A

A.1 Beweise der Hilfssätze aus Kapitel 2

Lemma A.1 *Für $\alpha, \beta \in \mathbb{N}$ mit $\alpha \leq \beta$ gilt*

$$\sum_{\nu=0}^{\ell} (-1)^{\nu} \binom{\alpha}{\nu} \binom{\beta + \ell - \nu}{\beta} = \binom{\ell + \beta - \alpha}{\beta - \alpha} \quad \textit{für alle } \ell \geq 0.$$

Beweis durch vollständige Induktion über ℓ.

- $\ell = 0$:

$$\sum_{\nu=0}^{0} (-1)^{\nu} \binom{\alpha}{\nu} \binom{\beta - \nu}{\beta} = 1 = \binom{\beta - \alpha}{\beta - \alpha}.$$

- $\ell \to \ell + 1$:

$$\begin{aligned}
\binom{\ell + 1 + \beta - \alpha}{\beta - \alpha} &= \frac{\ell + 1 + \beta - \alpha}{\ell + 1} \binom{\ell + \beta - \alpha}{\beta - \alpha} \\
&= \frac{\ell + 1 + \beta - \alpha}{\ell + 1} \sum_{\nu=0}^{\ell} (-1)^{\nu} \binom{\alpha}{\nu} \binom{\beta + \ell - \nu}{\beta} \\
&= \frac{1}{\ell + 1} \sum_{\nu=0}^{\ell} (-1)^{\nu} \left[\ell + \beta - \nu + 1 - (\alpha - \nu)\right] \binom{\alpha}{\nu} \binom{\beta + \ell - \nu}{\beta} \\
&= \frac{1}{\ell + 1} \sum_{\nu=0}^{\ell} (-1)^{\nu} \left[\binom{\alpha}{\nu} \frac{\ell + \beta - \nu + 1}{\ell - \nu + 1} (\ell - \nu + 1) \binom{\beta + \ell - \nu}{\beta} \right. \\
&\qquad \left. - (\alpha - \nu) \binom{\alpha}{\nu} \binom{\beta + \ell - \nu}{\beta} \right] \\
&= \frac{1}{\ell + 1} \sum_{\nu=0}^{\ell} (-1)^{\nu} \binom{\alpha}{\nu} (\ell - \nu + 1) \binom{\beta + \ell + 1 - \nu}{\beta} \\
&\qquad + \frac{1}{\ell + 1} \sum_{\nu=0}^{\ell} (-1)^{\nu + 1} \underbrace{(\alpha - \nu) \binom{\alpha}{\nu}}_{= (\nu + 1) \binom{\alpha}{\nu + 1}} \binom{\beta + \ell - \nu}{\beta} \\
&= \frac{1}{\ell + 1} \binom{\alpha}{0} (\ell + 1) \binom{\beta + \ell + 1}{\beta} \\
&\qquad + \frac{1}{\ell + 1} \sum_{\nu=1}^{\ell} (-1)^{\nu} \binom{\alpha}{\nu} (\ell - \nu + 1) \binom{\beta + \ell + 1 - \nu}{\beta}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\ell+1}(-1)^{\ell+1}(\ell+1)\binom{\alpha}{\ell+1}\binom{\beta}{\beta} \\
&+\underbrace{\frac{1}{\ell+1}\sum_{\nu=0}^{\ell-1}(-1)^{\nu+1}(\nu+1)\binom{\alpha}{\nu+1}\binom{\beta+\ell-\nu}{\beta}}_{=\sum_{\nu=1}^{\ell}(-1)^{\nu}\nu\binom{\alpha}{\nu}\binom{\beta+\ell-\nu+1}{\beta}} \\
=&\binom{\beta+\ell+1}{\beta}+(-1)^{\ell+1}\binom{\alpha}{\ell+1} \\
&+\frac{1}{\ell+1}\sum_{\nu=1}^{\ell}(-1)^{\nu}(\ell-\nu+\nu+1)\binom{\alpha}{\nu}\binom{\beta+\ell-\nu+1}{\beta} \\
=&\sum_{\nu=0}^{\ell+1}(-1)^{\nu}\binom{\alpha}{\nu}\binom{\beta+\ell-\nu+1}{\beta}. \quad \square
\end{aligned}
$$

Lemma A.2 *Sei K ein Körper, $a, b \in \mathrm{K}$ und $n, \ell \in \mathbb{N}$. Dann gilt für beliebige $q_{n,\ell} \in \mathrm{K}$ mit $0 \leq n \leq \ell$*

$$
\sum_{n=0}^{\ell} q_{n,\ell}a^n b^{\ell-n} = \sum_{n=0}^{\ell} c_{n,\ell}(a+b)^n a^{\ell-n},
$$

wobei

$$
c_{n,\ell} = \sum_{\nu=0}^{\ell-n}(-1)^{\nu}\binom{n+\nu}{\nu}q_{\ell-n-\nu,\ell}, \quad 0 \leq n \leq \ell.
$$

Beweis

$$
\begin{aligned}
\sum_{n=0}^{\ell} c_{n,\ell}(a+b)^n a^{\ell-n} &= \sum_{n=0}^{\ell} c_{n,\ell}a^{\ell-n}\sum_{t=0}^{n}\binom{n}{t}a^{n-t}b^t \\
&= \sum_{n=0}^{\ell}\sum_{t=0}^{n}\binom{n}{t}c_{n,\ell}a^{\ell-t}b^t = \sum_{t=0}^{\ell}\sum_{n=t}^{\ell}\binom{n}{t}c_{n,\ell}a^{\ell-t}b^t \quad (0 \leq t \leq n \leq \ell) \\
&= \sum_{t=0}^{\ell} a^{\ell-t}b^t\sum_{n=0}^{\ell-t}\binom{n+t}{t}c_{n+t,\ell} \\
&= \sum_{t=0}^{\ell} a^{\ell-t}b^t\sum_{n=0}^{\ell-t}\binom{n+t}{t}\sum_{\nu=0}^{\ell-t-n}(-1)^{\nu}\binom{n+t+\nu}{n+t}q_{\ell-n-t-\nu,\ell} \\
&= \sum_{t=0}^{\ell} a^{\ell-t}b^t\sum_{n=0}^{\ell-t}\binom{\ell-n}{t}\sum_{\nu=0}^{n}(-1)^{\nu}\binom{\ell-n+\nu}{\ell-n}q_{n-\nu,\ell}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{t=0}^{\ell} a^{\ell-t} b^t \sum_{\nu=0}^{\ell-t} \sum_{n=\nu}^{\ell-t} \binom{\ell-n}{t} (-1)^{\nu} \binom{\ell-n+\nu}{\ell-n} q_{n-\nu,\ell} \quad (0 \le \nu \le n \le \ell - t) \\
&= \sum_{t=0}^{\ell} a^{\ell-t} b^t \sum_{\nu=0}^{\ell-t} \sum_{n=0}^{\ell-t-\nu} \binom{\ell-n-\nu}{t} (-1)^{\nu} \binom{\ell-n}{\nu} q_{n,\ell} \\
&= \sum_{t=0}^{\ell} a^{\ell-t} b^t \sum_{\nu=0}^{\ell-t} \sum_{n=0}^{\nu} \binom{t-n+\nu}{t} (-1)^{\ell-t-\nu} \binom{\ell-n}{\ell-t-\nu} q_{n,\ell} \\
&= \sum_{t=0}^{\ell} a^{\ell-t} b^t \sum_{n=0}^{\ell-t} \sum_{\nu=n}^{\ell-t} \binom{t-n+\nu}{t} (-1)^{\ell-t-\nu} \binom{\ell-n}{\ell-t-\nu} q_{n,\ell} \\
&= \sum_{t=0}^{\ell} a^{\ell-t} b^t \sum_{n=0}^{\ell-t} q_{n,\ell} \sum_{\nu=0}^{\ell-t-n} \binom{t+\nu}{t} (-1)^{\ell-t-n-\nu} \binom{\ell-n}{t+\nu} \\
&= \sum_{t=0}^{\ell} a^{\ell-t} b^t \sum_{n=0}^{\ell-t} q_{n,\ell} \sum_{\nu=0}^{\ell-t-n} (-1)^{\ell-t-n-\nu} \binom{\ell-t-n}{\nu} \binom{\ell-n}{t} \\
&= \sum_{t=0}^{\ell} a^{\ell-t} b^t \sum_{n=0}^{\ell-t} q_{n,\ell} \binom{\ell-n}{t} \sum_{\nu=0}^{\ell-t-n} (-1)^{\ell-t-n-\nu} \binom{\ell-t-n}{\nu} \\
&= \sum_{t=0}^{\ell} a^{\ell-t} b^t q_{\ell-t,\ell}. \quad \square
\end{aligned}
$$

A.2 Beweise der Hilfssätze aus Kapitel 3

Zum Beweis des zentralen Lemmas wird das Folgende benötigt.

Lemma A.3 *Für alle $n \in \mathbb{N}$ gilt*

$$\binom{n}{k+1} = \sum_{p=0}^{k+1} (-1)^p \binom{n+1}{k+1-p}.$$

Beweis durch vollständige Induktion über n.

Für $n = 1$ ist die Aussage offenbar richtig. Weiter gilt:

$$\begin{aligned}\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k} &= \sum_{p=0}^{k+1} (-1)^p \binom{n+1}{k+1-p} + \sum_{p=0}^{k} (-1)^p \binom{n+1}{k-p} \\ &= \binom{n+1}{0} (-1)^{k+1} + \sum_{p=0}^{k} (-1)^p \binom{n+2}{k-p+1}. \quad \square\end{aligned}$$

Lemma A.4 *Für natürliche Zahlen $\mu_k \geq 3$ und $\mu_{\rho-1} \geq 1$ mit $\mu_k - 2 \geq \mu_{\rho-1}$ gilt*

$$\begin{aligned}&\frac{1}{\mu_{\rho-1}} + (-1)^{\mu_k-\mu_{\rho-1}-1} (\mu_{\rho-1}-1)!\mu_{\rho-1} \sum_{p=0}^{\mu_k-\mu_{\rho-1}-2} \binom{\mu_k - \mu_{\rho-1} - 1}{p} \\ &\qquad\qquad \times (-1)^{p-1} \frac{(\mu_k - \mu_{\rho-1} - p - 2)!}{(\mu_k - 1 - p)!} \\ &= (-1)^{\mu_k-\mu_{\rho-1}} (\mu_{\rho-1}-1)! \sum_{p=0}^{\mu_k-\mu_{\rho-1}-1} \binom{\mu_k - \mu_{\rho-1}}{p} \\ &\qquad\qquad \times (-1)^{p-1} \frac{(\mu_k - \mu_{\rho-1} - p - 1)!}{(\mu_k - 1 - p)!}.\end{aligned}$$

Beweis

Die Behauptung ist äquivalent zu

$$\begin{aligned}&\frac{(-1)^{\mu_k-\mu_{\rho-1}}}{\mu_{\rho-1}!} - \mu_{\rho-1} \sum_{p=0}^{\mu_k-\mu_{\rho-1}-2} \binom{\mu_k - \mu_{\rho-1} - 1}{p} \\ &\qquad\qquad \times (-1)^{p-1} \frac{(\mu_k - \mu_{\rho-1} - p - 2)!}{(\mu_k - 1 - p)!}\end{aligned}$$

$$= \sum_{p=0}^{\mu_k-\mu_{\rho-1}-1} \binom{\mu_k-\mu_{\rho-1}}{p} (-1)^{p-1} \frac{(\mu_k-\mu_{\rho-1}-p-1)!}{(\mu_k-1-p)!}$$

$\iff$

$$\frac{(-1)^{\mu_k-\mu_{\rho-1}}}{\mu_{\rho-1}!} - \mu_{\rho-1}\frac{(\mu_k-\mu_{\rho-1}-1)!}{(\mu_k-1)!} \sum_{p=0}^{\mu_k-\mu_{\rho-1}-2} \binom{\mu_k-1}{p} \frac{(-1)^{p-1}}{\mu_k-\mu_{\rho-1}-p-1}$$

$$= \frac{(\mu_k-\mu_{\rho-1})!}{(\mu_k-1)!} \sum_{p=0}^{\mu_k-\mu_{\rho-1}-1} \binom{\mu_k-1}{p} \frac{(-1)^{p-1}}{\mu_k-\mu_{\rho-1}-p}$$

$$= \frac{(\mu_k-\mu_{\rho-1})!}{(\mu_k-1)!} \sum_{p=0}^{\mu_k-\mu_{\rho-1}-2} \binom{\mu_k-1}{p} \frac{(-1)^{p-1}}{\mu_k-\mu_{\rho-1}-p} + (-1)^{\mu_k-\mu_{\rho-1}} \frac{\mu_k-\mu_{\rho-1}}{(\mu_{\rho-1})!}$$

$\iff$

$$\frac{(-1)^{\mu_k-\mu_{\rho-1}}}{\mu_{\rho-1}!}(1-\mu_k+\mu_{\rho-1})$$

$$= \frac{(\mu_k-\mu_{\rho-1})!}{(\mu_k-1)!} \sum_{p=0}^{\mu_k-\mu_{\rho-1}-2} \binom{\mu_k-1}{p} \frac{(-1)^{p-1}}{\mu_k-\mu_{\rho-1}-p}$$

$$+\mu_{\rho-1}\frac{(\mu_k-\mu_{\rho-1}-1)!}{(\mu_k-1)!} \sum_{p=0}^{\mu_k-\mu_{\rho-1}-2} \binom{\mu_k-1}{p} \frac{(-1)^{p-1}}{\mu_k-\mu_{\rho-1}-p-1}$$

$$= \frac{(\mu_k-\mu_{\rho-1}-1)!}{(\mu_k-1)!} \sum_{p=0}^{\mu_k-\mu_{\rho-1}-2} \binom{\mu_k-1}{p} (-1)^{p-1}$$

$$\times \left[\frac{\mu_k-\mu_{\rho-1}}{\mu_k-\mu_{\rho-1}-p} + \frac{\mu_{\rho-1}}{\mu_k-\mu_{\rho-1}-p-1}\right]$$

$$= \frac{(\mu_k-\mu_{\rho-1}-1)!}{(\mu_k-1)!} \sum_{p=0}^{\mu_k-\mu_{\rho-1}-2} \binom{\mu_k-1}{p} (-1)^{p-1}$$

$$\times \left[\frac{\mu_k}{\mu_k-\mu_{\rho-1}-p} + \frac{\mu_{\rho-1}}{(\mu_k-\mu_{\rho-1}-p)(\mu_k-\mu_{\rho-1}-p-1)}\right]$$

$\iff$

$$\frac{(-1)^{\mu_k-\mu_{\rho-1}-1}}{(\mu_k-\mu_{\rho-1}-2)!} \frac{(\mu_k-1)!}{\mu_{\rho-1}!}$$

$$= \sum_{p=0}^{\mu_k-\mu_{\rho-1}-2} \binom{\mu_k-1}{\mu_k-\mu_{\rho-1}-2-p} (-1)^{\mu_k-\mu_{\rho-1}-p-1}$$

$$\times \left[\frac{\mu_k}{p+2} + \frac{\mu_{\rho-1}}{(p+1)(p+2)}\right]$$

$\iff$

$$\frac{(-1)^{\mu_k-\mu_{\rho-1}-1}}{(\mu_k-\mu_{\rho-1}-2)!} \frac{(\mu_k-1)!}{\mu_{\rho-1}!}$$

$$= (-1)^{\mu_k-\mu_{\rho-1}-1} \sum_{p=0}^{\mu_k-\mu_{\rho-1}-2} \binom{\mu_k-1}{\mu_{\rho-1}+p+1}(-1)^p \times \left[\frac{\mu_k}{p+2}+\frac{\mu_{\rho-1}}{p+1}-\frac{\mu_{\rho-1}}{p+2}\right]$$

$\iff$

$$\binom{\mu_k-1}{\mu_{\rho-1}+1}(\mu_{\rho-1}+1) = \sum_{p=0}^{\mu_k-\mu_{\rho-1}-2} \binom{\mu_k-1}{\mu_{\rho-1}+p+1}(-1)^p \frac{\mu_k}{p+2}$$

$$+ \sum_{p=0}^{\mu_k-\mu_{\rho-1}-2} \binom{\mu_k-1}{\mu_{\rho-1}+p+1}(-1)^p \frac{\mu_{\rho-1}}{p+1}$$

$$+ \sum_{p=0}^{\mu_k-\mu_{\rho-1}-2} \binom{\mu_k-1}{\mu_{\rho-1}+p+1}(-1)^{p+1} \frac{\mu_{\rho-1}}{p+2}$$

$$= \sum_{p=0}^{\mu_k-\mu_{\rho-1}-2} \binom{\mu_k-1}{\mu_{\rho-1}+p+1}(-1)^p \frac{\mu_k}{p+2}$$

$$+ \sum_{p=1}^{\mu_k-\mu_{\rho-1}-2} \binom{\mu_k-1}{\mu_{\rho-1}+p+1}(-1)^p \frac{\mu_{\rho-1}}{p+1} + \binom{\mu_k-1}{\mu_{\rho-1}+1}\mu_{\rho-1}$$

$$+ \sum_{p=1}^{\mu_k-\mu_{\rho-1}-2} \binom{\mu_k-1}{\mu_{\rho-1}+p}(-1)^p \frac{\mu_{\rho-1}}{p+1} + (-1)^{\mu_k-\mu_{\rho-1}-1}\binom{\mu_k-1}{\mu_k-1}\frac{\mu_{\rho-1}}{\mu_k-\mu_{\rho-1}}$$

$$= \binom{\mu_k-1}{\mu_{\rho-1}+1}\mu_{\rho-1} + (-1)^{\mu_k-\mu_{\rho-1}-1}\frac{\mu_{\rho-1}}{\mu_k-\mu_{\rho-1}}$$

$$+ \sum_{p=1}^{\mu_k-\mu_{\rho-1}-2} \binom{\mu_k}{\mu_{\rho-1}+p+1}(-1)^p \frac{\mu_{\rho-1}}{p+1}$$

$$+ \sum_{p=1}^{\mu_k-\mu_{\rho-1}-1} \binom{\mu_k-1}{\mu_{\rho-1}+p}(-1)^{p-1} \frac{\mu_k}{p+1}$$

$$= \binom{\mu_k-1}{\mu_{\rho-1}+1}\mu_{\rho-1} + (-1)^{\mu_k-\mu_{\rho-1}-1}\frac{\mu_{\rho-1}}{\mu_k-\mu_{\rho-1}}$$

$$+ \sum_{p=1}^{\mu_k-\mu_{\rho-1}-2} \binom{\mu_k}{\mu_{\rho-1}+p+1}(-1)^p \frac{\mu_{\rho-1}}{p+1}$$

$$+ \sum_{p=1}^{\mu_k-\mu_{\rho-1}-1} \binom{\mu_k}{\mu_{\rho-1}+p+1}(-1)^{p-1} \frac{\mu_{\rho-1}+p+1}{p+1}$$

$$= \binom{\mu_k-1}{\mu_{\rho-1}+1}\mu_{\rho-1} + (-1)^{\mu_k-\mu_{\rho-1}-1}\frac{\mu_{\rho-1}}{\mu_k-\mu_{\rho-1}}$$

$$+\sum_{p=1}^{\mu_k-\mu_{\rho-1}-2}\binom{\mu_k}{\mu_{\rho-1}+p+1}(-1)^p\frac{\mu_{\rho-1}}{p+1}$$

$$+\sum_{p=1}^{\mu_k-\mu_{\rho-1}-2}\binom{\mu_k}{\mu_{\rho-1}+p+1}(-1)^{p-1}\frac{\mu_{\rho-1}+p+1}{p+1}$$

$$+(-1)^{\mu_k-\mu_{\rho-1}}\frac{\mu_k}{\mu_k-\mu_{\rho-1}}$$

$\Longleftrightarrow$

$$\begin{aligned}
&\binom{\mu_k-1}{\mu_{\rho-1}+1}(\mu_{\rho-1}+1)\\
&=\binom{\mu_k-1}{\mu_{\rho-1}+1}\mu_{\rho-1}+(-1)^{\mu_k-\mu_{\rho-1}-1}\frac{\mu_{\rho-1}}{\mu_k-\mu_{\rho-1}}+(-1)^{\mu_k-\mu_{\rho-1}}\frac{\mu_k}{\mu_k-\mu_{\rho-1}}\\
&\quad+\sum_{p=1}^{\mu_k-\mu_{\rho-1}-2}\binom{\mu_k}{\mu_{\rho-1}+p+1}(-1)^p\left[\frac{\mu_{\rho-1}}{p+1}-\frac{\mu_{\rho-1}+p+1}{p+1}\right]\\
&=\binom{\mu_k-1}{\mu_{\rho-1}+1}\mu_{\rho-1}+(-1)^{\mu_k-\mu_{\rho-1}-1}\frac{\mu_{\rho-1}}{\mu_k-\mu_{\rho-1}}+(-1)^{\mu_k-\mu_{\rho-1}}\frac{\mu_k}{\mu_k-\mu_{\rho-1}}\\
&\quad-\sum_{p=1}^{\mu_k-\mu_{\rho-1}-2}\binom{\mu_k}{\mu_{\rho-1}+p+1}(-1)^p
\end{aligned}$$

$\Longleftrightarrow$

$$\binom{\mu_k-1}{\mu_{\rho-1}+1}=(-1)^{\mu_k-\mu_{\rho-1}}-\sum_{p=1}^{\mu_k-\mu_{\rho-1}-2}\binom{\mu_k}{\mu_{\rho-1}+p+1}(-1)^p$$

$\Longleftrightarrow$

$$\binom{\mu_k-1}{\mu_{\rho-1}+1}=(-1)^{\mu_k-\mu_{\rho-1}}-\sum_{p=\mu_{\rho-1}+1}^{\mu_k-2}\binom{\mu_k}{p+1}(-1)^{p-\mu_{\rho-1}}$$

$\Longleftrightarrow$

$$\binom{\mu_k-1}{\mu_{\rho-1}+1}=(-1)^{\mu_k-\mu_{\rho-1}}+\sum_{p=\mu_{\rho-1}+2}^{\mu_k-1}\binom{\mu_k}{p}(-1)^{p-\mu_{\rho-1}}$$

$\Longleftrightarrow$

$$(-1)^{\mu_{\rho-1}}\binom{\mu_k-1}{\mu_{\rho-1}+1}=(-1)^{\mu_k}+\sum_{p=\mu_{\rho-1}+2}^{\mu_k-1}(-1)^p\binom{\mu_k}{p}$$

$\Longleftrightarrow$

$$(-1)^{\mu_{\rho-1}}\binom{\mu_k-1}{\mu_{\rho-1}+1}=\sum_{p=0}^{\mu_k}(-1)^p\binom{\mu_k}{p}-\sum_{p=0}^{\mu_{\rho-1}+1}(-1)^p\binom{\mu_k}{p}$$

$\Longleftrightarrow$

$$(-1)^{\mu_{\rho-1}+1}\binom{\mu_k-1}{\mu_{\rho-1}+1}=\sum_{p=0}^{\mu_{\rho-1}+1}(-1)^p\binom{\mu_k}{p}$$

$\Longleftrightarrow$

$$\binom{\mu_k - 1}{\mu_{\rho-1} + 1} = \sum_{p=0}^{\mu_{\rho-1}+1} (-1)^p \binom{\mu_k}{\mu_{\rho-1} + 1 - p}. \tag{75}$$

Für μ_k, $\mu_{\rho-1} \in \mathbb{N}$ mit $\mu_k - 2 \geq \mu_{\rho-1} \geq 1$ ist (75) aber eine direkte Konsequenz aus **Lemma A.3**. $\Box$