studies to explore the scope of control terms with distributed temporal and spatial delays.

All control terms in this thesis are linear. This is very convenient for calculations, but not always optimal: Already for ordinary differential equations, nonlinear control terms enhance chances of stabilization [7, 14]. We should therefore consider **control schemes of nonlinear type**,

$$u_t = u_{xx} + f(u) - cu_x + K(u(x,t), \Psi u(x-\xi, t-\tau)). \tag{8.9}$$

where $K : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is any (suitably smooth) function satisfying $K(y,y) = 0$. For example, we might use analytic functions

$$K(x,y) = \sum_{l,m=0}^{\infty} \kappa_{lm} x^l y^m, \tag{8.10}$$

where the coefficients $\kappa_{l,m} \in \mathbb{R}$ are chosen in such a way that the infinite series converges for all $x, y \in \mathbb{R}$. Since the control is noninvasive, we demand that for all $N \in \mathbb{N}$ the following condition holds:

$$\sum_{l+m=N} \kappa_{lm} = 0 \tag{8.11}$$

In particular, it follows that the constant κ_{00} is zero. For the linear term, the equality $\kappa_{01} = -\kappa_{10}$ is a necessary condition for the required noninvasiveness. Note the similarity to the control scheme by Pyragas. It will be interesting to see if stabilization regions can be enlarged by nonlinear control schemes compared to the linear schemes discussed in this thesis. Another important question is if Pyragas control can be saved using nonlinear control terms. We conjecture that this is not the case for similar reasons as in the linear analysis. This question should be the subject of further research.

In this thesis, we have set out to explore new methods of spatio-temporal feedback control for partial differential equations. We call the newly developed method the control triple method, and we have shown its success for scalar reaction-diffusion equations. Many new applications of the control triple method are waiting to be investigated.

Chapter A

Appendix

A.1. English Summary

Noninvasive time-delayed feedback control ("Pyragas control") has been investigated theoretically, numerically and experimentally during the last twenty years. Its success has been proven or experimentally demonstrated for numerous dynamical systems given by ordinary differential equations.
In this thesis we introduce new noninvasive spatio-temporal control terms for partial differential equations with the purpose to stabilize unstable equilibria and periodic orbits. We construct these successful control terms by introducing the notion of *control triples*. The control triple defines how we transform *output signal*, *space*, and *time* in the control term. This Ansatz, especially well-suited for the control of partial differential equations, does not exist in the literature so far. It incorporates the spatio-temporal patterns of the equilibria and periodic orbits into the control term.
We investigate the new control triple method in the context of scalar reaction-diffusion equations on the circle: For these equations we present two types of control schemes: *Control schemes of rotation type* combine rotations in space, which we interpret as a spatial delay, with a time delay and a sign change of the output signal, while *control schemes of reflection type* combine reflections in space, time delay and a sign change of the output signal.
For control schemes of rotation type it turns out that spatial delays of *half* the spatial period combined with a small time delay and a sign change in the output signal are successful in the stabilization of equilibria and periodic orbits. However, those control terms which use a *full* spatial period, and consequently no sign change of the output signal, fail their task of stabilization for every time delay. This failure includes the control terms of Pyragas type.
Using control schemes of reflection type, we are able to stabilize orbits with an *odd* reflection symmetry, but not those with an *even* symmetry. Here again, the sign change of the output signal decides whether the control is successful or not.
The proof of stabilization uses a modified version of Hill's equation with spatio-temporal delay. We combine Hill's equation with symmetry properties to obtain the results.
Finally, we present a detailed case study for a specific reaction-diffusion equation, namely the Chafee-Infante equation. We discuss possible extensions and limitations of our new control schemes.

A.2. Deutsche Zusammenfassung

Nichtinvasive zeitverzögerte Rückkopplungskontrolle („Pyragas-Kontrolle”) wurde während der letzten 20 Jahre intensiv theoretisch, numerisch und experimentell untersucht. Ihr Erfolg wurde für zahlreiche dynamische Systeme, die durch gewöhnliche Differentialgleichungen gegeben sind, bewiesen oder experimentell überprüft.
In dieser Arbeit führen wir neue nichtinvasive räumlich-zeitliche Kontrollterme für partielle Differentialgleichungen ein, mit dem Ziel, instabile Gleichgewichte und periodische Orbits zu stabilisieren. Wir konstruieren diese erfolgreichen Kontrollterme indem wir das Konzept der *Kontroll-Tripel* einführen. Das Kontroll-Tripel definiert, wie wir das *Ausgangssignal*, den *Raum* und die *Zeit* im Kontrollterm transformieren. Dieser Ansatz existiert bis jetzt nicht in der Literatur und er ist insbesondere für die Kontrolle von partiellen Differentialgleichungen konzipiert. Somit nutzen die neuen Kontrollterme auch die räumlich-zeitlichen Muster der Gleichgewichte und periodischen Orbits.
Wir untersuchen die neue Kontroll-Tripel-Methode für skalare Reaktions-Diffusions-Gleichungen. Für diese Gleichungen stellen wir zwei Kontrolltermtypen vor: Die *Kontrollschemata vom Rotationstyp* kombinieren Rotationen im Raum, die wir als räumliche Verzögerung interpretieren, mit einer Zeitverzögerung und einem Vorzeichenwechsel im Ausgangssignal. Dagegen kombinieren die *Kontrollschemata vom Reflektionstyp* Reflektionen im Raum, eine Zeitverzögerung und einen Vorzeichenwechsel im Ausgangssignal.
Bei den Kontrollschemata vom Rotationstyp stellt sich heraus, dass räumliche Verzögerungen, die die *Hälfte* der räumlichen Periode betragen, erfolgreich sind, wenn sie mit einer kleinen Zeitverzögerung und einem Vorzeichenwechsel im Ausgangssignal kombiniert werden. Dagegen scheitern diejenigen Kontrollterme, die eine *volle* räumliche Periode nutzen, und somit auch keinen Vorzeichenwechsel im Ausgangssignal haben, für beliebige Zeitverzögerungen.
Mit den Kontrollschemata vom Reflektionstyp können wir diejenigen Orbits mit einer *ungeraden* Reflektionssymmetrie stabilisieren, aber nicht diejenigen mit einer *geraden* Symmetrie. Auch hier entscheidet der Vorzeichenwechsel im Ausgangssignal, ob die Kontrolle erfolgreich ist oder nicht.
Für den Beweis der Stabilisierung nutzen wir eine erweiterte Version von Hills Gleichung mit räumlich-zeitlicher Verzögerung. Wir kombinieren Hills Gleichung mit Symmetrieeigenschaften, um die Ergebnisse zu erhalten.
Unsere Ergebnisse präsentieren wir auch im Rahmen einer detaillierten Fallstudie für eine bestimmte Reaktions-Diffusions-Gleichung, und zwar für die Chafee-Infante-Gleichung. Außerdem diskutieren wir mögliche Erweiterungen sowie Einschränkungen unserer neuen Kontrollterme.

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Curriculum vitae

Der Lebenslauf ist in der Online-Version aus Gründen des Datenschutzes nicht enthalten.

Publikationen und Preprints

I. Schneider, B. Fiedler (2016). “Symmetry-breaking control of rotating waves”. *Control of Self-Organizing Nonlinear Systems.* Ed. by E. Schöll, S. Klapp, P. Hövel. Springer-Verlag, pp. 105-126.

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Selbstständigkeitserklärung

Hiermit bestätige ich, Isabelle Schneider, dass ich die vorliegende Dissertation mit dem Thema

**Spatio-temporal feedback control
of partial differential equations**

selbstständig angefertigt und nur die genannten Quellen und Hilfen verwendet habe. Die Arbeit ist erstmalig und nur an der Freien Universität Berlin eingereicht worden.

Berlin, den 7. Juli 2016