Thermalization in Nature and on a Quantum Computer

Arnau Riera, Christian Gogolin, and Jens Eisert
Dahlem Center for Complex Quantum Systems, Freie Universität Berlin, 14195 Berlin, Germany
Institute for Physics and Astronomy, University of Potsdam, 14476 Potsdam, Germany
(Received 23 May 2011; published 23 February 2012)

In this work, we show how Gibbs or thermal states appear dynamically in closed quantum many-body systems, building on the program of dynamical typicality. We introduce a novel perturbation theorem for physically relevant weak system-bath couplings that is applicable even in the thermodynamic limit. We identify conditions under which thermalization happens and discuss the underlying physics. Based on these results, we also present a fully general quantum algorithm for preparing Gibbs states on a quantum computer with a certified runtime and error bound. This complements quantum Metropolis algorithms, which are expected to be efficient but have no known runtime estimates and only work for local Hamiltonians.

DOI: 10.1103/PhysRevLett.108.080402 PACS numbers: 05.30.−d, 03.65.Yz, 05.70.Ln

How do thermal quantum states—cornerstones of a description in canonical ensembles in quantum statistical physics—arise from the underlying theory of quantum physics? This question, a long tradition as it obviously has, is in many ways still surprisingly wide open. Indeed, much progress was made only recently [1–11]; this is motivated and triggered both by new mathematical [5–9,11,12] and numerical [13] techniques becoming available, as well as by new experiments with quantum many-body systems in nonequilibrium [14].

In this work we present a set of precise sufficient conditions for the emergence of Gibbs states from the underlying microscopic theory of quantum mechanics. Our results go beyond previous approaches in that they apply in a physically relevant weak coupling limit and constitute the key insight leading to the invention of a quantum algorithm that prepares Gibbs states with certified precision and runtime.

The three ingredients that enter the standard textbook proof of the canonical ensemble in classical statistical physics are (i) the *equal a priori probability postulate* (also known as microcanonical ensemble) and an equilibration postulate (such as the second law), (ii) the assumption of *weak coupling*, and (iii) an assumption about the *density of states* of the bath, namely, that it grows faster than exponentially with the energy and that it can be locally well approximated by an exponential [15]. Here each of these steps is translated to the pure state quantum statistical mechanics approach [1–8]. In particular, (i) can be replaced by either a typicality argument or a statement about dynamical relaxation that follows directly from quantum mechanics and (ii) is made precise by proving a novel perturbation theorem that has applications far beyond the scope of the present Letter.

Our new technical results allow us to design a quantum algorithm preparing Gibbs states with explicit error and runtime bounds, invoking a new variant of phase estimation. Our algorithm complements another algorithm with certified runtime that was proposed in Ref. [16] and recent developments on quantum Metropolis algorithms [11].

Setting and notation.—We consider a system S weakly coupled to an environment B. The Hilbert space reads $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$, where $\mathcal{H}_S$ and $\mathcal{H}_B$ are the Hilbert spaces of the subsystem and the "bath" (with finite dimensions d_S and d_B). The evolution of the total system is governed by the Hamiltonian $H = H_0 + V$, with eigenvalues and eigenvectors $\{E_k\}$ and $\{|E_k\rangle\}$ consisting of an uncoupled Hamiltonian $H_0 = H_S + H_B$, with eigenvalues and eigenvectors $\{E_k^{(0)}\}$ and $\{|E_k^{(0)}\rangle\}$, and a coupling Hamiltonian V. We give conditions under which the reduced state $\psi_t^S = \mathrm{Tr}_B \psi_t$, with $\psi_t = |\psi_t\rangle\langle\psi_t|$, of the subsystem S relaxes for most times to a Gibbs state $\rho_{\mathrm{Gibbs}}^S := e^{-\beta H_S}/\mathrm{Tr} e^{-\beta H_S}$ with inverse temperature β under unitary time evolution $|\psi_t\rangle = e^{-iHt}|\psi_0\rangle$. By this we mean that for most times their trace distance $\mathcal{D}(\psi_t^S, \rho_{\mathrm{Gibbs}}^S)$, which measures the physical distinguishability [17], is small. Note that the decomposition of a given H into H_S, H_B, and V is not unique. This freedom can be used to optimize the bounds in our results, and the correct H_S naturally results from this optimization. We assume that the Hamiltonians H and H_0 are nondegenerate such that time averaging and dephasing in the eigenbasis give the same result

$$\omega := \overline{\psi_t} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \psi_t dt = \sum_k |E_k\rangle\langle E_k|\psi_0|E_k\rangle\langle E_k|.$$

Whenever an expectation value equilibrates, it does so to the expectation value in ω [17].

"Natural thermalization": Conditions for Gibbs states to appear.—In this section, we go through points (i)–(iii). The final conclusion is summarized in Corollary 1. The central point of the argument is a novel perturbation theorem that relates spectral projectors of weakly interacting

and noninteracting Hamiltonians in a physically relevant weak coupling limit. It allows us to connect results on dynamical equilibration and measure concentration with classical counting arguments and thereby prove a set of natural sufficient conditions for thermalization in quantum mechanics.

A stepping stone in the argument will be states that have an energy distribution that is flat in an interval $[E, E + \Delta]$ and vanishes otherwise. We indicate such states, and their dephased states, by a subscript $\sqcap$ like in $\psi_\sqcap$ or $\omega_\sqcap$ and call them rectangular states. This class of states includes both mixed states (in particular, the microcanonical state $\omega_\sqcap$) and pure states and thus, because of the freedom to choose the phases, usually also initial states that can be locally out of equilibrium.

The equal *a priori* probability postulate (i) can be replaced by a typicality argument using results from Refs. [1,2,5]. In Ref. [1], powerful concentration of measure techniques are used to show that almost all states from a microcanonical subspace corresponding to a microcanonical energy window $[E, E + \Delta]$ locally look like the reduction of the corresponding microcanonical state; i.e., $\mathcal{D}(\psi^S, \omega_\sqcap^S)$ is small for all but exponentially few of the states ψ from the subspace, where $\omega_\sqcap$ is the microcanonical state on the subspace. Alternatively, one can use the results concerning the dynamics of states with a high effective dimension of Refs. [5,6]. Under one assumption on the spectrum of the Hamiltonian (nondegenerate energy gaps), it is shown that all reduced states on small subsystems of such states tend to the time-averaged equilibrium state ω^S and stay close to it for most times. In many-body systems, natural initial states have a high effective dimension, and this is provably true for all but exponentially few states from a microcanonical subspace [5].

The delicate issue, which has up to now not been addressed in the literature in a general and rigorous way, is the weak coupling approximation (ii) [6,18]. The problem is that due to the exponential growth of the Hilbert space dimension and the at-most polynomial growth of the energy content, the spectrum of the noninteracting Hamiltonian H_0 becomes exponentially dense with increasing bath size. Therefore, the perturbative limit, in which the coupling V is weak compared to the gaps of the noninteracting Hamiltonian H_0, and in which it can be guaranteed that the energy eigenvectors $|E_k\rangle$ of the full Hamiltonian $H = H_0 + V$ are close to product states, is arguably not the physically relevant weak coupling limit. Even worse, in this limit memory effects provably prevent thermalization [8]. As in the classical setting, a coupling should be considered to be weak as long as it does not change the total energy in a noticeable way. That is to say, the energy stored in the interaction is much less than our (microcanonical) uncertainty about the energy of the system, i.e., $\|V\|_\infty \ll \Delta$, or for thermalizing systems much less than the thermal energy $1/\beta$. This is the relevant weak coupling limit in which we prove equilibration towards a Gibbs state. We do this by relating the dephased or microcanonical state $\omega_\sqcap$ to the state $\omega_\sqcap^{(0)}$ dephased with respect to the noninteracting Hamiltonian, for which we can easily perform the partial trace to obtain $\omega_\sqcap^{S(0)}$ and thereby an approximation to $\omega_\sqcap^S$.

Theorem 1 (interacting vs noninteracting case).—Let $\omega_\sqcap^{(0)}$ and $\omega_\sqcap$ be the dephased and microcanonical states belonging to the interval $[E, E + \Delta]$ with respect to H_0 and $H = H_0 + V$; then for every $\varepsilon < \Delta/2$

$$\mathcal{D}(\omega_\sqcap^S, \omega_\sqcap^{S(0)}) \leq \mathcal{D}(\omega_\sqcap, \omega_\sqcap^{(0)}) \leq \frac{\|V\|_\infty}{\varepsilon} + \frac{\Delta\Omega + \Omega_\varepsilon}{2\Omega_{\max}}, \quad (1)$$

where $\Omega_{\max}$ and $\Delta\Omega$ are the maximum and the difference, respectively, of the dimensions of the supports of $\omega_\sqcap^{(0)}$ and $\omega_\sqcap$ and Ω_ε is the total number of eigenstates of H and H_0 in the intervals $[E, E + \varepsilon]$ and $[E + \Delta - \varepsilon, E + \Delta]$.

The theorem shows that, for any two initial (possibly pure) states that have a flat energy distribution in the interval $[E, E + \Delta]$ with respect to the Hamiltonians H_0 and H with $\|V\|_\infty \ll \Delta$, the distance of their reduced dephased states $\omega_\sqcap^{S(0)}$ and $\omega_\sqcap^S$ is small. In particular, assuming an approximately constant density of states such that $\Omega_\varepsilon/(2\Omega_{\max}) \approx 2\varepsilon/\Delta$ and $\Delta\Omega/(2\Omega_{\max}) \lesssim \|V\|_\infty/\Delta$, the best choice for ε is $\varepsilon \approx \sqrt{\|V\|_\infty \Delta/2}$, which gives

$$\mathcal{D}(\omega_\sqcap^S, \omega_\sqcap^{S(0)}) \lesssim 4\sqrt{\frac{\|V\|_\infty}{\Delta}}. \quad (2)$$

In cases with an exponential density of states, for which we will get equilibration towards $\rho_{\text{Gibbs}}^S \propto e^{-\beta H_S}$, we can guarantee that $\mathcal{D}(\omega_\sqcap^S, \omega_\sqcap^{S(0)})$ is small whenever $\|V\|_\infty \ll 1/\beta$ (compare Appendix H in Supplemental Material [19]).

Proof.—First note that by monotonicity of the trace distance and the triangle inequality

$$\mathcal{D}(\omega_\sqcap^S, \omega_\sqcap^{S(0)}) \leq \frac{1}{2}\|\omega_\sqcap - \omega_\sqcap^{(0)}\|_1 \leq \frac{\|G - F\|_1 + \Delta\Omega}{2\Omega_{\max}}, \quad (3)$$

where G and F are the projectors onto the support of $\omega_\sqcap$ and $\omega_\sqcap^{(0)}$, respectively, $\Omega_{\min/\max} = \min/\max[\text{rank}(G), \text{rank}(F)]$, and $\Delta\Omega = \Omega_{\max} - \Omega_{\min}$. It remains to bound $\|G - F\|_1$. Let $\overline{G} = \mathbb{1} - G$ and $\overline{F} = \mathbb{1} - F$; then $G - F = G\overline{F} - \overline{G}F$, and thus $\|G - F\|_1 \leq \|G\overline{F}\|_1 + \|\overline{G}F\|_1$. To bound $\|G\overline{F}\|_1$ we decompose $G = G_i + G_e$ into an interior part G_i, which is the projector onto the eigenstates from the interval $[E + \varepsilon, E + \Delta - \varepsilon]$, and the exterior part G_e and find $\|G\overline{F}\|_1 \leq \|G_i\overline{F}\|_1 + \|G_e\|_1$ (see Fig. 1). By using the inequality $\|\cdot\|_1 \leq \text{rank}(\cdot)\|\cdot\|_\infty$, submultiplicativity of the rank, and that $\text{rank}(G_i) \leq \Omega_{\max}$, this can be recast into $\|G\overline{F}\|_1 \leq \Omega_{\max}\|G_i\overline{F}\|_\infty + \text{rank}(G_e)$. Finally, from Theorem V.II.3.1 in Ref. [20], it follows that $\|G_i\overline{F}\|_\infty \leq \|V\|_\infty/\varepsilon$. Repeating the argument for $\|\overline{G}F\|_1$, introducing the notation $\Omega_\varepsilon = \text{rank}(G_e) + \text{rank}(F_e)$, and putting everything together gives the desired result. ■

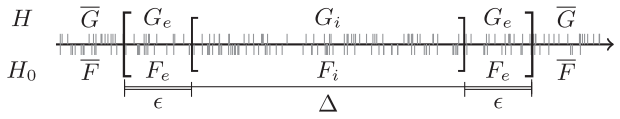

FIG. 1. Definition of the projectors used in the proof of Theorem 1.

The level counting argument (iii)—which is ultimately the reason for the exponential form of $\rho^S_{\mathrm{Gibbs}} \propto e^{-\beta H_S}$—carries over to the quantum case in a straightforward way in the absence of coupling between the system and bath [2,6], and with a bit more work one can also obtain a rigorous trace norm error bound. If the number of states of the bath $\Omega^B_\Delta(E^B)$ in the interval $[E^B, E^B + \Delta]$ is such that the proportion $\Omega^B_\Delta(E - E^S_k)/\sum_l \Omega^B_\Delta(E - E^S_l)$ is close to $e^{-\beta E^S_k}/\sum_l e^{-\beta E^S_l}$ for the given E and Δ and some β, then the distance of $\mathcal{D}(\omega^{S(0)}_\sqcap, \rho^S_{\mathrm{Gibbs}})$ is small. This can be guaranteed under a set of natural assumptions that are satisfied by a wide range of natural quantum many-body systems and that resemble the ones commonly used in classical statistical physics, such as an exponential increase of the density of states (Appendix A [19]). In particular, for a bath consisting of m noninteracting spin-1/2 particles with a slightly varying on-site field strength and average local energies of 0 and η, one finds (Appendix B [19])

$$\mathcal{D}(\omega^{S(0)}_\sqcap, \rho^S_{\mathrm{Gibbs}}) \leq \tfrac{1}{2}(e^{2^{\frac{\|H_S\|^2_\infty}{\eta^2 m}}} - 1) + C \qquad (4)$$

with C exponentially small in the bath size. We will later use this bath in our algorithm. In summary, Eq. (4), Theorem 1, and the results on dynamical equilibration and random states from the unitary invariant measure derived in Refs. [1,5] lead to the following conclusions.

Corollary 1.—(Kinematic) Almost all pure states ψ from a microcanonical subspace corresponding to an energy interval $[E, E + \Delta]$ of a weakly interacting, sufficiently large quantum system are locally close to a Gibbs state in the sense that for every gaps$(H_0) \ll \varepsilon < \Delta/2$ the probability that

$$\mathcal{D}(\psi^S, \omega^{S(0)}_\sqcap) \geq \frac{2d_S}{\sqrt{\Omega_{\min}}} + \frac{\|V\|_\infty}{\varepsilon} + \frac{\Delta\Omega + \Omega_\varepsilon}{2\Omega_{\max}} + \varepsilon' \qquad (5)$$

drops off exponentially with $\Omega_{\min}\varepsilon'^2$. (Dynamic) Moreover, if the Hamiltonian in addition has nondegenerate energy gaps [5], all initial states $\psi_{\sqcap,t=0}$, even those locally out of equilibrium, with a flat energy distribution in the interval, locally equilibrate towards ρ^S_{Gibbs} in the sense that

$$\overline{\mathcal{D}(\psi^S_{\sqcap,t}, \omega^{S(0)}_\sqcap)} \leq \frac{d_S}{2\sqrt{\Omega_{\min}}} + \frac{\|V\|_\infty}{\varepsilon} + \frac{\Delta\Omega + \Omega_\varepsilon}{2\Omega_{\max}}. \qquad (6)$$

Both inequalities are robust against deviations from the rectangular distribution. If the bath has an exponentially increasing density of states, only a region of bounded variation followed by a sharp cutoff towards higher energies should be sufficient (for details, see Appendix C [19]).

"Artificial thermalization": A quantum algorithm for Gibbs state preparation.—It follows from Eq. (4) and Theorem 1 that all one has to do to prepare a Gibbs state is to prepare a state close to $\omega_\sqcap$ or $\omega^{(0)}_\sqcap$ on a suitable combination of the system plus bath. This is the central idea behind the quantum circuit shown in Fig. 2, which prepares thermal states without using any knowledge about the eigenstates of the Hamiltonian.

Quantum algorithms that prepare thermal states have several advantages over classical simulation methods: Quantum Monte Carlo methods offer a way to, for example, estimate correlation functions of thermal states on a classical computer. However, such methods are restricted to certain types of Hamiltonians as they suffer from the sign problem. A procedure that certifiably prepares Gibbs states in a quantum computer not only overcomes the sign problem but, moreover, makes it possible to use the thermal state in experiments addressing questions of nonequilibrium dynamics in quantum simulators, for example, to study quenches.

Our algorithm requires two registers (see Fig. 2). The first register R consists of r qubits initially in $|0\rangle$ and is used to perform quantum phase estimation. The second register Q holds the quantum system plus bath and is put into a rectangular state by performing the following steps. (i) Initialization.—The register Q is initialized into the completely mixed state $\rho_1 = \frac{1}{d}\sum_{k=1}^d |E_k\rangle\langle E_k| \otimes |0\rangle\langle 0|^r$. (ii) Partial quantum phase estimation.—A new form of quantum phase estimation is performed, which comprises three steps: the application of r Hadamard gates on the qubits of R, the application of r controlled-U operations (with U raised to successive powers of two), and an inverse Fourier transform on R. After this operation, the state of the registers is

$$\rho_2 = \frac{1}{d}\sum_{s,s'=0}^{2^r-1}\sum_{k=1}^{d} \alpha_s(\varphi_k)\alpha^*_{s'}(\varphi_k)|E_k\rangle\langle E_k| \otimes |s\rangle\langle s'|, \qquad (7)$$

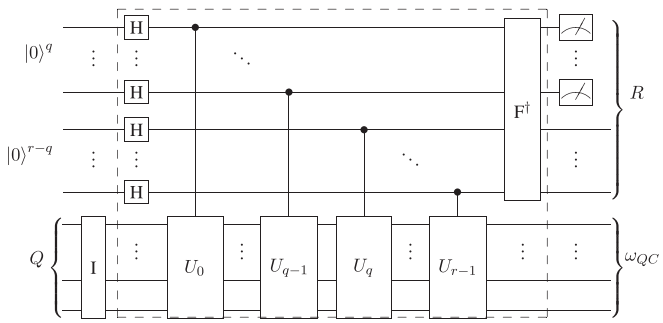

FIG. 2. Quantum circuit that generates a dephased rectangular state $\omega^{(0)}_\sqcap$. I is the initialization gate, H are Hadamard gates, $U_\tau = U^{2^r}$, $U = \exp(-iH_0/\|H_0\|_\infty)$ with $H_0 = H_S + H_B$, and $F^\dagger$ is the inverse Fourier transform.

where $\varphi_k := E_k/\|H\|_\infty$ and

$$\alpha_s(\varphi_k) := \frac{1}{2^r}\frac{1-\exp[2\pi i(2^r\varphi_k - s)]}{1-\exp[2\pi i(\varphi_k - s/2^r)]}.$$

Note that $|\alpha_s(\varphi_k)|^2$ is a probability distribution that becomes more and more peaked around $s/2^r$ as r increases. (iii) Measurement.—By measuring the first q qubits of R, some binary string s_* of length q is obtained, and the system is left in the state

$$\rho_3 \propto \sum_{s,s'=s_*\Delta_*}^{(s_*+1)\Delta_*}\sum_{k=1}^{d}\alpha_s(\varphi_k)\alpha^*_{s'}(\varphi_k)|E_k\rangle\langle E_k|\otimes|s\rangle\langle s'|, \quad (8)$$

where $\Delta_* := 2^{r-q}$ is the number of states of the ancilla register R compatible with the measurement. By choosing r, one can determine the width $\Delta = \|H\|_\infty 2^{-r}\Delta_*$ of the rectangular state that is prepared. The measured value of s_* determines the energy $E = \|H\|_\infty 2^{-q}s_*$ of the rectangular state and, thereby, the inverse temperature β of the Gibbs state. To thermalize the subsystem at some particular temperature, the previous steps must be repeated until the desired energy is measured. The number of runs increases exponentially with the inverse temperature β. This prevents us from preparing thermal states at very low temperatures (see Appendix D [19]). This is not a deficit of the algorithm, for otherwise QMA-hard problems (the quantum analog of NP [21]) could be efficiently solved. Any general algorithm will presumably have this feature [21]. The final state of Q is

$$\omega_{QC} := \mathrm{Tr}_R\rho_3 \propto \sum_{k=1}^{d}\Big(\sum_{s=s_*\Delta_*}^{(s_*+1)\Delta_*}|\alpha_s(\varphi_k)|^2\Big)|E_k\rangle\langle E_k|. \quad (9)$$

For large enough r, this state is close to the desired state $\omega_\sqcap$ with $E = \|H\|_\infty 2^{-q}s_*$ and $\Delta = \|H\|_\infty 2^{-r}\Delta_*$. The precise deviation of ω^S_{QC} from ρ^S_{Gibbs},

$$\mathcal{D}(\omega^S_{QC},\rho^S_{\mathrm{Gibbs}}) \le \mathcal{D}(\omega_{QC},\omega^{(0)}_\sqcap) + \mathcal{D}(\omega^{S(0)}_\sqcap,\rho^S_{\mathrm{Gibbs}}), \quad (10)$$

depends on the density of states of the system plus bath. A good candidate for the bath is the system of m noninteracting spin-1/2 particles discussed before (Appendix B [19]), and we give explicit results for the errors and the complexity for this bath.

Algorithm.—For any chosen $\lambda > 0$, any given inverse temperature β, and system Hamiltonian H_S, the algorithm presented in Fig. 2, using the bath with m spin-1/2 particles and energy scale $\eta = \sqrt{\lambda/m}\|H_S\|_\infty$ discussed before (Appendix B [19]), prepares the system S of n qubits in a state within trace norm distance bounded by

$$\mathcal{D}(\omega^S_{QC},\rho^S_{\mathrm{Gibbs}}) \le 2^{q-r+2}[1 + \ln(2^{r-q})/\pi^2]e^{2/\lambda+\beta\|H_S\|_\infty+\lambda\|H_S\|^2_\infty\beta^2/8} + \tfrac{1}{2}(e^{2/\lambda}-1) + C \quad (11)$$

with C exponentially small in m, to a Gibbs state ρ^S_{Gibbs} with a temperature in the interval $[\beta-\delta\beta,\beta+\delta\beta]$, where

$$\delta\beta \le 2^{2-q}\sqrt{\frac{\lambda}{m}}\frac{1}{\|H_S\|_\infty}\Big(1+\frac{1}{\sqrt{m\lambda}}\Big). \quad (12)$$

This is achieved by using r ancilla qubits and running the algorithm an average number of

$$\overline{\#\mathrm{runs}} \le 2^q\sqrt{\frac{\pi}{2m}}e^{2/\lambda+\beta\|H_S\|_\infty+\lambda\|H_S\|^2_\infty\beta^2/8} \quad (13)$$

times, where each run requires the application of $n+2r$ Hadamard gates, r controlled single qubit gates, $n+q$ (with $q \le r$) single qubit measurements, and 2^r controlled unitary time evolutions under $H_0 = H_S + H_B$ for a time $1/\|H_0\|_\infty$.

Notice that the time evolution under H_B can be implemented with m gates as the bath is a model of uncoupled spins. In practice, in the absence of an oracle for the Hamiltonian of the system, the error produced to perform the U gate carries a second source of error that comes from the Trotter-Suzuki approximation. Nevertheless, this error can be suppressed at a polynomial cost for local Hamiltonians [16,22].

The two contributions to the trace distance error (10) are computed in Appendixes B and F, the average number of runs is computed in Appendix F, and the error in the temperature comes from the discrete nature of the energy measurement via quantum phase estimation and is calculated in Appendix E [19]. As is clear from Fig. 2, we need $\sum_{\tau=0}^{r-1}2^\tau = 2^r$ of the U gates, and the part of the circuit that does not correspond to the controlled time evolution, i.e., the initialization and the inverse Fourier transform, requires only the implementation of $n+m+2r$ Hadamard gates, r controlled single qubit gates, and $n+m+q$ single qubit measurements.

For a fixed large λ, the error of the algorithm can be made small by choosing an ancilla register of size $O(\beta^2\|H_S\|^2_\infty)$. The exponential scaling of the runtime with $\beta\|H_S\|_\infty$ caused by this is not a deficit of the algorithm, as any general efficient algorithm would contradict hardness results such as the local Hamiltonian problem [21]. Unlike our approach, Metropolis algorithms [11] are expected to be efficient in some cases, but no rigorous runtime estimates are known and they are applicable only to local Hamiltonians. An interesting step towards constructing an efficient (in system size) and certified algorithm for local systems was recently made in Ref. [23] (compare Appendix G [19]).

Conclusions.—A set of sufficient conditions for thermalization in quantum mechanics has been presented. The conditions are a natural translation of the standard assumptions from classical statistical physics. Along the way, a perturbation argument for realistic weak coupling has been proven that we expect to have significant applications

beyond the scope of this Letter. By using our technical results, we are able to design a quantum algorithm preparing thermal states of arbitrary Hamiltonians with rigorous runtime and error bounds.

We thank M. P. Müller for discussions, P. Reimann for comments, the anonymous referee for the well justified criticism, and the EU (Qessence, Compas, Minos), and the EURYI for support.

[1] S. Popescu, A. J. Short, and A. Winter, Nature Phys. **2**, 754 (2006).
[2] S. Goldstein, J. L. Lebowitz, R. Tumulka, and N. Zanghì, Phys. Rev. Lett. **96**, 050403 (2006).
[3] H. Tasaki, Phys. Rev. Lett. **80**, 1373 (1998).
[4] H. Dong, S. Yang, X. F. Liu, and C. P. Sun, Phys. Rev. A **76**, 044104 (2007).
[5] P. Reimann, Phys. Rev. Lett. **101**, 190403 (2008); N. Linden, S. Popescu, A. J. Short, and A. Winter, Phys. Rev. E **79**, 061103 (2009); A. J. Short, arXiv:1012.4622.
[6] P. Reimann, New J. Phys. **12**, 055027 (2010).
[7] N. Linden, S. Popescu, A. J. Short, and A. Winter, New J. Phys. **12**, 055021 (2010); C. Gogolin, Phys. Rev. E **81**, 051127 (2010).
[8] C. Gogolin, M. P. Müller, and J. Eisert, Phys. Rev. Lett. **106**, 040401 (2011); O. Lychkovskiy, Phys. Rev. E **82**, 011123 (2010).
[9] M. Cramer, C. M. Dawson, J. Eisert, and T. J. Osborne, Phys. Rev. Lett. **100**, 030602 (2008); T. Barthel and U. Schollwöck, *ibid.* **100**, 100601 (2008).
[10] M.-H. Yung, D. Nagaj, J. D. Whitfield, and A. Aspuru-Guzik, Phys. Rev. A **82**, 060302(R) (2010).
[11] K. Temme *et al.*, Nature (London) **471**, 87 (2011).
[12] M. P. Müller, D. Gross, and J. Eisert, Commun. Math. Phys. **303**, 785 (2011).
[13] C. Kollath, A. Läuchli, and E. Altman, Phys. Rev. Lett. **98**, 180601 (2007); S. Manmana *et al.*, *ibid.* **98**, 210405 (2007); A. Flesch *et al.*, Phys. Rev. A **78**, 033608 (2008); M. Rigol, Phys. Rev. Lett. **103**, 100403 (2009); M. C. Banuls, J. I. Cirac, and M. B. Hastings, *ibid.* **106**, 050405 (2011).
[14] M. Greiner *et al.*, Nature (London) **415**, 39 (2002); L. E. Sadler *et al.*, *ibid.* **443**, 312 (2006); S. Hofferberth *et al.*, *ibid.* **449**, 324 (2007); S. Trotzky *et al.*, arXiv:1101.2659 [Nature Phys. (to be published)].
[15] L. D. Landau and E. M. Lifschitz, *Course of Theoretical Physics* (Pergamon, Oxford, 1980), 3rd ed.
[16] D. Poulin and P. Wocjan, Phys. Rev. Lett. **103**, 220502 (2009).
[17] See the introduction of Ref. [8].
[18] Reference [2] deals only with the noninteracting case. In Ref. [3], weak means that $\|V\|_\infty$ is much smaller than the level spacing of H_S, which is well defined in the limit of large baths, but still breaks down quickly even for relatively small subsystems, and, moreover, the rigorous results are limited to cases where V couples only neighboring eigenstates. Reference [4] contains a nonrigorous discussion of the general case and numerical results suggesting that couplings much larger than the level spacing can be sufficiently weak for Gibbs states to emerge; a fact made rigorous in this work.
[19] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.108.080402 for detailed proofs.
[20] R. Bhatia, *Matrix Analysis* (Springer-Verlag, New York, 1997).
[21] J. Kempe, A. Kitaev, and O. Regev, SIAM J. Comput. **35**, 1070 (2006).
[22] D. W. Berry *et al.*, Commun. Math. Phys. **270**, 359 (2006).
[23] E. Bilgin and S. Boixo, Phys. Rev. Lett. **105**, 170405 (2010).