Chapter 3

The Weak Freese-Nation Property

While in the last chapter I considered tightly κ-filtered Boolean algebras, in this chapter I will drop 'tightly'. Moreover, I will only consider σ-filtered partial orders, mostly complete Boolean algebras. This chapter will be much more set-theoretic than the last one.

3.1 WFN($\mathfrak{P}(\omega)$) in forcing extensions

In this section I will show that WFN($\mathfrak{P}(\omega)$) is very fragile in the sense that in typical forcing extensions which have a larger continuum than the ground model and which are not Cohen extensions, WFN($\mathfrak{P}(\omega)$) fails. The following notions are crucial.

3.1.1. Definition. A notion of forcing P *yields a σ-extension of* $\mathfrak{P}(\omega)$ iff $\Vdash_P \mathfrak{P}(\omega) \cap \check{V} \leq_\sigma \mathfrak{P}(\omega)$. Similarly, *$P$ yields a non-σ-extension of* $\mathfrak{P}(\omega)$ iff $\Vdash_P \mathfrak{P}(\omega) \cap \check{V} \not\leq_\sigma \mathfrak{P}(\omega)$. □

Typically, when enlarging the continuum by forcing, one uses some kind of long iteration of rather small forcings which add new reals. Very popular examples are countable support iterations of length ω_2 of proper forcings of size $\aleph_1$ over a model of CH or finite support iterations of length $> 2^{\aleph_0}$ of forcings satisfying c.c.c. These examples can be treated using

3.1.2. Lemma. *Let A be a partial order such that for every A-generic filter G over the ground model M every countable set of ordinals in $M[G]$ is covered*

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*by some countable set in M. Assume that forcing with A does not collapse
cardinals. Suppose $\lambda := |A|$ is regular. Let $S \subseteq \lambda$ be a stationary set of
ordinals of uncountable cofinality. Suppose A is the union of an increasing
chain $(A_\alpha)_{\alpha < \lambda}$ of completely embedded suborders which is continuous at limit
ordinals in S. Assume for each $\alpha < \lambda$, $\Vdash_{A_\alpha} 2^{\aleph_0} < \lambda$. Suppose for each $\alpha \in S$,
$\Vdash_{A_{\alpha+1}} \mathfrak{P}(\omega) \cap \check{V}[\dot{G} \cap \check{A}_\alpha] \not\leq_\sigma \mathfrak{P}(\omega)$. Then $\Vdash_A \neg\,\mathrm{WFN}(\mathfrak{P}(\omega))$.*

Proof. Let G be A generic over M. I argue in $M[G]$. For $\alpha < \lambda$ let $P_\alpha :=
\mathfrak{P}(\omega) \cap M[G \cap A_\alpha]$ and let $P_\lambda := \mathfrak{P}(\omega)$. For convenience, let $A_\lambda := A$.

By Lemma 2.5.11, $(P_\alpha)_{\alpha \leq \lambda}$ is continuous at limit ordinals in S and at λ.
Note that the continuum is at least λ since $|S| = \lambda$ and $P_{\alpha+1} \setminus P_\alpha \neq \emptyset$ for
$\alpha \in S$. By assumption, $|P_\alpha| < \lambda$ for each $\alpha < \lambda$. Therefore $2^{\aleph_0} = \lambda$. For each
$\alpha < \lambda$ let $Q_{\alpha+1} := P_{\alpha+1}$ and for every limit ordinal $\delta < \lambda$ let $Q_\delta := \bigcup_{\alpha < \delta} P_\alpha$.
Now $(Q_\alpha)_{\alpha < \lambda}$ is continuously increasing and agrees with $(P_\alpha)_{\alpha < \lambda}$ on S. Let
$f : \mathfrak{P}(\omega) \to [\mathfrak{P}(\omega)]^{\aleph_0}$ be any function. Since S is stationary, there is $\alpha \in S$
such that Q_α is closed under f. Now $Q_\alpha = P_\alpha$ and

$$M[G \cap A_{\alpha+1}] \models \mathfrak{P}(\omega) \cap M[G \cap A_\alpha] \not\leq_\sigma \mathfrak{P}(\omega).$$

Therefore there is $x \in P_{\alpha+1} \subseteq \mathfrak{P}(\omega)$ such that no in $M[G \cap A_{\alpha+1}]$ countable
set includes a cofinal subset of $Q_\alpha \restriction x$. Since $[2^{\aleph_0}]^{\aleph_0} \cap M$ is cofinal in $[2^{\aleph_0}]^{\aleph_0}$,
$Q_\alpha \restriction x$ really has uncountable cofinality. Thus f is not a WFN-function for
$\mathfrak{P}(\omega)$. Since f was arbitrary, $\mathrm{WFN}(\mathfrak{P}(\omega))$ fails. $\qquad\square$

A characterization of the forcings that yield σ-extensions of $\mathfrak{P}(\omega)$

The following lemma characterizes those proper notions of forcing which yield
σ-extensions of $\mathfrak{P}(\omega)$. I am not going to introduce properness, since I will
only use the following property of proper forcing extensions:

Every countable set of ordinals in the extension is included in a countable
set in the ground model.

In particular, $\aleph_1$ of the ground model remains a cardinal in the extension.
Note that all c.c.c. forcings as well as many other forcing notions, especially

the standard forcing notions used for manipulating the cardinal invariants of the continuum, are proper.

It is convenient to introduce some additional notions first.

3.1.3. Definition. Let P be a partial order and C be a subset of P. Then $S(C)$ denotes the set of all greatest upper bounds of subsets of C that exist in P. A subset Q of P is *predense* below $p \in P$ iff for each $r \leq p$ there is $q \in Q$ such that $q \leq p$ and q is compatible with r. ◻

3.1.4. Lemma. *Let M be a transitive model of* ZFC* *. Let $A \in M$ be a complete Boolean algebra such that for each A-generic filter over M every countable set of ordinals in $M[G]$ is included in a countable set of ordinals which is an element of M. Then the following are equivalent:*

(i) *For each A-generic filter G, $M[G] \models \mathfrak{P}(\omega) \cap M \leq_\sigma \mathfrak{P}(\omega)$.*

(ii) *In M: For every countable subset C of A there is a dense set of $a \in A$ such that $(a \cdot S(C))^+$ has a countable subset B which is in A predense below each element of $(a \cdot S(C))^+$. Here $a \cdot S(C)$ means $\{a \cdot s : s \in S(C)\}$.* ◻

The formulation in (ii) sounds exceedingly strange. The problem is that the algebra generated by $S(C)$ does not have to be a regular subalgebra of A. I wanted a formulation that does not use generic filters. The property of the countable set B can be described as follows: For any generic filter G containing $a \cdot c$ for some $c \in S(C)$ there is $b \in B$ such that $b \leq a \cdot c$ and $b \in G$. In particular, B is dense in $(a \cdot S(C))^+$.

Proof of the lemma. Suppose (i) holds for A. Let C be any countable subset of A and $b \in A$. I have to show that there is $a \in A^+$ with $a \leq b$ such that $(a \cdot S(C))^+$ has a countable subset which is in A predense below every element of $(a \cdot S(C))^+$.

Let $\sigma : \omega \to C$ be onto. I regard σ as a name for a subset of ω. Let G be an A-generic filter over M containing b. By (i), there is a countable set D of subsets of ω such that $I := (\mathfrak{P}(\omega) \cap M) \restriction \sigma_G$ is generated by a subset of D. By the properties of the extension, I may assume that D is an element

of M. Now there is $a \in G$ such that a forces that I is generated by a subset of D. Since G is a filter, I can choose a below b. I may assume that for each $d \in D$ there is a non-zero $a_d \leq a$ such that $a_d \Vdash \check{d} \subseteq \sigma$ since only these d's are interesting. Let $[\![\check{d} \subseteq \sigma]\!]$ denote the truth value of the statement '$\check{d} \subseteq \sigma$'.

Claim 1. $[\![\check{d} \subseteq \sigma]\!] = \prod\{\sigma(n) : n \in d\}$.

This is easily seen. In particular, $[\![\check{d} \subseteq \sigma]\!] \in S(C)$. By enlarging the a_d's if necessary, I may assume that for each $d \in D$, a_d is the product of a and $[\![\check{d} \subseteq \sigma]\!]$. Now $a_d \in (a \cdot S(C))^+$ for every $d \in D$.

Claim 2. $\{a_d : d \in D\}$ is predense below each $c \in (a \cdot S(C))^+$.

Proof of Claim 2: Let C' be a subset of C such that $c := a \cdot \prod C' > 0$. Let $c' \leq c$ be such that $c' > 0$. Let $e := \{n \in \omega : \sigma(n) \geq c'\}$. Now c' forces that I is generated by a subset of D and $e \subseteq \sigma$. Let H be generic containing c'. Then there is $d \in D$ such that $e \subseteq d$ and $a_d \in H$. Since $a_d \leq \sigma(n)$ for each $n \in e$, $a_d \leq c$. Since a_d and c' both are elements of H, they are compatible. This proves Claim 2 and hence one direction of the equivalence.

For the other direction suppose (ii) holds. Let G be A-generic over M and let σ be a name for a subset of ω. I may assume that σ is a function from ω to A. Let $C := \operatorname{Im} \sigma$. By (ii), there is $a \in G$ such that $(a \cdot S(C))^+$ has a countable subset B which is predense below every non-zero element of $a \cdot S(C)$. For each $b \in B$ let $d_b := \{n \in \omega : \sigma(n) \geq b\}$. Let $D := \{d_b : b \in B\}$.

Claim 3. $I := (\mathfrak{P}(\omega) \cap M) \restriction \sigma_G$ is generated by a subset of D.

Proof of Claim 3: Let $e \in I$. Then there is $c \leq a$ with $c \in G$ such that $c \Vdash \check{e} \subseteq \sigma$. By Claim 1, $c \Vdash \check{e} \subseteq \sigma$ holds precisely if for each $n \in e$, $\sigma(n) \geq c$. Hence $a \cdot \prod\{\sigma(n) : n \in e\}$ also forces $\check{e} \subseteq \sigma$ and is an element of G. Since B is predense below $a \cdot \prod\{\sigma(n) : n \in e\}$, there is $b \in B \cap G$ such that $b \Vdash \check{e} \subseteq \sigma$. But now $e \subseteq d_b \subseteq \sigma_G$. This shows Claim 3 and therefore finishes the proof the lemma.   □

This lemma is quite abstract and technical, but it has interesting consequences. For example, it follows that any proper $^\omega\omega$-bounding forcing notion which adds a new real gives a non-σ-extension of $\mathfrak{P}(\omega)$. This can be seen as follows: For complete Boolean algebras $^\omega\omega$-boundingness is the same as weak (ω, ω)-distributivity. Let A be a proper complete weakly (ω, ω)-distributive

Boolean algebra and let $C \leq A$ be countable such that an enumeration of C is a name for a new real. Suppose A yields a σ-extension of $\mathfrak{P}(\omega)$. Then by (ii) of the preceding lemma, there is a dense set of $a \in A$ such that $(S(C) \cdot a)^+$ has a countable dense subset D_a. I may assume that D_a contains $C \cdot a$. Let $B_a := \langle D_a \rangle_{A \restriction a}$. Then D_a is dense in B_a and B_a is atomless. Let $\{b_n : n \in \omega\}$ be a maximal antichain in D_a. Since D_a is dense in B_a, this antichain is a maximal antichain in B_a. For each $n \in \omega$ pick $\{c_n^m : m \in \omega\} \subseteq C$ such that $b_n = \prod \{c_n^m : m \in \omega\}$. Now

$$b := a - \sum \{b_n : n \in \omega\} = a \cdot \prod_{n \in \omega} \sum_{m \in \omega} -c_n^m$$

$$= a \cdot \sum_{f : \omega \to \omega} \prod_{n \in \omega} \sum_{m < f(n)} -c_n^m.$$

Since C is a subalgebra of A, $\prod_{n \in \omega} \sum_{m < f(n)} -c_n^m \in S(C)$ for every $f : \omega \to \omega$. Thus b is zero or else there is some element of D_a below b, contradicting the maximality of the antichain. Therefore B_a is a regular subalgebra of $A \restriction a$. But this contradicts weak (ω, ω)-distributivity.

However, later I will prove a much more general result. But the argument above is still useful, as it leads to

3.1.5. Remark. Let A be a weakly (ω, ω)-distributive complete c.c.c. Boolean algebra and let C be a subalgebra of A which completely generates A. Then $S(C)$ is dense in A.

Proof. First assume that C is countable. Let B be the subalgebra of A that is generated by $S(C)$ (using only finite operations). Since C is a subalgebra of A, $S(C)$ is dense in B. Let K be a maximal antichain in B. By c.c.c., K is countable. By the same argument as above, it follows that K is already maximal in A. Thus B is a regular subalgebra of A. This means that B is dense in the complete subalgebra of A generated by B. But B completely generates A and thus B is dense in A. Therefore $S(C)$ is dense in A.

Now let C be arbitrary. Let $a \in A^+$. I have to show that there is $b \in S(C)$ such that $0 < b \leq a$. By c.c.c., there is a countable subalgebra C' of C such that a is contained in the complete subalgebra of A generated by C'. By

the first part of the proof, there is $b \in S(C')$ such that $0 < b \le a$. Clearly, $b \in S(C)$.   □

In order to show that certain forcing notions yield non-σ-extensions of $\mathfrak{P}(\omega)$, it is usually sufficient to apply the following version of the preceding lemma:

3.1.6. Lemma. *Let M be a transitive model of a sufficiently large part of ZFC. Let $A \in M$ be a complete atomless Boolean algebra such that for each A-generic filter G over M every countable set of ordinals in $M[G]$ is included in a countable set of ordinals which is an element of M. Suppose A has a countable subset C such that $S(C)$ is dense in A. Assume that for no $a \in A$ the algebra $A \restriction a$ has a countable dense subset. (Note that this holds in particular if forcing extensions obtained using A cannot be obtained by just adding one Cohen real.) Then for every A-generic filter G over M,*

$$M[G] \models \mathfrak{P}(\omega) \cap M \nleq_\sigma \mathfrak{P}(\omega).$$

Proof. W.l.o.g. I may assume that C is a subalgebra of A. It is easy to show that (ii) of the lemma above does not hold for A:

Suppose it does. Let a be such that $(a \cdot S(C))^+$ has a countable subset B which is predense below every non-zero element of $a \cdot S(C)$. In particular, such a set B is dense in the set $(a \cdot S(C))^+$. Since $S(C)$ is dense in A, $a \cdot S(C)$ is dense below a. Hence B is dense below a. But now $A \restriction a$ has a countable dense subset. A contradiction.   □

Many examples

In this section I show that many forcing notions meet the conditions in Lemma 3.1.6. It follows that they yield non-σ-extensions of $\mathfrak{P}(\omega)$. For most of these forcings it will turn out later that it is not necessary to apply Lemma 3.1.6 to show that they yield non-σ-extensions of $\mathfrak{P}(\omega)$. In the section 3.2 I will collect some purely combinatorial criteria for when an extension N of some model M yields a non-σ-extension of $\mathfrak{P}(\omega)$. These criteria work even if the extensions are not obtained by forcing. However, for forcing extensions,

this section provides a more uniform approach and some of the results of this section do not follow from the combinatorial criteria mentioned above.

3.1.7. Definition. A forcing notion P *meets the conditions in Lemma 3.1.6* iff $\mathrm{ro}(P)$ has a countable subset C such that $S(C)$ is dense in $\mathrm{ro}(P)$ and for every P-generic filter G over the ground model M every in $M[G]$ countable set of ordinals is covered by some set in M which is countable in M and there is no Cohen real $x \in M[G]$ over M such that $M[G] = M[x]$. □

I need some additional forcing theoretic notions.

3.1.8. Definition. For every partial order $(P, \leq)$ and all $p, q \in P$ let $p \leq^* q$ iff there is no $r \leq p$ such that $r \perp q$. A subset D of P is **dense* iff D is dense in P with respect to $\leq^*$. Similarly, $D \subseteq P$ is **dense below $p \in P$* iff D is dense below p in P with respect to $\leq^*$.

Clearly, $\leq^*$ extends $\leq$. Note that $p \leq^* q$ iff the image of p in $\mathrm{ro}(P)$ under the canonical mapping is smaller or equal to the image of q under this mapping. Using $\leq^*$, I can argue in P itself rather than in $\mathrm{ro}(P)$.

3.1.9. Lemma. *Let $(P, \leq)$ be a partial order and let $e : P \to \mathrm{ro}(P)$ be the canonical mapping. Let $C \subseteq P$. If*

$$\forall p, q \in P(\forall c \in C(p \leq c \Rightarrow q \leq c) \Rightarrow q \leq^* p),$$

then $S(e[C])$ is dense in $\mathrm{ro}(P)$. In particular, if $S(C)$ is dense in P, then $S(e[C])$ is dense in $\mathrm{ro}(P)$.

Proof. Easy, using the fact that $e[P]$ is dense in $\mathrm{ro}(P)$. □

In the following, I will sometimes use this lemma without referring to it. I first consider the measure algebra of the Cantor space and Sacks forcing.

3.1.10. Definition. *Random forcing* is the measure algebra $\mathbb{R}(\omega)$ of the Cantor space ${}^\omega 2$ which already has been introduced. *Sacks forcing* is the partial order $\mathbb{S}$ consisting of all perfect subsets of the unit interval ordered by inclusion. □

Note that the generic objects for these forcings can be coded by single reals. A Sacks real is the unique element of the intersection of all elements of an $\mathbb{S}$-generic filter. The real coding an $\mathbb{R}(\omega)$-generic filter is obtained in a similar way and is called a random real.

3.1.11. Lemma. $\mathbb{R}(\omega)$ *and* $\mathbb{S}$ *meet the conditions in Lemma 3.1.6.*

Proof. Random forcing and Sacks forcing are both proper and ${}^\omega\omega$-bounding. As mentioned above, for complete Boolean algebras the latter property is equivalent to weak (ω, ω)-distributivity, which is hereditary with respect to regular subalgebras and relative algebras. Cohen forcing does not share this property. Hence random forcing and Sacks forcing both do not add Cohen reals. Let $C_{\mathbb{R}(\omega)}$ be the subset of $\mathbb{R}(\omega)$ that consists of equivalence classes of clopen sets. Then $S(C_{\mathbb{R}(\omega)})$ consists of the equivalence classes of closed subsets of ${}^\omega 2$. Since every subset of ${}^\omega 2$ of positive measure includes a closed set of positive measure, the set $S(C_{\mathbb{R}(\omega)})$ is dense in $\mathbb{R}(\omega)$.

For Sacks forcing let $C_{\mathbb{S}}$ be the set of finite unions of infinite closed intervals with rational endpoints. Clearly, this set is countable. Also, it is easy to see that $S(C_{\mathbb{S}})$ is dense. Hence random forcing and Sacks forcing both meet the conditions in Lemma 3.1.6.   □

Similarly, Lemma 3.1.6 applies to amoeba forcing.

3.1.12. Definition. *Amoeba forcing* is the partial order $\mathbb{A}$ consisting of all open subsets of ${}^\omega 2$ of measure $< \frac{1}{2}$ ordered by reverse inclusion.   □

Amoeba forcing is σ-linked and thus proper. Another notion of forcing is connected with amoeba forcing, localization forcing $\mathbb{LOC}$.

3.1.13. Definition. *Localization forcing* is the partial order $\mathbb{LOC}$ consisting of all $s \in {}^\omega([\omega]^{<\aleph_0})$ such that $\forall n \in \omega(|s(n)| \leq n)$ and $\exists k \in \omega \forall^\infty n(|s(n)| \leq k)$. The order is componentwise inclusion.   □

$\mathbb{LOC}$ is also proper and $\mathbb{A}$ completely embeds into $\mathbb{LOC}$. Both forcings are treated in [1]. Again the respective generic filters can be coded by a single real.

3.1.14. Lemma. $\mathbb{A}$ *and* $\mathbb{LOC}$ *both meet the conditions in Lemma 3.1.6.*

Proof. Consider $\mathbb{A}$ first. It is easy to see that for the set $C_{\mathbb{A}}$ of clopen subsets of the Cantor space with measure $< \frac{1}{2}$ the set $S(C_{\mathbb{A}})$ is dense in $\mathbb{A}$. Thus it remains to show that for no $p \in \mathbb{A}$ there is a countable set *dense below p. Let $p \in \mathbb{A}$ and suppose $D = \{d_n : n \in \omega\}$ is a countable set of conditions in $\mathbb{A}$. Let $\varepsilon > 0$ be such that $\varepsilon < \frac{1}{2} - \mu(p)$. For each $n \in \omega$ pick an open set $p_n \subseteq {}^{\omega}2$ which is disjoint from d_n such that $\mu(p_n) < \frac{\varepsilon}{2^{n+17}}$. Now $q := p \cup \bigcup_{n \in \omega} p_n$ is a condition below p such that for no $n \in \omega$, $d_n \leq^* q$. Hence D is not *dense below p.

Now consider $\mathbb{LOC}$. Since $\mathbb{A}$ completely embeds into $\mathbb{LOC}$, a generic extension obtained by adding an $\mathbb{LOC}$-generic filter cannot be obtained by adding a Cohen real. The set $C_{\mathbb{LOC}}$ of sequences in $\mathbb{LOC}$ that are eventually constant with value $\emptyset$ is countable and $S(C_{\mathbb{LOC}})$ is easily seen to be dense in $\mathbb{LOC}$. $\qquad\blacksquare$

Next I consider Hechler forcing and eventually different forcing.

3.1.15. Definition. *Hechler forcing* is the partial order $\mathbb{D}$ consisting of all conditions $p = (f_p, F_p)$ where f_p is a finite sequence of natural numbers and F_p is a finite set of (total) functions from ω to ω. The order is defined as follows: For all $p, q \in P$, $p \leq q$ iff $f_q \subseteq f_p$, $F_q \subseteq F_p$, and for all $n \in \mathrm{dom}(f_p \setminus f_q)$ and all $f \in F_q$, $f_p(n) \geq f(n)$.

Eventually different forcing is the the partial order $\mathbb{E}$ having the same conditions as $\mathbb{D}$ and the following order: For all $p, q \in \mathbb{E}$, $p \leq q$ iff $f_q \subseteq f_p$, $F_q \subseteq F_p$, and for all $n \in \mathrm{dom}(f_p \setminus f_q)$ and all $f \in F_q$, $f_p(n) \neq f(n)$. $\qquad\blacksquare$

$\mathbb{D}$ is frequently called dominating forcing since it adds a function from ω to ω which dominates all the functions from the ground model. In order to avoid confusion, in the following by a dominating real I mean an element of ${}^{\omega}\omega$ that eventually dominates all functions from ω to ω in the ground model. The dominating real added by Hechler forcing is a Hechler real.

$\mathbb{E}$ adds a real which is eventually different from all functions from ω to ω in the ground model. The generic filter is coded by such a real. $\mathbb{E}$ behaves similarly to $\mathbb{D}$. Like $\mathbb{D}$, it is σ-centered and adds Cohen reals. But it does

not add a dominating real. An elegant proof of the latter fact can be found in [1]. An element of $^\omega\omega$ that is eventually different from all functions from ω to ω in the ground model is an eventually different real. I will not use a special name for the eventually different real added by $\mathbb{E}$.

3.1.16. Lemma. $\mathbb{D}$ *and* $\mathbb{E}$ *both meet the conditions in Lemma 3.1.6.*

Proof. I have to struggle with the fact that neither $\mathbb{D}$ nor $\mathbb{E}$ is separative. This means that certain conditions will be identified when passing to the completion of the respective partial order. Call two conditions p and q equivalent if $p \leq^* q$ and $q \leq^* p$ and write $p \sim q$ in this case. With this definition, two conditions are equivalent iff they will be identified in the completion of the respective partial order. For $p \in \mathbb{D}$ or $p \in \mathbb{E}$ (well, actually the underlying sets of both partial orders are the same) such that $p = (f, F)$ write f_p for f and F_p for F.

Claim 1. Two conditions p and q are equivalent in $\mathbb{D}$ iff $f_p = f_q$ and

$$\forall n \in \omega \setminus \operatorname{dom} f_p(\max\{g(n) : g \in F_p\} = \max\{g(n) : g \in F_q\}).$$

Proof of Claim 1: Note that p and q are equivalent iff $\{r \in \mathbb{D} : r \perp p\} = \{r \in \mathbb{D} : r \perp q\}$. Now the claim follows from the fact that $r, s \in \mathbb{D}$ are compatible iff $f_r \subseteq f_s$ or $f_s \subseteq f_r$ and the condition with the larger first coordinate, say r, satisfies

$$\forall n \in \operatorname{dom}(f_r \setminus f_s)\forall g \in F_s(f_r(n) \geq g(n)).$$

Almost the same argument works for $\mathbb{E}$, only the $\geq$ in the last line has to be replaced by $\neq$. Thus the followings holds:

Claim 2. Two conditions p and q are equivalent in $\mathbb{E}$ if $f_p = f_q$ and

$$\forall n \in \omega \setminus \operatorname{dom} f_p(\{g(n) : g \in F_p\} = \{g(n) : g \in F_q\}).$$

For $P = \mathbb{D}$ or $P = \mathbb{E}$ consider the countable set C_P consisting of those conditions $p \in P$ for which $F_p = \{g\}$ for some g that is eventually constant with value 0.

Claim 3. For $P = \mathbb{D}, \mathbb{E}$ the set $S(e[C_P])$ is dense in $\mathrm{ro}(P)$, where $e : P \to \mathrm{ro}(P)$ is the canonical mapping.

Proof of Claim 3: First let $p \in \mathbb{D}$. For $m, n \in \omega$ define

$$g_p^n(m) := \begin{cases} \max\{g(m) : g \in F_p\} & \text{for } m < n, \\ 0 & \text{otherwise.} \end{cases}$$

Having the proof of Claim 1 in mind, it is not difficult to see that p is the greatest lower bound of the set $\{(f_p, \{g_p^n\}) : n \in \omega\}$ in $\mathrm{ro}(\mathbb{D})$.

Now let $p \in \mathbb{E}$. For $n \in \omega$ let

$$F_p^n := \{g \in {}^\omega\omega : \exists h \in F_p(h \restriction n = g \restriction n) \wedge \forall m \geq n(g(m) = 0)\}.$$

Let $z : \omega \to \omega$ be the function which is constant with value 0. Using the arguments from the proof of Claim 2, it is not difficult to see that $(f_p, F_p \cup \{z\}) \leq p$ is the greatest lower bound of the set $\{(f_p, F_p^n) : n \in \omega\}$ in $\mathrm{ro}(\mathbb{E})$, which proves Claim 3.

It remains to show that forcing extensions obtained using $\mathbb{D}$ or $\mathbb{E}$ cannot be obtained by adding one Cohen real. This is immediate for $\mathbb{D}$ since it is well known that Cohen forcing does not add a dominating real. For $\mathbb{E}$ I only have to show that adding a Cohen real does not add an eventually different real. Note that the fact that Cohen forcing does not add a dominating real follows from this since a dominating real is eventually different.

Let $(p_n)_{n \in \omega}$ be an enumeration of a countable dense subset of the Cohen algebra such that every condition is listed infinitely often. Let σ be a name for a function from ω to ω. For each $n \in \omega$ let $U_n := \{m \in \omega : p \Vdash \sigma(n) \neq m\}$. Pick a function $g : \omega \to \omega$ such that $g(n) \notin U_n$ for all n. Let G be a Cohen-generic filter. I show that σ_G is not eventually different from g. Suppose it is. Then there are $n \in \omega$ and $m \geq n$ such that $p_m \Vdash \forall k \geq n(\sigma(k) \neq g(k))$. But now $g(m) \in U_m$, a contradiction. □

I will quickly sketch how to prove similar results for some other notions of forcing.

3.1.17. Definition. The underlying set of *Miller forcing* $\mathbb{M}$ is the set of

superperfect trees, i.e. subtrees of $^{<\omega}\omega$ in which beyond every node there is one with infinitely many immediate successors. The order is inclusion. □

Miller forcing is proper. A Miller-generic filter can be coded by a single real. Note that Sacks forcing can also be considered as a partial order made up from trees, namely the set of perfect trees. That is, subtrees of $^{<\omega}\omega$ in which beyond every node there is one with at least two immediate successors.

3.1.18. Lemma. *Miller forcing meets the conditions in Lemma 3.1.6.*

Proof. For a tree $T \subseteq {}^{<\omega}\omega$ and $s \in T$ let $\mathrm{succ}_T(s) := \{n \in \omega : s^\frown(n) \in T\}$. Consider the set C of trees T in which up to some finite level only finite or cofinite sets occur as $\mathrm{succ}_T(s)$ and beyond that level only ω occurs as $\mathrm{succ}_T(s)$. Clearly, C is countable. It is easy to see that every superperfect tree is the greatest lower bound of some subset of C. For a given superperfect tree T and a countable set D of superperfect trees below T one can inductively thin out T in order to obtain a superperfect tree $T' \subseteq T$ such that no tree from D lies below T'. Thus Lemma 3.1.6 applies to Miller forcing as well. □

3.1.19. Definition. *Grigorieff reals.* A filter $\mathcal{F}$ on ω is called a p-filter if for every countable set $\mathcal{G} \subseteq \mathcal{F}$ there is a set $a \in \mathcal{F}$ which is almost included in every element of $\mathcal{G}$. A filter $\mathcal{F}$ on ω is called unbounded if the set of monotone enumerations of elements of $\mathcal{F}$ is unbounded in $^{\omega}\omega$. Note that every ultrafilter is unbounded and that CH (as well as MA) implies that there are p-ultrafilters, i.e. p-points. For an unbounded p-filter $\mathcal{F}$ containing all cofinite sets let Grigorieff forcing $\mathbb{G}_{\mathcal{F}}$ be the set of partial functions f from ω to 2 such that $\omega \setminus \mathrm{dom}\, f \in \mathcal{F}$. The order is reverse inclusion. $\mathbb{G}_{\mathcal{F}}$ is proper and $^{\omega}\omega$-bounding.

Prikry-Silver reals. Priky-Silver forcing is the set of partial functions from ω to 2 with co-infinite domains ordered by reverse inclusion. Prikry-Silver forcing is proper.

Infinitely equal forcing. The conditions of infinitely equal forcing $\mathbb{EE}$ are partial functions p from ω to $^{<\omega}2$ such that for all $n \in \omega$ the sequence $p(n)$ is an element of $^n 2$ and $\omega \setminus \mathrm{dom}(p)$ is infinite. The order is reverse inclusion. $\mathbb{EE}$ is proper and $^{\omega}\omega$-bounding. □

3.1.20. Lemma. *Let P be either Grigorieff forcing, Prikry-Silver forcing, or $\mathbb{EE}$. Then P meets the conditions in Lemma 3.1.6.*

Proof. P is proper. For each condition $p \in P$ there is a condition $q \in P$ such that $p \subseteq q$ and $\mathrm{dom}(q \setminus p)$ is infinite. In fact, there is an uncountable antichain below p. In particular, there is no countable set dense below p. Thus it only remains to construct a suitable subset C_P of P in order to show that P meets the conditions in Lemma 3.1.6. Let C_P consist of all conditions in P that have finite domain. Clearly, C_P is countable. Clearly, every $p \in P$ is the greatest lower bound of some subset of C_P. $\qquad\square$

Finally, Lemma 3.1.6 also applies to the countable support iteration of Cohen forcing of length ω.

3.1.21. Lemma. *Let $P := (\mathrm{Fn}(\omega, 2))^\omega$ be ordered componentwise. Then P meets the conditions in Lemma 3.1.6.*

Proof. Since Cohen forcing is absolute, P is equivalent to the countable support iteration of Cohen forcing of length ω. Thus P is proper. Let C_P be the set of all conditions with finite support. Clearly, C_P is countable. It is easily seen that $S(C_P)$ is dense in P. However, below every element of P there is an uncountable antichain. Thus a generic extension obtained by adding a P-generic filter cannot be obtained by adding a Cohen real. This proves the lemma. $\qquad\square$

It follows that Lemma 3.1.6 applies to all notions of forcing mentioned so far, except for Cohen forcing, of course.

3.1.22. Corollary. *Random forcing, Sacks forcing, amoeba forcing, localization forcing, Hechler forcing, eventually different forcing, Miller forcing, Grigorieff forcing, Prikry-Silver forcing, infinitely equal forcing and the countable support iteration of Cohen forcing of length ω yield non-σ-extensions of $\mathfrak{P}(\omega)$.* $\qquad\square$

It follows from this corollary together with Lemma 3.1.2 that $\mathfrak{P}(\omega)$ does not have the WFN in many popular models of set theory. I only mention one example.

3.1.23. Corollary. *Let M be a model of* $\mathrm{ZFC}^* +$ *CH. In M let P be the measure algebra on* $^{\omega_2}2$. *Suppose G is P-generic over M. Then*

$$M[G] \models \neg \mathrm{WFN}(\mathfrak{P}(\omega)).$$

Proof. For $\alpha < \omega_2$ let P_α be the measure algebra on $^{\omega \cdot \alpha}2$. P_α can be considered as a complete subalgebra of P in a natural way. The sequence $(P_\alpha)_{\alpha < \omega_2}$ is continuous at limit ordinal of uncountable cofinality. For $\alpha < \omega_2$ and a $P_{\alpha+1}$-generic filter G, $M[G] = M[G \cap P_\alpha][r]$ where r is a random real over $M[G \cap P_\alpha]$. Thus it follows from the last corollary that

$$M[G] \models \mathfrak{P}(\omega) \cap M[G \cap P_\alpha] \not\leq_\sigma \mathfrak{P}(\omega).$$

Therefore Lemma 3.1.2 applies. Thus $\Vdash_P \neg \mathrm{WFN}(\mathfrak{P}(\omega))$. $\square$

Note that if N is a proper forcing extension of the ground model M, then in N, $\mathbb{R}(\omega)^M$ still meets the conditions in Lemma 3.1.6. It follows that even forcing with a side-by-side product of $\aleph_2$ copies of random forcing over a model of CH gives a model of $\neg \mathrm{WFN}(\mathfrak{P}(\omega))$.

The latter model is especially interesting since Fuchino has recently observed that for every regular $\kappa > \aleph_1$ a combinatorial principle called $C^S(\kappa)$ holds in this model ([8]). This principle was introduced in [25] and implies among other things that there is no increasing chain with respect to $\subseteq^*$ in $\mathfrak{P}(\omega)$ of ordertype ω_2 and that there is no socalled $\aleph_2$-Luzin gap. It was shown in [16] respectively in [19] that under $\mathrm{WFN}(\mathfrak{P}(\omega))$, there is no increasing chain with respect to $\subseteq^*$ in $\mathfrak{P}(\omega)$ of ordertype ω_2 and there is no $\aleph_2$-Luzin gap. Fuchino and Soukup asked whether the latter two statements imply $\mathrm{WFN}(\mathfrak{P}(\omega))$. They do not.

It should also be noted that Corollary 3.1.23 can be obtained in a different way. After forcing with P over a model of CH, the covering number of the ideal of measure zero subsets of the Cantor space, $\mathrm{cov}(\mathcal{N})$, is $\aleph_2$. But $\mathrm{WFN}(\mathfrak{P}(\omega))$ implies that $\mathrm{cov}(\mathcal{N})$ is $\aleph_1$. This can be seen as follows. Assume $\mathrm{WFN}(\mathfrak{P}(\omega))$. Let M be some $V_{\aleph_1}$-like elementary submodel of H_χ for sufficiently large χ. M has size $\aleph_1$. If $\mathrm{cov}(\mathcal{N})$ is larger than $\aleph_1$, then the

measure zero subsets of the Cantor space which are in M do not cover the whole space. By Solovay's characterization of random reals, there is a random real x over M. Now $M[x] \models \mathfrak{P}(\omega) \cap M \not\leq_\sigma \mathfrak{P}(\omega)$. By $V_{\aleph_1}$-likeness of M, every countable subset of M is covered by a countable element of M. It follows that in the real world, $\mathfrak{P}(\omega) \cap M \not\leq_\sigma \mathfrak{P}(\omega)$. But this contradicts WFN($\mathfrak{P}(\omega)$). However, Section 3.2 on cardinal invariants of the continuum contains a stronger result, due to Soukup. He showed that even non($\mathcal{M}$), the smallest cardinality of a non-meager subset of the Cantor space, is $\aleph_1$ under WFN($\mathfrak{P}(\omega)$). It is well-known that non($\mathcal{M}$) is larger or equal to cov($\mathcal{N}$).

Adding a Hechler real over ω_2 Cohen reals to a model of CH gives a model of $\neg$ WFN($\mathfrak{P}(\omega)$)

Soukup pointed out to me that forcing with Fn($\omega, 2$) preserves WFN($\mathfrak{P}(\omega)$), but cannot introduce it. Koppelberg and Shelah ([31]) constructed a complete subalgebra A of $\mathbb{C}(\aleph_2)$ which is not a Cohen algebra. Soukup also pointed out to me that forcing with A introduces an $\aleph_2$-Luzin gap. As I have mentioned earlier, he and Fuchino have shown in [19] that WFN($\mathfrak{P}(\omega)$) fails if there is an $\aleph_2$-Luzin gap. It follows that forcing with A over a model of CH gives a model of $\neg$ WFN($\mathfrak{P}(\omega)$), but WFN($\mathfrak{P}(\omega)$) can be introduced by some cardinal preserving notion of forcing, namely the quotient forcing $\mathbb{C}(\aleph_2) : A$. Of course, collapsing the continuum to $\aleph_1$ always introduces WFN($\mathfrak{P}(\omega)$). This is the reason why only cardinal preserving notions of forcing are interesting here.

It is clear from the results in the previous section that adding many reals can destroy WFN($\mathfrak{P}(\omega)$). In [37] Shelah proved that $0^\sharp$ is needed in order to destroy CH by adding a single real without collapsing $\aleph_1$. This means that typically, adding only one real to a model of CH preserves WFN($\mathfrak{P}(\omega)$), simply because it preserves CH. Soukup asked me whether adding only one real by some proper, c.c.c., or even σ-centered forcing can destroy WFN($\mathfrak{P}(\omega)$). It can. A Hechler real is sufficient. It would be nice to know some cardinal preserving generic reals, apart from Cohen reals, which preserve WFN($\mathfrak{P}(\omega)$). But I guess these are hard to find. (Provided they exist at all.) Note that

adding a Hechler real gives a model where the popular cardinal invariants of the continuum have the same values as in a model with the same size of the continuum that is obtained by adding Cohen reals to a model of CH. This was shown by Brendle, Judah, and Shelah ([7]).

The strategy to show that adding a Hechler real over ω_2 Cohen reals gives a model in which $\mathrm{WFN}(\mathfrak{P}(\omega))$ fails is the following:

First decompose the forcing for first adding ω_2 Cohen reals and then adding a Hechler real into a chain of small forcings indexed by ω_2 instead of $\omega_2 + 1$. Now this iteration can be handled using the techniques developed in the last sections.

I will use an alternative definition of Hechler forcing now, which yields a partial order that is forcing equivalent to the Hechler forcing $\mathbb{D}$ defined before. The definition above was chosen in order to make the similarity between Hechler forcing and eventually different forcing apparent.

3.1.24. Definition. For two partial functions $f, g \subseteq \omega \times \omega$ let $f \leq g$ iff for every $n \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$, $f(n) \leq g(n)$.

Let $\mathbb{D}' := \{(\sigma, f) : \sigma \in {}^{<\omega}\omega \wedge f \in {}^{\omega}\omega\}$. For $(\sigma, f), (\tau, g) \in \mathbb{D}'$ let $(\sigma, f) \leq (\tau, g)$ iff $\sigma \supseteq \tau$, $f \geq g$, and $\sigma \setminus \tau \geq g$. $\square$

For a set $F \in [{}^{\omega}\omega]^{<\aleph_0}$ let $\max(F) : \omega \to \omega; n \mapsto \max\{f(n) : f \in F\}$. The mapping $\varphi : \mathbb{D} \to \mathbb{D}'; (\sigma, F) \mapsto (\sigma, \max(F))$ is easily seen to induce an isomorphism between $\mathrm{ro}(\mathbb{D})$ and $\mathrm{ro}(\mathbb{D}')$. This justifies calling $\mathbb{D}'$ Hechler forcing as well.

3.1.25. Definition. Let

$\dot{\mathbb{D}} := \{(\sigma, \dot{f}) : \sigma \in {}^{<\omega}\omega$ and $\dot{f}$ is an $\mathrm{Fn}(\omega, 2)$-name

$\qquad\qquad\qquad\qquad\qquad\qquad$ for a function from ω to $\omega\}$. $\square$

$\dot{\mathbb{D}}$ can be regarded as an $\mathrm{Fn}(\omega, 2)$-name for Hechler forcing in a straightforward way.

3.1.26. Definition. Let $\mathbb{P} := \mathrm{Fn}(\omega, 2) * \dot{\mathbb{D}}$ and let

$$\mathbb{Q} := \{(1_{\mathrm{Fn}(\omega, 2)}, (\sigma, \check{f})) : \sigma \in {}^{<\omega}\omega \wedge f \in {}^{\omega}\omega\} \subseteq \mathbb{P}.   \square$$

$\mathbb{Q}$ is equivalent to ordinary Hechler forcing. When analyzing the relation between $\mathbb{P}$ and $\mathbb{Q}$, it will be necessary to approximate functions in $^{\omega}\omega$ in a generic extension by ground model functions.

3.1.27. Definition. Let P be any notion of forcing. Suppose $\dot{f}$ is a P-name for an element of $^{\omega}\omega$ and $p \in P$. Then a function $g \in {}^{\omega}\omega$ is *possible* for $\dot{f}$ and p iff for all $n \in \omega$, $p \nVdash \dot{f} \restriction n \neq \check{g} \restriction n$. $\square$

Note that for any name $\dot{f}$ for a function from ω to ω and any condition $q \in Q$ there is a possible function $g \in {}^{\omega}\omega$ for $\dot{f}$. Using this notion, one can show that $\mathbb{Q}$ behaves reasonably well with respect to $\mathbb{P}$.

3.1.28. Lemma. $\mathbb{Q}$ *is completely embedded into* $\mathbb{P}$.

Proof. According to Kunen's book ([32]), the following points have to be checked:

  (i) $\forall p, q \in \mathbb{Q}(p \perp_{\mathbb{Q}} q \Leftrightarrow p \perp_{\mathbb{P}} q)$

  (ii) $\forall p \in \mathbb{P} \exists q \in \mathbb{Q} \forall r \in \mathbb{Q}(r \leq q \Rightarrow r \not\perp_{\mathbb{P}} p)$

(i) is easily seen. Therefore in the following I will omit the subscripts on $\perp$. For (ii) let $p \in \mathbb{P}$, say $p = (s, (\sigma, \dot{f}))$ for $s \in \text{Fn}(\omega, 2)$, $\sigma \in {}^{<\omega}\omega$, and an $\text{Fn}(\omega, 2)$-name $\dot{f}$ for a function from ω to ω. Let g be a possible function for $\dot{f}$ and s. $q := (\emptyset, (\sigma, \check{g}))$ works for (ii):

Let $r \in \mathbb{Q}$ be such that $r \leq q$, say $r = (\emptyset, (\tau, \dot{h}))$. Let $v \leq s$ be a condition in $\text{Fn}(\omega, 2)$ which forces that $\dot{f}$ and g are equal on $\text{dom}(\tau)$. This is possible since g is possible for $\dot{f}$ and s. Let $\max(\dot{f}, \dot{h})$ be an $\text{Fn}(\omega, 2)$-name for a function such that for all $n \in \omega$, $\Vdash \max(\dot{f}, \dot{h})(n) = \max(\dot{f}(n), \dot{h}(n))$. Since r extends q, for all $n \in \text{dom}(\tau \setminus \sigma)$, $\tau(n) \geq g(n)$. Thus $(v, (\tau, \max(\dot{f}, \dot{h})))$ is a common extension of r and p. $\square$

Since $\mathbb{Q}$ is completely embedded into $\mathbb{P}$, it makes sense to consider the quotient $\mathbb{P} : \mathbb{Q}$.

3.1.29. Definition. Let $\dot{H}$ be the canonical $\mathbb{Q}$-name for the $\mathbb{Q}$-generic filter and let $\mathbb{P} : \mathbb{Q}$ be a $\mathbb{Q}$-name for a subset of $\mathbb{P}$ s.t.

$$\Vdash_{\mathbb{Q}} \mathbb{P} : \mathbb{Q} = \{p \in \check{\mathbb{P}} : \forall q \in \dot{H}(p \text{ and } q \text{ are compatible})\}.$$

If H is a $\mathbb{Q}$-generic filter, let $\mathbb{P} : H := (\mathbb{P} : \mathbb{Q})_H$.   □

It is well-known that forcing with $\mathbb{Q} * (\mathbb{P} : \mathbb{Q})$ is equivalent to forcing with $\mathbb{P}$. The proof of this fact really gives the following:

3.1.30. Lemma. *Let H be $\mathbb{Q}$-generic over the ground model M. If G is $\mathbb{P} : H$-generic over $M[H]$, then G as a subset of $\mathbb{P}$ is $\mathbb{P}$-generic over M and contains H.*

Proof. Let H be $\mathbb{Q}$-generic over M and let G be $\mathbb{P} : H$-generic over $M[H]$. For every $q \in H$, $q \in \mathbb{P} : H$ and q is compatible with every element of G. The $\mathbb{P} : H$-genericity of G implies $q \in G$. Thus $H \subseteq G$. It is clear that G is a filter.

Consider $A := \mathrm{ro}(\mathbb{P})$ and $B := \mathrm{ro}(\mathbb{Q})$. The complete embedding from $\mathbb{Q}$ into $\mathbb{P}$ induces a complete embedding from B into A. Thus I may think of B as a complete subalgebra of A. Clearly, forcing with B is equivalent to forcing with $\mathbb{Q}$. Let $H' \subseteq B$ be the B-generic filter induced by H and let $A : H := \{a \in A : \forall b \in H'(a \cdot b \neq 0)\}$. For each $a \in A$ let $\pi(a) := \prod(B \uparrow a)$. Note that for every $a \in A$, $a \in A : H$ iff $\pi(a) \in H'$. Let $f : \mathbb{P} \to A$ be the canonical mapping. Now $f[\mathbb{P} : H] \subseteq A : H$ and $f^{-1}[A : H] = \mathbb{P} : H$. Suppose $D \in M$ is a dense subset of $\mathbb{P}$.

Claim. $D \cap (\mathbb{P} : H)$ is dense in $\mathbb{P} : H$.

Let $p \in \mathbb{P} : H$. Then $\pi(f(p)) \in H'$. Let $b \in B$ be such that $b \leq \pi(f(p))$. By the definition of π, $b \cdot f(p) \neq 0$. Since $\mathbb{D}$ is dense in $\mathbb{P}$ and by the properties of f, $\{f(p') : p' \in D \wedge p' \leq p)\}$ is dense below $f(p)$ in A. Therefore there is $p' \leq p$ such that $f(p') \leq b$ and $p' \in D$. Clearly, $\pi(f(p')) \leq b$. It follows that the set $\{\pi(f(p')) : p' \in D \wedge p' \leq p\}$ is dense below $\pi(f(p))$ in B. Therefore there is $p' \in D$ such that $p' \leq p$ and $\pi(f(p')) \in H'$. Now $f(p') \in A : H$ and thus $p' \in \mathbb{P} : H$. This proves the claim.

Since G is $\mathbb{P} : H$ generic, the claim implies that G intersects D. Since D was arbitrary, it follows that G is $\mathbb{P}$-generic over M.   □

$\mathrm{ro}(\mathbb{P} : H)$ is generated by a name for a real in a nice way.

3.1.31. Lemma. *If H is $\mathbb{Q}$-generic, then in $M[H]$, $\mathrm{ro}(\mathbb{P} : H)$ has a countable subset C such that $S(C)$ is dense in $\mathrm{ro}(\mathbb{P} : H)$.*

Proof. For a function $f : \omega \to \omega$ and a condition $p \in \mathrm{Fn}(\omega, 2)$ let $\dot{f}_p$ be an $\mathrm{Fn}(\omega, 2)$-name for a function such that $p \Vdash (\dot{f}_p = \check{f})$ and for each $q \in \mathrm{Fn}(\omega, 2)$ such that $q \perp p$, $q \Vdash_{\mathrm{Fn}(\omega,2)} (\dot{f}_p$ is constant with value 0). Let

$$C' := \{(p, (\sigma, \dot{f}_q)) \in \mathbb{P} : H : p, q \in \mathrm{Fn}(\omega, 2),\ \sigma \in {}^{<\omega}\omega$$
$$\text{and } f \in {}^{\omega}\omega \text{ is eventually constant}\}.$$

Clearly, C' is countable. The image C of C' under the embedding of $\mathbb{P} : H$ into $\mathrm{ro}(\mathbb{P} : H)$ works for the lemma:

Let $(q, (\tau, \dot{g})) \in \mathbb{P} : H$. Let $(r, (\rho, \dot{h})) \in \mathbb{P} : H$ be such that for all $c \in C'$ with $(q, (\tau, \dot{g})) \leq c$, $(r, (\rho, \dot{h})) \leq c$ holds. Now $r \leq q$ and $\rho \supseteq \tau$. I will be done if I can show

$$r \Vdash_{\mathrm{Fn}(\omega,2)} \forall n \in \omega(\dot{h}(n) \geq \dot{g}(n)) \wedge \forall n \in \mathrm{dom}(\rho \setminus \tau)(\rho(n) \geq \dot{g}(n)).$$

But here it is sufficient to prove

$$(*) \quad \forall n \in \omega(r \Vdash_{\mathrm{Fn}(\omega,2)} \dot{h}(n) \geq \dot{g}(n))$$

and

$$(**) \quad \forall n \in \mathrm{dom}(\rho \setminus \tau)(r \Vdash_{\mathrm{Fn}(\omega,2)} \rho(n) \geq \dot{g}(n)).$$

Let $n \in \omega$ and let $s \in \mathrm{Fn}(\omega, 2)$ be such that $s \leq r$ and $s \Vdash (\dot{g}(n) = m)$ for some $m \in \omega$. Let $f : \omega \to \omega$ be the function that has the value m at the place n and is 0 everywhere else. Then $(q, (\tau, \dot{g})) \leq (q, (\tau, \dot{f}_s))$ and $(q, (\tau, \dot{f}_s)) \in C$. Thus $r \Vdash_{\mathrm{Fn}(\omega,2)} (\dot{h}(n) \geq \dot{g}(n))$ since $(r, (\rho, \dot{h})) \leq (q, (\tau, \dot{f}_s))$ for a set of s's dense below r. This shows $(*)$. The proof of $(**)$ is practically the same. $\square$

Next I am going to show that for no $q \in \mathrm{ro}(\mathbb{P} : H)$ there is a countable subset of $\mathrm{ro}(\mathbb{P} : H)$ that is dense below q. This needs some combinatorial preparation.

3.1.32. Lemma. *Let $n \in \omega$. Then ω^n ordered componentwise is wellfounded and every set $A \subseteq \omega^n$ consisting of pairwise incomparable elements is finite. In particular, every subset of ω^n has only finitely many minimal elements.*

Proof. For wellfoundedness let $S \subseteq \omega^n$. For each $b \in \omega^n$ and every $i < n$ let b_i be the i-th coordinate of b. Inductively for $i < n$ pick $a_i \in \omega$ minimal with the property $\exists b \in S \forall j \leq i(b_j = a_j)$. $(a_i)_{i<n}$ is minimal in S.

Now let $A \subseteq \omega^n$ consist of pairwise incomparable elements and assume for contradiction that A is infinite. Let $(a^k)_{k \in \omega}$ be an one-one-enumeration of A. Thinning out this sequence in n steps using the wellfoundedness of ω, one can find an infinite subset S of ω such that for each $i \in n$ the sequence $(a_i^k)_{k \in S}$ is strictly increasing or constant. Since the enumeration of A was chosen to be one-one, $\{a^k : k \in S\}$ is an infinite linearly ordered subset of A. A contradiction. $\qquad\square$

Note that this proof works for any other wellordered set instead of ω as well. However, I am not going to use this.

Let me collect some additional facts on $\mathbb{P} : H$.

3.1.33. Lemma. *a) Let H be $\mathbb{Q}$-generic. The* dominating real *added by H is $d := \bigcup\{\tau : \exists g((\emptyset, (\tau, \check{g})) \in H)\}$. For all $(p, (\sigma, \dot{f})) \in \mathbb{P}$, $(p, (\sigma, \dot{f})) \in \mathbb{P} : H$ iff $\sigma \subseteq d$ and for no $n \in \omega$, $p \Vdash \dot{f} \not\leq d \restriction n \setminus \mathrm{dom}(\sigma)$.*

b) Let $(\emptyset, (\tau, \check{g})) \in \mathbb{Q}$ and $(p, (\sigma, \dot{f})) \in \mathbb{P}$. If $\sigma \subseteq \tau$ and

$$p \not\Vdash \dot{f} \restriction \omega \setminus \mathrm{dom}(\sigma) \not\leq \tau \vee \dot{f} \restriction \omega \setminus \mathrm{dom}(\tau) \not\leq \check{g},$$

then $(\emptyset, (\tau, \check{g})) \Vdash_{\mathbb{Q}} (p, (\sigma, \dot{f})) \in \mathbb{P} : \mathbb{Q}$.

If $(\emptyset, (\tau, \check{g})) \Vdash_{\mathbb{Q}} (p, (\sigma, \dot{f})) \in \mathbb{P} : \mathbb{Q}$, then $\sigma \subseteq \tau$.

c) Let H be $\mathbb{Q}$-generic and $(p, (\sigma, \dot{f})) \in \mathbb{P} : H$. Let $\dot{g}$ be an $\mathrm{Fn}(\omega, 2)$-name in the ground model for an element of $^\omega\omega$. Then there is $(q, (\tau, \dot{h})) \in \mathbb{P} : H$ such that $(q, (\tau, \dot{h})) \leq (p, (\sigma, \dot{f}))$ and $\Vdash \dot{g} \leq \dot{h}$.

Proof. For a) let $(p, (\sigma, \dot{f})) \in \mathbb{P} : H$. Suppose $\sigma \not\subseteq d$. Then there is $(\emptyset, (\tau, \check{g})) \in H$ such that $\tau \cup \sigma$ is not a function. Clearly, $(\emptyset, (\tau, \check{g}))$ and $(p, (\sigma, \dot{f}))$ are incompatible in $\mathbb{P}$. A contradiction. Thus $\sigma \subseteq d$. Now suppose for some $n \in \omega \setminus \mathrm{dom}(\sigma)$, $p \Vdash \dot{f} \not\leq d \restriction n \setminus \mathrm{dom}(\sigma)$. Let G be $\mathbb{P}$ generic over the ground model such that $H \subseteq G$ and $(p, (\sigma, \dot{f})) \in G$. G exists by Lemma 3.1.30. By genericity of G, there is $q \in \mathrm{Fn}(\omega, 2)$ such that $q \leq p$, $(q, (\sigma, \dot{f})) \in G$, and for some $m \in n \setminus \mathrm{dom}(\sigma)$, $q \Vdash \dot{f}(m) > d(m)$. There

is $g : \omega \to \omega$ in the ground model such that $(\emptyset, (d \restriction n, \check{g})) \in H$. Now $q \Vdash (\sigma, \dot{f}) \perp_{\dot{\mathbb{D}}} (d \restriction n, \check{g})$ and thus $(q, (\sigma, \dot{f})) \perp_{\mathbb{P}} (\emptyset, (d \restriction n, \check{g}))$. But this is impossible since $(\emptyset, (d \restriction n, \check{g})), (q, (\sigma, \dot{f})) \in G$.

For the other direction of a) let $(p, (\sigma, \dot{f})) \in \mathbb{P}$ be as in the right-hand-side of the statement. Let $(\emptyset, (\tau, \check{g})) \in H$. There is $q \in \mathrm{Fn}(\omega, 2)$ such that $q \leq p$ and $q \Vdash \dot{f} \restriction \mathrm{dom}(\tau \setminus \sigma) \leq \tau$. Let $\max(\dot{f}, \check{g})$ be an $\mathrm{Fn}(\omega, 2)$-name for an element of ${}^\omega \omega$ such that for all $n \in \omega$, $\Vdash \max(\dot{f}, \check{g})(n) = \max(\dot{f}(n), \check{g}(n))$. Since $\tau \subseteq d$, $\sigma \cup \tau$ is a function. Now $(q, (\tau \cup \sigma, \max(\dot{f}, \check{g})))$ is a common extension of $(p, (\sigma, \dot{f}))$ and $(\emptyset, (\tau, \check{g}))$. It follows that $(p, (\sigma, \dot{f}))$ is compatible with all elements of H and therefore $(p, (\sigma, \dot{f})) \in \mathbb{P} : H$.

For b) note that $(\emptyset, (\tau, \check{g}))$ forces that the dominating real added by the $\mathbb{Q}$-generic filter starts with τ and is larger or equal to g on $\omega \setminus \mathrm{dom}(\tau)$. Together with a) this implies the first part of b). The second part is straightforward and uses arguments already given above.

Finally let H, $(p, (\sigma, \dot{f}))$, and $\dot{g}$ be as in c). Let G be a $\mathbb{P}$-generic filter extending H that contains $(p, (\sigma, \dot{f}))$. The set of conditions $(q, (\tau, \dot{h})) \leq (p, (\sigma, \dot{f}))$ such that $\Vdash \dot{g} \leq \dot{h}$ is dense below $(p, (\sigma, \dot{f}))$. Thus G contains such a condition. This condition is compatible with all elements of H and therefore lies in $\mathbb{P} : H$. $\qquad\square$

By Lemma 3.1.31, a $\mathbb{P} : H$-generic filter can be coded by a single real. But it cannot be coded by a Cohen real.

3.1.34. Lemma. *Let H be $\mathbb{Q}$-generic. For no $(p, (\sigma, \dot{f})) \in \mathbb{P} : H$ there is a countable subset of $\mathbb{P} : H$ which is *dense below it. In particular, for no $a \in \mathrm{ro}(\mathbb{P} : H)^+$, $\mathrm{ro}(\mathbb{P} : H) \restriction a$ has a countable dense subset.*

Proof. Assume on the contrary that $(p_n, (\sigma_n, \dot{f}_n))_{n \in \omega}$ enumerates a subset of $\mathbb{P}$ which contains a subset of $\mathbb{P} : H$ that is dense below $(p, (\sigma, \dot{f}))$. Since the formulation here is carefully chosen and $\mathbb{P}$ has c.c.c., I may assume that $(p_n, (\sigma_n, \dot{f}_n))_{n \in \omega}$ is an element of the ground model. By part c) of Lemma 3.1.33, I may also assume that each $\dot{f}_n$ is a name for a new function. Let

$(\emptyset, (\tau, \check{g}))$ be a condition in H such that

$$(\emptyset, (\tau, \check{g})) \Vdash_{\mathbb{Q}} (p, (\sigma, \dot{f})) \in \mathbb{P} : \mathbb{Q} \text{ and } \{(p_n, (\sigma_n, \dot{f}_n)) : n \in \omega\}$$

$$\text{contains a set *dense below } (p, (\sigma, \dot{f})) \text{ in } \mathbb{P} : \mathbb{Q}.$$

By part b) of Lemma 3.1.33, $\tau \supseteq \sigma$. By Lemma 3.1.32, for each $n \in \omega$ there are only finitely many minimal restrictions of possible functions for $\dot{f}_n$ and p_n to $X := (n \cup \mathrm{dom}(\tau)) \setminus \mathrm{dom}(\sigma_n)$. For each such restriction fix an element of $\mathrm{Fn}(\omega, 2)$ below p_n deciding $\dot{f}_n$ on X accordingly and let A_n be the set of the chosen conditions. A_n is a finite antichain in $\mathrm{Fn}(\omega, 2)$. For each $a \in A_n$ let $p_{n,0}^a := a$ and let f_n^a be a possible function for $\dot{f}_n$ and a. Suppose $p_{n,m}^a$ has been constructed for some $m \in \omega$. Let $q_{n,m}^a, p_{n,m+1}^a \leq p_{n,m}^a$ be such that f_n^a is possible for $\dot{f}_n$ and $p_{n,m+1}^a$ and such that $q_{n,m}^a$ and $p_{n,m+1}^a$ decide a larger initial segment of $\dot{f}_n$ than $p_{n,m}^a$ does, but the way $p_{n,m+1}^a$ decides an initial segment of $\dot{f}_n$ is inconsistent with the way in which $q_{n,m}^a$ decides an initial segment of $\dot{f}_n$. This can be done since $\dot{f}_n$ is a name for a new function. Let $f_{n,m}^a$ be possible for $\dot{f}_n$ and $q_{n,m}^a$. Now $(q_{n,m}^a)_{m \in \omega}$ is an antichain below a for each $a \in A_n$.

Note that for all $m \in \omega$ and all $k \geq m$, $f_{n,k}^a \restriction m + 1 = f_n^a \restriction m + 1$. Let $h \in {}^\omega\omega$ be defined as follows:

$$\forall k \in \omega(h(k) := \max(\{f_{n,m}^a(k) : n, m \leq k \wedge a \in A_n\} \cup \{g(k)\}))$$

Clearly, $(\emptyset, (\tau, \check{h})) \leq (\emptyset, (\tau, \check{g}))$.

Claim. $(\emptyset, (\tau, \check{h})) \not\Vdash_{\mathbb{Q}} \{(p_n, (\sigma, \dot{f}_n)) : n \in \omega\}$ contains a subset of $\mathbb{P} : \mathbb{Q}$ which is *dense below $(p, (\sigma, \dot{f}))$.

Proof of the claim: Pick $q \leq p$ such that $q \Vdash \dot{f} \restriction \mathrm{dom}(\tau \setminus \sigma) \leq \tau$. This is possible since $(\emptyset, (\tau, \check{g}))$ and $(p, (\sigma, \dot{f}))$ are compatible. Fix a partition $(X_n^a)_{n \in \omega}$ of ω into infinite pieces. Define a $\mathrm{Fn}(\omega, 2)$-name $\dot{e}$ for a function from ω to ω in such a way that

$$\forall n, m \in \omega \forall a \in A_n \forall k \in X_n^a(q_{n,m}^a \Vdash_{\mathrm{Fn}(\omega,2)} \dot{e}(k) \geq m).$$

By the choice of q, $(q, (\tau, \dot{e})) \leq (p, (\sigma, \dot{f}))$. $(q, (\tau, \dot{e}))$ and $(\emptyset, (\tau, \dot{h}))$ are compatible, so let G be a $\mathbb{P}$-generic generic filter containing both conditions. Let $H' := G \cap \mathbb{Q}$. Let $n \in \omega$ be such that the dominating real d added by H' extends σ_n and $p_n \leq q$. For all n which do not satisfy these conditions, $(p_n, (\sigma_n, \dot{f}_n)) \notin \mathbb{P} : H'$ or $(p_n, (\sigma_n, \dot{f}_n)) \not\leq^* (q, (\tau, \dot{e}))$. Let $a \in A_n$ be such that a forces $\dot{f}_n$ to be below τ on $n \setminus \mathrm{dom}(\sigma_n)$. This is possible if $(p_n, (\sigma_n, \dot{f}_n)) \in \mathbb{P} : H'$. Again, if the latter does not hold, this n is not interesting.

Subclaim. For all $m \in \omega$, $(q^a_{n,m}, (\sigma_n, \dot{f}_n)) \in \mathbb{P} : H'$.

Proof of the subclaim: Let $m \in \omega$. For $k \geq n$, $h(k) \geq f^a_{n,m}(k)$ since $f^a_{n,m}(k) = f^a_n(k)$ for $m \geq k$. By construction, $f^a_{n,m}$ is possible for $\dot{f}_n$ and $q^a_{n,m}$. By choice of a, $f^a_{n,m} \upharpoonright (\mathrm{dom}(\tau) \cup n) \setminus \mathrm{dom}(\sigma_n) \leq d$. By part b) of Lemma 3.1.33,

$$(\emptyset, (\tau, \check{h})) \Vdash_{\mathbb{Q}} (q^a_{n,m}, (\sigma_n, \dot{f}_n)) \in \mathbb{P} : \mathbb{Q}.$$

Now the subclaim follows from $(\emptyset, (\tau, \check{h})) \in \mathbb{P} : H'$.

Pick $k \in X^a_n \setminus \mathrm{dom}(\tau)$. Then $(q^a_{n,d(k)+1}, (\sigma_n, \dot{f}_n)) \leq (p_n, (\sigma_n, \dot{f}_n))$, but $(q^a_{n,d(k)+1}, (\tau, \dot{e})) \notin \mathbb{P} : H'$ since $q^a_{n,d(k)+1} \Vdash \dot{e} \upharpoonright \omega \setminus \mathrm{dom}(\tau) \not\leq d$. Thus $(q^a_{n,d(k)+1}, (\sigma_n, \dot{f}_n)) \perp_{\mathbb{P}:H'} (q, (\tau, \dot{e}))$ and therefore $(p_n, (\sigma_n, \dot{f}_n)) \not\leq^* (q, (\tau, \dot{e}))$. This proves the claim and the claim contradicts the choice of $(\emptyset, (\tau, \check{g}))$. $\square$

Using Lemma 3.1.6 and Lemma 3.1.2, it is now easy to prove

3.1.35. Theorem. *Adding a Hechler real over ω_2 Cohen reals to a model of CH gives a model of $\neg\,\mathrm{WFN}(\mathfrak{P}(\omega))$.*

Proof. For $X \subseteq \omega_2$ let $\mathbb{P}_X := \mathrm{Fn}(X, 2) * \dot{\mathbb{D}}_X$, where

$$\dot{\mathbb{D}}_X := \{(\sigma, \dot{f}) : \sigma \in {}^\omega\omega \text{ and } \dot{f} \text{ is an } \mathrm{Fn}(X, 2)\text{-name for an element of } {}^\omega\omega\}$$

is considered as an $\mathrm{Fn}(X, 2)$-name for Hechler forcing. By the same argument as in Lemma 3.1.28, for $X \subseteq Y \subseteq \omega_2$, $\mathbb{P}_X$ is completely embedded into $\mathbb{P}_Y$. The sequence $(\mathbb{P}_{\omega \cdot \alpha})_{\alpha \leq \omega}$ is increasing and continuous at limits of uncountable cofinality. Each $\mathbb{P}_{\omega \cdot \alpha}$ is of size $\leq \aleph_1$ and satisfies c.c.c. Applying Lemma 3.1.6 together with Lemma 3.1.31 and Lemma 3.1.34, it follows that for each

$\alpha < \omega_2$,

$$\Vdash_{P_{\omega \cdot (\alpha+1)}} \mathfrak{P}(\omega) \cap M[\dot{G} \cap \check{P}_\alpha] \not\leq_\sigma \mathfrak{P}(\omega).$$

Now the theorem follows from Lemma 3.1.2. $\square$

A characterization of Cohen forcing

In this section I consider σ-extensions of $\mathfrak{P}(2^{\aleph_0})$ since this will give a characterization of Cohen forcing.

3.1.36. Theorem. *Let M be a transitive model of* ZFC* *and let A be an atomless complete c.c.c. Boolean algebra in M. Then the following are equivalent:*

(i) *For any A-generic filter G over M,*

$$M[G] \models (\mathfrak{P}(2^{\aleph_0}))^M \leq_\sigma \mathfrak{P}((2^{\aleph_0})^M).$$

(ii) *A is isomorphic to $\mathbb{C}(\omega)$.* $\square$

For the proof of this theorem it is convenient to introduce the cardinal invariant τ of complete Boolean algebras.

3.1.37. Definition. For a complete Boolean algebra A let $\tau(A)$ be the least cardinal λ such that A is completely generated by a subset of size λ. $\square$

The first approximation of the theorem is

3.1.38. Lemma. *Let M be a transitive model of* ZFC* *and let A be an atomless complete c.c.c. Boolean algebra in M. Then the following statements are equivalent:*

(i) *For any A-generic filter G over M,*

$$M[G] \models (\mathfrak{P}(2^{\aleph_0}))^M \leq_\sigma \mathfrak{P}((2^{\aleph_0})^M).$$

(ii) *For any A-generic filter G over M and any $x \in M[G] \setminus M$ such that $x \subseteq (2^{\aleph_0})^M$ there is a Cohen real r over M such that $x \in M[r]$.*

   (iii) Any complete subalgebra B of A with $\tau(B) \leq 2^{\aleph_0}$ has a countable dense subset.

Proof. (i) implies (iii): First note that any subalgebra of A which is completely generated by a set X of size at most $2^{\aleph_0}$ has size at most $2^{\aleph_0}$. This is because the closure of X under countable operations has size at most $2^{\aleph_0}$ and is already complete since A satisfies c.c.c.

   Claim. Let B be a complete subalgebra of A with $\tau(B) \leq 2^{\aleph_0}$. Then B has a dense subset of elements a such that $B \upharpoonright a$ has a countable dense subset.

   First I show how (iii) follows from the claim: Take a maximal antichain K consisting of elements $a \in B$ such that $B \upharpoonright a$ has a countable dense subset. K is countable since B satisfies c.c.c. For each $a \in K$ let D_a be a countable dense subset of $B \upharpoonright a$. Now $\bigcup \{D_a : a \in K\}$ is a countable dense subset of B.

   Proof of the claim: I argue like in the proof of Lemma 3.1.6. Let $\sigma : 2^{\aleph_0} \to B$ be onto. Consider σ as a name for a subset of $2^{\aleph_0}$. Let G be B-generic over M. Since $I := \{x \in (\mathfrak{P}(2^{\aleph_0}))^M : x \subseteq \sigma_G\}$ is countably generated and B satisfies c.c.c., there is a countable set $C \in M$ such that $C \cap I$ is cofinal in I. It is forced by some $a \in G$ that C has this property. For each $x \in C \cap I$ there is some $b_x \in B$ such that $b_x \leq a$ and $b_x \Vdash x \subseteq \sigma$. W.l.o.g. I may assume that for each $x \in C$ there exists $b_x \leq a$ such that $b_x \Vdash x \subseteq \sigma$. I may also assume that $b_x = a \cdot \prod \{\sigma(\alpha) : \alpha \in x\}$ and $x = \{\alpha \in 2^{\aleph_0} : \sigma(\alpha) \geq b_x\}$ hold for all $x \in C$. Now suppose that $\{b_x : x \in C\}$ is not dense below a. Then there is $b \leq a$ such that no element of $\{b_x : x \in C\}$ lies below b. Now $b \Vdash \sigma^{-1}(b) \subseteq \sigma$, but no $x \in C$ includes $\sigma^{-1}(b)$. This contradicts the fact that a forces that $C \cap I$ is cofinal in I. Hence $B \upharpoonright a$ has a countable dense subset.

   It follows that the set D of $a \in B$ such that $B \upharpoonright a$ has countable dense subset is predense in B. But since every relative algebra of a Boolean algebra with a countable dense subset has a countable dense subset as well, D is even dense in B.

   (iii) implies (ii): Let G be A-generic over M an let $x \in M[G] \setminus M$ such that $x \subseteq (2^{\aleph_0})^M$. Let σ be a name for X. By c.c.c., I may assume that σ uses only $2^{\aleph_0}$ conditions. Let B be the complete subalgebra of A that is completely

generated by the conditions used by σ. Let $a \in B$ be the complement of the sum of all atoms in B. Since x is a new subset of $(2^{\aleph_0})^M$, G does not contain an atom of B. Thus B is not atomic and therefore $a \neq 0$. By (iii), the algebra B has a countable dense subset. Hence $B \upharpoonright a$ has a countable dense subset. Since $B \upharpoonright a$ is atomless, $G \cap B \upharpoonright a$ is a Cohen-generic filter which can be coded by a Cohen real $r \in M[G]$. Clearly, one can recover $G \cap B$ from $G \cap B \upharpoonright a$. Thus $x \in M[G \cap B] = M[r]$.

(ii) implies (i): Let G be A-generic over M and let $x \in M[G] \setminus M$ such that $x \subseteq (2^{\aleph_0})^M$. Pick a Cohen real $r \in M[G]$ over M such that $x \in M[r]$. By the same argument as for $\mathfrak{P}(\omega)$ in the proof of Lemma 3.1.4 or in [16], one can see that $I := (\mathfrak{P}(2^{\aleph_0}))^M \upharpoonright x$ is countably generated in $M[r]$. Hence I is countably generated in $M[G]$. $\qquad\square$

Koppelberg noticed that statement (iii) in Lemma 3.1.38 already characterizes $\mathbb{C}(\aleph_0)$. I give a slight generalization of her argument.

3.1.39. Definition. For a complete Boolean algebra A let

$$\sigma_\tau(A) := \{\tau(B) : B \text{ is a complete subalgebra of } A\}$$

be the τ-*spectrum* of A. $\qquad\square$

3.1.40. Lemma. *Let A be a complete Boolean algebra and let κ be an uncountable regular cardinal such that A satisfies the κ-c.c. Suppose there is $\lambda \in \sigma_\tau(A)$ such that $\kappa \leq \lambda$. Then $\kappa \in \sigma_\tau(A)$.* $\qquad\square$

The proof of this lemma uses

3.1.41. Lemma. *(Vladimirov, see [30].) Let A be complete and B a complete subalgebra of A. Assume that for no $b \in B^+$, $B \cap A \upharpoonright b$ is dense in $A \upharpoonright b$. Then there is $a \in A$ such that a is independent over B, i.e. for all $b \in B^+$, $a \cdot b \neq 0$ and $b - a \neq 0$.* $\qquad\square$

Proof of Lemma 3.1.40. By passing from A to a complete subalgebra of A if necessary, I may assume $\lambda := \tau(A) \geq \kappa$. Note that τ is monotone in the sense that $\tau(A \upharpoonright a) \leq \tau(A)$ for every $a \in A^+$. Call $a \in A^+$ τ-*homogeneous* iff

for all $b \in (A \restriction a)^+$, $\tau(A \restriction b) = \tau(A \restriction a)$. Since the cardinals are wellfounded, the set of τ-homogeneous elements of A is dense in A. Let C be a maximal antichain in A consisting of τ-homogeneous elements. By κ-c.c., $|C| < \kappa$. By $\kappa \leq \lambda$ and since κ is regular, there is $a \in C$ such that $\tau(A \restriction a) \geq \kappa$. Define a chain $(B_\alpha)_{\alpha < \kappa}$ of complete subalgebras of $A \restriction a$ as follows:

Let $B_0 := \{0, a\}$. Let $\alpha < \kappa$ and assume for all $\beta < \alpha$, B_β has been defined such that $\tau(B_\beta) < \kappa$. Let B_α' be the complete subalgebra of $A \restriction a$ generated by $\bigcup_{\beta < \alpha} B_\beta$. Since κ is regular, $\tau(B_\alpha') < \kappa$. Now for all $b \in (B_\alpha')^+$, $B_\alpha' \restriction b$ is not dense in $A \restriction b$ since $\tau(A \restriction b) \geq \kappa$ while $\tau(B_\alpha' \restriction b) \leq \tau(B_\alpha') < \kappa$. Therefore Vladimirov's Lemma applies. Let $a_\alpha \in A \restriction a$ be such that a_α is independent over B_α' in $A \restriction a$. Let B_α be the complete subalgebra of $A \restriction a$ generated by B_α' and a_α.

Let $B := \bigcup_{\alpha < \kappa} B_\alpha$. By κ-c.c., B is a complete subalgebra of $A \restriction a$. Since $\tau(B_\alpha) < \kappa$ for every $\alpha < \kappa$, $\tau(B) \leq \kappa$. Since every subset of B of size less than κ is included in some B_α and $B_\alpha \neq B$ for all $\alpha < \kappa$, $\tau(B) = \kappa$. Let B' be the complete subalgebra of A generated by B. Now $\tau(B') = \kappa$ and thus $\kappa \in \sigma_\tau(A)$. $\qquad\square$

Using Lemma 3.1.40, it is now easy to finish the

Proof of Theorem 3.1.36. (ii)$\Rightarrow$(i) follows immediately from (iii)$\Rightarrow$(i) in Lemma 3.1.38.

For (i)$\Rightarrow$(ii) it is sufficient to show that (iii) of Lemma 3.1.38 already characterizes $\mathbb{C}(\omega)$. Let A be a complete c.c.c. Boolean algebra as in (iii) of Lemma 3.1.38. Suppose $\tau(A) > \aleph_0$. Then by c.c.c. and Lemma 3.1.40, A has a complete subalgebra B with $\tau(B) = \aleph_1$. By the properties of A, B has a countable dense subset and therefore $\tau(B) = \aleph_0$. A contradiction. Hence $\tau(A) = \aleph_0$. Again by the properties of A, A itself has a countable dense subset and thus $A \cong \mathbb{C}(\omega)$. $\qquad\square$

3.2 WFN($\mathfrak{P}(\omega)$) and cardinal invariants of the continuum

The reason for studying the question whether certain forcing extensions yield σ-extension or not is to provide an easy way to recognize those models of ZFC in which WFN($\mathfrak{P}(\omega)$) fails. But this only works well for models which have been obtained by adding reals to some model and thereby enlarging the continuum. Another approach is to determine the values of cardinal invariants of the continuum under the assumption WFN($\mathfrak{P}(\omega)$). The arguments here often can be phrased in terms of σ-embeddedness of $\mathfrak{P}(\omega) \cap M$ in $\mathfrak{P}(\omega)$ for some model M of ZFC*. Here M will be either an elementary submodel of H_χ for sufficiently large χ or a transitive class. In order to spare notation, I take the following definition:

3.2.1. Definition. A pair (M, N) is *convenient* iff one of the following holds:

(i) N and M are transitive classes satisfying ZFC* such that $M \subseteq N$, M is a definable class in N, M and N have the same ordinals, and every in N countable set of ordinals is covered by a set in M which is countable in M.

(ii) N is a (possibly class-) model of ZFC* and M is an elementary submodel of H_χ^N for some sufficiently large χ such that $M \cap [M]^{\aleph_0}$ is cofinal in $[M]^{\aleph_0}$. □

Cichoń's diagram: The small cardinals

The first explicit result on the effect of WFN($\mathfrak{P}(\omega)$) on cardinal invariants of the continuum was the result of Fuchino, Koppelberg, and Shelah ([16]) that the unboundedness number $\mathfrak{b}$ is $\aleph_1$ under WFN($\mathfrak{P}(\omega)$). This can also be proved in the following way: Using the argument in the proof of Lemma 1.4.11, it is not difficult to see that if (M, N) is a convenient pair and $(^\omega\omega)^N$ contains a function dominating $(^\omega\omega)^M$, then $N \models \mathfrak{P}(\omega) \cap M \not\leq_\sigma \mathfrak{P}(\omega)$. Now if M is a $V_{\aleph_1}$-like elementary submodel of H_χ for some sufficiently large χ and $\mathfrak{b} > \aleph_1$, then (M, V) is convenient and there is a function $f : \omega \to \omega$

dominating $^\omega\omega \cap M$. Thus $\mathfrak{P}(\omega) \cap M \not\leq_\sigma \mathfrak{P}(\omega)$ and therefore WFN($\mathfrak{P}(\omega)$) fails.

However, one can do better. Soukup proved the following ([14]):

3.2.2. Theorem. *Assume* WFN($\mathfrak{P}(\omega)$). *Let M be a $V_{\aleph_1}$-like elementary submodel of H_χ for some sufficiently large χ. Then $^\omega 2 \cap M$ is not meager. In particular, the minimal cardinality of a non-meager subset of $^\omega 2$ is $\aleph_1$.*

Proof. I show that for every countable family $\mathcal{I}$ of dense ideals of clop($^\omega 2$), $\bigcap\{\bigcup I : I \in \mathcal{I}\}$ intersects M. This implies that $M \cap {}^\omega 2$ is not meager. Let $\mathcal{I}$ be a countable family of dense ideals. By WFN($\mathfrak{P}(\omega)$) and Theorem 1.4.4, $M \cap \mathfrak{P}(\text{clop}(^\omega 2)) \leq_\sigma \mathfrak{P}(\text{clop}(^\omega 2))$. Thus for each $I \in \mathcal{I}$, $(\mathfrak{P}(\text{clop}(^\omega 2)) \cap M) \uparrow I$ has a countable coinitial subset. By $V_{\aleph_1}$-likeness of M, there is a countable family $\mathcal{J} \in M$ of dense open subsets of clop($^\omega 2$) such that for each $I \in \mathcal{I}$ and each $I' \in \mathfrak{P}(\text{clop}(^\omega 2))$ with $I \subseteq I'$, there is $J \in \mathcal{J}$ such that $I \subseteq J \subseteq I'$. Since $\mathcal{J} \in M$ and by Baire's Theorem, $\bigcap\{\bigcup J : J \in \mathcal{J}\} \cap M \neq \emptyset$. Let $x \in M \cap \bigcap\{\bigcup J : J \in \mathcal{J}\}$ and assume $x \notin \bigcap\{\bigcup I : I \in \mathcal{I}\}$. Let $I' := \{a \in \text{clop}(^\omega 2) : x \notin a\}$. Clearly, $I' \in M$. Since $x \notin \bigcap\{\bigcup I : I \in \mathcal{I}\}$, there is $I \in \mathcal{I}$ such that $I \subseteq I'$. By the choice of $\mathcal{J}$, there is $J \in \mathcal{J}$ such that $I \subseteq J \subseteq I'$. This implies $x \notin \bigcap\{\bigcup J : J \in \mathcal{J}\}$, a contradiction. It follows that $\bigcap\{\bigcup J : J \in \mathcal{J}\} \cap M$ is non-empty. $\square$

It follows that WFN($\mathfrak{P}(\omega)$) implies that all cardinal invariants in the left half of Cichoń's diagram are $\aleph_1$. Recall that $^\omega\omega$ is homeomorphic to the space of irrational numbers of the unit interval and the unit interval is homeomorphic to $^\omega 2/\sim$, where $\sim$ identifies every sequence that is eventually, but not everywhere 1 with its successor with respect to the lexicographical order on $^\omega 2$. Looking at these homeomorphisms more closely, it follows that $^\omega\omega$ is homeomorphic to $^\omega 2 \backslash X$ for a countable set X. Since there is a definable homeomorphism proving this, $^\omega 2 \cap M$ is meager iff $^\omega\omega \cap M$ is. Observing that for a function $f : \omega \to \omega$ the set of functions in $^\omega\omega$ which are eventually different from f is meager in $^\omega\omega$ and using Borel codes and the absoluteness of their elementary properties, it turns out that the proof of Theorem 3.2.2 gives the following:

If (M, N) is a convenient pair of models of ZFC^* and $({}^\omega\omega)^N$ contains a function that is eventually different from every function in $({}^\omega\omega)^M$, then $N \models \mathfrak{P}(\omega) \cap M \not\leq_\sigma \mathfrak{P}(\omega)$. This shows that adding an eventually different real yields a non-σ-extension of $\mathfrak{P}(\omega)$.

Cichoń's diagram: The big cardinals

On the other hand, $\mathrm{WFN}(\mathfrak{P}(\omega))$ implies that all cardinal invariants on the right half of Cichoń's diagram are large. My first argument along this line only showed that $\mathrm{WFN}(\mathfrak{P}(\omega))$ implies that the dominating number is large and was derived from the proof of Lemma 3.1.4. The argument used some tree of closed subsets of ${}^\omega 2$. Soukup noticed that this tree could be replaced by a certain family of closed covers of ${}^\omega 2$, simplifying my original proof, and that his argument even gives that the eventually different number, which is just $\mathrm{cov}(\mathcal{M})$, is large under $\mathrm{WFN}(\mathfrak{P}(\omega))$. The dual of an eventually different real is an infinitely equal real.

3.2.3. Definition. Let M be a set or a class. $f \in {}^\omega\omega$ is an *infinitely equal real* over M iff for all $g \in {}^\omega\omega \cap M$, $\{n \in \omega : f(n) = g(n)\}$ is infinite. $\square$

Using this notion, I can state the key lemma for determining $\mathrm{cov}(\mathcal{M})$ under $\mathrm{WFN}(\mathfrak{P}(\omega))$.

3.2.4. Lemma. *Let (M, N) be convenient. Suppose $({}^\omega 2)^N \setminus M$ is non-empty and $N \models \mathfrak{P}(\omega) \cap M \leq_\sigma \mathfrak{P}(\omega)$. Then for every real $x \in N$ there is an infinitely equal real f over M such that $M[x] = M[f]$.* $\square$

Note that one half of '$M[x] = M[f]$' is cheating since it is well-known that an infinitely equal real can code every other real:

3.2.5. Lemma. *Let M be either an elementary submodel of H_χ for some sufficiently large χ or a transitive model of ZFC^*. If there is an infinitely equal real over M, then for every $g \in {}^\omega\omega$ there is an infinitely equal real h over M such that $g \in M[h]$.*

Proof. Let f be infinitely equal over M and $g \in {}^\omega\omega$. For every $n \in \omega$ let $h(2n) := f(n)$ and $h(2n + 1) := g(n)$. Clearly, $g \in M[h]$. It remains to

show that h is infinitely equal over M. Let $e \in {}^\omega\omega \cap M$. For each $n \in \omega$ let $e'(n) := e(2n)$. By the choice of f, e' and f agree on an infinite subset of ω. By the definition of h, for every $n \in \omega$, $e'(n) = f(n)$ iff $e(2n) = h(2n)$. Therefore e and h agree on an infinite set. It follows that h is infinitely equal over M. $\qquad\square$

Proof of Lemma 3.2.4. I argue in N and pretend that M is an elementary submodel of H_χ^N for some sufficiently large χ. But the whole argument can be done using Borel codes instead of subsets of ${}^\omega 2$ as well. Let $x \in {}^\omega 2 \setminus M$. By Lemma 3.2.5, it is sufficient to show that there is an infinitely equal real f over M such that $f \in M[x]$.

Consider $F_x := \{a \in \mathrm{clop}({}^\omega 2) : x \in a\}$. Since $\mathfrak{P}(\omega) \cap M \leq_\sigma \mathfrak{P}(\omega)$, also $\mathfrak{P}(\mathrm{clop}({}^\omega 2)) \cap M \leq_\sigma \mathfrak{P}(\mathrm{clop}({}^\omega 2))$. By convenience, there is $A \in M$ such that $A \subseteq \mathfrak{P}(\mathrm{clop}({}^\omega 2))$ and for all $G \in M$ with $G \subseteq F_x$ there is $F \in A$ such that $G \subseteq F \subseteq F_x$. W.l.o.g. I may assume that A consists of filters. Let $C := \{\bigcap F : F \in A\}$. Then $C \in M$ is a set of closed subsets of ${}^\omega 2$ with the following property:

(∗) Whenever $a \in M$ is a closed subset of ${}^\omega 2$ containing x, then there is $c \in C$ such that $x \in b \subseteq c$.

Since $x \notin M$, I may assume that all members of C are infinite. Let $(c_n)_{n \in \omega}$ be an enumeration of C in M. For each $n \in \omega$ pick a family $(U_n^m)_{m \in \omega}$ of pairwise disjoint open sets intersecting c_n and covering ${}^\omega 2$ except for one point $y \in M$. (In fact, since c_n is closed, $y \in c_n \cap M$.) This is possible by infinity of c_n. Now let $f : \omega \to \omega$ be the function such that for each $n \in \omega$ the point x is contained in $U_n^{f(n)}$. This is possible since $x \notin M$. Clearly, $f \in M[x]$.

Suppose f is not infinitely equal over M. Let $g \in {}^\omega\omega \cap M$ be eventually different from f. Since ${}^\omega\omega \cap M$ is closed under finite changes, I may even assume that g is everywhere different from f. Thus $a := \bigcap_{n \in \omega}({}^\omega 2 \setminus U_n^{g(n)})$ is a closed set in M containing x, but not including any $c \in C$. This contradicts (∗). $\qquad\square$

Lemma 3.2.4 easily gives

3.2.6. Theorem. *Assume* WFN($\mathfrak{P}(\omega)$).

    a) If $0^\sharp$ does not exist, then $\mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}$.

    b) If $\kappa < 2^{\aleph_0}$ is such that $\mathrm{cf}([\kappa]^{\aleph_0}) = \kappa$, *then the* $\mathrm{cov}(\mathcal{M}) > \kappa$. *In particular, if $n \in \omega$ is such that $\aleph_n < 2^{\aleph_0}$, then $\mathrm{cov}(\mathcal{M}) > \aleph_n$.*

Proof. First note that by a result of Bartoszyński ([1]), $\mathrm{cov}(\mathcal{M})$ is the minimal cardinality of a family $E \subseteq {}^\omega\omega$ such that for every function $f : \omega \to \omega$ there is $g \in E$ such that g is eventually different from f. The latter cardinal invariant is the eventually different number. E is called an eventually different family. a) and b) are handled by the same argument. Let $\kappa < 2^{\aleph_0}$ be such that $\mathrm{cf}([\kappa]^{\aleph_0}) = \kappa$. If $0^\sharp$ does not exist, then by Jensens covering lemma, any κ with $\mathrm{cf}\,\kappa > \omega$ has this property. Let χ be a sufficiently large cardinal. For $\alpha < \omega_1$ let M_α be an elementary submodel of H_χ of size κ including κ such that $M_\alpha \cap [\bigcup_{\beta<\alpha} M_\beta]^{\aleph_0}$ is cofinal in $[\bigcup_{\beta<\alpha} M_\beta]^{\aleph_0}$. Let $M := \bigcup_{\alpha<\omega_1} M_\alpha$. Then (M, V) is convenient. By WFN($\mathfrak{P}(\omega)$), $\mathfrak{P}(\omega) \cap M \leq_\sigma \mathfrak{P}(\omega)$ and thus, using Lemma 3.2.4, there is an infinitely equal real over M. Thus $M \cap {}^\omega\omega$ is not an eventually different family. Assume $\mathrm{cov}(\mathcal{M}) \leq \kappa$. By elementarity, M contains an enumeration of an eventually different family. But since κ is a subset of M, M includes an eventually different family. A contradiction. $\square$

    It should be pointed out that the argument Bartoszyński used for showing that the eventually different number equals $\mathrm{cov}(\mathcal{M})$ does not give a direct correspondence between infinitely equal reals and Cohen reals. From a Cohen real one can easily define an infinitely equal real, but according to Blass ([4]), it is an open problem whether forcing notions adding an infinitely equal real also add a Cohen real. However, it is known that if x is infinitely equal over M and y is infinitely equal over $M[x]$, then $M[x][y]$ contains a real that is Cohen over M. There seems to be no simple way to strengthen Lemma 3.2.4 by replacing the infinitely equal real by a Cohen real. But of course, a large value of $\mathrm{cov}(\mathcal{M})$ implies that there are Cohen reals over small sets.

    Modulo the assumption $\neg 0^\sharp$ used in the last theorem, this closes the book on the effect of WFN($\mathfrak{P}(\omega)$) on cardinal invariants in Cichoń's diagram. Fuchino proved that the minimal size $\mathfrak{a}$ of a maximal almost disjoint family of subsets of ω is $\aleph_1$ under WFN($\mathfrak{P}(\omega)$) ([14]). Investigating the various

diagrams in Blass' article ([4]), it turns out that there is one cardinal invariant defined in that paper for which no bounds have been determined here yet, and that is

The groupwise density number $\mathfrak{g}$

3.2.7. Definition. The *standard* topology on $\mathfrak{P}(\omega)$ is the topology $\mathfrak{P}(\omega)$ inherits from ${}^{\omega}2$ when each subset of ω is identified with its characteristic function. A family $\mathcal{G} \subseteq [\omega]^{\aleph_0}$ is called *groupwise dense* if $\mathcal{G}$ is non-meager with respect to the standard topology on $\mathfrak{P}(\omega)$ and closed under taking almost subsets. $\mathfrak{g}$ is the smallest number of groupwise dense families with empty intersection. □

Actually, Blass uses a different definition of groupwise dense families, but he proves that the two definitions are equivalent. He has shown that $\mathfrak{g}$ is $\aleph_1$ in the Cohen model ([5]). Thus it should be $\aleph_1$ under WFN($\mathfrak{P}(\omega)$). And indeed, this is true.

3.2.8. Theorem. WFN($\mathfrak{P}(\omega)$) *implies that the groupwise density number* $\mathfrak{g}$ *is* $\aleph_1$.

Proof. Let M be an $V_{\aleph_1}$-like submodel of H_χ for some sufficiently large χ. Let $x \in [\omega]^{\aleph_0}$. By WFN($\mathfrak{P}(\omega)$), there is a countable set $A \subseteq [\omega]^{\aleph_0}$ in M such that for each $y \in \mathfrak{P}(\omega) \cap M$ with $x \subseteq y$ there is $a \in A$ such that $x \subseteq a \subseteq y$. I may assume that A is closed under finite changes. Let $\mathcal{G}_A := \{z \in [\omega]^{\aleph_0} : \exists c \in \mathfrak{P}(\omega) \cap M(z \subseteq^* c \wedge \forall a \in A(a \not\subseteq^* c))\}$. Obviously, $\mathcal{G}_A$ is closed under taking almost subsets. $\mathcal{G}_A$ does not contain x by the choice of A. From Theorem 3.2.2 it follows that $[\omega]^\omega \cap M$ is non-meager. For each $a \in A$ let $F_a := \{b \subseteq \omega : a \subseteq b\}$. Each F_a is closed and nowhere dense by infinity of a. Thus $C := ([\omega]^{\aleph_0} \cap M) \setminus \bigcup_{a \in A} F_a$ is non-meager. Since A is closed under finite changes, $C \subseteq \mathcal{G}_A$. Hence $\mathcal{G}_A$ is groupwise dense. Now $\bigcap_{A \in [[\omega]^{\aleph_0}]^{\aleph_0} \cap M} \mathcal{G}_A = \emptyset$ and thus $\mathfrak{g} = \aleph_1$. □

Assuming $\neg 0^\sharp$, WFN($\mathfrak{P}(\omega)$) therefore implies that the values of all cardinal invariants of the continuum considered in [4] are the precisely as in

the Cohen model, that is, $2^{\aleph_0}$ for all invariants $\geq \mathrm{cov}(\mathcal{M})$ and $\aleph_1$ for all invariants below $\mathrm{non}(\mathcal{M})$, $\mathfrak{a}$, or $\mathfrak{g}$.

3.3   More complete Boolean algebras with the WFN

This section contains some results which show that at least assuming $\neg 0^\sharp$, $\mathrm{WFN}(\mathfrak{P}(\omega))$ implies the WFN of several complete c.c.c. Boolean algebras, among them the measure algebras. However, Soukup ([14]) has shown that if the existence of a supercompact cardinal is consistent with ZFC, then the existence of a complete c.c.c. Boolean algebra without the WFN is consistent with ZFC+GCH. Moreover, he proved that adding $\aleph_2$ Cohen reals to a model of CH gives a model where there is a complete c.c.c. Boolean algebra of size $2^{\aleph_0}$ without the WFN while $\mathrm{WFN}(\mathfrak{P}(\omega))$ holds. Lemma 3.3.2 below gives that in that model there is even a countably generated complete c.c.c. Boolean algebra without the WFN. These examples show that it is not possible to extend the results of this section very far.

The measure algebra of the reals

To commence, I show that $\mathrm{WFN}(\mathfrak{P}(\omega))$ implies $\mathrm{WFN}(\mathbb{R}(\omega))$.

3.3.1. Lemma. *The measure algebra $\mathbb{R}(\omega)$ is an order retract of $\mathfrak{P}(\omega)$. In particular, if $\mathrm{WFN}(\mathfrak{P}(\omega))$ holds, then so does $\mathrm{WFN}(\mathbb{R}(\omega))$.*

Proof. By Corollary 1.4.9, it is sufficient to construct an order embedding e from $\mathbb{R}(\omega)$ into $\mathfrak{P}(\omega)$.

In order to construct e it is convenient to replace $\mathfrak{P}(\omega)$ by the isomorphic algebra $\mathfrak{P}(\mathrm{clop}(^\omega 2) \times \omega)$. As usual, I identify $\mathrm{clop}(^\omega 2)$ with a subalgebra of $\mathbb{R}(\omega)$ in the obvious way. For $a \in \mathbb{R}(\omega)$ let

$$e(a) := \{(c, n) \in \mathrm{clop}(^\omega 2) \times \omega : \mu(c - a) < \frac{1}{2^n}\}.$$

It is clear that e is monotone. Let $a, b \in \mathbb{R}(\omega)$ such that $a \not\leq b$. Let $n \in \omega$ be such that $\frac{1}{2^n} < \mu(a - b)$. There is a clopen set $c \subseteq {}^\omega 2$ such that

$\mu((a-c)+(c-a)) < \frac{1}{2^{n+1}}$. In particular $(c, n+1) \in e(a)$. But $\mu(c-b) > \frac{1}{2^{n+1}}$ and thus $(c, n+1) \notin e(b)$. Therefore $e(a) \nsubseteq e(b)$. This shows that e is an order embedding. $\square$

Getting the WFN from the WFN of small complete subalgebras

To extend the last result to larger measure algebras, I need the following theorem which is already interesting at its own. The argument in the proof of the $\neg 0^\sharp$-case is basically the same as an argument used by Fuchino and Soukup in an older, unpublished version of [19] that was kindly explained to me by Soukup. However, the theorem stated here does not seem to follow easily from their results.

3.3.2. Theorem. *Let A be a complete c.c.c. Boolean algebra.*

a) If A is completely generated by a set of less than $\aleph_\omega$ generators, then A has the WFN iff every countably generated complete subalgebra B of A does.

b) Assume $0^\sharp$ does not exist. Then A has the WFN iff every countably generated complete subalgebra B of A does.

The proof of the $\neg 0^\sharp$-part of the theorem uses

3.3.3. Lemma. *Let μ be a singular cardinal of cofinality κ with $\mathrm{cf}([\mu]^\kappa) = \mu^+$. Let X be a set of size μ. Assume $\square_\mu$ holds. Then there is a matrix $(X_{\alpha,\nu})_{\alpha<\mu^+,\nu<\kappa}$ of subsets of X s.t.*

(i) *$(X_{\alpha,\nu})_{\nu<\kappa}$ is increasing for all $\alpha < \mu^+$;*

(ii) *$|X_{\alpha,\nu}| < \mu$ for all $\alpha < \mu^+$ and all $\nu < \kappa$;*

(iii) *For $\alpha < \mu^+$ let $\mathcal{X}_\alpha := \bigcup_{\nu<\kappa}[X_{\alpha,\nu}]^{\leq\kappa}$. Then $(\mathcal{X}_\alpha)_{\alpha<\mu^+}$ is increasing and continuous at limit ordinals with cofinality $> \kappa$;*

(iv) *Every $Y \in [X]^\kappa$ is included in some $X_{\alpha,\nu}$.*

Proof. Let $\{Y_\alpha : \alpha < \mu^+\}$ be a cofinal subset of $[X]^\kappa$. Let *lim* be the class of limit ordinals. By $\square_\mu$, there is a sequence $(\mathcal{C}_\alpha)_{\alpha<\mu^+,\alpha\in lim}$ such that the following hold for all limit ordinals $\alpha < \mu^+$:

(1) $\mathcal{C}_\alpha$ is club in α,

(2) $\text{otp}(\mathcal{C}_\alpha) < \mu$,

(3) If $\beta < \alpha$ is a limit point of $\mathcal{C}_\alpha$, then $\mathcal{C}_\beta = \beta \cap \mathcal{C}_\alpha$.

Note that (2) usually reads '$\text{cf}(\alpha) < \mu \Rightarrow \text{otp}(\mathcal{C}_\alpha) < \mu$', but this is not necessary here, since μ is singular. Fix an increasing cofinal sequence $(\mu_\nu)_{\nu<\kappa}$ of regular cardinals larger than κ in μ. Define $(X_{\alpha,\nu})_{\alpha<\mu^+,\nu<\kappa}$ as follows:

For $\nu < \kappa$ let $X_{0,\nu} := \emptyset$. For $\alpha = \beta + 1 < \mu^+$ and $\nu < \kappa$ let $X_{\alpha,\nu} := X_{\beta,\nu} \cup Y_\beta$. For a limit ordinal $\alpha < \mu^+$, $\nu < \kappa$ let $X_{\alpha,\nu} := \emptyset$ if $\mu_\nu < |\mathcal{C}_\alpha|$ and $X_{\alpha,\nu} := \bigcup_{\beta\in\mathcal{C}_\alpha} X_{\beta,\nu}$ if $\mu_\nu \geq |\mathcal{C}_\alpha|$.

It is clear from the construction that the matrix $(X_{\alpha,\nu})_{\alpha<\mu^+,\nu<\kappa}$ satisfies (iv).

Claim 1. $|X_{\alpha,\nu}| \leq \mu_\nu$ for all $\alpha < \mu^+$ and $\nu < \kappa$.

The proof proceeds by induction on α. For $\alpha = 0$ the statement is true since $X_{0,\nu}$ is empty. Let $\alpha = \beta + 1$. By the inductive hypothesis, $|X_{\beta,\nu}| \leq \mu_\nu$. By construction, $X_{\alpha,\nu} = X_{\beta,\nu} \cup Y_\beta$ and $|Y_\beta| = \kappa$. Since μ_ν was chosen to be larger than κ, it follows that $|X_{\alpha,\nu}| \leq \mu_\nu$. Finally let α be a limit ordinal. If $|\mathcal{C}_\alpha| > \mu_\nu$, then $X_{\alpha,\nu}$ is empty. If $|\mathcal{C}_\alpha| \leq \mu_\nu$, then $X_{\alpha,\nu} = \bigcup_{\beta\in\mathcal{C}_\alpha} X_{\beta,\nu}$ and thus, by the inductive hypothesis, $|X_{\alpha,\nu}| \leq \mu_\nu$.

This claim immediately gives (ii). (i) is easily seen by induction on α. In order to show (iii), I need

Claim 2. For $\alpha \leq \beta < \mu^+$ and $\nu < \kappa$ there is $\rho \in [\nu, \kappa)$ such that $X_{\alpha,\rho} \subseteq X_{\beta,\rho}$.

The proof proceeds by induction on β, parallel for all ν. For $\alpha = \beta$ there is nothing to show. Suppose $\beta > \alpha$ and $\beta = \gamma + 1$. By the inductive hypothesis, there is $\rho \in [\nu, \kappa)$ such that $X_{\alpha,\rho} \subseteq X_{\gamma,\rho}$. By construction, $X_{\gamma,\rho} \subseteq X_{\beta,\rho}$. Now suppose β is a limit ordinal and $\beta > \alpha$. Pick $\gamma \in \mathcal{C}_\beta$ such that $\alpha \leq \gamma$. By the inductive hypothesis, there is $\rho \in [\nu, \kappa)$ such that $X_{\alpha,\rho} \subseteq X_{\gamma,\rho}$ and $|\mathcal{C}_\gamma| \leq \mu_\rho$. By construction, $X_{\beta,\rho} := \bigcup_{\delta\in\mathcal{C}_\beta} X_{\delta,\rho}$. Thus $X_{\gamma,\rho} \subseteq X_{\beta,\rho}$.

Now let $(\mathcal{X}_\alpha)_{\alpha<\mu^+}$ be defined as in (iii). Suppose $\alpha \leq \beta < \mu^+$ and $Y \in \mathcal{X}_\alpha$. Pick $\nu < \kappa$ with $Y \subseteq X_{\alpha,\nu}$. By Claim 2, there is $\rho \in [\nu, \kappa)$ such that $X_{\alpha,\rho} \subseteq X_{\beta,\rho}$. By (i), $Y \subseteq X_{\alpha,\rho} \subseteq X_{\beta,\rho}$ and thus $Y \in \mathcal{X}_\beta$. This shows that $(\mathcal{X}_\alpha)_{\alpha<\mu^+}$ is increasing.

Suppose $\alpha < \mu^+$ is a limit ordinal of cofinality $> \kappa$ and $Y \in \mathcal{X}_\alpha$. Fix $\nu < \kappa$ such that $Y \subseteq X_{\alpha,\nu}$. Since $X_{\alpha,\nu}$ is nonempty, $X_{\alpha,\nu} = \bigcup_{\beta \in \mathcal{C}_\alpha} X_{\beta,\nu}$ and $|\mathcal{C}_\alpha| \le \mu_\nu$. Since $\mathrm{cf}(\alpha) > \kappa$, there is a limit $\beta < \alpha$ of $\mathcal{C}_\alpha$ such that $Y \subseteq \bigcup_{\gamma \in \mathcal{C}_\alpha \cap \beta} X_{\gamma,\nu}$. Now $\mathcal{C}_\beta = \mathcal{C}_\alpha \cap \beta$ and $|\mathcal{C}_\beta| \le |\mathcal{C}_\alpha| \le \mu_\nu$. Therefore $Y \subseteq \bigcup_{\gamma \in \mathcal{C}_\beta} X_{\gamma,\nu} = X_{\beta,\nu}$. Hence $Y \in \mathcal{X}_\beta$. This shows that $(\mathcal{X}_\alpha)_{\alpha < \mu^+}$ is continuous at limit ordinals of cofinality $> \kappa$ and thus establishes (iii). $\qquad\square$

Proof of the theorem. The proof of part b) does not use $\neg 0^\sharp$ unless A is not completely generated by a subset of size less than $\aleph_\omega$. Therefore a) will follow from the proof of b). Every complete subalgebra of A is a retract of A and thus has the WFN if A does. This shows the easy direction of a) and b).

The proof of the other direction proceeds by induction on the size of a set completely generating A. If A is countably generated, then there is nothing to prove. Let A be completely generated by a subset $X = \{a_\alpha : \alpha < \mu\}$ for some uncountable cardinal μ and assume that for each subset Y of X of size less than μ the subalgebra A_Y of A completely generated by Y has the WFN. If $\mathrm{cf}\,\mu > \omega$, then by c.c.c., $A = \bigcup_{\alpha < \mu} A_{\{a_\beta : \beta < \alpha\}}$. Every $A_{\{a_\beta : \beta < \alpha\}}$ is a σ-subalgebra of A and $\mathrm{WFN}(A_{\{a_\beta : \beta < \alpha\}})$ holds by the inductive hypothesis. This implies $\mathrm{WFN}(A)$.

Now assume $\mathrm{cf}\,\mu = \aleph_0$. By $\neg 0^\sharp$ and Jensen's Covering Lemma, $\mathrm{cf}([\mu]^{\aleph_0}) = \mu^+$ and $\square_\mu$ holds. (See [10] for these things.) So let $(X_{\alpha,\nu})_{\alpha < \mu^+, \nu < \omega}$ be the matrix of subsets of X guaranteed by Lemma 3.3.3. For all $\alpha < \mu^+$ and $\nu < \omega$ let $A^{\alpha,\nu} := A_{X_{\alpha,\nu}}$. For each $\alpha < \mu^+$ let $A^\alpha := \bigcup_{\nu < \omega} A^{\alpha,\nu}$. By property (i) of the matrix, A^α is a subalgebra of A. Note that A^α is even a σ-subalgebra of A, because it is a countable union of complete subalgebras. By property (ii) of the matrix together with the inductive hypothesis, $\mathrm{WFN}(A^{\alpha,\nu})$ holds for all α and ν. Thus for every α, $\mathrm{WFN}(A^\alpha)$ holds. By c.c.c., there is a function $\mathrm{supp} : A \to [X]^{\aleph_0}$ such that for all $a \in A$, $a \in A_{\mathrm{supp}(a)}$. Since $\mathrm{supp}(a)$ is included in some $X_{\alpha,\nu}$ for each $a \in A$, $A = \bigcup_{\alpha \le \mu^+} A^\alpha$. By property (iii) of the matrix, $(A^\alpha)_{\alpha < \mu}$ is increasing and continuous at limit ordinals of cofinality $> \aleph_0$. This implies $\mathrm{WFN}(A)$. $\qquad\square$

The larger measure algebras

Under $\neg 0^\sharp$ the last theorem together with Lemma 3.3.1 and Maharam's Theorem will give complete information on the WFN of measure algebras. Note that it was already proved in [19] that under $\neg 0^\sharp$ every Cohen algebra has the WFN iff WFN($\mathfrak{P}(\omega)$) holds. This also immediately follows from the last theorem since every countably generated complete subalgebra of a Cohen algebra is a complete subalgebra of a countably generated Cohen algebra.

3.3.4. Definition. A *measure algebra* is a complete Boolean algebra A together with a function $\mu : A \rightarrow [0,1]$ such that

(i) forall $A \in A$, $\mu(a) = 0$ iff $a = 0$ and

(ii) for every countable antichain $C \subseteq A$, $\mu(\sum C) = \sum\{\mu(a) : a \in C\}$. $\square$

A Boolean algebra A is called *measurable* iff there is a function $\mu : A \rightarrow [0,1]$ such that (A, μ) is a measure algebra. By the usual abuse of notation, I will write only 'measure algebra' when I mean 'measurable algebra'.

Note that the measure algebras in the definition above are frequently called totally finite measure algebras. For measure algebras usually μ is not assumed to be bounded. However, since I will consider only totally finite measure algebras, I call them just measure algebras.

3.3.5. Corollary. *Let A be an infinite measure algebra.*

a) If A is completely generated by strictly less than $\aleph_\omega$ generators, then WFN(A) *holds iff* WFN($\mathfrak{P}(\omega)$) *does.*

b) If $0^\sharp$ does not exist, then WFN(A) *holds iff* WFN($\mathfrak{P}(\omega)$) *does.*

Proof. The 'only if'-part of a) and b) follows from the fact that $\mathfrak{P}(\omega)$ is a retract of every infinity complete Boolean algebra. The proof of the 'if'-part is the almost same for a) and b), too. The only difference is that for a) part a) of Theorem 3.3.2 is used, and for b) part b) of Theorem 3.3.2 is used. So let B be a countably generated complete subalgebra of A. The restriction of the measure on A to B is a measure on B. By Maharam's Theorem, there are $\nu \leq \omega$ and a sequence $(B_n)_{n<\nu}$ of measure algebras such that $B \cong \prod_{n<\nu} B_n$,

where B_n is isomorphic to the measure algebra $\mathbb{R}(\omega)$ or trivial, i.e. $= \{0,1\}$. it follows that B trivial or isomorphic to a product $C \times D$ where C is either trivial or isomorphic to $\mathbb{R}(\omega)$ and D is the powerset of an at most countable set. Assume $\mathrm{WFN}(\mathfrak{P}(\omega))$. By Lemma 3.3.1, C has the WFN. Obviously, D has the WFN. It follows that B has the WFN. Now $\mathrm{WFN}(A)$ follows from Theorem 3.3.2 $\qquad\square$